Existence of Nonclassical Solutions in a Pedestrian Flow Model

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Abstract

The main result of this note is the existence of nonclassical solutions to the Cauchy problem for a scalar conservation law modeling pedestrian flow. From the physical point of view, the main assumption of this model was recently experimentally confirmed in [15]. From the analytical point of view, this model is an example of a conservation law in which nonclassical solutions have a physical motivation and a global existence result for the Cauchy problem with large data is available.

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1 Introduction

This paper is concerned with existence and qualitative properties of nonclassical solutions of a Cauchy problem for a scalar conservation law motivated by pedestrian flow. Consider the equation for the conservation of mass

\[ \partial_t \rho + \partial_x q(\rho) = 0, \quad (1.1) \]

where \( \rho = \rho(t, x) \) denotes the pedestrian density at time \( t \in [0, +\infty) \) and at point \( x \in \mathbb{R} \). The function \( q \) quantifies the flow of pedestrian. Recall that \( q = \rho v \), \( v \) being the pedestrians’ speed. In the case of car traffic, assuming that \( v = v(\rho) \) leads to the usual Lighthill–Whitham and Richards (LWR) model. Here, we too postulate that \( v \) is a function of \( \rho \), but with two key differences with respect to the LWR model: the speed law is qualitatively different and, more important, we assign to (1.1) nonclassical solutions, so that the usual standard theory of conservation laws does not apply.
The particular nonclassical solution here chosen are justified by their providing a description of the dynamics of crowds. Indeed, pedestrian models are currently under intense investigation in the specialized literature, see [14, 15, 16, 17, 18, 19, 22, 23, 24] and the references therein. Often, these models are of a microscopic nature, i.e. they postulate rules for the individual behavior and then consider many individuals, as in [14, 16, 17, 22, 23, 24].

On the contrary, here we study a continuum, or macroscopic, model, as for example in [11, 18, 19]. The use of continuum models in the context of pedestrian flows is not justified, a priori, by the number of individuals, obviously far lower than the typical number of molecules in fluid dynamics. However, the availability of reliable continuum models allows to state and possibly solve optimal management problems.

Standard solutions to (1.1) enjoy the “Maximum Principle”, i.e. if the initial data $\rho$ satisfies the bounds $\rho(x) \in [\rho_{\min}, \rho_{\max}]$ for all $x \in \mathbb{R}$, then the corresponding solution satisfies the same bounds, i.e. $\rho(t, x) \in [\rho_{\min}, \rho_{\max}]$ for all $t \geq 0$ and $x \in \mathbb{R}$. This property holds also in the multidimensional case, see [13, Theorem 6.2.2] or [20, Chapter IV, Theorem 2.1(a)]. The maximum principle prevents any increase in the initial maximal density, contrary to crowd dynamics, where a sort of “overcompression” is a well known phenomenon often causing major incidents.

If a boundary constraint is present, then the maximum principle may fail. Nevertheless, a unilateral constraint may not describe the well known phenomenon of the fall in a door outflow typical of panic situations, see [8] for a study of unilateral constraint for conservation laws.

In [11], the overcompressed densities $[R, R^*]$ are introduced next to the standard ones $[0, R]$. Under usual circumstances, $\rho$ varies in $[0, R]$, while the rise of panic forces $\rho$ to enter $[R, R^*]$. To describe the rise of panic, suitable nonclassical solutions to (1.1) are selected in [11], where the corresponding Riemann problem is completely solved under mild assumptions on the flow function $q$. The Riemann solver so defined yields solutions that do not satisfy to the Maximum Principle. A key role in this framework is played by the fundamental diagram, i.e. the curve $q$ vs. $\rho$. The one postulated in [11] has been recently experimentally confirmed in [15].

Remark that the present analytical structure does not fit in the results on nonclassical solutions to conservation laws in [20, Chapter III, § 4]. Indeed, the flow function here considered needs not be polynomial. Moreover, it can be qualitatively approximated only through 4th order polynomials, while the theory in [20] applies to 3rd order polynomials.

In the next section we state the assumptions on $q$ and define the Riemann solver for (1.1). Here, we consider a class of flow functions slightly smaller than that in [11] in order to reduce the number of cases to be considered, while always comprising the qualitative behavior experimentally observed in [15]. Section 3 is devoted to the the existence of nonclassical solutions to the Cauchy problem for (1.1) generated by this Riemann solver. Technical
proofs are deferred to the final Section 4.

2 The Model and the Riemann Problem

The following properties of $q$ are assumed, see Figure 1. First, the Lipschitzeanity of $q$ is a minimal regularity requirement:

(Q.1) $q \in W^{1,\infty}([0, R^*]; [0, +\infty])$.

The flow vanishes if and only if the density is either zero or maximal:

(Q.2) $q(\rho) = 0$ if and only if $\rho \in \{0, R^*\}$.

Concavity is a standard technical assumption that avoids mixed waves:

(Q.3) The restrictions $q|_{[0,R]}$ and $q|_{[R,R^*]}$ are strictly concave.

Hence, there exists a unique $R_M \in [0, R]$ and a unique $R_M^* \in [R, R^*]$ such that $q(R_M) = \max \{q(\rho): \rho \in [0, R]\}$ and $q(R_M^*) = \max \{q(\rho): \rho \in [R, R^*]\}$. Figure 1: The flow function $q$ and notations.

