Dynamical Chaos in a One-Dimensional Bose System

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We study the many-body quantum dynamics of δ-interacting bosons confined in a one-dimensional ring. We study the transition from the mean-field to the Tonks-Girardeau regime from the viewpoint of quantum chaos. The latter manifests itself in the model by a linear increase with time of the Shannon entropy of the wavefunction. Correspondingly, the momentum distribution of the system reveals transitions from periodic behavior, to a statistical relaxation and finally to a steady state distribution. This transition can be observed experimentally by studying the interference fringes obtained after releasing the trap and letting the boson system expand ballistically.

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The study of the quantum many-body dynamics of a Bose-Einstein condensate (BEC) is a major challenge in the physics of trapped gases. The mean-field Gross-Pitaevskii framework has successfully described an impressive set of experimental data [1], but it fails when genuine quantum many-body (QMB) correlations are important (e.g., in quantum phase transitions [2, 3], in the design of quantum information devices [4] and interferometers [5], and near dynamically unstable regions where quantum corrections appear on a logarithmically short time-scale [6].

QMB effects also become particularly striking in low-dimensional systems, where the competition between quantum fluctuations and statistical properties is particularly enhanced. Recent experimental achievements in effective one-dimensional (1D) harmonically confined quantum degenerate systems [7] have stimulated several efforts to understand their main properties [8]. The 1D regime can be achieved in optical/magnetic traps when the radial degrees of freedom are frozen by tight transverse confinement. The Hamiltonian of the boson gas is given by the Lieb-Liniger model [9], where the two-body interaction is assumed to be point–like. The thermodynamical properties of this Hamiltonian, and its excitation spectra have been calculated analytically by Lieb and Liniger [9]. An unexpected feature, predicted by Girardeau [10], is the onset of fermionization when $n/g \to 0$ (here n is the particle density, and g is the interatomic coupling constant which is inversely proportional to the 1D interatomic scattering length [11]). In this Tonks-Girardeau (TG) regime, the density of the interacting bosons becomes identical to that of non-interacting fermions (while, of course, the wave-function keeps the bosonic symmetry). On the other hand, in the opposite limit, $n/g \to \infty$, the system is described in the mean-field (MF) approximation as a weakly interacting boson gas. The crossover between these two regimes occurs near $n/g \sim 1$. In this region the MF approach breaks down and more complicated two-body correlations become crucial.

While the thermodynamics of a 1D boson gas is fairly well understood, only a few studies have been devoted so far to explore the many-body dynamical properties of this system. In this context, one should mention the investigation of general quantum (beyond MF) properties of the dynamics of trapped bosons, which can be considered as the next frontier in BEC studies. In pursuit of this goal, in this Letter we study numerically the quantum dynamics of N interacting Bose particles on a one-dimensional ring of length L. Our main result is the discovery of a dynamical transition from regular to (quantum) chaotic dynamics when $n/g \to 1$. This transition is accompanied by a linear increase of the Shannon entropy of the wave packets, that is directly related to the vanishing of the interference fringes which occurs after releasing the confinement and letting the bosons expand ballistically.

Our model is specified by the Hamiltonian,

$$\hat{H} = \sum_k \epsilon_k \hat{n}_k + \frac{g}{2L} \sum_{k,q,p,r} \hat{a}_k^\dagger \hat{a}_q^\dagger \hat{a}_p \hat{a}_r \delta(k+q-p-r). \quad (1)$$

Here $\hat{n}_k = \hat{a}_k^\dagger \hat{a}_k$ is the occupation number operator; $\hat{a}_s^\dagger$ and $\hat{a}_s$ are the creation-annihilation operators; and $\epsilon_k = 4\pi^2 k^2/L^2$. In what follows, we explore the situation when all bosons initially occupy the single-particle level with the angular momentum $k = 0$ or, equivalently, at $t = 0$ the system is in the unperturbed ($g = 0$) ground state $|\Psi_g\rangle$. Note that the total momentum is conserved in time. Our main interest is the evolution of the system for different values of the control parameter n/g. This type of experiment can be realized since the ratio n/g can be manipulated by tuning the interatomic scattering length.

It is convenient to label *single-particle states* $|k\rangle$ according to their momentum $k = 0, \pm 1, \pm 2 ... \pm m, ...$ Then, any *many-body* state $|j\rangle$ can be represented as

$|..., n^j_{-m}, ..., n^j_0, ..., n^j_m, ...\rangle$, where n^j_k represents the number of particles in the single-particle level characterized by the momentum k of the j-th state. In our numerical studies we need to take a finite number of particles N in a finite number $\ell = 2m + 1$ of single-particle states. Clearly, the number N of atoms and ℓ of levels should be chosen in a consistent way in order to have results which can be extrapolated to an arbitrary number of atoms. In a 1D geometry on a ring, N particles define the smallest spacing, which corresponds to the largest value of the momentum $m \approx N$. In our numerical simulations, we satisfy the latter relation when choosing changing values of N and m.

