Robustness of quantum effects under environment-induced decoherence in open quasi-classical nonlinear systems: a coherent state representation approach

G.P. Berman,[1] F. Borgonovi,[2,3] and D.A.R. Dalvit[1]

[1]*Theoretical Division, MS B213, Los Alamos National Laboratory, Los Alamos, NM 87545, USA*
[2]*Dipartimento di Matematica e Fisica, Universitá Cattolica, via Musei 41, 25121 Brescia, Italy*
[3]*INFN, Sezione di Pavia, Italy*
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We review our results on a mathematical dynamical theory for observables for open many-body quantum nonlinear bosonic systems for a very general class of Hamiltonians. We argue that for open quantum nonlinear systems in the deep quasi-classical region, important quantum effects survive even after decoherence and relaxation processes take place. Estimates are derived which demonstrate that for a wide class of nonlinear quantum dynamical systems interacting with the environment, and which are close to the corresponding classical systems, quantum effects still remain important and can be observed, for example, in the frequency Fourier spectrum of the dynamical observables and in the corresponding spectral density of noise. Preliminary estimates are presented for Bose-Einstein condensates, low temperature mechanical resonators, and nonlinear optical systems prepared in large amplitude coherent states.

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I. INTRODUCTION

Real physical systems are not isolated, they are coupled to external degrees of freedom. The classical and quantum dynamics of these open systems are especially complex for nonlinear systems (with non-quadratic Hamiltonians) that exhibit several phenomena, including quantum revivals, decoherence, and dissipation. Recently much theoretical and experimental effort has been devoted to study the open dynamics of nonlinear quantum systems, with the aim of understanding the quantum to classical transition in a controlled way.

Standard mathematical treatments of open quantum nonlinear systems suffer from problems arising from the interplay between the nonlinearity and the openness of the system. Usually the dynamics of open quantum systems is studied using different mathematical approaches, such as the master equation for the reduced (averaged over the environmental variables) density matrix [1], and quasi-probability distributions (e.g. the so-called Q-function [2], the Wigner function [3], etc). Although all of these approaches allow one, in principle, to calculate the time evolution of the average values of the dynamical variables (observables), they have significant drawbacks. In particular, they may not be positively defined; they may be inconsistent for certain density matrices; it may be difficult to extract physical information from these distributions, especially in the context of quantum nonlinear open systems; in the "deep" quasi-classical region of parameters, $\epsilon = \hbar/J \ll 1$ (where $\hbar$ is Planck constant and J is a characteristic action of the corresponding classical system) these quasi-probability distributions exhibit fast oscillations due to phases like $\exp(iS(t)/\hbar)$, with $|S(t)| \simeq J$. Therefore, it is difficult to separate the physical effects for dynamical observables (requiring an additional multi-dimensional integration of quasi-distribution densities) from the effects of errors related to a concrete mathematical approach.

We are approaching these problems using an alternative strategy that starts from a mathematical dynamical theory based on exact, linear, partial differential equations (PDEs) for the observables of open many-body quantum nonlinear bosonic systems governed by a very general class of Hamiltonians. The key advantage of this method is that it leads to a well-behaved asymptotic theory for open quantum systems in the quasi-classical region of parameters. This approach is a generalization to the open case of the asymptotic theory for bosonic and spin closed quantum systems [4–6], and it can be applied to general open quantum nonlinear bosonic (and spin) systems for a large range of parameters, including the deep quasi-classical region.

In this contribution we review our first studies [7, 8] of this new approach to quantum nonlinear systems interacting with an environment. As will be discussed below, certain quantum effects present in the dynamics of these nonlinear systems are robust to the influence of the environment, and can survive long after decoherence and relaxation processes take place. In order to observe these effects experimentally it is necessary to have quasi-classical systems in certain region of parameters. Many quasi-classical systems have the drawback that they are either "too classical" (i.e., they have a large J so that the quasi-classical parameter ϵ is extremely small), or they interact too strongly with the environment (i.e., their effective temperature is so high that quantum effects are killed). Only recently have adequate open nonlinear quasi-classical systems become available, including Bose-Einstein condensates (BEC) with a large number of atoms and thermally well isolated, high frequency cantilevers with large nonlinearities and at sufficiently low temperatures, and nonlinear optical systems in high Q resonators. We will present estimates on the parameter regions where survival of certain quantum effects to environment-induced decoherence can be observed in these systems.

II. DYNAMICS OF QUANTUM OBSERVABLES FOR CLOSED QUANTUM NONLINEAR SYSTEMS

We first consider closed nonlinear systems. As a simple example we take the one-dimensional quantum nonlinear oscillator (QNO) described by the Hamiltonian [4, 6]

$$H_s = \hbar\omega a^\dagger a + \mu\hbar^2 (a^\dagger a)^2 \ , \ [a^\dagger, a] = 1, \tag{1}$$

