Time Scale for Chaotic Driven Magnetic Reversal

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We analyze an anisotropic classical Heisenberg model with infinite range couplings. Below an ergodicity threshold, the energy surface is disconnected in two components with positive and negative magnetizations respectively. Above this threshold, in a fully chaotic regime, magnetization changes sign in a stochastic way and its behavior can be fully characterized by an average magnetization reversal time. We show that statistical mechanics predicts a phase–transition at an energy higher than the ergodicity threshold. We assess the dynamical relevance of the latter for finite systems through numerical simulations and analytical calculations. In particular, we derive an explicit expression of the time scale for magnetic reversal. As for standard phase transitions, this time-scale has a power law divergence at the ergodicity threshold.

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In the standard Ising model with short range interactions, below the critical temperature, reversing the order parameter (magnetization) requires a time of order $O(\exp(\sqrt{N}))$, where N is the total number of spins. Broken ergodicity appears in the thermodynamic limit, due to the exponential divergence of the reversing time [1]. Another type of broken ergodicity, not induced by the thermodynamic limit, can be caused by disconnections of the energy surface. Although it can generically appear only in low-dimensional phase-spaces, this effect is uncommon at large N. However, it has been recently observed [2] that, below a given specific energy, an anisotropic classical Heisenberg model with all-to-all spin coupling exhibits this type of broken ergodicity for all N. Moreover, it is well known that, for systems with long-range interactions [3], the escape time from metastable states diverges like $\exp(N)$ [4–6]. Comparing with the reversing time quoted above for the Ising model, ergodicity breaking due to the large N limit is stronger for long-range interactions. Being related with the infinite range nature of the interaction, the presence of a *non-ergodicity threshold* [2] could appear, at a first glance, a purely theoretical issue. Nevertheless, it could also have an experimental relevance since all-to-all interacting systems can be realized using modern experimental techniques [7]. Thus, it is important to find characteristic signatures of this threshold.

In this Letter, we investigate the classical Hamiltonian dynamics of an anisotropic Heisenberg model, for which a finite number N of spins interact with all-to-all couplings. The aim is to establish the main physical effects associated with the non-ergodicity threshold with respect to a phase-transition appearing at a higher energy. The phase-transition is studied in the microcanonical ensemble, applying a recently developed solution method of mean-field Hamiltonians based on large-deviation theory [8]. These two transitions remain distinct in the thermodynamic limit. We focus on finite size systems, because they allow a precise study of the interplay between dynamics and statistics. Moreover, it has been remarked that, due to dynamical chaos, even systems with a small number of degrees of freedom can acquire a statistical behavior [9].

A first result presented in this Letter is that the ferromagnetic/paramagnetic transition can be *dynamically driven* below the statistical phase-transition. This happens if the spin coupling strength is large enough to produce fully chaotic motion. This means that, for all finite N, an *observation time* exists for which magnetization vanishes in an energy region above the non-ergodicity threshold, and thus below the statistical one. A further result reported in this Letter is the derivation of an explicit expression for the time scale of *magnetization reversal.* At the ergodicity threshold, this reversal time diverges as a power law, with a characteristic exponent proportional to the number of spins N.

The Hamiltonian of the model is

$$H = B\sum_{i=1}^{N} S_i^z + \frac{J}{2}\sum_{i=1}^{N}\sum_{j\neq i}(S_i^x S_j^x - S_i^y S_j^y), \qquad (1)$$

where $\vec{S}_i = (S_i^x, S_i^y, S_i^z)$ is the spin vector with continuous components, N is the number of spins, B is the rescaled external magnetic field strength and J the all-to-all coupling strength (the summation is extended over all pairs). The equations of motion are derived in a standard way from this Hamiltonian. The total energy $E = H$ and the spin moduli $|\vec{S}_i|^2 = 1$ are constants of the motion. Dynamics has been already studied in a similar model [2]. It was found to be characterized by chaotic motion (positive maximal Lyapunov exponent) for not too small energy values and spin coupling constants. For $J = 0$ the model is exactly integrable, while for generic J and B there is a mixed phase space with prevalently chaotic motion for $|E| \lesssim JN$.