The following three conditions were not assumed in [11] and they are not strictly necessary. Nevertheless, they reduce the amount of technicalities and have a clear physical meaning. First, the maximal flow in standard situations exceeds that in panic. Besides, when entering the panic states, there is a small increase in the flow.

(Q.4) $\max \{q(\rho): \rho \in [0, R]\} > \max \{q(\rho): \rho \in [R, R^*]\}$.

(Q.5) $q$ has a local minimum at $R$.

(Q.6) $q(R) < \min \{q'(R^+)R, -q'(R^-)(R^* - R)\}$.

The latter condition corresponds to the fact that the flow $q(R)$, i.e. the flow at the standard maximal density, is very small. Furthermore, by (Q.6), the line through the origin and $(R, q(R))$ intersects $q = q(\rho)$ at a point $(R_4, q(R_4))$ with $R_4 \in [R, R^*]$, while the line through $(R^*, 0)$ and $(R, q(R))$ intersects $q = q(\rho)$ at a point $(R_1, q(R_1))$ with $R_1 \in [0, R]$, see Figure 1, right. We concentrate our attention on the cases in which

(Q.7) $R_4 \in [R_T^*, R^*]$ and $R_1 \in [0, R_T]$.

It is of use to further introduce the auxiliary functions $\Psi$ and $\Phi$. First, see Figure 2, let $\Psi(R) = R$ and, for $\rho \neq R$, let $\Psi(\rho)$ be such that the
straight line through \((\rho, q(\rho))\) and \(\left(\Psi(\rho), q(\Psi(\rho))\right)\) is tangent to the graph of \(q\) at \(\left(\Psi(\rho), q(\Psi(\rho))\right)\). By (Q.6), \(\Psi\) is well defined and \(\Psi(\rho) \neq R\) for all \(\rho \in [0, R^*] \setminus \{R\}\). We assume also that there exists only one couple \((R_T, R_T^*) \in ]R_M, R[ \times ]R_M^*, R^*[\) such that \(\Psi(R_T) = R_T^*\) and \(\Psi(R_T^*) = R_T\), see Figure 1, i.e.

\[(Q.8)\quad q(\Psi(R^*)) < -q'(R^*) (R^* - \Psi(R^*)).\]

Then, the line through \((R_T, q(R_T))\) and \((R_T^*, q(R_T^*))\) is the unique tangent to \(q = q(\rho)\) in two (distinct) points. These assumptions imply that \(\Psi\) is increasing in \([0, R_T]\) \cup \([R^*_T, R^*]\) and decreasing in \([R_T, R^*_T]\) while \(\Psi'(R_T) = \Psi'(R_T^*) = 0\), see Figure 2, right. Moreover, \(R_T < \Psi(R^*) < R < \Psi(0) < R^*_T\).

Finally, assume also that

\[(Q.9)\quad \rho \mapsto \Psi(\rho) - \rho\] is non-increasing in \([0, R]\).

Let \(\rho \in [0, R_T]\), then by (Q.6) the line through \((\tilde{\rho}, q(\tilde{\rho}))\) and \((\Psi(\tilde{\rho}), q(\Psi(\tilde{\rho}))\) has a further intersection with the graph of \(q\), which we call \((\Phi(\tilde{\rho}), q(\Phi(\tilde{\rho}))\).

Following [11], we now define nonclassical solutions to Riemann problems for (1.1). As usual, for any pair \((\rho', \rho'') \in [0, R^*]^2\), we denote by \(R(\rho', \rho'')\) the self similar weak solution to the Riemann problem

\[
\begin{cases}
\partial_t \rho + \partial_x q(\rho) = 0 \\
\rho(0, x) = \begin{cases}
\rho^l & \text{for } x < 0 \\
\rho^r & \text{for } x > 0
\end{cases}
\end{cases}
\] (2.1)

computed at time, say, \(t = 1\). Introduce two thresholds \(s\) and \(\Delta s\) such that

\[s > 0\quad \Delta s > 0, \quad s < R_M\quad \text{and}\quad R > s + \Delta s \geq \Phi(s) > R_T > R - \Delta s.\] (2.2)

Sufficient conditions for (2.2) to be satisfied are in Lemma 4.1. The solution to Riemann problems with data in \([0, R^*]\) are selected through the following conditions, see Figure 3:
Figure 3: Left: The Riemann solver selected by (R.1)–(R.4). Here, \( N\mathcal{R}^\perp \) indicates that \( \mathcal{R}(\rho^l, \rho^r) \) is a nonclassical shock followed by a decreasing rarefaction. Right: The weighted total variation TV\(_w\). The notation 1\( W \) means that the first wave has weight 1 and the second wave has weight \( W \).

(R.1) If \( \rho^l, \rho^r \in [0, R] \), then \( \mathcal{R}(\rho^l, \rho^r) \) selects the classical solution unless

\[
\rho^l > s \quad \text{and} \quad \rho^r - \rho^l > \Delta s. 
\]

In this case, \( \mathcal{R}(\rho^l, \rho^r) \) consists of a nonclassical shock between \( \rho^l \) and \( \Psi(\rho^l) \), followed by the classical solution between \( \Psi(\rho^l) \) and \( \rho^r \).

(R.2) If \( \rho^r < \rho^l \), then \( \mathcal{R}(\rho^l, \rho^r) \) is the classical solution.

(R.3) If \( R \leq \rho^l < \rho^r \) or \( \rho^l < R < \rho^r \) and the segment between \((\rho^l, q(\rho^l))\) and \((\rho^r, q(\rho^r))\) does not intersect \( q = q(\rho) \), then the solution is a shock between \( \rho^l \) and \( \rho^r \).

