

Adiabaticity conditions for volatility smile in Black-Scholes pricing model

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Abstract. Our derivation of the distribution function for future returns is based on the risk neutral approach which gives a functional dependence for the European call (put) option price $C(K)$ given the strike price K and the distribution function of the returns. We derive this distribution function using for $C(K)$ a Black-Scholes expression with volatility σ in the form of a volatility smile. We show that this approach based on a volatility smile leads to relative minima for the distribution function (“bad” probabilities) never observed in real data and, in the worst cases, negative probabilities. We show that these undesirable effects can be eliminated by requiring “adiabatic” conditions on the volatility smile.

1 Introduction

One of the simplest “products” on the derivative financial market is the European call (put) option [1,2]. Considering the risk neutral approach, the price of the European call option $C \equiv C(S_T, K, T, r)$ is defined by

$$C = e^{-rT} \int_K^\infty (S_T - K) P(S_T) dS_T, \quad (1)$$

where S_T is the stock price at time $t = T$, K is the strike price of the option, T is the expiration time (time to maturity) of the option, r is the interest rate and $P(S_T) \geq 0$ is the probability density function (PDF) of the stock prices in a “risk-neutral world” ($\int_0^\infty P(S_T) dS_T = 1$).

Equation (1) is too general since it does not place any restrictions on the underlying stock price distribution function $P(S_T)$. To calculate explicitly the option price C using equation (1), one must know the PDF $P(S_T)$. Consequently, one must make some assumptions about the stock prices. An important achievement in the theory of option pricing is the Black-Scholes (BS) theory which gives analytic solutions for the European call and put options [3].

In particular, for the European call option, a solution of the BS equation is given by equation (1), if one assumes for the PDF $P(S_T)$ a log-normal distribution,

$$P(x) = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \exp \left[-\frac{(x + \sigma^2(T-t)/2)^2}{2\sigma^2(T-t)} \right], \quad (2)$$

where $x = \ln(S_T/S(t)) - r(T-t)$ is the logarithmic return deprived of the risk-free component, $S(t)$ is the stock price

at time t and σ is the stock price volatility. For seek of simplicity, in the following we consider $t = 0$ and we define $S_0 \equiv S(t = 0)$. Substituting equation (2) in equation (1) an explicit expression for the price of the European call option which satisfies the BS equation [3] is obtained,

$$C^{\text{BS}} = S_0 N(d_1) - K e^{-rT} N(d_2), \quad (3)$$

where

$$\begin{aligned} d_1 &= \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \\ d_2 &= d_1 - \sigma\sqrt{T}, \\ N(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x dz e^{-z^2/2}. \end{aligned} \quad (4)$$

The PDF (2) follows from a stochastic model for stock prices,

$$dS = rSdt + \sigma Sdz, \quad (5)$$

where dz is a Wiener increment. The behavior of the Wiener process can be described on the finite time-interval Δt as $\Delta z = \gamma\sqrt{\Delta t}$, where the random variable γ has the distribution function: $g(\gamma) = 1/\sqrt{2\pi} \exp(-\gamma^2/2)$. It can be shown it is never optimal to exercise an American call option on a non-dividend-paying stock early [1,4]; therefore equation (3) can also be used to estimate the fair value for this kind of options.

There are some problems with the expressions for C given by equations (1–3). Indeed, one can derive *any* option price from equation (1), using different assumptions about the PDF $P(S_T)$. To derive from equation (1) a result for C which will even approximately coincide with

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the real market price C^M one must specify a distribution function for *future* stock prices $P(S_T)$. On the other hand, the expression given by equation (3) is (a) too specific, and (b) derived using rather strong restrictions. Namely, the stochastic process equation (5) does not account for correlations of returns x and, moreover, the volatility σ and the interest rate r are not well-defined parameters (given the actual data). As a result, the expression C^{BS} often does not coincide (even approximately) with the corresponding market option price C^M . Useful approaches have been developed which partially solve the problems mentioned above.