where $a, a^\dagger$ are the annihilation and creation operators, ω is the frequency of linear oscillations, and μ is a dimensional parameter of nonlinearity. We assume that initially the QNO is prepared in a coherent state $|\alpha\rangle$ ($a|\alpha\rangle = \alpha|\alpha\rangle$). In the classical limit ($a \to \alpha, a^\dagger \to \alpha^*, |\alpha|^2 \to \infty, \hbar|\alpha|^2 = J$, the classical action of the linear oscillator) the Hamiltonian (1) becomes $H_{\rm cl} = \omega J + \mu J^2$. Below we use the following dimensionless notation: $\tau \equiv \omega t$, $\bar\mu \equiv \hbar\mu/\omega$, and $\mu_{\rm cl} \equiv \mu J/\omega$. The quantum parameter of nonlinearity $\bar\mu$ can be presented as the product of two parameters, quantum and classical, $\bar\mu = \epsilon\mu_{\rm cl}$. The parameter $\mu_{\rm cl}$ characterizes the nonlinearity in the classical nonlinear oscillator (BEC, cantilever, optical field, etc) and can be written as $\mu_{\rm cl} = (J/2\omega)(d\omega_{\rm cl}/dJ)$, where $\omega_{\rm cl} = dH_{\rm cl}/dJ = \omega + 2\mu J$ is the classical frequency of nonlinear oscillations. The limit $\mu_{\rm cl} \ll 1$ corresponds to weak nonlinearity, while $\mu_{\rm cl} \simeq 1$ corresponds to strong nonlinearity. As was mentioned above, ϵ is the quasi-classical parameter. Namely, $\epsilon \simeq 1$ corresponds to the pure quantum system, and $\epsilon \ll 1$ corresponds to the quasi-classical limit, which is the subject of our interest.

The time evolution of the expectation value of any observable of the system can be easily calculated when the system is initially populated in a coherent state $|\alpha\rangle$. For an arbitrary operator function $f = f(a, a^\dagger)$, the time-dependent expectation value (observable) of such a function,

$$f(\alpha^*, \alpha, \tau) = \langle\alpha| e^{iHt/\hbar} f e^{-iHt/\hbar} |\alpha\rangle,$$

satisfies a PDE of the form

$$\partial f/\partial\tau = \hat K f,$$

where $\hat K = \hat K_{\rm cl} + \epsilon\mu_{\rm cl}\hat K_{\rm q}$ (see [4] and references therein). Here the operator $\hat K_{\rm cl}$ includes only the first order derivatives and describes the corresponding classical limit, while the operator $\hat K_{\rm q}$ includes higher-order derivatives and describes the quantum effects. For the Hamiltonian (1) we have

$$\begin{aligned}\frac{\partial f}{\partial\tau} \ &= \ i(1+\bar\mu+2\bar\mu|\alpha|^2)\left(\alpha^*\frac{\partial}{\partial\alpha^*} - \alpha\frac{\partial}{\partial\alpha}\right) \\ &\quad + i\bar\mu\left((\alpha^*)^2\frac{\partial^2}{\partial(\alpha^*)^2} - \alpha^2\frac{\partial^2}{\partial\alpha^2}\right) f. \end{aligned} \tag{2}$$

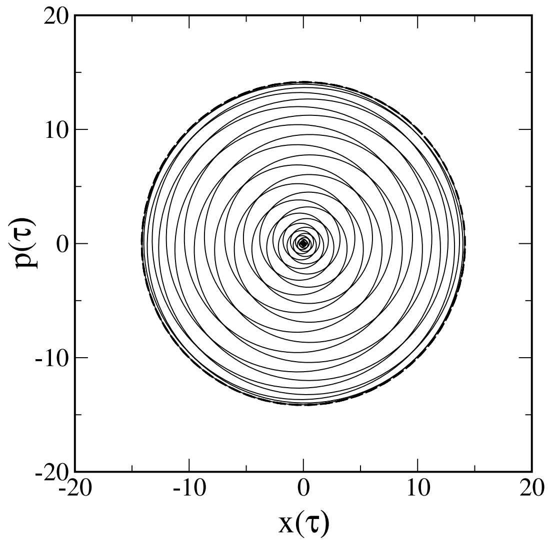

FIG. 1: Quasi-classical dynamics as described by the observable in Eq. (3). Parameters are $\epsilon = 0.02$, $\bar\mu = 0.01$, $\tau_{\rm E} = 5$, $\tau_{\rm R} = 314$, $\tau_{\rm cl} = 2.09$, $|\alpha|^2 = 100$, $\mu_{\rm cl} = 1$. Hence $\tau_{\rm cl} < \tau_{\rm E} < \tau_{\rm R}$.

In particular, for the operator function $f(\tau = 0) = a$ the evolution of $f(\tau)$ corresponds to the evolution of $\alpha(\tau) = \langle\alpha|a(\tau)|\alpha\rangle$, with the initial condition $\alpha(\tau = 0) = \alpha$. In this case Eq.(2) can be solved exactly [4, 6]

$$\alpha(\tau) = \alpha e^{-i(1+\bar\mu)\tau} \ e^{|\alpha|^2(e^{-2i\bar\mu\tau}-1)}. \tag{3}$$

Figure 1 depicts the dynamics described by the observable Eq. (3) in the coordinate-momentum plane. The effective coordinate is defined as

$$x(\tau) = (\alpha^*(\tau) + \alpha(\tau))/\sqrt{2},$$

and the effective momentum is defined as

$$p(\tau) = i(\alpha^*(\tau) - \alpha(\tau))/\sqrt{2}.$$

The classical dynamics is described by the function

$$\alpha_{\rm cl}(\tau) = e^{-i(1+2\mu_{\rm cl})\tau},$$

which corresponds to the circumference in Fig. 1.

