Point charges between two grounded conducting intersecting planes: a dynamical systems approach

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Abstract. The classical electrostatic problem of a point charge q between two semi-infinite conducting planes intersecting at a given angle α is approached using methods from dynamical system theory. In this way the standard solution is recovered: the image method works if and only if $\alpha = \pi/n$ with n a positive integer; the number of image charges is $2n-1$ ($n-1$ having the same charge of the physical one, n with opposite charge) and they must be disposed at the vertices of two regular polygons having n-sides inscribed inside the circle of radius equal to the distance between the physical charge and the intersection point of the two planes.

Riassunto. In questo lavoro viene affrontato il problema elettrostatico classico relativo ad una carica puntiforme tra due semipiani conduttori intersecantesi con un certo angolo α usando i metodo tipici della teoria dei sistemi dinamici. In questo modo si riottene la soluzione usuale, ovvero è possibile utilizzare il metodo delle cariche immagine se e solo se $\alpha = \pi/n$ con n intero positivo. Il numero delle cariche immagine è $2n-1$ (di cui $n-1$ aventi lo stesso segno della carica fisica e n segno opposto) e devono essere disposte ai vertici di due poligoni regolari aventi n lati inscritti nel cerchio di raggio pari alla distanza tra la carica fisica ed il punto di intersezione tra i due piani.

1. Introduction

The image charge method is a beautiful and powerful mathematical trick which permits one to find exact solutions of partial differential equations with appropriate boundary conditions. One of the most well known fields of application comes from electrostatics. Every graduate student in physics or mathematics knows how to explicitly find out the potential generated by an infinite grounded plane and a point charge q, a distance d apart [1] by simply adding a negative charge in the hidden region of the space at the same distance d from the plane. Of course this solution only holds in the space where the ‘physical’ charge is located as can be easily observed. In other words, image charges can only be eliminated in the distant region where the potential is screened, in such a way as to realize suitable boundary conditions.

The problem addressed here is a very simple generalization of this: two intersecting semi-infinite conducting earthed planes and a point charge. This is a well known classical problem, see for instance [2]. In this paper I will find the standard conditions of validity of the image charge method mapping this problem to a dynamical system and using results well known in this field. I will then show that the image charge method can be applied, and the solution easily found, if and only if the planes intersect each other at an angle $\alpha = \pi/n$, n being an positive integer. It would be interesting to show whether this new formulation is able to solve the case $\alpha \neq \pi/n$ where an analytical solution is still lacking.

2. Formulation of the problem

Consider the simple electrostatic problem of two conducting earthed planes intersecting at a given angle α and a point charge q at a given distance r_0 from the origin of the intersection and at angle θ_0, as indicated in figure 1. This problem is easily found to be a 2D one and it is sufficient to consider the projection onto the plane $z = 0$ containing the point charge (plane in figure 1).

From the mathematical point of view, one should solve the Poisson equation,

$$\nabla^2\phi = q\delta(\vec{r} - \vec{r}_0), \qquad (2.1)$$

with appropriate boundary conditions,

$$\phi|_{\Gamma_1} = \phi|_{\Gamma_2} = 0, \qquad (2.2)$$

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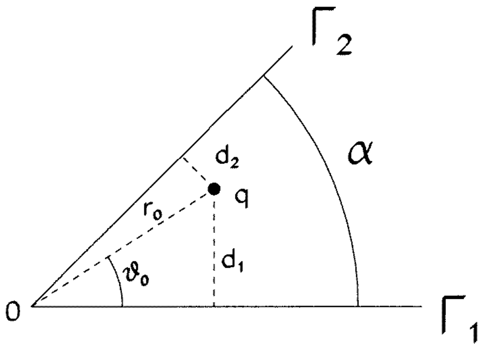

Figure 1. Conducting grounded planes Γ_1 and Γ_2 and the point charge q having polar coordinates (r_0, θ_0) at distance d_1, d_2 from the planes.

inside the physical region $S = \{(r,\theta)|r > 0; 0 < \theta < \alpha\}$. It is well known that the potential inside the physical region S can be obtained, without solving the Poisson equation, by placing appropriate image charges in the unphysical region, $\tilde{S} = \{(r,\theta)|r > 0; \alpha < \theta < 2\pi\}$, in such a way as to satisfy the boundary conditions (2.2).