We shall mention here one analysis which is related to that presented below. This analysis deals with building “implied trees” [5–9]. There are many variations of this approach, but the main idea is based on the solution of the inverse problem: a search for a stock price model that corresponds to the real market prices of options C^M . A more restricted problem is to search for a stock price model that effectively deals with the volatility smile. In this case, one starts with the BS formula (3), (even for American options) but instead of choosing a fixed volatility $\sigma = \text{constant}$, one uses the dependence $\sigma = \sigma(K)$ (volatility smile). To some extent, this dependence corresponds to the “real behavior” of the volatility σ if one wants to use equation (3) as the “zeroth approximation” for option pricing. The name “volatility smile” is due to the typical parabolic shape of the curve $\sigma(K)$ that is recurrent in the pricing of many call options. The reason of this shape is related to the non Gaussian nature of actual financial returns distributions in contrast to the model hypothesis; to compensate this model deficiency one needs to increase the implied volatility when the options is out of the money (and decrease it when in the money) generating the typical parabolic shape, see Section 3.

There are still some problems with this implied volatility. For example, the corresponding “implied” stock prices S_T can have “bad” (negative) probabilities which must be eliminated. Despite the simplicity of the calibration procedure, a solution to this kind of problem for trees generation was proposed [10,11]. In general, another way to avoid the problem of “bad” probabilities is to find some constraint on the general shape of $\sigma(K)$ via a nonparametric or parametric specification of the functional form of the volatility smile. In the first case, volatility smiles are computed requiring some smoothness conditions derived from arbitrage-free considerations [12–14]. This kind of algorithms has the advantage to be based on more general assumptions on volatility smile and PDFs of return but, as a consequence, they increase the complexity of the problem and the computational time of the calibration. On the contrary, using a parametric approach, one needs to specify the expression of the volatility smile to get a parametric restriction on the implied volatility space that can be used straightforwardly in the fitting procedure, generally decreasing the computational time for calibration.

In this paper, we discuss the inverse problem for equation (1) using the following parametric approach. First, using equation (1), we build the PDF for future stocks prices

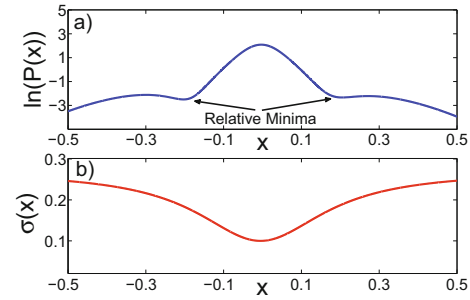


Fig. 1. (Color online) (a) Volatility smile given by equation (19), with the following parameters: $g = 0.1$, $T = 0.5$, $n = 0.04$, $\chi = 2.7$. (b) Log-distribution given by equation (13) for the volatility smile (a).

and returns from the empirical data for the market option prices $C(K)$. Second, we build the returns PDF using for $C(K)$ a BS expression C^{BS} with a volatility $\sigma = \sigma(K)$ in the form of the volatility smile. In particular, we show that the condition of the absence for relative minima in the PDF of returns (Fig. 1), (or elimination of “bad” probabilities) leads to the condition of “adiabaticity” for the volatility smile. The term “adiabatic” comes from statistical physics and is related to the slowness of the variation of a parameter λ that specifies the properties of a system or an external field. In fact from a physical point of view, it can be shown that if in a system one introduces a small perturbation (λ) compared to the characteristic period of the motion T , namely

$$T \frac{d\lambda}{dt} \ll \lambda, \quad (6)$$

the rate of the change of the energy of the system will be also small [15]. In the same spirit we assume that our parameter λ is represented by the implied volatility σ and we look for an adiabatic condition that leads to a small perturbation of the implied Gaussian distribution that derives from the BS model with constant volatility, eliminating the problem of negative probabilities.