The solution (3) has three characteristic time-scales. In the limit $\bar\mu\tau \ll 1$, Eq. (3) can be written in the form

$$\alpha(\tau) = \alpha_{\rm cl} e^{-\tau^2/2\tau_{\rm E}^2} \ \left[1 + O(\bar\mu\tau) + O(|\alpha|^2\bar\mu^3\tau^3)\right]. \tag{4}$$

Thus, the first time-scale is the characteristic classical time-scale, which can be chosen as the period of classical nonlinear oscillations,

$$\tau_{\rm cl} = 2\pi\omega/\omega_{\rm cl} = 2\pi/(1+2\mu_{\rm cl}).$$

The second time-scale is a characteristic time of departure of the quantum dynamics from the corresponding classical one. We call this time the Ehrenfest time-scale,

$$\tau_{\rm E} = 1/2\bar\mu|\alpha|$$

(a similar time-scale for quantum systems which are classically chaotic was introduced in [9], and was widely discussed. See, for example, [10, 11]). The amplitudes of quantum and classical observables coincide at multiple times of the quantum recurrence time-scale, which is the third characteristic time-scale,

$$\tau_{\rm R} = \pi/\bar{\mu}.$$

Since we are interested in the quasi-classical region of parameters, it is reasonable to impose the following inequalities on these three characteristic time-scales: $\tau_{\rm cl} < \tau_{\rm E} \ll \tau_{\rm R}$. In our case, $\tau_{\rm cl}/\tau_{\rm E} = 4\pi\bar{\mu}|\alpha|/(1+2\mu_{\rm cl}) \approx \pi\sqrt{\epsilon} \ll 1$, and $\tau_{\rm E}/\tau_{\rm R} \approx \sqrt{\epsilon}/\pi \ll 1$. When deriving the first inequality, we used the conditions $|\alpha|^2 \simeq J/\hbar = 1/\epsilon$ and $\mu_{\rm cl} \simeq 1$, which corresponds to the condition of strong nonlinearity. Note that the condition $|\alpha|^2\bar{\mu}^3\tau^3 \simeq 1$ (see the third term in (4) in the square brackets in the expression for $\alpha(\tau)$) gives the characteristic times $\tau \gg \tau_{\rm E}$, namely $\tau/\tau_{\rm E} = 2/\epsilon^{1/6} \gg 1$. This means that the third term in Eq. (4) is small on the time scale $\tau_{\rm E}$. For the values of parameters in Fig. 1, the inequalities $\tau_{\rm cl} < \tau_{\rm E} \ll \tau_{\rm R}$ are satisfied.

A. Quantum effects as a singular perturbation to the classical solution

As was mentioned above, the general form of the differential operator $\hat{K}$ is

$$\hat{K} = \hat{K}_{\rm cl} + \epsilon\mu_{\rm cl}\hat{K}_{\rm q}.$$

The operator $\hat{K}_{\rm cl}$ includes only the first order derivatives and describes the classical dynamics of the system. Usually, the corresponding classical solution can be found by the method of characteristics, or some alternative well-developed methods. Note that even this part of the solution can be rather complicated, especially for classically unstable and chaotic systems, and usually requires large-scale numerical simulations. (See details for closed quantum nonlinear systems and quantum nonlinear systems interacting with the time-periodic fields [4]). Another example is the classical mean field GP equation for BECs, which is also described by the differential operator $\hat{K}_{\rm cl}$ [12]. For quantum linear systems ($\mu_{\rm cl} = 0$) the quantum effects vanish (except for renormalization of parameters) for any values of the quasi-classical parameter ϵ. The differential operator $\hat{K}_{\rm q}$ includes second and higher order derivatives, and it describes quantum effects. The solutions of these PDEs are well behaved in the quasi-classical region, $\epsilon \ll 1$, and in contrast to the fast oscillating WKB solutions (typical of standard methods based on quasi-probability distributions), our method leads to the so-called Laplace-type expansions [5]. The crucial property of the Laplace asymptotics is that the dynamical observables are exponentially localized in phase space around coherent states.

As it follows from our considerations, quantum effects for observables represent a singular perturbation to the classical solution. Indeed, in the quasi-classical region, quantum terms in our PDEs are represented by the product of the small parameter ϵ times high order derivatives. Consequently, these quantum terms lead to a secular behavior of the solution, which diverges in time from the solution describing the corresponding classical world. Only the case $\epsilon = 0$ (for finite $\mu_{\rm cl}$) corresponds to the exact classical limit. But the problem with this limit is that for any real system $\epsilon \neq 0$ (because $\hbar \neq 0$ and $J \neq \infty$). Then, even a very small value of ϵ still results in a singular perturbation to the classical solution due to the quantum terms.

The singularity arising from the quantum terms reminds, up to some extent, of the singularity provided by a "small" viscosity in the Navier-Stokes (NS) equation, describing the dynamics of liquid and gas flows. Indeed, in the NS equation a small viscosity multiplies the higher order spatial derivatives. Then, even for very large Reynolds numbers (when the nonlinear terms are very large compared to the viscous ones), the viscosity (which formally represents a "small" perturbation) plays a crucial role in the dynamics of the flow. Similarly, a small parameter ϵ multiplies the higher order derivatives resulting in a quantum singular perturbation for observables, even in a "deep" quasi-classical region. It is this singularity that leads to a significant difference from the classical solution.