For example in the case of figure 1, if one put a charge $q_1 = -q$ at a distance d_1 from Γ_1, below Γ_1 and at a distance r_0 from O, the potential due to q, q_1 is found to be zero on the plane Γ_1. Adopting the same line of reasoning one could make $\phi = 0$ on Γ_2 by placing a charge $q_2 = -q$ above Γ_2, at a distance d_2 from Γ_2 and r_0 from O. Problems arise since each of these two image charges 'perturbs' the other plane, namely q_1 perturbs Γ_2 and q_2 perturbs Γ_1. This can in turn be adjusted by adding two other image charges $q_3 = q_4 = q$ using reflections around the planes Γ_1 and Γ_2. The procedure can be iterated and one obtains, in general, a series of image charges. The question which I address is the following: when is it possible to obtain a finite or infinite number of image charges lying only in the unphysical region? In fact the method works only if the image charges lie in the unphysical region and only if they exactly balance themselves in order to realize the boundary conditions (2.2).

For the time being, it is sufficient to consider $0 < \alpha < \pi$ and the point charge inside the smallest angle generated by the intersection of the two planes Γ_1 and Γ_2. The relevant variables are the polar coordinates (r,θ) and the equation of the Γ_1, Γ_2 planes are respectively $\theta = 0$ and $\theta = \alpha$. Let (r_0, θ_0) be the coordinates of the physical point charge. Image charges will then be located by subsequent reflections around the planes Γ_1 and Γ_2, and better, around their extension Γ'_1 and Γ'_2 (infinite planes).

Let us introduce the reflection operator around the x-axis (Γ'_1 plane),

$$\mathcal{R}_0(r,\theta) = (\bar{r},\bar{\theta}) = (r, 2\pi - \theta) \tag{2.3}$$

and the reflection around the extended straight line $\theta = \alpha$ (Γ'_2 plane):

$$\mathcal{R}_\alpha(r,\theta) = (\bar{r},\bar{\theta}) = (r, 2(\pi+\alpha) - \theta \bmod -2\pi). \tag{2.4}$$

In equation (2.3) $0 \le \bar{\theta} < 2\pi$ for any $0 \le \theta < 2\pi$ and the operation mod $-\,2\pi$ is not strictly necessary. Anyway $\mathcal{R}_{\alpha=0} = \mathcal{R}_0$ as one can easily check.

A 'first generation' of image charges will be obtained starting from (r_0, θ_0) and reflecting by $\mathcal{R}_0$ and $\mathcal{R}_\alpha$ in sequence, that is

$$\begin{aligned}
(r_1,\theta_1) &= \mathcal{R}_0(r_0,\theta_0)\\
(r_2,\theta_2) &= \mathcal{R}_\alpha(r_1,\theta_1) = \mathcal{R}_\alpha\mathcal{R}_0(r_0,\theta_0)\\
(r_3,\theta_3) &= \mathcal{R}_0(r_2,\theta_2) = \mathcal{R}_0\mathcal{R}_\alpha\mathcal{R}_0(r_0,\theta_0)\\
&\dots
\end{aligned} \tag{2.5}$$

The sign of the image charge changes at each reflection. The 'second generation' set of image charges is obtained in the same way, exchanging $\mathcal{R}_\alpha$ with $\mathcal{R}_0$:

$$\begin{aligned}
(r'_1,\theta'_1) &= \mathcal{R}_\alpha(r_0,\theta_0)\\
(r'_2,\theta'_2) &= \mathcal{R}_0(r'_1,\theta'_1) = \mathcal{R}_0\mathcal{R}_\alpha(r_0,\theta_0) && (2.6)\\
(r'_2,\theta'_2) &= \mathcal{R}_\alpha(r'_2,\theta'_2) = \mathcal{R}_\alpha\mathcal{R}_0\mathcal{R}_\alpha(r_0,\theta_0)\\
&\dots && (2.7)
\end{aligned}$$