This condition can be introduced in the fitting procedure of the volatility smile to get a returns PDF closer to the actual one. In this way one can avoid to generate arbitrage opportunities (negative probabilities) in the option pricing methodology (exotic derivatives) and can get a more reliable estimation of the implied volatility. The latter has a key role in the scenario generation and in value at risk (VaR) estimation and has application in the risk management activities.

2 The inverse problem for the stock price distribution function

In this Section, we derive an explicit expression for the PDF for the future stock prices $P(S_T)$ and for logarithmic returns $P(x)$. In equation (1) the PDF $P(S_T)$ can be rather arbitrary but it is natural to assume that $P(S_T)$ does not depend on the strike price K . According to equation (1), the option price C is expressed explicitly through

the strike price K . Differentiating C in equation (1) twice with respect to K , we have [16]

$$P(S_T) = e^{rT} \frac{\partial^2 C(K)}{\partial K^2} \Big|_{K=S_T}. \quad (7)$$

In equation (7), we indicate only the dependence $C(K)$ in the option prices. In particular, applying equation (7) to C^{BS} given in equation (3) we derive a distribution function $P^{\text{BS}}(S_T)$ which we present in the form

$$P^{\text{BS}}(S_T) \equiv e^{rT} \frac{\partial^2 C^{\text{BS}}(K)}{\partial K^2} \Big|_{K=S_T} = \frac{1}{\sqrt{2\pi\sigma^2 T} S_T} \times \exp \left(-\frac{(\ln(S_T/S_0) - (rT - \sigma^2/2T))^2}{2\sigma^2 T} \right). \quad (8)$$

Analogously, the distribution of returns for the Black-Scholes model is Gaussian, as expected:

$$P^{\text{BS}}(x) = \frac{1}{\sqrt{2\pi\sigma^2 T}} \exp \left[-\frac{(x + x_0)^2}{2\sigma^2 T} \right], \quad (9)$$

where $x_0 = \sigma^2/2T$.

We can try to consider the inverse problem substituting the dependence $\sigma = \sigma(K)$ in equation (3) and evaluating the distribution of future stocks price and returns, applying equation (7). After differentiation we get

$$P(S_T) = \frac{F(S_T; S_0, r, T, \sigma)}{\sqrt{2\pi\sigma^2 T} S_T} \times \exp \left[-\frac{(\ln(S_T/S_0) - (rT - x_0))^2}{2\sigma^2 T} \right], \quad (10)$$

where we defined

$$\begin{aligned} F(S_T; S_0, r, T, \sigma) &= \left[1 + S_T \frac{\dot{\sigma}}{\sigma} (rT - \ln(S_T/S_0)) \right]^2 \\ &\quad - \frac{(\dot{\sigma} \sigma T S_T)^2}{4} + \dot{\sigma} \sigma T S_T + S_T^2 \sigma \ddot{\sigma} T, \\ \dot{\sigma} &= \frac{\partial \sigma}{\partial K} \Big|_{K=S_T}, \\ \ddot{\sigma} &= \frac{\partial^2 \sigma}{\partial K^2} \Big|_{K=S_T}. \end{aligned} \quad (11)$$

Obviously, we can get the expression for the distribution of returns by a simple change of variable

$$x \equiv \ln \left(\frac{K}{S_0} \right) \Big|_{K=S_T} - rT, \quad (12)$$

so that

$$P_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma^2 T}} \exp \left[-\frac{(x + x_0)^2}{2\sigma^2 T} \right] F(x; T, \sigma), \quad (13)$$

where, in a similar way, we have defined:

$$\begin{aligned} F(x; T, \sigma) &= \left(1 - \frac{\sigma'}{\sigma} x \right)^2 - \frac{(\sigma' \sigma T)^2}{4} + \sigma \sigma'' T, \\ \sigma' &= \frac{\partial \sigma}{\partial x}, \\ \sigma'' &= \frac{\partial^2 \sigma}{\partial x^2}. \end{aligned} \quad (14)$$

From equation (14) it is clear that if σ is constant equation (10–13) are the distributions for the standard Black-Scholes model; we will call them “zeroth-approximation” distributions. If $\sigma \neq \text{const.}$, the term $F(x; r, T, \sigma)$ could “perturb” the relative zero approximation (Gaussian) giving rise to distributions that cannot fit real data. As we will show here, it is possible to get distributions with relative minima (not observed in real returns distributions) and, in the worst case, negative probability.