(R.4) If \( \rho^l < R < \rho^r \) and the segment between \((\rho^l, q(\rho^l))\) and \((\rho^r, q(\rho^r))\) intersects \( q = q(\rho) \), then \( \mathcal{R}(\rho^l, \rho^r) \) consists of a nonclassical shock between \( \rho^l \) and a panic state followed by a possibly null classical wave. More precisely,

\[
\rho^r \in ]R, \Psi(\rho^l)[ : \mathcal{R}(\rho^l, \rho^r) \text{ consists of a nonclassical shock between } \rho^l \\
\quad \text{and } \Psi(\rho^l), \text{ followed by a decreasing rarefaction between } \Psi(\rho^l) \\
\quad \text{and } \rho^r; \\
\rho^r \in [\Psi(\rho^l), R^*[, \mathcal{R}(\rho^l, \rho^r) \text{ consists of a single nonclassical shock}. 
\]

For the sake of completeness, we recall here the main result of [11, Theorem 2.1] concerning the solution to Riemann problems. To state it, the
following subsets of the square $[0, R^*]^2$ are of use, see Figure 3:

\[ \mathcal{C}_N = \left\{ (\rho_l, \rho_r) \in [0, R^*]^2 : \rho_l \geq \rho_r \geq R \right\}, \]
\[ \mathcal{N}_C = \left\{ (\rho_l, \rho_r) \in [0, R]^2 : \rho_l > s \text{ and } \rho_r - \rho_l > \Delta s \right\}, \]
\[ \mathcal{C} = \left( [0, R^*] \times [0, R] \right) \cup \mathcal{C}_N \setminus \mathcal{N}_C, \]
\[ \mathcal{N} = \left( [0, R^*] \times [R, R^*] \right) \cup \mathcal{N}_C \setminus \mathcal{C}_N. \]

**Theorem 2.1** Let $q : [0, R^*] \to [0, +\infty]$ satisfy Assumptions (Q.1)–(Q.8). Choose thresholds $s$ and $\Delta s$ such that (2.2) holds. Then, there exists a unique Riemann solver $\mathcal{R} : [0, R^*]^2 \to BV(R)$ satisfying (R1)–(R4) and such that $\rho(t, x) = \left( \mathcal{R}(\rho^l, \rho^r) \right) (x/t)$ is a weak solution to (2.1). Moreover,

- $\mathcal{R}$ is consistent in $\mathcal{C}$ and separately, in $\mathcal{N}$,
- $\mathcal{R}$ is $L^1_{\text{loc}}$-continuous in $\mathcal{C}$, in $\mathcal{N}$ and also along the segment $\rho^l = \rho^r$ for $\rho^l \in [R, R^*]$.

Recall that a Riemann Solver $\tilde{\mathcal{R}}$ is consistent if the following two conditions hold:

(C1) $\tilde{\mathcal{R}}(u^l, u^m)(\bar{x}) = u^m \Rightarrow \tilde{\mathcal{R}}(u^l, u^r)(\bar{x}) = u^m$

(C2) $\tilde{\mathcal{R}}(u^l, u^r)(\bar{x}) = u^m \Rightarrow \left\{ \begin{array}{ll} \tilde{\mathcal{R}}(u^l, u^m) = u^m & \text{if } x \leq \bar{x} \\ \tilde{\mathcal{R}}(u^r, u^r) = u^m & \text{if } x > \bar{x} \end{array} \right.$

Both these properties are enjoyed by the standard Lax solver. Moreover, if the RS $\tilde{\mathcal{R}}$ generates a SRS, then $\tilde{\mathcal{R}}$ needs to satisfy (C1). Essentially,

![Figure 4: Consistency of a Riemann solver.](image)

(C1) states that whenever two solutions to two Riemann problems can be placed side by side, then their juxtaposition is again a solution to a Riemann problem, see Figure 4. (C2) is the viceversa.

We recall that in [11, Lemma 4.1, Lemma 4.2] it is also proved that both regularity properties are lost on the boundary separating $\mathcal{C}$ and $\mathcal{N}$. 
3 The Cauchy Problem

This section is concerned with the Cauchy problem for the equation (1.1). The availability of a Riemann solver allows to tackle the Cauchy problem through wave front tracking, see [4, 6].

Let \( \rho \in L^1 \cap BV (\mathbb{R}; [0, R^*]) \) and consider the Cauchy problem associated to the equation (1.1) with initial condition

\[
\rho(0, x) = \bar{\rho}(x) \quad x \in \mathbb{R}.
\]

Recall that a function \( \rho \in C^0(\mathbb{R}_+; L^1_{\text{loc}}(\mathbb{R}; [0, R^*])) \) is a weak solution to the Cauchy problem (1.1), (3.1) if the initial condition (3.1) holds and

\[
\int_0^{+\infty} \int_{-\infty}^{+\infty} (\rho \partial_t \varphi + q(\rho) \partial_x \varphi) \, dx \, dt = 0
\]

for every \( \varphi \in C^1_c (\mathbb{R}_+ \times \mathbb{R}; \mathbb{R}) \), see [4, Section 4.1]. The existence of weak solutions to (1.1), (3.1) will be proved by the method of wave front tracking.