The fact that for quantum nonlinear systems the terms with high-order derivatives in the evolution equations for the density matrix and for the Wigner function represent a singular perturbation to the classical limit (Liouville function) is well known. However, in spite of a large number of papers on this subject, from this fact it is still unclear what are the conditions for the quantum-classical correspondence for observables. The important question is: Under what conditions does the environment kill (if at all) the quantum corrections which represent a singular perturbation to the *observables* of the classical world? Our answer to this question is the following: Generally, for open quantum nonlinear systems in the deep quasi-classical region of parameters, quantum effects survive after decoherence and relaxation processes took place.

B. Frequency Fourier spectrum of the effective momentum

The observable $\alpha(\tau)$ can be written in the form

$$\alpha(\tau) = \alpha e^{-i(1+\bar{\mu})\tau - i|\alpha|^2\sin(2\bar{\mu}\tau)}\, e^{-2|\alpha|^2\sin^2(\bar{\mu}\tau)}. \qquad (5)$$

The first exponent in Eq.(5) is responsible for phase modulations of the classical dynamics, while the second one is responsible for amplitude modulations. The characteristic time-scale of the amplitude modulations, $\tau_{\rm am}$, is defined by the condition $|\alpha|^2\bar{\mu}^2\tau_{\rm am}^2 \approx 1$, or by the Ehrenfest time scale $\tau_{\rm am} \approx \tau_{\rm E}$. The time-scale of phase modulations of the classical dynamics is defined by the condition $|\alpha|^2\bar{\mu}\tau_{\rm ph} \approx 1$, or $\tau_{\rm ph} \approx \tau_{\rm E}/\sqrt{\epsilon} \gg \tau_{\rm E}$. Thus, the shortest

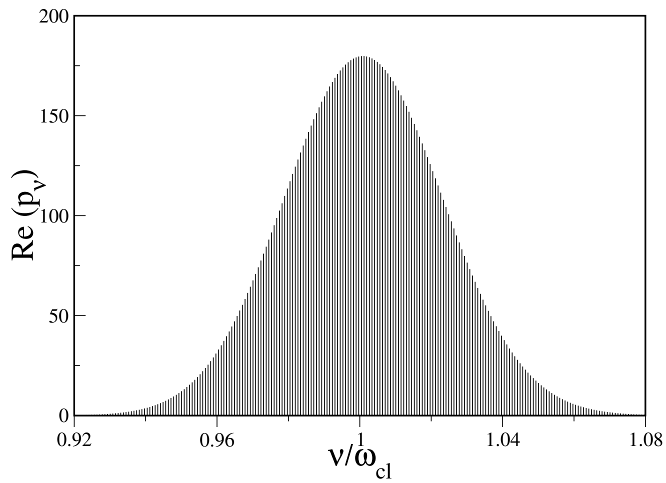

FIG. 2: Frequency spectrum of the effective momentum $p(\tau)$. Parameters are $\epsilon = 1/900$, $\bar{\mu} = 1/900$, $\tau_{\rm E} \approx 15$, $\tau_{\rm R} \approx 900\pi$, $\tau_{\rm cl} \approx 2\pi/3$, and $|\alpha|^2 = 900$. Hence $\tau_{\rm cl} < \tau_{\rm E} < \tau_{\rm R}$.

time-scale which characterizes the deviation of the quantum dynamics from the corresponding classical one is the Ehrenfest time. Moreover, this time-scale is responsible for the finite width of the spectral line $\Delta\nu_{\rm E} \approx 2\sqrt{2}/\tau_{\rm E}$.

Figure 2 depicts the frequency Fourier spectrum of the effective momentum $p(\tau)$, with initial condition $p(0) = 0$. One can see that the frequency spectrum consists of one central line with $\nu = \omega_{\rm cl} = 1 + 2\mu_{\rm cl}$, and a width which is approximately equal to $\Delta\nu \approx \Delta\nu_{\rm E}$. In our case the analytical estimate gives $\Delta\nu_{\rm E} \approx 2\sqrt{2}/\tau_{\rm E} \approx 0.19$, which is very close to the numerical results presented in Fig. 2, $\Delta\nu \approx 0.183$. The fine structure of the frequency spectrum is provided by the characteristic revival time scale $\tau_{\rm R} = \pi/\bar{\mu}$, or by the frequencies $\nu_n = 2\bar{\mu}n$, which are responsible for the complicated dynamics of quantum recurrences.

III. DYNAMICS OF QUANTUM OBSERVABLES FOR OPEN QUANTUM NONLINEAR SYSTEMS

The Hamiltonian of open quantum nonlinear system interacting with an environment contain three terms,

$$\hat{H} = \hat{H}_{\rm S} + \hat{H}_{\mathcal{E}} + \hat{H}_{\rm int}.$$

The first term is typically a time-independent polynomial Hamiltonian of a general form which describes the self evolution of the closed system,

$$\hat{H}_{\rm S} = \sum_{l,s} H_{l,s} a_1^{\dagger l_1} \dots a_N^{\dagger l_N} a_1^{s_1} \dots a_N^{s_N},$$

where $H_{l,s} = H^*_{l,s}$, $l = (l_1, \dots, l_N) \in Z_+^N$, and $s = (s_1, \dots, s_N) \in Z_+^N$. The operators a_l and $a_k^\dagger$ satisfy bosonic commutation relations, $[a_l, a_k^\dagger] = \delta_{l,k}$. A particular system corresponds to a particular choice of the coefficients $H_{l,s}$ in $\hat{H}_{\rm S}$. The second term is the Hamiltonian of the environment, which, for example, can be modeled by a collection of harmonic oscillators,