The negative probability aspect in option pricing theory is particularly relevant because it is related to arbitrage opportunity, namely the opportunity to get a risk free profit. In fact even if from a mathematical point of view negative probabilities are impossible, from a financial perspective it can be shown from arbitrage theorem that there is a strong relation between negative probability and arbitrage opportunity. A simple example of this relation is given considering the discrete approximation of the second derivative in equation (7):

$$\frac{\partial^2 C^{\text{BS}}(K)}{\partial K^2} = \frac{C(K) - 2C(K + \Delta K) + C(K + 2\Delta K)}{\Delta K^2}. \quad (15)$$

At the numerator on the right hand of equation (15) we have an imaginary portfolio of three call options of different strikes: $K, K + \Delta K, K + 2\Delta K$. Considering that at the time to maturity T the payoff of a call option is $\max(S(T) - K, 0)$, it is clear that at that time the value of the portfolio is greater or equal to zero. If for some $t < T$ the implied distribution is negative as well the value of the imaginary portfolio, we know for sure that this portfolio will increment its value when it reaches the time to maturity passing from a negative value to a positive one. This gain can be greater to the one allowed by the risk-free rate, so it is, in principle, a simple example of static arbitrage opportunity.

The rest of the paper is organized as follows. In Section 3, we analyze FX market data and we propose a suitable function to fit the volatility smile. We also determine the typical range of the parameters we use to perform the fit. We will use these ranges to check our adiabatic condition. In Section 4, we show qualitatively the reason why we have the presence of the relative minima in the returns distribution and why an adiabatic approach can be suitable to describe the problem and used to avoid these “bad” probabilities. In Section 5, we show our numerical and theoretical results about the adiabaticity parameter. We also show that there is a critical value of the adiabatic parameter which can be used to determine whether the returns PDF will have relative minima. We discuss the

relation of this critical value to the other parameters of the fit. Finally, in Section 6 we present our conclusions.

3 The volatility smile: real market data

Typically, traders on option markets and practitioners consider the BS model as a zeroth order approximation that takes into account the main features of options prices. To get a pricing closer to the actual data, they consider the volatility as a parameter that can be adjusted considering the inverse problem given by equation (3) and the real price of call and put options. In this way a more reliable value of the volatility (*implied volatility*) can be obtained and it can be used to price more complex options for which analytical solutions are not available. The value of the implied volatility depends on the value of the strike K in a well-known characteristic curve called the *smile volatility* (typically for foreign currency options) whose shape is approximately parabolic and symmetric, or *skew volatility* (typically for equity options) when asymmetric effects dominate [17–20].

An intuitive explanation of this shape can be found if an actual returns distribution is considered. In fact, it is well known that the tails of the returns PDF are not Gaussian but exhibit a power law decay (fat tails) [21,22]. On the contrary, BS model assumes that the PDF of returns is Gaussian thus underestimating the actual probability of rare events. To compensate for this model deficiency, one has to consider the greater implied volatility for strike out of the money then for strike at the money.