Now we can start the standard wave front tracking procedure [3, 7, 9], see also [1, 2, 4, 5, 6, 10, 12], to construct an approximate solution to (1.1), (3.1). First, fix \( n \in \mathbb{N} \) and define the mesh \( M_n = \{ i2^{-n} R^* : i = 0, \ldots, 2^n \} \cup \{ R_M, R_T, R, R_M^*, R_T^* \} \). Approximate the flow \( q \) by means of a piecewise linear and continuous flow \( q_n \) such that \( q_n (\rho) = q(\rho) \) for all \( \rho \in M_n \). Introduce the functions \( q_n^+ : [0, R^*] \times M_n \to [0, +\infty[ \) and \( \Psi_n : [0, R^*] \to [R_T, R_T^*] \) defined as follows:

\[
q_n^+(\rho, \bar{\rho}) = \begin{cases} 
\inf \left\{ Q(\rho) : Q \in \text{PLC} \text{ is concave, and} \right. \\
Q(\rho) \geq q_n(\rho) \quad \forall \rho \in [R, R^*] \cup \{0, \bar{\rho}\} \left. \right\} & \text{if } \bar{\rho} \in [0, R] \\
\inf \left\{ Q(\rho) : Q \in \text{PLC} \text{ is concave, and} \right. \\
Q(\rho) \geq q_n(\rho) \quad \forall \rho \in [R, R^*] \cup \{\bar{\rho}, R^*\} \left. \right\} & \text{if } \bar{\rho} \in [R, R^*] 
\end{cases}
\]

\[
\Psi_n(\rho) = \begin{cases} 
\inf \{ r \in ]R, R^*] : q_n(r) = q_n^+(r, \rho) \} & \text{if } \rho \in ]0, R[ \\
R & \text{if } \rho = R \\
\sup \{ r \in [0, R] : q_n(r) = q_n^+(r, \rho) \} & \text{if } \rho \in ]R, R^*].
\end{cases}
\]

Finally, define also \( \Phi_n : [0, R_T[ \cap M_n \to ]R_T, R] \cap M_n \) as follows

\[
\Phi_n(\rho) = \sup \left\{ r \in ]R_T, R[ \cap M_n : \frac{q_n(r) - q_n(\rho)}{r - \rho} \geq \frac{q_n(\Psi_n(\rho)) - q_n(\rho)}{\Psi_n(\rho) - \rho} \right\}.
\]

Observe that \( q_n \) induces through (R1)–(R4) a Riemann solver \( R_n \) that assigns to any Riemann data \( (\rho^l, \rho^r) \in (M_n)^2 \) a self similar, piecewise constant, weak and nonclassical solution \( R_n(\rho^l, \rho^r) \) with range in \( M_n \) for all time \( t \geq 0 \). More precisely, the following discretized version of Theorem 2.1 holds true.
Lemma 3.1 Under the same assumptions of Theorem 2.1, for all $n \in \mathbb{N}$ there exists a unique Riemann solver $R_n: (\mathcal{M}_n)^2 \to BV(\mathbb{R}; \mathcal{M}_n)$ that to any pair $(\rho^l, \rho^r) \in \mathcal{M}_n^2$ associates a weak solution, possibly nonclassical, to the Riemann problem
\[
\begin{cases}
\partial_t \rho + \partial_x q_n(\rho) = 0 \\
\rho(0, x) = \begin{cases}
\rho^l & \text{if } x < 0 \\
\rho^r & \text{if } x > 0
\end{cases}
\end{cases}
\]
that satisfies (R1)–(R4) with $q$, $\Psi$ and $\Phi$ respectively replaced by $q_n$, $\Psi_n$ and $\Phi_n$.

The proof is a straightforward adaptation from the analogous result in [11] and is, therefore, omitted.

Consider the Cauchy problem (1.1), (3.1) with $\bar{\rho} \in L^1 \cap BV(\mathbb{R}; [0, R^*])$. By [4, Lemma 2.2] we can approximate the initial datum $\bar{\rho}$ by means of a sequence $\{\bar{\rho}_n\}_{n \in \mathbb{N}}$ of piecewise constant functions such that for all $n \in \mathbb{N}$,
\[
\bar{\rho}_n(\mathbb{R}) \subseteq \mathcal{M}_n, \quad TV(\bar{\rho}_n) \leq TV(\bar{\rho}) \quad \text{and} \quad \lim_{n \to +\infty} \|\bar{\rho}_n - \bar{\rho}\|_{L^1} = 0. \tag{3.3}
\]
Recall that by piecewise constant we mean also that $\rho_n$ has compact support. Consider the approximating Cauchy problem
\[
\begin{cases}
\partial_t \rho + \partial_x q_n(\rho) = 0 & (t, x) \in \mathbb{R}_+ \times \mathbb{R} \\
\rho(0, x) = \bar{\rho}_n & x \in \mathbb{R}.
\end{cases} \tag{3.4}
\]
Each point of jump in $\rho_n$ gives rise to a Riemann problem which we solve by means of $R_n$. Gluing these solutions, we obtain an approximate solution $\rho_n$ defined up to the first time at which two or more discontinuities in $\rho_n$ collide. $\rho_n$ can be extended beyond this interaction time solving the Riemann problem arising at the interaction point. This procedure allows to extend the approximate solution to $[0, +\infty[ \times \mathbb{R}$, provided $t \mapsto TV(\rho_n(t))$ remains uniformly bounded. As it is standard in this context, the key difficulty is in providing a uniform bound for the total variation of $\rho_n$. To this aim, we introduce a weighted total variation as follows. Let
\[
\rho_n(t) = \sum_k \rho_n(x_{[x_{n-1}^k, x_n^k]}^k) \quad \text{with} \quad x_{n-1}^k < x_n^k \quad \text{for all } k
\]
be the approximate solution at time $t$. Define
\[
\mathcal{I}_n = \left\{ k: \rho_n^k < \rho_n^{k+1} \text{ and } \rho_n^{k+1} > R \right\}
\]
\[
TV_w(\rho_n) = \sum_{k \in \mathcal{I}_n} \left| \rho_n^{k+1} - \rho_n^k \right| + W \cdot \sum_{k \notin \mathcal{I}_n} \left| \rho_n^{k+1} - \rho_n^k \right|.
\]
In other words, all classical and nonclassical shocks with right state in the panic interval $[R, R^*]$ have weight 1, all the other classical waves have weight $W$. In [21] it is proved the following lemma.
Lemma 3.2 Assume that
\[ \frac{R_2}{\Psi(0)} \leq \frac{\Delta s}{\Psi(s) - s}, \] (3.5)
and consider a constant \( W \) such that
\[ W > 1 \quad \text{and} \quad \frac{R_2}{\Psi(0)} \leq \frac{W + 1}{2W} \leq \frac{\Delta s}{\Psi(s) - s}. \] (3.6)
Then the map \( t \mapsto TV_w(\rho_n(t)) \) is a non increasing function.