$$\hat{H}_{\mathcal{E}} = \sum_{j=1}^{N} \hbar\omega_j b_j^\dagger b_j.$$

Usually, the oscillators of the environment are assumed to be initially in thermal equilibrium,

$$\rho_{\mathcal{E}}(t=0) = Z_{\mathcal{E}}^{-1} e^{-\hat{H}_{\mathcal{E}}/k_{\rm B}T},$$

where $Z_{\mathcal{E}} = {\rm Tr}[e^{-\hat{H}_{\mathcal{E}}/k_{\rm B}T}]$ is the partition function of the environment, T is the temperature of the environment, and $k_{\rm B}$ is Boltzmann constant. The third term is the interaction Hamiltonian between the system and the environment. Prototype examples are the dipole-dipole interaction Hamiltonian,

$$\hat{H}_{\rm int} = \sum_{n,j} g_{n,j}[(a_n^\dagger + a_n)(b_j^\dagger + b_j)],$$

the density-density interaction Hamiltonian [7], etc.

A. The differential operator $\hat{K}$ for many-body systems

In a general many-body system the differential operator $\hat{K}$ can formally be written as

$$\begin{aligned} \hat{K} \;=\; & \frac{i}{\hbar} e^{-\sum(|\alpha_n|^2+|\beta_j|^2)} \sum \left[H\left(\alpha_l^*, \beta_q^*, \frac{\partial}{\partial\alpha_l^*}, \frac{\partial}{\partial\beta_q^*}\right) \right. \\ & \left. - H\left(\alpha_l, \beta_q, \frac{\partial}{\partial\alpha_l}, \frac{\partial}{\partial\beta_q}\right)\right] e^{-\sum(|\alpha_n|^2+|\beta_j|^2)}. \end{aligned} \quad (6)$$

Note that after explicit differentiations, exponents in $\hat{K}$ vanish. Specific examples considered in our previous works include: (i) the closed quantum one-dimensional nonlinear system in the vicinity of an elliptic point [4, 6]; (ii) the closed quantum one-dimensional nonlinear system in the neighborhood of a hyperbolic point [6, 12]; (iii) chaotic systems describing the interaction of atoms with radiation and external *rf* fields [4]; (iv) finally, in [7, 8] we considered the open system of a QNO interacting with different types of environments.

B. Frequency Fourier spectrum of $p(\tau)$ in the presence of an environment

Now we introduce formally a relaxation (dissipation) term into Eq. (3). Namely, we consider the function

$$\alpha(\tau) = \alpha e^{-\gamma\tau - i(1+\bar{\mu})\tau}\, e^{|\alpha|^2(e^{-2i\bar{\mu}\tau}-1)}, \quad (7)$$

where the parameter γ plays the role of an effective relaxation. Then, the characteristic time scale of relaxation is

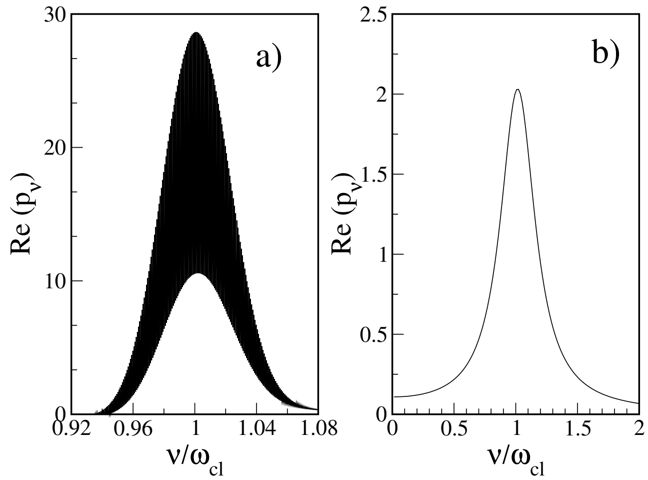

FIG. 3: Fourier frequency spectrum of the momentum $p(\tau)$ obtained from Eq. (7). Parameters are: a) $\gamma = 0.0005$, $\tau_\gamma = 2000 \gg \tau_{\rm E} \approx 15$; b) $\gamma = 0.5$, $\tau_\gamma = 2 < \tau_{\rm E} \approx 15$; all other parameters are the same as in Fig. 2.

$\tau_\gamma = 1/\gamma$. We consider the frequency Fourier spectrum of the momentum

$$p(\tau) = i(\alpha^*(\tau) - \alpha(\tau))/\sqrt{2},$$

with $\alpha(\tau)$ from Eq. (7), for two cases: (i) $\tau_\gamma \gg \tau_{\rm E}$ (Fig. 3a), and (ii) $\tau_\gamma < \tau_{\rm E}$ (Fig. 3b) (similar dependences can be built for the effective coordinate $x(\tau)$). As one can see, when the influence of the effective dissipation is small (Fig. 3a), the width of the Gaussian spectral line (at the level e^{-1}) is still determined by the Ehrenfest time-scale ($\Delta\nu_{\rm E} \simeq 2\sqrt{2}/\tau_{\rm E} \approx 0.19$), and not by the environment ($\Delta\nu_\gamma \simeq 2\gamma = 0.001$). The numerical results give $\Delta\nu \approx 0.186$. Note that in this case the fine structure of the spectral line is not completely destroyed, as both time-scales, $\tau_{\rm R} \approx 2826$ and $\tau_\gamma = 2000$, are of the same order. In the case of strong dissipation (Fig. 3b), the width of the spectral line has a Lorentzian form,