In this paper we focus our attention on the volatility smile of foreign currency options and we neglect the skew effect [23]. To perform our analysis we consider the volatility smile as a function of the Δ of the option (defined by Eq. (16)), the time to maturity T and the currency considered. We consider specific days, for which volatility is not affected by the skew effect, and we use Bloomberg as data provider. In the BS model, the Δ of a call option is defined as

$$\Delta = \frac{\partial C}{\partial S_T} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\Delta_1} dz e^{-z^2/2}. \quad (16)$$

Inverting this relation it is possible to get an expression for x :

$$x = \sigma^2/2T - \sigma\sqrt{T}\Phi^{-1}(\Delta), \quad (17)$$

where $\Phi^{-1}(x)$ is the inverse of the error function

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt. \quad (18)$$

In Figure 2 we show an example of volatility smile in terms of our variables and a suitable fit given by the function

$$\sigma(x) = g \left[1 + (\chi - 1) \frac{(x + g^2T/2)^2}{(x + g^2T/2)^2 + n} \right] \quad (19)$$

where g, χ, n are fitting parameters. In this case, g represents the minimum of the volatility smile, $\sqrt{n}$ is the half

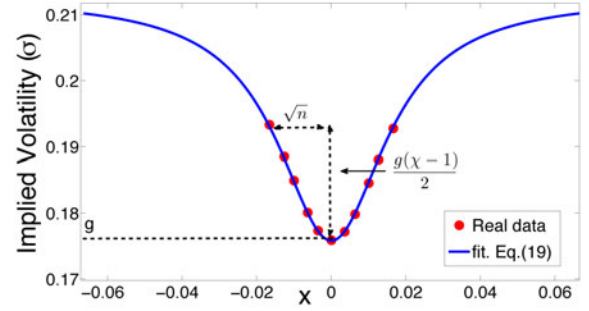


Fig. 2. (Color online) Typical volatility smile and the relative fit obtained with equation (19). The parameters of the fit are: $g = 0.1758(5)$, $\chi = 1.20(9)$, $n = 0.00030(9)$. We get the real data using Bloomberg provider and they refer to the AUDUSD currency with time to maturity $T = 1/365$ years.

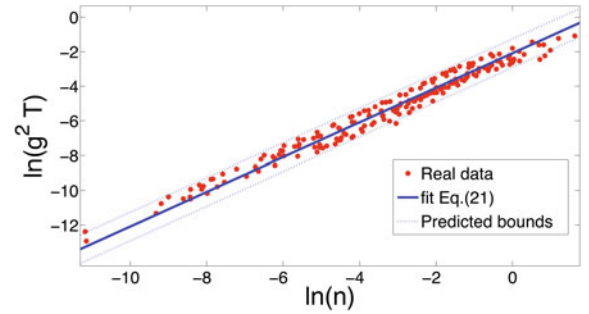


Fig. 3. (Color online) Relation between the parameters n, g, T . We fit 72 volatility symmetrical smiles (Bloomberg) considering different currency (EURUSD, AUDUSD, EURCHF, EURGBP, EURJPY, GBPJPY, GBPUSD, USDCAD, USDCHEF, USDJPY) and time to maturity (1 day, 1–3 weeks, 1, 2, 3, 4, 6, 9 months, 1, 1.5, 2, 3, 4, 5 years) with the function (19). The data are from the days 21/10/2009 and 01/02/2010. We also show the best linear fit $\ln(g^2T) = \ln(n) + c$, where $c = -1.95(12)$.

width at the half height, while $g(\chi - 1)$ represents the height of the smile. In particular χ is the ratio between the limiting value of σ and g as x approaches ∞ . In this way the variation of σ is bounded between g and $g\chi$. In the light of the intuitive explanation of the volatility smile proposed above and since from equation (9) it follows that the average value is

$$\langle x \rangle = -\frac{\sigma^2 T}{2}, \quad (20)$$

one expects that the minimum of the implied volatility occurs at $x = -g^2T/2$ as required by our fitting function.

Repeating many times the interpolation procedure considering different values for T and currencies, we can determine typical parameters that can fit a wide range of volatility smiles; in the following we will use this information to check our results.

Let us notice that the following relation between n, g and T holds

$$n \propto Tg^2, \quad (21)$$

as shown in Figure 3. This gives a scaling rule that can be used to determine the range of n , fixing T and g .