We will first determine the statistical phase-transition energy of the model in the microcanonical ensemble. The Hamiltonian can be rewritten as

$$H = BNm_z + \frac{J}{2}N^2\left(m_x^2 - m_y^2\right), \qquad (2)$$

where $\vec{m} = (m_x, m_y, m_z) = 1/N\sum_i \vec{S}_i$ and we neglect the term $J/2\sum_i (S_i^y)^2 - (S_i^x)^2$, which is unimportant as far as the statistical properties , in the $N \to \infty$ limit, are concerned. The most remarkable difference of Hamiltonian (2) with respect to (1) is that a new constant of the motion appears: the modulus of the total angular momentum $M^2 = m_x^2 + m_y^2 + m_z^2$. In numerical simulations we always find a vanishing Lyapunov exponent: hence, we presume that model (2) becomes exactly integrable. The dynamics of the global magnetization is indeed integrable, but this does not obviously imply the integrability of Hamiltonian (2). The entropy s as a function of the order parameter $\vec{m}$ can be exactly calculated in the large N limit, using large deviation techniques [8].

Remarking that in the negative energy range a non-zero value of m_x is ruled out by entropic considerations, and expressing m_z, using relation (2), as a function of m_y and ϵ, we obtain the specific entropy as a function of m_y and ϵ only. Then, imposing the vanishing of the second derivative of $s(m_y, \epsilon)$ at $m_y = 0$ (a signal of the second order phase-transition) we extract the specific transition energy ϵ_{stat}. At the phase–transition the entropy as a function of m_y changes from a single peaked function to a double-peaked one. For large N the phase–transition energy density is

$$\epsilon_{stat} = -\frac{B^2}{JN}. \qquad (3)$$

This value is in good agreement with numerical results obtained using the full Hamiltonian (1).

The non-ergodicity energy density ϵ_{ne} for the Hamiltonian (1) can be obtained in the same way as Ref. [2]. Even in this case, it is possible to show [10, 11] that the phase space of the system is topologically disconnected below ϵ_{ne}. From symmetry considerations both positive and negative regions of m_y exist on the same energy surface. Switching from a negative m_y value to a positive one requires, for continuity, to pass through $m_y = 0$. Hence, for all energy values above $\epsilon_{ne} = min(\epsilon|m_y = 0)$ magnetization reversal is possible, while below this value magnetization cannot change sign. Computing the minimum, we get [12],

$$\epsilon_{ne} = \begin{cases} -B \text{ for } J \leq B \\ \\ -(\frac{B^2}{2J} + \frac{J}{2}) \text{ for } J > B. \end{cases} \qquad (4)$$

The existence of ϵ_{ne} does not represent a sufficient condition in order to demagnetize a sample for $\epsilon > \epsilon_{ne}$. Regular structures indeed appear that prevent most of trajectories to cross the $m_y = 0$ plane. The sufficient condition can only be given by chaos, presumably in the same way as chaos provides the way to break the last golden circle in the standard map, thus allowing transition to global stochasticity[13].