$${\rm Re}(p_\nu) = \gamma^2 {\rm Re}(p_0)/(\gamma^2 + \nu^2),$$

with a width (at ${\rm Re}(p_\nu) = 1/2$) determined by the dissipation parameter γ ($\Delta\nu_\gamma \approx 2\gamma = 1$). The numerical results are in good agreement, $\Delta\nu \approx 1$. Also, the fine structure is destroyed, as in this case $\tau_\gamma = 2 \ll \tau_{\rm R} \approx 2826$. Similar dependences of the frequency spectrum on the parameters are given in [8] for a concrete example of the QNO interacting with the environment.

IV. AN EXACTLY SOLVABLE EXAMPLE OF AN OPEN QUANTUM NONLINEAR SYSTEM

Although the PDEs described above look rather complicated (especially for open quantum nonlinear systems), we have found the exact solution for a QNO interacting with the environment in the special case of a density-density type of interaction [7]. These results demonstrate that, for some region of parameters, quantum effects survive the effects of the environment, and the corresponding observables do not have a classical limit. We present here the results of [7] in the context of the quantum-classical transition for observables and the frequency Fourier spectrum. Following [7], we choose $\hat{H}_{\rm S}$ as Eq. (1), and

$$\begin{aligned} \hat{H}_{\mathcal{E}} &= \sum_{j=1}^{N} \hbar\omega_j b_j^\dagger b_j \\ \hat{H}_{\rm int} &= a^\dagger a \sum_{j=1}^{N} g_j b_j^\dagger b_j. \end{aligned} \quad (8)$$

As was discussed above, for the simple closed quantum nonlinear system given by Eq. (1) there are three characteristic time-scales (see [5] for details on multidimensional systems). Due to the interaction with the environment, two new time-scales appear: $\tau_{\rm d}$ - a very short decoherence time, and τ_γ -the relaxation time. All of these five time-scales depend on the parameters of the system and the environment. The typical region of parameters in which one can observe quantum effects after decoherence and relaxation is $\tau_{\rm d} \ll \tau_{\rm cl} < \tau_{\rm E} < \tau_\gamma < \tau_{\rm R}$. The key inequality is $\tau_{\rm E} < \tau_\gamma$. In this case, the deviation of the quantum dynamics from the classical one formally works as an effective "quantum relaxation" (or a "quantum amplitude modulation"), which gives the main contribution to the frequency spectral line width. The relations between $\tau_{\rm cl}$ and $\tau_{\rm E}$, and between $\tau_{\rm R}$ and τ_γ are not so important. There can be additional time-scales related to accumulation of quantum phases [8], multidimensionality [5], etc. The details for a one-dimensional case were presented in [7, 8].

For the model under consideration the interaction Hamiltonian commutes with Hamiltonian of the system. Thus, the operators $a^\dagger a$ and $\hat{H}_{\rm int}$ are integrals of motion. At the same time, such dynamical observables as $x(\tau)$ or $p(\tau)$ experience the influence of the environment. Note that the dynamics of the observables for the Hamiltonians Eqs. (1), (8) can be calculated in the Schrödinger representation. At the same time, this model system is useful as an easy demonstration of our approach. Following our previous results [7] it is possible to write an exact linear PDE for any quantum dynamical observable

$$f(\alpha^*, \alpha; \beta^*_{\bf j}, \beta_{\bf j}; t) = \langle \alpha, \beta_{\bf j} | \hat{f}(t) | \alpha, \beta_{\bf j} \rangle,$$

where

$$\hat{f}(t) = f(a^\dagger(t), a(t); b^\dagger_{\bf j}(t), b_{\bf j}(t))$$

is a generic Heisenberg operator function, and $|\alpha, \beta_{\bf j}\rangle$ is an initial coherent state of the system and the environment. Here $a^\dagger(t)$, $a(t)$, $b^\dagger_{\bf j}(t)$, and $b_{\bf j}(t)$ are the Heisenberg bosonic creation and annihilation operators for the system and the environment, respectively, and