The transition from the MF to the TG regime can be understood using the following handwaving argument. Having all particles initially in the lowest state with $k = 0$, let us estimate the strength of the interaction necessary to move two particles from the unperturbed ground state to the upper (and lower) single particle levels $k = \pm m$. One can expect that this interaction will result in an ergodic filling (in time) of all single-particle states. The energy required for the shift is approximately $m^2/L^2 \approx N^2/L^2$, and the matrix element of the interaction between the corresponding states is $< j|\hat{V}|j' >\sim g\sqrt{N(N-1)}/L$. Equating these values one can obtain $n = N/L \sim g$ that is associated with the crossover from the bosonic to fermionic regime. Dynamically this crossover is reflected by a rapid depletion of the fraction of particles in the lowest single-particle state with $k = 0$. Note that at $n/g \sim 1$, the ratio $N_0/N \sim \frac{1}{2}$, with $N_0 = \langle \hat{a_0}^\dagger \hat{a_0} \rangle$.

One of the most appropriate quantities to characterize the dynamical properties of this system is the Shannon entropy of the wave packet [12],

$$S(t) = -\sum_j |\Psi_j(t)|^2 \ln |\Psi_j(t)|^2 \qquad (2)$$

Here $\Psi_j(t) = \langle j|\Psi(t)\rangle$ is the projection of the wave function onto the noninteracting many-body basis. After switching on the interaction, the wave function evolves according to the total Hamiltonian, $\Psi(t) = e^{-iHt}\Psi_g(0)$ and spreads over the unperturbed basis.

A few examples of the time dependence of the entropy are reported in Fig.1. One can see that for $n/g \gg 1$ the entropy oscillates in a regular way, while for $n/g \ll 1$ there is a generic linear time dependence for a short time followed by a saturation at later times. While saturation is due to the finiteness of our Hilbert space, the linear growth in time of the entropy was shown to be associated with the onset of quantum chaos [12]. Quantum chaos is a situation when both the energy spectrum and eigenstates manifest specific properties described by the random matrix theory. (See, for example, [13]).

The Shannon entropy, defined by Eq.(2) allows one to estimate the effective number of unperturbed many-body states that are involved in the dynamics due to the inter–particle interaction, $N_{eff}(t) \approx \exp\{S(t)\}$.

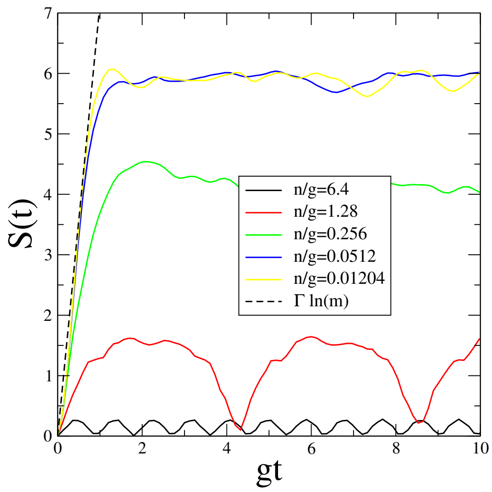

FIG. 1: Entropy as a function of the rescaled time gt for fixed $m = 6$ and different values of n/g. Data are given for $N = 6$. The saturation of $S(t)$ for small values of n/g manifests the onset of the Tonks-Girardeau regime. The dashed line corresponds to the theoretical prediction (3).

The general problem of the entropy increase has been studied in detail for the two-body random interaction model (TBRM). (See [12] and references therein). As was analytically shown, for a strong enough interaction between particles, the linear increase of $S(t)$ is due to an exponential increase in time of the number of many-body basis states involved in the dynamics. Although our model is not random, off-diagonal matrix elements determined by the particle interaction strongly fluctuate in time due to the different number of particles occupying single particle levels. This fact allows us to apply the results obtained in [12] for our model.

For the case when the probability of remaining in the originally excited state decreases exponentially, $W_0(t) = \exp(-\Gamma t)$, it was found that the entropy increases linearly,

$$S(t) = \Gamma t \ln M \ , \qquad (3)$$

apart from a generic quadratic increase that occurs at small times. Here the value of Γ is determined by the decay of the probability to stay in the unperturbed ground state, and $M = m$ is the number of many-body unperturbed states directly coupled to the initial state. As was shown in [12], the linear growth of entropy appears in the regime for which Γ is proportional to perturbation g.