Sufficient conditions for (3.5) to be satisfied are in Lemma 4.1.

As a consequence of the above lemma the total variation of \( \rho_n(t) \) is bounded for all \( t \in \mathbb{R}_+ \), and therefore the total number of interactions is finite. Thus the above method of wave front tracking defines a piecewise constant solution \( \rho_n(t) \) to (3.4) for all \( t \in \mathbb{R}_+ \), which has a finite number of jumps and satisfies
\[ TV(\rho_n(t)) \leq TV_w(\rho_n(t)) \leq TV_w(\bar{\rho}_n) \leq W \cdot TV(\bar{\rho}_n) \leq W \cdot TV(\bar{\rho}). \] (3.7)

Lemma 3.3 For every times \( t, s \in \mathbb{R}_+ \)
\[ \|\rho_n(t) - \rho_n(s)\|_{L^1} \leq W \cdot TV(\bar{\rho}) \cdot \text{Lip}(q) \cdot |t - s|. \] (3.8)

The proof is deferred to Section 4. By compactness arguments, we prove the global existence of entropy weak solutions to (1.1), (3.1), within a class of functions with bounded variation. More precisely, the following adaptation of [4, Theorem 6.1] holds true.

Theorem 3.4 Let \( q \) satisfy (Q.1)–(Q.9), let \( s, \Delta s \) satisfy (2.2) and assume that there exists a \( W \) satisfying (3.5). For any initial datum \( \bar{\rho} \in (L^1 \cap BV)(\mathbb{R}; [0, R^*]) \), the Cauchy problem (1.1), (3.1) admits a nonclassical weak solution \( \rho = \rho(t, x) \) generated by the nonclassical Riemann solver \( R \) and defined for all \( t \in \mathbb{R}_+ \). Moreover:
\[ TV(\rho(t)) \leq W \cdot TV(\bar{\rho}), \quad \text{for all} \ t \in \mathbb{R}_+. \] (3.9)

Further qualitative properties of the solutions constructed above are difficult to prove analytically. As an example we note the following straightforward consequence of the maximum principle and of the diminishing of the total variation that hold for classical scalar conservation laws.

Proposition 3.5 Fix an initial datum \( \bar{\rho} \in (L^1 \cap BV)(\mathbb{R}; [0, R^*]) \). Then, the solution \( \rho = \rho(t, x) \) exhibited from Theorem 3.4 satisfies
\[ \rho(t, x) \leq \max \left\{ \|\bar{\rho}\|_{L^\infty}, R_T^* \right\} \quad \text{for all} \ (t, x) \in \mathbb{R}_+ \times \mathbb{R}. \]
Furthermore, if
\[ \bar{\rho}(\mathbb{R}) \subseteq [0, R] \quad \text{and} \quad \text{TV}(\bar{\rho}) < \Delta s \]
then, the solution \( \rho = \rho(t, x) \) exhibited from Theorem 3.4 coincides with the classical solution and does not attain values among the panic states. Hence, it is a Lipschitz function of the initial data with respect to the \( L^1 \) norm.

As a numerical example, we consider the specific flow function
\[ q(\rho) = \max \left\{ \frac{\rho(7 - \rho)}{6}, \frac{3(\rho - 6)(2\rho - 21)}{20(\rho - 12)} \right\} \tag{3.10} \]
inspired by [15, Figure 1, top right], see Figure 5. Note that if we consider the Cauchy problem (1.1), (3.1) associated to the flow \( q \) represented in the Figure 5 right and the initial data \( \bar{\rho} \) takes values in [0, 10], then also the corresponding solution takes values in [0, 10]. For this reason we are allowed to prolong the flow represented in the Figure 5 left beyond 10 as in the Figure 5 right. Geometrical considerations on Figure 5 and elementary computations show that the assumptions in Theorem 3.4 are satisfied, provided
\[ s \in \left(0, \frac{1}{25}\right], \quad \Delta s \in \left[\frac{34}{5}, \frac{67}{10}\right] \quad \text{and} \quad W \cdot 10^4 \in [18563, 18599]. \]

4 Technical Proofs

Throughout this section, following (Q.1)–(Q.8), \( q' \) is not necessarily defined at \( R \). Let \( \text{PLC} \) denote the set of all piecewise linear and continuous functions \([0, R^*] \to \mathbb{R} \).