Since the reflection does not change the radius r_0, all the image charges, if any, will be distributed on the circle of radius r_0. For this reason the r coordinate will be omitted hereafter. First generation points (2.6) can be arranged in the following way :

$$\begin{aligned}
\theta_{2n} &= (\mathcal{R}_\alpha\mathcal{R}_0)^n\theta_0\\
\theta_{2n+1} &= (\mathcal{R}_0\mathcal{R}_\alpha)^n\theta_1
\end{aligned} \tag{2.8}$$

for any $n \ge 1$, and $\theta_1 = \mathcal{R}_0\theta_0$. The same can be done for second generation points (2.7):

$$\begin{aligned}
\theta'_{2n} &= (\mathcal{R}_0\mathcal{R}_\alpha)^n\theta_0\\
\theta'_{2n+1} &= (\mathcal{R}_\alpha\mathcal{R}_0)^n\theta'_1
\end{aligned} \tag{2.9}$$

for any $n \ge 1$, with $\theta'_1 = \mathcal{R}_\alpha\theta_0$. The image charges method will work if and only if all these points will be eliminated outside the physical region. The elements of the two sets $\{\theta_{2n}\}_{n\ge1}$ and $\{\theta'_{2n}\}_{n\ge1}$ are image charges of the same sign and value of the physical charge since an even number of reflections is involved. They will be called 'positive' charges. For the same reason the elements of the sets $\{\theta_{2n+1}\}_{n\ge0}$ and $\{\theta'_{2n+1}\}_{n\ge0}$, involving an odd number of reflections have opposite sign of the physical charge and they will be called 'negative'. It is easy to show that the operator $\mathcal{T}_\alpha = \mathcal{R}_\alpha\mathcal{R}_0$ is the shift operator on the arc of a circle of angle 2α. It follows that all the positive image charges of the first generation set are separated one from each other by an angle 2α. The same holds true for the sets, (2.8) and (2.9), of image charges.

It is convenient to consider the set $\{\theta_{2n}\}_{n\ge1}$ as a dynamical orbit on the plane, n being the time and $\mathcal{R}_\alpha\mathcal{R}_0$ the evolution operator in discrete time (one period evolution operator). In this way the system can be thought of as a dynamical system and its proper theoretical formalism can be applied.

3. The irrational case

The irrational case $\alpha = \pi\nu$, with $0 < \nu < 1$ irrational number, can be treated at once. In fact $\mathcal{T}_\alpha$ becomes the ergodic shift on the circle and it is easy to prove (see for instance [3], chapter V) that the set of points $\{\mathcal{T}_\alpha^n\theta_0\}_{n\geq 1}$ has an infinite number of elements which are uniformly distributed on the circle of radius r_0. There are then an infinite amount of points inside the physical region S and this excludes the possibility of using the image method approach.

4. The rational case

Let then assume $\alpha = \pi p/n$ with p, n prime integers, $p < n$. The orbit $\{\mathcal{T}_\alpha^n\theta_0\}_{n\geq 1}$, from the dynamical point of view, is a periodic orbit, and this means that one has four sets of periodic orbits. I will show that this is not the case and only two sets of periodic orbits need to be taken into account.

The periodicity of one of these orbits can be stated as:

$$(\mathcal{R}_{\pi p/n}\mathcal{R}_0)^n = (\mathcal{R}_0\mathcal{R}_{\pi p/n})^n = 1. \tag{4.1}$$

From condition (4.1) it follows that

$$\begin{aligned} \theta_{2n} &= \theta_0 \\ \theta'_{2n} &= \theta_0. \end{aligned} \tag{4.2}$$

Moreover $\theta_{2i} - \theta_{2(i-1)} = 2\alpha$ and $\theta'_{2i} - \theta'_{2(i-1)} = -2\alpha$, $\forall i = 1, ..., n$ which means that they are the same orbit. Furthermore the total angle covered by the orbit is given by $2\alpha n = 2(\pi p/n)n = 2p\pi$: this means that p crossings of the $\theta = 0$ half line have to be done in order to close the orbit.