We have carefully checked the predictions for the time-dependence of $S(t)$ given in Ref.[12]. In particular, we have confirmed that the number of excited basis states grows in time exponentially fast before saturation. The saturation occurs because the total number of many-body states that we had in our numerical simulation is finite. We also have found that the probability $W_0(t)$ decays exponentially in time. As a result, we were able to verify the increase of entropy given by Eq.(3) and indicated as a dashed line in Fig.1.

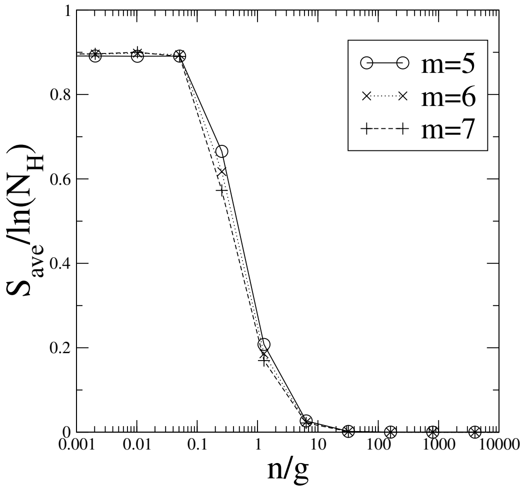

FIG. 2: Rescaled entropy as a function of n/g for $N = 6$ particles and different m as indicated in the caption.

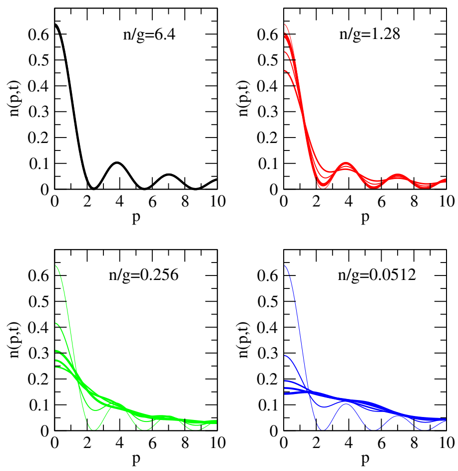

FIG. 3: Particle density in the momentum representation for four values of the interaction n/g, and for different rescaled times $gt = 0, 0.2, 0.4, 0.6, 0.8$ (lines are the thicker for later times).

We summarize our main results in Fig.2. Here S_{ave} is the mean value of the entropy averaged over time after the saturation, and N_H is the dimension of the Hilbert space (the total number of many-body states defined by m and N). Entropy rescaling has been done in order to avoid its trivial dependence on the size of the Hilbert space. In this case the plateau for $n/g \ll 1$ is independent of the choice of the number of particles and the number of levels. One can see that the renormalized entropy is a quantity that is very sensitive to the transition MF-TG regime. Also, as one can see, slight variations in $m \approx N$ value does not change the results. This indicates the universal character of the entropy in the transition region. Note that the limiting value of the renormalized entropy is not 1, that would correspond to the prediction of random matrix theory. This deviation clearly indicates the influence of the dynamical nature of the Hamiltonian.

We now discuss how our predictions can be tested experimentally. Since the entropy of wave functions is not an easy quantity to measure experimentally, we need to relate it to an observable which can be measured directly. In cylindrical coordinates the boson field operator (defined on a ring of radius R in the plane $z = 0$) has the form, $\hat{\psi}(\rho, \theta, z) = \hat{\delta}(z)\hat{\delta}(\rho - R)\hat{\psi}(\theta)$. Here the angle-dependent part can be expressed in terms of plane waves, $\hat{\psi}(\theta) = \sum_k \hat{a}_k e^{ik\rho\theta}$. Fourier transforming, $\hat{\chi}(\vec{p}) = \int d_3\vec{r}\ \hat{\psi}(\rho, \theta, z)\ e^{-i\vec{p}\cdot\vec{r}}$, and taking into account that $\vec{p} = p\hat{e}_y$ and $\vec{p}\cdot\vec{r} = p\rho\sin\theta$, we have,

$$\hat{\chi}(p) = C \sum_k \hat{a}_k \int_{\pi}^{\pi} d\theta e^{ikR\theta} e^{-ipR\sin(\theta)} \tag{4}$$

where C is a normalization constant. Using periodic boundary conditions, $k = \frac{2\pi}{L} n = \frac{n}{R}$ with $n = 0, \pm 1 \pm 2....$, one obtains

$$\hat{\chi}(p) = C \sum_{n=-\infty}^{+\infty} J_n(pR)\ \hat{a}_n \tag{5}$$

where $J_n(x)$ is the Bessel function of order n.