Lemma 4.1 Assume that
\[ \Psi(0) \leq 2R_2 \quad \text{and} \quad \frac{\Psi'(0) - 1}{1 - \Phi'(0)} > \frac{\Psi(0)}{\Phi(0)}. \tag{4.1} \]
Then, there exists $\rho^* \in ]0, R_T[$ such that for any $s \in ]0, \rho^*[ $ there exists $\Delta s$ and $W$ satisfying (2.2) and (3.6).

**Proof.** Introduce the function $\sigma: [0, R_T] \to \mathbb{R}_+$ by

$$\sigma(\rho) = \frac{\Phi(\rho) - \rho}{\Psi(\rho) - \rho}.$$ 

By definition, $\sigma(0) = R_2/\Psi(0)$ and $\sigma(R_T) = 0$. Furthermore, by (4.1), we have $\sigma'(0) > 0$. Therefore, there exists $\rho^* \in ]0, R_T[$ such that $\sigma(\rho) > R_2/\Psi(0)$ for all $\rho \in ]0, \rho^*[ $ and $\sigma(\rho^*) = R_2/\Psi(0)$. Fix $s \in ]0, \rho^*[ $. Then, it is possible to choose $\Delta s > 0$ such that $\max\{R - R_T, \Phi(s) - s\} < \Delta s < R - s$. Clearly (2.2) holds true. Concerning (3.6), observe that

$$\frac{\Delta s}{\Psi(\rho) - s} \geq \sigma(s) > \frac{R_2}{\Psi(0)}, \quad \frac{1}{2} < \frac{R_2}{\Psi(0)} < 1.$$ 

To complete the proof, observe that $\tau(W) = (W + 1)/(2W)$ is a decreasing continuous function for $W > 0$, $\tau(1) = 1$ and $\lim_{W \to +\infty} \tau(W) = 1/2$. \qed

**Proof of Lemma 3.2.** The proof amounts to consider all possible interactions between two waves in the $n$-approximate solution $\rho_n$, showing that, in each of these cases, the weighted total variation may not increase. The theory for standard Riemann solver, see for instance [4, 13], ensures that the total variation does not increase after an interaction of classical waves. Thus, it is not restrictive to consider only the interactions which involves waves with different weights or nonclassical shocks. We limit the presentation below to the most representative situations, leaving the detailed description of all the other cases to [21].

Consider a left wave connecting two states $\rho^l$ and $\rho^m$ interacting with a right wave connecting two states $\rho^m$ and $\rho^r$. We use, for instance, the usual notation $\mathcal{R}\mathcal{S} - \mathcal{N}$ when the left incoming wave is a rarefaction, the right incoming wave is a classical shock, and the outgoing wave is a nonclassical shock. We indicate whether the wave is increasing or decreasing by adding an up arrow or a down arrow.

Consider the following cases:

1. If $\mathcal{R}(\rho^l, \rho^m) = \mathcal{R}^\uparrow$ with $\rho^l \leq R$ and $\mathcal{R}(\rho^m, \rho^r) = \mathcal{S}^\uparrow$ with $\rho^r \leq R$, $\rho^m \leq s < \rho^l$ and $\rho^r - \rho^l > \Delta s$, see Figure 6 left, then the outgoing wave is a...
nonclassical shock from $\rho^l$ to $\Psi(\rho^l)$, followed by a rarefaction from $\Psi(\rho^l)$ to $\Psi(\rho^r)$ and by a possible null shock from $\Psi(\rho^r)$ to $\rho^r$. In this case it results that the weighted total variation decreases if and only if

$$(W + 1)(\Psi(\rho^l) - \rho^l) + 2W \rho^m \leq 2W \rho^r.$$

The condition $\rho^r \in ]s + \Delta s, R[ \text{ implies } 2W \rho^r \geq 2W(s + \Delta s)$. On the other hand, by $\rho^m \leq s < \rho^l$ and (Q.9), $(W + 1)\left(\Psi(\rho^l) - \rho^l\right) + 2W \rho^m \leq (W + 1)\Psi(s) + (W - 1)s$. Therefore, the weighted total variation decreases by (3.6).

2. If $\mathcal{R}(\rho^l, \rho^m) = \mathcal{R}^\dagger$ with $\rho^l \leq R$ and $\mathcal{R}(\rho^m, \rho^r) = \mathcal{N}$ with $\rho^l \leq R_T$ and $\rho^r < \Psi(\rho^l)$, see Figure 6 right, then the outgoing wave is a nonclassical shock from $\rho^l$ to $\Psi(\rho^l)$, followed by a rarefaction from $\Psi(\rho^l)$ to $\rho^r$. In this case, the weighted total variation decreases if and only if $\Psi(\rho^l) - \rho^l \leq \rho^r - \rho^m$. Since $\rho^r \geq \Psi(\rho^m)$ and $\rho^m < \rho^l$, by (Q.9) we obtain $\rho^r - \rho^m \geq \Psi(\rho^m) - \rho^m \geq \Psi(\rho^l) - \rho^l$. Therefore, the weighted total variation decreases.