The same holds for the negative orbits. In fact the negative elements of the first generation set can be written as $(\alpha = p\pi/n)$:

$$\mathcal{R}_0\theta_0, \quad (\mathcal{R}_0\mathcal{R}_\alpha)\mathcal{R}_0\theta_0, \ \ldots, \ (\mathcal{R}_0\mathcal{R}_\alpha)^{n-1}\mathcal{R}_0\theta_0 \tag{4.3}$$

which exactly coincide with those of the second generation, conveniently rearranged:

$$(\mathcal{R}_\alpha\mathcal{R}_0)^{n-1}\mathcal{R}_\alpha\theta_0, \quad (\mathcal{R}_\alpha\mathcal{R}_0)^{n-2}\mathcal{R}_\alpha\theta_0, \ \ldots, \ \mathcal{R}_\alpha\theta_0. \tag{4.4}$$

Even in the case of negative charges the points of the orbit have the same interspace $2\alpha = 2\pi p/n$ on the circle, starting from $\mathcal{R}_0\theta_0$ which is the symmetric, with respect to the x-axis, of the physical charge.

4.1. The solvable case: $\alpha = \pi/n$

Let us assume $p = 1$. In such a case only one crossing of the $\theta = 0$ line must be done. Consider the $n - 1$ positive image charges. Since they are separated, on the circle, by an angle 2α, none of them are inside the physical region S. The same holds for the negative image charges, since the 'starting' point is θ_1 (symmetric with respect to the x-axis of the physical charge) which satisfies $2\pi - \alpha < \theta_1 < 2\pi$.

This implicitly suggests the way to put the image charges for $\alpha = \pi/n$.

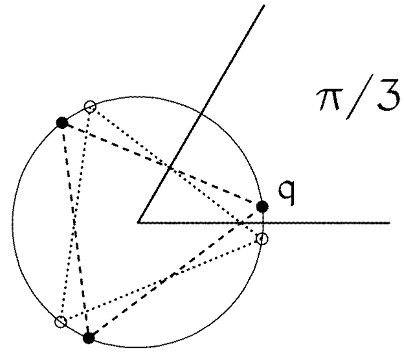

Figure 2. Distribution of the image charges for $\alpha = \pi/3$. Full circles have the same sign of the physical point charge. Open circles have opposite charge. Dashed and dotted lines are shown to indicate explicitly the way to eliminate the image charges (in this case at the vertices of two equilateral triangles).

(i) Draw the circle with the centre at the origin of the conducting planes and radius r_0.
(ii) Start with the physical charge and draw a regular polygon with n sides and a vertex at the point where the physical charge is located. Put positive image charges at the vertices of the polygon.
(iii) Consider the symmetric point, with respect to the x-axis of the physical charge. Starting from this vertex draw the regular polygon with n sides inside the same circle. Put the negative charges at it's vertices.

In figure 2 the system of charges constructed in the way indicated above is drawn for $\alpha = \pi/3$. Full circles indicate the positive charges, open circles the negative ones. The equilateral triangles are shown by dotted and dashed lines. The system in a certain way can compensate itself, that is each point has a twin charge of different sign in front of it with respect to the Γ'_1, Γ'_2 planes. This is clearly shown in figure 3 where the symmetry around the two planes has been indicated by dashed and dotted lines.

With increasing n the system becomes of course more complicated. In figure 4 the set of thirteen image charges necessary to solve the problem when $\alpha = \pi/7$ has been drawn.

4.2. The insolvable case: $\alpha = p\pi/n$

The last point concerns $\alpha = p\pi/n$ with p, n prime integers, $1 < p < n$. I will prove that, in this case, at least one point of the orbits $\{\theta_{2i}\}_{i=1,\ldots,n-1}$ or $\{\theta_{2i+1}\}_{i=1,\ldots,n}$ (the point θ_0 has been excluded) exists inside the physical region S. Defining the following region:

$$S' = \{(r, \theta) | \alpha < \theta < 2\alpha\} \tag{4.5}$$

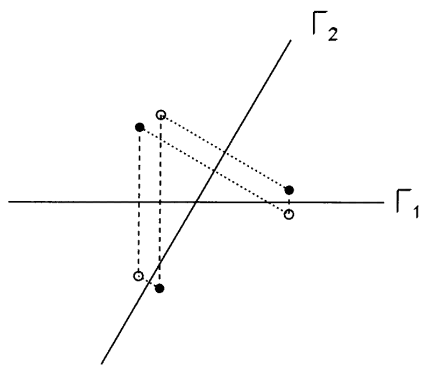

Figure 3. Image charges and symmetry lines around the Γ'_2 plane (dotted) and Γ'_1 (dashed), for the same case as figure 2.