Expresssing the wave function at time t in terms of the basis states $|j\rangle \equiv |n^j_{-\infty}, ...n^j_0, ...n^j_{+\infty}\rangle$, we obtain an expression for the occupation number distribution $n(p, t) = \langle\psi(t)|\hat{\chi}^{\dagger}(p)\hat{\chi}(p)|\psi(t)\rangle$. This quantity gives the probability density for finding a boson with momentum p along the y axis at time t. Using our expression (5), we obtain,

$$n(p,t) = \sum_{j,j'} \bar{\psi}_j(t)\psi_{j'}(t) \sum_{l,s=-\infty}^{+\infty} J_l(pR)J_s(pR)\langle j|\hat{a}^{\dagger}_l\hat{a}_s|j'\rangle \tag{6}$$

Since the total momentum is zero, the selection rules, $\langle j|\hat{a}^{\dagger}_l\hat{a}_s|j'\rangle = n^j_l\delta_{j,j'}\delta_{l,s}$ hold, where n^j_l is the number of particles occupying the single-particle level with momentum l in the j-th basis state. Finally, we obtain,

$$n(p,t) = C \sum_j |\psi_j(t)|^2 \sum_{l=-\infty}^{+\infty} J^2_l(pR)\ n^j_l \tag{7}$$

where $C^{-1} = N^{-1}\int\ dp\ n(p,t)$.

Results are reported in Fig.3. When $n/g >> 1$, the momentum distribution does not practically change in time. All particles remain mainly in the condensate during the evolution. Close to the transition point $n/g \leq 1$, the distribution begins to be flat, without relaxing to an asymptotic distribution. In this intermediate region, the occupation number distributions reveal a mixture of very slow relaxation and strong oscillations in time that was first observed[14] in a similar model and termed as "non-ergodic behaviour of interacting bosons."

Contrary to this, for $n/g \ll 1$, one can see a clear relaxation of the occupation number distribution to a steady state distribution. This fact was shown [15] to be directly related to the onset of quantum chaos in the TBRM, with a linear increase of entropy in time. In fact,

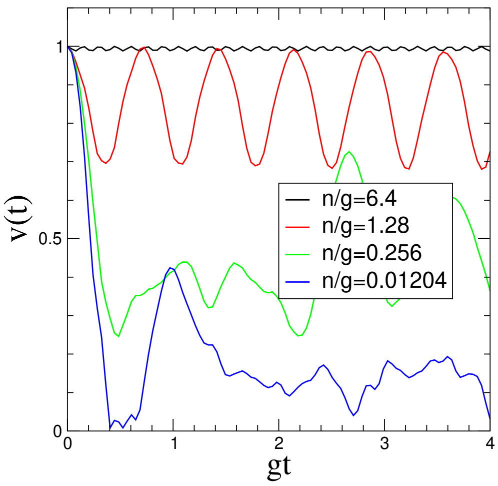

FIG. 4: Visibility as a function of time, for four values of rescaled interaction n/g. Here is $N = 6$ and $m = 6$.

the onset of relaxation to the steady distribution means that the behavior of the model can be well described by statistical methods [16].

In order to characterize qualitatively the onset of relaxation in the TG regime, we propose that one observe experimentally the *fringe visibility*, defined as

$$v(t) = |I_{max} - I_{min}|/(I_{max} + I_{min}) \tag{8}$$

Here we chose $I_{max} = n(p = 0, t)$, $I_{min} = n(p_0, t)$ where $p_0 \simeq 2.41$ is the first zero of J_0. See Fig.4. This quantity allows one to distinguish between periodic and aperiodic behavior of the occupation number distribution. These different time behaviors are shown in Fig.4.

As one can see, the TG regime is characterized by a fast decay of the visibility with subsequent fluctuations that persist in time. These fluctuations are additional evidence of the statistical nature of the dynamics when the interparticle interaction is strong. When $n/g \gg 1$ (MF regime), the visibility oscillates, revealing collapses and revivals of the condensate fraction population, as the entropy does in the same regime. See Fig.1. Therefore, there is a strict correspondence between the dynamical evolution of the entropy and the visibility of the momentum distribution. The interfernce properties of a thermal Tonks gas localized on one side of a ring has been studied in [17].

In conclusion, we have studied the dynamics of a Bose-Einstein condensate model on a torus by paying attention to the role of the interparticle interaction. Specifically, we considered the situation in which initially the ground state only, defined in the absence of the interaction, is populated. By switching on the interaction we have found different regimes depending on the strength of interaction. The first regime (MF regime, $n/g \gg 1$) is characterized by regular dynamics associated with periodic oscillations in time of the Shannon entropy Eq.(2) and of the fringe visibility Eq.(8) of particle density. In the Tonks-Girardeau regime, $n/g \ll 1$, the Shannon entropy grows linearly in time in accordance with our analytical estimates. Even if these estimates are obtained for completely random models, they give remarkable agreement with our data. Moreover, this approach allows one to understand the mechanism of the transition from the mean field to Tonks gas regime.

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