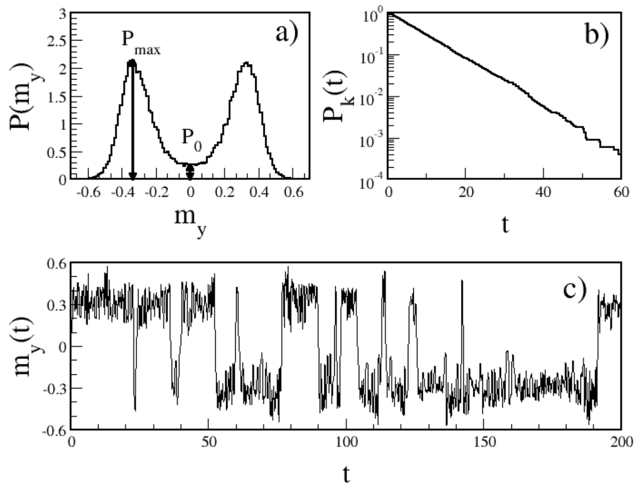

FIG. 1: a) Probability distribution of m_y for $\epsilon = -0.9$. The maximal probability, P_{max}, and $P_0 = P(m_y = 0)$ are indicated by vertical arrows. b) Probability of keeping magnetization sign up to time t *vs* time. c) Magnetization m_y *vs* time. In this figure all data refer to the $N = 6$, $B = 1$, $J = 3$ case.

In the following, we will study the dynamics of the full Hamiltonian (1), which, at variance with (2), is non–integrable and displays chaotic motion. Moreover, we will restrict ourselves to the case $J > B/N$ for which $\epsilon_{stat} > \epsilon_{ne}$. The two thresholds ϵ_{ne} and ϵ_{stat} define three energy regions which show different dynamical and statistical properties.

1) For $\epsilon < \epsilon_{ne}$, the probability distribution of m_y, $P(m_y)$, obtained by a random sampling of constant energy surface [14], shows two separate peaks, with $P(m_y = 0) = 0$, so that m_y cannot change sign in time.

2) For $\epsilon_{ne} < \epsilon < \epsilon_{stat}$, the probability distribution is double peaked around the most probable values of the magnetization. These two peaks are not separated and $P(m_y = 0) \neq 0$, see Fig. (1a). What actually happens dynamically depends on the relative strength of the coupling J with respect to B. For J large enough the behaviour of $m_y(t)$ resembles a random telegraph noise [15], (Fig. (1c)): magnetization switches randomly between its two most probable values. The probability distribution of magnetization reversal times follows a Poissonian law, $P_k(t) \sim e^{-t/\tau}$, see Fig. (1b), for any initial conditions. Hence, we can characterize the behavior of the system through an average magnetization reversal time τ. The Poissonian distribution of the reversal times is a consequence of strong chaos: the system looses its memory due to sensitivity to initial conditions and the reversal probability per unit time becomes time independent. On

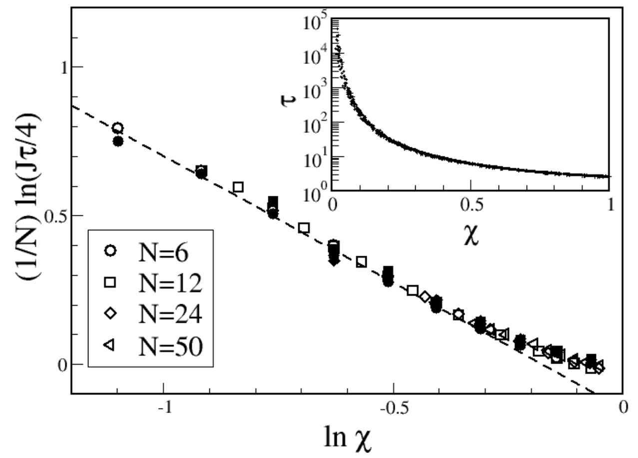

FIG. 2: Average magnetization reversal times τ obtained dynamically (open symbols) and statistically (full symbols), *vs* the energy scaling parameter $\chi = (\epsilon - \epsilon_{ne})/(\epsilon_{stat} - \epsilon_{ne})$ for different N values and $B = 0$. The dynamical determination of τ has been obtained by iterating 10^4 randomly chosen trajectories for each energy and computing the average reversal time explicitly. The statistical determination of τ is obtained from formula (6), where P_{max}/P_0 is numerically determined for each energy density. The dashed line is $\ln(J\tau/4) = -0.15 - 0.85 \ln \chi$, where the constants have been obtained through a fitting. Inset : divergence of τ at $\chi = 0$ for $N = 5$ and $J = B = 1$.

the contrary, for small J, we observe a quasi-integrable behavior almost everywhere in the energy range: reversal times strongly depend on initial conditions. Therefore we will limit our considerations to the large J case.
3) Finally, for $\epsilon_{stat} \le \epsilon \le 0$, m_y quickly changes sign and $P(m_y)$ is peaked at $m_y = 0$.