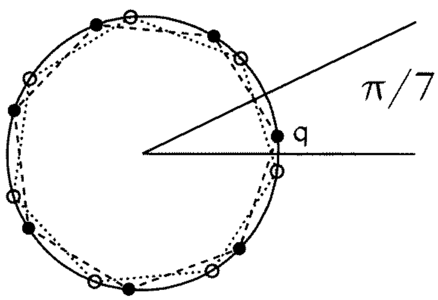

Figure 4. The same as figure 2 for $\alpha = \pi/7$.

which is the symmetric with respect to the Γ_2 plane, of the physical region S, such that

$$\mathcal{R}_\alpha S' = S. \tag{4.6}$$

It is clear that applying $\mathcal{R}_\alpha$ or $\mathcal{R}_0$ to one point of the two orbits above, a point of the other orbit is obtained. For instance, given θ_{2i}, one has :

$$\begin{aligned}\mathcal{R}_\alpha\theta_{2i} &= \mathcal{R}_\alpha(\mathcal{R}_\alpha\mathcal{R}_0)^i\theta_0 \\ &= \mathcal{R}_\alpha^2(\mathcal{R}_0\mathcal{R}_\alpha)^{i-1}\mathcal{R}_0\theta_0 \\ &= (\mathcal{R}_0\mathcal{R}_\alpha)^{i-1}\theta_1 = \theta_{2i-1}.\end{aligned} \tag{4.7}$$

Let us consider the positive orbit $\{\theta_{2i}\}_{i=1,\ldots,n-1}$. These points cannot coincide with θ_0 since this would imply $n = pm$ with m integer; but this is against the hypothesis that p and n are prime integers.

Then we are left with two possibilities : (i) at least one point is inside S; (ii) none of them are in S.

In case (i) the theorem holds.

In case (ii) it has been shown that above the p crossing of the $\theta = 0$ line have to be done in order to close the orbit. Moreover the points which cross this line cannot be in S (case (i)) and, at the same time, every point is separated from each other by a 'distance' of 2α. This means that these $p-1$ points (and not p since the last crossing is necessary to return back to θ_0) should accumulate in the region S'. But then, due to (4.6), there are $p-1$ points of the second orbit which lie in the region S. Since $p > 1$ this completely proves the theorem.

In general, there exist s positive image charges and r negative image charges inside S, depending on the initial position θ_0, with $s \geq 0$, $r \geq 0$ and $s + r = p - 1$. Since $p > 1$ there is always at least one image charge inside the physical region.

The case $n = 1$, even if excluded from the very beginning $(0 < \alpha < \pi)$ is, naturally valid (the two planes become only one plane) and the polygon degenerates in a point.

What would happen if $\pi < \alpha < 2\pi$? The demonstration for $\alpha = \nu\pi$ with $1 < \nu < 2$ irrational number is of course still valid. The case $\alpha = p\pi/n$ with $n < p < 2n$ needs a little comment since, now, $S \bigcap S' \neq \emptyset$. In this case it is sufficient to define $S' = \tilde{S}$ (the unphysical space). Then one has $\mathcal{R}_\alpha S' \subset S$ and the demonstration follows along the lines indicated in subsection 4.2.

Since the possibility to solve the problem using the image charge method is related to the angle α between the two conducting planes and not to the position θ_0 of the physical charge, this result can be straightforwardly generalized, via the superposition principle, to a generic number N of physical charges.

Acknowledgments

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References

[1] Jackson J D 1975 *Classical Electrodynamics* (New York: Wiley)
[2] Coulson C A 1951 *Electricity* (New York: Oliver and Boyd)
[3] Lichtenberg A J and Lieberman M A 1983 Regular and Stochastic Motion *Appl. Math. Series* **38**