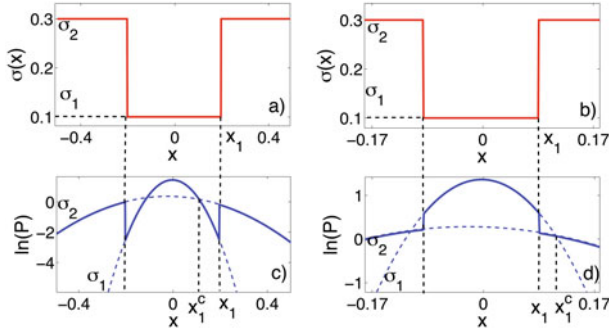


Fig. 4. (Color online) (a, b) Squared well as a fit of the volatility smile. (c, d) The corresponding PDF with (c) or without (d) spurious minima.

4 First approximation of the volatility smile: the squared well

In this section we show qualitatively the reason why there is a relative minimum in the returns distribution and why an adiabatic approach can describe the problem of avoiding these “bad” probabilities. To keep the problem simple, we consider, as a first step, a volatility smile modelled by a squared well defined as follows

$$\sigma(x) = \begin{cases} \sigma_1 & \text{for } |x| < x_1 \\ \sigma_2 & \text{otherwise,} \end{cases} \quad (22)$$

where $\sigma_2 > \sigma_1$ and x_1 are positive constants. See Figures 4a, 4b. The distribution functions corresponding to two values of σ_i , $i = 1, 2$ are shown in Figures 4c, 4d. Indicating as $\pm x_1^c$ the abscissa of the intersections between the two distributions, it is clear that a sufficient condition for avoiding spurious minima is $x_1 < x_1^c$. A rough estimation of x_1^c , ignoring the term x_0 , usually small, is

$$x_1^c = \sigma_1 \sqrt{T} \sqrt{\frac{2\chi^2 \ln \chi}{\chi^2 - 1}}, \quad (23)$$

where $\chi = \sigma_2/\sigma_1$. Therefore, a sufficient condition to avoid minima in the PDF is to use, as a fitting function a square well depending on the parameters x_1, σ_1, σ_2 , such that $x_1 < x_1^c$. Therefore, a standard fitting procedure of the volatility smile with a square well, constrained by the condition $x_1 < x_1^c$ solves the problem of avoiding spurious minima, even if it is very rough.

If we consider a volatility smile with a continuous variation from σ_1 to σ_2 , we can get, instead of a discontinuity, the onset of a relative minimum. The latter can be avoided if the variation between σ_1 and σ_2 is slow enough so that the connection between the two PDFs takes place keeping constant the sign of the first derivative of the distribution during the whole transition. So there will be a critical “speed” of the transition that will generate zero derivative points which will not correspond to the maximum of the distribution. In this case the variable related to the time is x_1 , while $\chi = \sigma_2/\sigma_1$ can be identified as a “distance”. To be more precise, one should consider that

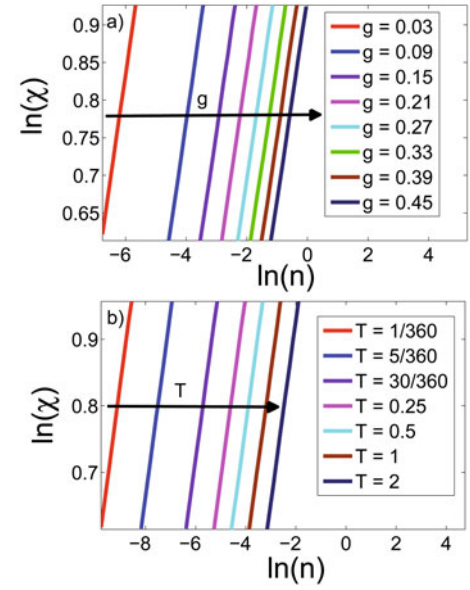


Fig. 5. (Color online) (a) Critical ratio $\chi_c = \sigma_2/\sigma_1$ as a function of n for fixed $T = 0.5$, and different g as indicated in the legend. The arrow indicates the direction of growing g . In the region to the right of the lines the PDF do not have minima, while in the left hand region it has. (b) The same as (a) but for fixed $g = 0.1$ and different T values. The arrow indicates the direction of growing T .

for $x < x_1^c$, $P_{\sigma_1}(x) \geq P_{\sigma_2}(x)$, so that the effective “time” should be: $x_1 - x_1^c$.