3. If $\mathcal{R}(\rho^l, \rho^m) = \mathcal{R}^\dagger$ with $\rho^l \leq R$ and $\mathcal{R}(\rho^m, \rho^r) = \mathcal{N}$ with $\rho^m < R_T < \rho^r$ and $\rho^r \leq R_T^*$, see Figure 7, then the outgoing wave is a rarefaction from $\rho^l$ to $R_T$ followed by a nonclassical shock from $R_T$ to $R_T^*$ and by a possible null rarefaction from $R_T^*$ to $\rho^r$. In this case it results that the weighted total variation decreases if and only if $R_T^* - R_T \leq \rho^r - \rho^m$. Since $\rho^r \geq \Psi(\rho^m)$ and $\rho^m < R_T$, by (Q.9), $\rho^r - \rho^m \geq \Psi(\rho^m) - \rho^m \geq R_T - R_T$. Therefore, the weighted total variation decreases.

4. If $\mathcal{R}(\rho^l, \rho^m) = \mathcal{S}^\dagger$ with $\rho^m \leq R$ and $\mathcal{R}(\rho^m, \rho^r) = \mathcal{S}^\dagger$ with $\rho^r \leq R$, $s < \rho^l < \rho^r - \Delta s \leq \rho^m$, see Figure 8 left, then the outgoing wave is a nonclassical shock form $\rho^l$ to $\Psi(\rho^l)$ followed by a rarefaction form $\Psi(\rho^l)$ to $\Psi(\rho^r)$ and by a possible null shock form $\Psi(\rho^r)$ to $\rho^r$. In this case it results that the weighted total variation decreases if and only if $(W + 1)(\Psi(\rho^l) - \rho^l) + 2W(\rho^r - \rho^l) \leq 2W(\Delta s)$. Since $s < \rho^l$, by (Q.9), $\Psi(\rho^l) - \rho^l \leq \Psi(s) - s$ and the weighted total variation decreases because of condition (3.6).

5. If $\mathcal{R}(\rho^l, \rho^m) = \mathcal{N}$ and $\mathcal{R}(\rho^m, \rho^r) = \mathcal{S}^\dagger$ with $\rho^l < R_T$, $\rho^m \geq \Psi(\rho^l)$ and $q(\rho^l) - q(\rho^m) < q(\rho^l) - q(\rho^m)$, see Figure 8 right, then the outgoing wave is a shock form $\rho^l$ to $\rho^r$. In this case it results that the weighted total variation decreases if and only if $2W(\rho^r - \rho^l) \leq (W + 1)(\rho^m + (W - 1)\rho^l)$. Since $\rho^l \geq 0$, $\rho^m \geq \Psi(0)$ and $\rho^r < R_2$, by condition (3.6), the weighted total variation decreases.
Proof of Lemma 3.3. (This proof is inspired by [20]). Let \([t_1, t_2]\) be an interval containing no interaction, and \(y_k = y_k(t), k = 1, 2, \ldots, N\) be the propagating fronts in \(\rho_n\). Then,

\[
\|\rho_n(t_2) - \rho_n(t_1)\|_{L^1} \leq \sum_{k \geq 1} \left| \rho_n^+(y_k(t_1), t_1) - \rho_n^-(y_k(t_1), t_1) \right| \cdot |y_k(t_2) - y_k(t_1)|,
\]

where \(\rho_n^-\) and \(\rho_n^+\) are the left and right traces of \(\rho_n\). Since each \(y_k'\) is constant in \([t_1, t_2]\), we have

\[
|y_k(t_2) - y_k(t_1)| = |y_k'|(t_2 - t_1) \leq \text{Lip}(q) \cdot (t_2 - t_1).
\]

By using these two estimates we find that

\[
\|\rho_n(t_2) - \rho_n(t_1)\|_{L^1} \leq W \cdot \text{TV}(\bar{\rho}) \cdot \text{Lip}(q) \cdot (t_2 - t_1).
\]

Therefore, if \([s, t]\) is an interval containing no interaction, then there is nothing else to prove.

Let us now consider an interval \([s_1, s_2]\) containing one interaction at time \(\xi \in ]s_1, s_2[\). It is not restrictive to assume that the interaction involves exactly two incoming waves \(y_k = y_k(t), t \in ]s_1, \xi[, k = 1, 2\), i.e. \(y_1(\xi) = y_2(\xi)\). Let \(z_j = z_j(t), t \in ]\xi, s_2[, j = 1, \ldots, h\), be the outgoing waves generated by the interaction, i.e. \(z_j(\xi) = y_1(\xi), j = 1, \ldots, h\). Denote by \(H_n\) the maximal number of outgoing waves after an interaction between two waves. Clearly \(h \leq H_n \leq 4M_n\). Let \(y_k = y_k(y), t \in ]s_1, s_2[, k \geq 3\), be the others non-interacting waves. Then,

\[
\|\rho_n(s_2) - \rho_n(s_1)\|_{L^1} \leq \sum_{j \geq 3} \left| \rho_n^+(s_1, y_j(s_1)) - \rho_n^-(s_1, y_j(s_1)) \right| \cdot |y_j(s_2) - y_j(s_1)|
\]

\[
+ \sum_{k=1}^{h} \sum_{j=1}^{2} \left| \rho_n^+(s_1, y_j(s_1)) - \rho_n^-(s_2, z_k(s_2)) \right| \cdot |z_k(s_2) - y_j(s_1)|
\]

\[
+ \sum_{k=1}^{h} \sum_{j=1}^{2} \left| \rho_n^-(s_1, y_j(s_1)) - \rho_n^+(s_2, z_k(s_2)) \right| \cdot |z_k(s_2) - y_j(s_1)|
\]