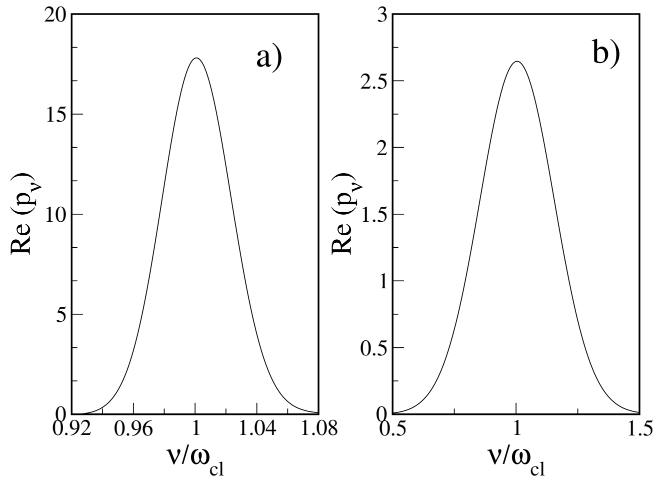

FIG. 4: Fourier frequency spectrum of the effective momentum $p(\tau)$ obtained from Eq. (10) for different values of $\gamma_{\mathcal{E}}$. All other parameters are the same as in Fig. 2. (a) $\gamma_{\mathcal{E}} = 5\times10^{-5}$. The characteristic relaxation time due to the interaction with the environment is $\tau_{\gamma_{\mathcal{E}}} = 200$. The characteristic width of the spectral line due to the interaction with the environment is $\Delta\nu_{\gamma_{\mathcal{E}}} \approx 2\sqrt{2\gamma_{\mathcal{E}}} = 0.02$. The Ehrenfest time scale is $\tau_{\mathrm{E}} \approx 15$, hence $\tau_{\gamma_{\mathcal{E}}} \gg \tau_{\mathrm{E}}$. The width of the spectral line due to the Ehrenfest time scale is $\Delta\nu_{\mathrm{E}} \approx 0.19$, and the numerical result give $\Delta\nu \approx 0.183$; (b) $\gamma_{\mathcal{E}} = 0.2$ ($\tau_{\gamma_{\mathcal{E}}} = 3.16$ and $\Delta\nu_{\gamma_{\mathcal{E}}} \approx 1.26$). The Ehrenfest time scale is $\tau_{\mathrm{E}} \approx 15$, hence $\tau_{\gamma_{\mathcal{E}}} < \tau_{\mathrm{E}}$. The numerical result for the width of the line is approximately 1.29, that corresponds to $\Delta\nu_{\gamma_{\mathcal{E}}}$.

$\mathbf{j} = (j_1,\ldots,k_N)$. Consequently, the physical interpretation of the solution is straightforward and does not require the computation of multi-dimensional integrals over rapidly oscillating functions in order to calculate physical quantities. The corresponding PDE has the form

$$\frac{\partial}{\partial t} f(\alpha^*,\alpha;\beta^*_{\mathbf{j}},\beta_{\mathbf{j}}) = \hat{K} f(\alpha^*,\alpha;\beta^*_{\mathbf{j}},\beta_{\mathbf{j}};t), \tag{9}$$

where the differential operator $\hat{K}$ includes the derivatives of different orders over α^*, α, $\beta^*_{\mathbf{j}}$, and $\beta_{\mathbf{j}}$, and depends on the explicit form of the corresponding Hamiltonian,

$$\hat{H} = \hat{H}_{\mathrm{S}} + \hat{H}_{\mathcal{E}} + \hat{H}_{\mathrm{int}}.$$

As before, the general form of the differential operator $\hat{K}$ is $\hat{K} = \hat{K}_{\mathrm{cl}} + \hat{K}_{\mathrm{q}}$. The operator $\hat{K}_{\mathrm{cl}}$ includes only the first order derivatives and describes the classical dynamics of the system and environment. The operator $\hat{K}_{\mathrm{q}}$ (now it includes all small parameters) describes the quantum effects of the system and the environment. The explicit expressions for both these operators are given in [7].

The function $f(\alpha^*,\alpha,\beta^*_{\mathbf{j}},\beta_{\mathbf{j}})$ has to be traced over the variables of the environment $\beta^*_{\mathbf{j}},\beta_{\mathbf{j}}$. We have assumed above that initially each jth environmental oscillator is populated initially in the coherent state $|\beta_j\rangle$. Let us now assume that the each environmental oscillator is initially in a mixed (thermal) state at temperature T. Then we should perform an additional averaging of the environmental oscillators over the thermal distribution. The corresponding procedure is thoroughly explained in [7]. We have the following exact solution for the observable $\langle\alpha(\tau)\rangle_{\mathcal{E}}$, averaged over the environmental variables,

$$\langle\alpha(\tau)\rangle_{\mathcal{E}} = e^{-\gamma_{\mathcal{E}}\tau^2/2}\, e^{-i\delta\bar{\omega}\tau}\, \alpha(\tau), \tag{10}$$

where the thermal relaxation $\gamma_{\mathcal{E}}$, and the thermal phase shift $\delta\bar{\omega}$ are given by :

$$\begin{aligned} \gamma_{\mathcal{E}} &= (1/\hbar^2\omega^2)\sum_j g_j^2[\langle n_j^2\rangle - \langle n_j\rangle^2] \\ \delta\bar{\omega} &= (1/\hbar\omega)\sum_j g_j\langle n_j\rangle \end{aligned}$$

The function $\alpha(\tau)$ in Eq. (10) coincides with Eq. (3). It is clear from Eq. (10) that under the condition

$$\frac{1}{\tau_{\mathrm{E}}^2} > \gamma_{\mathcal{E}}, \tag{11}$$

the width of the frequency spectrum of $\langle\alpha(\tau)\rangle_{\mathcal{E}}$ is defined by the Ehrenfest time-scale τ_{E} and not by the interaction with the environment (see Fig. 4a.). In the opposite case, $1/\tau_{\mathrm{E}}^2 < \gamma_{\mathcal{E}}$, the width of the spectral line is determined by the interaction with the environment (see Fig. 4b). A similar result was obtained in [8] for the QNO interacting via the dipole-dipole interaction with the environment.