We will define an adiabatic transition as one whose passage from σ_1 to σ_2 is sufficiently slow, so that relative minima in the distribution are not generated. In this same spirit, we define the critical adiabatic parameter $\chi_c(n, g, T)$ as the minimum value of χ that generates a minimum in the distribution, equation (13). This effect has been shown in Figure 5, modelling the volatility smile using equation (19), so that $\sigma_1 = g$, $\sigma_2 = g\chi$. We fix the parameters g, T and we vary n and χ , seeking relative minima in the PDF. The lines in Figure 5, obtained respectively at fixed T (a) and fixed g (b), divide the plane of parameters into two regions: to the left of the lines the PDF has spurious minima, while this does not happen in the region to the right of it. It is then clear that for a given set of fixed parameters (g, T), there is a relation between χ and n that allow one to obtain a PDF without minima (minima are not observed in real data). The main goal of this paper is to determine a simple relation that determines whether or not the parameters of a volatility smile fit are consistent with real returns distribution and if they could give a reliable option pricing.

5 Numerical and theoretical results

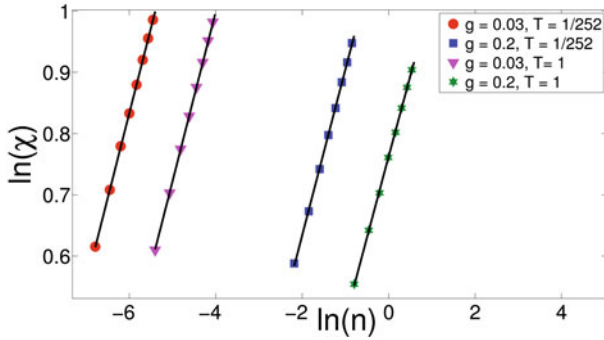
In this section we show our numerical and theoretical results about the relation between the set of parameters n, g, T and the critical adiabatic parameter χ_c . Using a numerical simulation, we kept fixed n, g, T and we

Table 1. Range of the parameters of the numerical simulations.

	Min	Max
g	0.03	0.5
ρ	2.5	10
T (years)	1/365	4

Table 2. Fitting parameters and relative errors.

α	1.4373 ± 0.0002
β	0.2787 ± 0.0006
γ	0.1738 ± 0.0002
δ	0.4683 ± 0.0006
Mean squared errors	1×10^{-5}

**Fig. 6.** (Color online) Critical adiabatic parameter as a function of n for few selected pairs of values of (g, T) as indicated in the legend. The points are numerical data, straight lines are the result of fitting procedure.

continuously increased the parameter χ until we found a zero-derivative point for some $x \neq -g^2T/2$. In this way we could determine numerically the critical χ_c . We repeated this approach for a wide range of the parameters values, as shown in Table 1, where we used the parameter $\rho = n/g^2T$ instead of n , due to the scaling relation (21). In order to obtain the relation $\chi_c = f_T(n, g)$, we use the following fit function

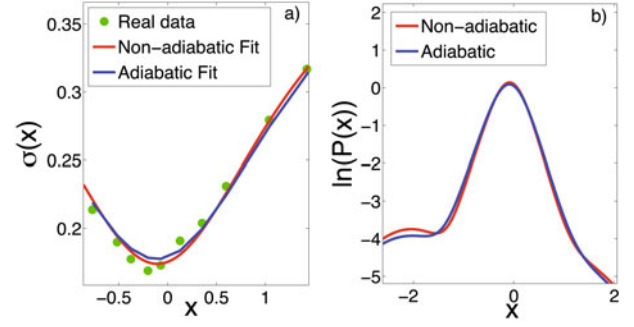
$$f_T(n, g) = \alpha \left(\frac{n}{g^2T} \right)^\beta - \gamma \sqrt{T} g \left(\frac{n}{g^2T} \right)^\delta. \quad (24)$$