\[
\leq (5h - 1) \cdot W \cdot \text{TV}(\bar{\rho}) \cdot \text{Lip}(q) \cdot (s_2 - s_1),
\]
since the equalities $y_j(t) = y_j'(t - \xi) + y_1(\xi)$, $z_k(t) = z_k'(t - \xi) + y_1(\xi)$ imply

$$|z_k(s_2) - y_j(s_1)| \leq |z_k'(s_2 - \xi)| + |y_j'(\xi - s_1)| \leq \text{Lip}(q) \cdot (s_2 - s_1),$$

and the equalities

$$\rho_n^- (s_1, y_1(s_1)) = \rho_n^- (s_2, z_1(s_2)), \quad \rho_n^+ (s_1, y_1(s_1)) = \rho_n^+ (s_1, y_2(s_1)),
\rho_n^+ (s_1, y_2(s_1)) = \rho_n^+ (s_2, z_k(s_2)), \quad \rho_n^+ (s_2, z_k(s_2)) = \rho_n^+ (s_2, z_{k+1}(s_2)),$$

for $k = 1, \ldots, h - 1$, imply that

$$\sum_{k=1}^h \sum_{j=1}^2 \left| \rho_n^+(s_1, y_j(s_1)) - \rho_n^-(s_2, z_k(s_2)) \right| + \left| \rho_n^-(s_1, y_j(s_1)) - \rho_n^+(s_2, z_k(s_2)) \right|$$

$$= \left| \rho_n^+(s_1, y_1(s_1)) - \rho_n^-(s_1, y_1(s_1)) \right| + \sum_{k=2}^h \left| \rho_n^+(s_1, y_1(s_1)) - \rho_n^-(s_2, z_k(s_2)) \right|$$

$$+ \sum_{k=1}^h \left| \rho_n^-(s_2, z_1(s_2)) - \rho_n^+(s_2, z_k(s_2)) \right| + \left| \rho_n^+(s_2, z_1(s_2)) - \rho_n^-(s_2, z_k(s_2)) \right|$$

$$+ \sum_{k=1}^{h-1} \left| \rho_n^-(s_1, y_2(s_1)) - \rho_n^+(s_2, z_k(s_2)) \right| + \left| \rho_n^-(s_1, y_2(s_1)) - \rho_n^+(s_1, y_2(s_1)) \right|$$

$$\leq h \cdot \text{TV} (\rho_n(s_1)) + 2(2h - 1) \cdot \text{TV} (\rho_n(s_2)) \leq (5h - 2) \cdot W \cdot \text{TV}(\bar{\rho}).$$

Therefore, if $t > s$ and $[s, t]$ contains some interactions, then it is not restrictive to assume that at times $s$ and $t$ no interaction occurs, and that each interaction involves exactly two entering waves. Let $t_k, k = 1, \ldots, K$, the times corresponding to these interactions. If we denote $t_o = s$ and $t_{K+1} = t$, then, for what we proved before

$$\| \rho_n (t) - \rho_n (s) \|_{L^1(\mathbb{R})} \leq \lim_{\varepsilon \to 0+} \sum_{k=0}^K \| \rho_n(t_{k+1} - \varepsilon) - \rho_n(t_k + \varepsilon) \|_{L^1(\mathbb{R})}$$

$$+ \lim_{\varepsilon \to 0+} \sum_{k=1}^K \| \rho_n(t_k + \varepsilon) - \rho_n(t_k - \varepsilon) \|_{L^1(\mathbb{R})}$$

$$\leq \lim_{\varepsilon \to 0+} \sum_{k=0}^K W \cdot \text{TV}(\bar{\rho}) \cdot \text{Lip}(q) \cdot (t_{k+1} - t_k - 2\varepsilon)$$

$$+ \lim_{\varepsilon \to 0+} 2\varepsilon K(5H_n - 1) \cdot W \cdot \text{TV}(\bar{\rho}) \cdot \text{Lip}(q)$$

$$\leq W \cdot \text{TV}(\bar{\rho}) \cdot \text{Lip}(q) \cdot (t - s)$$

completing the proof. □
Proof of Theorem 3.4. By (3.8) and the obvious estimate $\|\rho_n(t)\|_{L^\infty} \leq R^*$, $t \in \mathbb{R}^+$, we can apply the Helly’s Theorem, see for instance [4, Theorem 2.3], and deduce the existence of a subsequent $\{\rho_m\}_{m \in \mathbb{N}}$ which converges to some function $\rho$ in $L^1_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R})$. Clearly (3.7) implies (3.9). Observing that the convergence $q_m \to q$ is uniform on the interval $[0, R^*]$, and recalling that $\rho_m$ is an entropy solution to (3.4), we obtain

$$\int \int (|\rho - k|\partial_t \varphi + (q(\rho) - q(k)) \text{sgn}(u - k)\partial_x \varphi) \, dx \, dt = \lim_{m \to \infty} \int \int (|\rho_m - k|\partial_t \varphi + (q_m(\rho_m) - q_m(k)) \text{sgn}(\rho_m - k)\partial_x \varphi) \, dx \, dt \geq 0$$

for all nonnegative function $\varphi \in C^1$ with compact support contained in $\{(t, x) \in \mathbb{R}^2 : t > 0\}$. Finally, (3.8) and (3.3) imply that the initial condition (3.1) is attained. This proves that $\rho$ is an entropy weak solution to (1.1), (3.1).

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References