V. CONCLUSIONS

Our main statement is that generally there is no classical limit for quantum nonlinear systems interacting with the environment, even when these systems are in the deep quasi-classical region of parameters (the quasi-classical parameter ϵ is small but finite). In this context we note that most classical systems surrounding us represent a very particular exception due to (i) either an extremely deep quasi-classicality (extremely small value of ϵ) and/or (ii) a very strong interaction with the environment. At the same time, the general belief in the recent scientific literature is that after the process of decoherence, the quasi-classical system can be described by using classical probabilistic approaches. According to the results discussed here it appears to be true only for quantum linear systems (with quadratic Hamiltonians). In the general case of quantum nonlinear systems, in the deep quasi-classical region, quantum effects survive after the processes of decoherence and relaxation took place. Moreover, these quantum effects make a crucial contribution to the dynamics of observables. This observation will have a significant influence, for example, on our understanding of the properties of noise in complex quantum technological systems and nano-devices. In particular, the performance of future BEC based interferometers and nano-devices will be limited by the level of noise. Thus, understanding the properties of noise is a very important issue for quantum technologies. A generalization of this result to many-body systems is straightforward. At the same time, many additional details will appear in the many-body systems.

The key condition for survival of quantum effects for observables related to the Ehrenfest time-scale is $\tau_{\rm E} < \tau_\gamma$, which can be written in the form:

$$\Theta \equiv \frac{\tau_\gamma}{\tau_{\rm E}} = 2\mu_{\rm cl}\sqrt{\epsilon}\tau_\gamma \gg 1. \qquad (12)$$

We now present estimates on different quasi-classical nonlinear physical systems that may satisfy the above condition, and therefore may lead to the observation of certain quantum effects that survive the process of environment-induced decoherence and dissipation.

We start by considering Bose-Einstein condensates (BECs). The quantum many-body BEC Hamiltonian can be reduced to the QNO Hamiltonian in the so-called single mode approximation. This is valid as long as the total number of depleted atoms from the condensate mode is small. This condition holds in the limit $N \gg 1$, and $Na = {\rm const}$, where a is the s-wave scattering length. The quasiclassical parameter is $\epsilon = 1/N$, and the classical parameter of nonlinearity is $\mu_{\rm cl} = 4\pi\hbar Na/m\omega V_{\rm eff}$. Here m is the mass of the atoms, ω is the trapping frequency (we are assuming an isotropic harmonic 3D trap, with a potential $V(\mathbf{x}) = m\omega^2|\mathbf{x}|^2/2$), and $V_{\rm eff}^{-1} = \int d^3x|\Psi(\mathbf{x})|^4$, where $\Psi(\mathbf{x})$ is the condensate Gross-Pitaevskii wave function. The condition (12) can be satisfied by a 3D BEC in the Thomas-Fermi (TF) limit, in which the nonlinearity outweighs the kinetic energy. This limit corresponds to $\kappa \equiv Na/l_{\rm osc} \gg 1$, where $l_{\rm osc} = \sqrt{\hbar/m\omega}$ is the width of the harmonic oscillator ground state wave function. The TF condensate wave function is $\Psi(\mathbf{x}) = \sqrt{(\mu_{\rm chem} - V(\mathbf{x}))/Ng}$, where $\mu_{\rm chem}$ is the chemical potential and $g = 4\pi\hbar^2 a/m$. Computing the effective volume one gets

$$V_{\rm eff} = \frac{56}{4\cdot 15^{2/5}}(Na)^{3/5}\left(\frac{\hbar}{m\omega}\right)^{6/5}.$$

Therefore, the dimensionless Ehrenfest time in this case is $\tau_{\rm E} = \omega t_{\rm E} = 56\sqrt{N}/(8\pi 15^{2/5}\kappa^{2/5})$. Using typical parameters for ^{87}Rb, $a = 5$ nm, $m = 1.5\times10^{-25}$ kg, $\omega/2\pi = 100$ Hz, and $N = 10^4$ we have $\kappa \approx 50 \gg 1$ and $\tau_{\rm E} \approx 16$. Assuming $t_\gamma \approx 1$ sec (i.e., $\tau_\gamma \approx 600$), the condition $\tau_{\rm E} \ll \tau_\gamma$ would be satisfied.

For a cantilever (or a mechanical resonator) the quasi-classical parameter is $\epsilon = 1/n$, where n is the average number of levels involved in the coherent state of the cantilever. For the dimensionless relaxation time we take $\tau_\gamma = 2Q$, where Q is the cantilever quality factor. Then, for a cantilever the condition Eq. (12) takes the form

$$\Theta_{\rm cantilever} = \frac{4\mu_{\rm cl}Q}{\sqrt{n}} \gg 1. \qquad (13)$$

Different aspects of cantilevers, from kilohertz to gigahertz frequencies, including their nonlinear properties, are discussed, for example, in [13, 14]. A condition similar to Eq. (13) holds for quantum nonlinear optical systems in high quality resonators. In this case, is the average number of photons in the initially coherent state of the cavity resonance mode, and a classical parameter of nonlinearity can be written as $\mu_{\rm cl} = \chi J/\omega_{\rm cav}$, where χ is the nonlinear susceptibility, and $\omega_{\rm cav}$ is the cavity resonance frequency [15].

We hope that the condition (12) can be experimentally realized, and quantum effects related to the Ehrenfest time-scale can be observed in the quasi-classical region of parameters.

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