This has been obtained assuming that the value of the critical parameter χ_c depends on the rescaled “time” of the transition (in our model given by ρ). We also consider a further term $\gamma \sqrt{T} g \rho^\delta$ to take into account the time correction x_c as explained in Section 4. In this case we make explicit the dependence of the time correction on T and g as suggested by σ_2 in equation (23). In equation (24), $\alpha, \beta, \gamma, \delta$ are the fitting parameters whose values are given in Table 2.

In Figure 6 we show the result of our fit for a few selected values of g and T .

The whole procedure can thus be summarized as follows:

- The real volatility smile, usually given for a fixed T can be fitted by a function dependent on three parameters g ,

**Fig. 7.** (Color online) (a) Volatility smile as a function the returns x . Dots indicate real data, red curve is the non adiabatic fit, while the blue one represents the adiabatic (constrained) fit. (b) PDF of returns for the two curves indicated in (a).

n, χ , as indicated in equation (19) and the optimal values $g^{\text{opt}}, n^{\text{opt}}, \chi^{\text{opt}}$ are returned.

- The optimal values $g^{\text{opt}}, n^{\text{opt}}$ are inserted in equation (24), with $\alpha, \beta, \gamma, \delta$ given in Table 2 and a critical $\chi_c = f_T(n^{\text{opt}}, g^{\text{opt}})$ obtained.
- If $\chi^{\text{opt}} < \chi_c$ then we know that relative minima in the PDF do not exist. Otherwise we should perform a fitting procedure to the volatility smile using equation (19), constrained by $\chi \leq \chi_c$.

An example of the previous procedure has been shown in Figure 7 where the PDF with unwanted minima and the “corrected” one is shown together with the corresponding fitting curve to the volatility smile. As one can see the price to pay in order to get a smooth PDF is very small: the two fitting curves for the real volatility smile are similar, but the PDF has, in the latter case a more realistic behavior.

6 Conclusions

We started from the pricing equation of the Black-Scholes model for a European call and we analyzed a suitable generalization to include the volatility smile effect. Then we considered the inverse problem and the relative returns distribution, equation (13), varying the typical parameters of the volatility smile. We showed that, for some values of the parameters, it is possible to get relative minima in the returns distribution (bad distribution) that are never observed in real distributions. We demonstrated that bad distributions can be eliminated by requiring adiabatic constraints (intuitively justified with the example of the squared well) on the volatility smile and we gave a numerical formula to determine the value of the adiabatic critical parameter, χ_c . In this way we provide an easy-to-use tool to determine if a volatility smile fit is consistent with the general requirement for probabilities ($P(x) \geq 0$) and if it can generate a suitable returns distribution. This kind of problems has also been discussed using a nonparametric approach [12–14]. In these cases volatility smiles are computed requiring some smoothness conditions derived from arbitrage-free considerations. These algorithms

have on one side the advantage to be based on more general assumptions on volatility smile and PDFs of return, while, on the other, the disadvantage to increase the complexity of the problem and the computational time of the calibration. Moreover our approach is based on the explicit knowledge of the fitting function of the volatility smile and the implied PDF of returns. In this way it is possible to get a globally parametric restriction on the implied volatility space that can be used straightforwardly in the fitting procedure, simplifying the calibration procedure. Interestingly, this restriction is derived using theoretical principles that come from physics.

A reliable estimate of the implied volatility has application in the risk management activities and in the pricing of exotic derivatives, where, in general, the implied volatility is an input of more complex models.

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