For all energies in the range $(\epsilon_{ne}, \epsilon_{stat})$, we find that the reversal time τ grows exponentially with the number of spins for sufficiently large N. In the following we give a theoretical justification of this dependence. Indeed, we are even able to derive empirically an explicit formula for the dependence of τ on the parameter $\chi = (\epsilon - \epsilon_{ne})/(\epsilon_{stat} - \epsilon_{ne})$

$$\tau \sim \chi^{-\alpha N}. \tag{5}$$

Eq. (5) is valid above the non-ergodicity threshold and not too close to the statistical border (observe that χ varies in this range between 0 and 1). The comparison of this formula with numerical results is shown in Fig.(2). Let us remark that the reversal time diverges at ϵ_{ne}, see the inset of Fig.(2), as a power law, showing that this energy threshold shares many peculiarities with standard second order phase transitions. In the case $B = 0$, we find $\alpha = 0.85$, which is at variance with the value $\alpha = 1$ obtained, for $N \to \infty$, in the mean field approximation (see below). This small discrepancy can be attributed to a finite N effect. Extensions of these results to the $B \neq 0$ case show additional dependences of α on the parameters B and J [16]. From Eq. (5) it is also clear that the infinite time average of the magnetization is zero above the non-ergodicity threshold and different from zero below, due to the divergence of the reversal time. Nevertheless, this is not what we obtain during a finite observational time τ_{obs}. In Fig.(3) we show the time-averaged magnetization

$$\langle m_y \rangle_{obs} = (1/\tau_{obs}) \int_0^{\tau_{obs}} dt\ m_y(t)$$

vs the specific energy ϵ for $N = 5$ (Fig.3a) and $N = 50$ (Fig.3b) spins during a fixed observational time. While in (a) $\langle m_y \rangle_{obs}$ is zero just above ϵ_{ne}, in (b) it vanishes at a value ϵ_{obs} located between ϵ_{ne} and ϵ_{stat}. Indeed, if $\tau_{obs} \gg \tau$, the magnetization has time to flip between the two opposite states and, as a consequence, $\langle m_y \rangle_{obs} \simeq 0$. On the contrary, if $\tau_{obs} \ll \tau$ the magnetization keeps its sign and cannot vanish during τ_{obs}. Defining an effective transition energy ϵ_{obs} from $\tau_{obs} = \tau(\epsilon_{obs})$, one gets, inverting Eq. (5), the value indicated by the vertical arrow in Fig. (3b). This is, *a posteriori*, a further demonstration of the validity of Eq. (5) for any N.

From a theoretical point of view, it is interesting to note that, for any fixed N, and sufficiently large J, $\epsilon_{obs} \to \epsilon_{ne}$ when $\tau_{obs} \to \infty$. On the other side, in agreement with statistical mechanics, for any finite τ_{obs}, $\epsilon_{obs} \to \epsilon_{stat}$ when $N \to \infty$. This implies that the limits $\tau_{obs} \to \infty$ and $N \to \infty$ do not commute.

Usually, for long-range interactions, the interaction strength is rescaled in order to keep an extensive energy[17]. In our case this can be done setting $J = I/N$. With this choice of J as $N \to \infty$, at fixed I, J becomes much smaller than B, then Eq. (5) looses its validity and the quasi–integrable regime sets in. The presence of the non-ergodicity threshold is therefore hidden.

A justification of Eq. (5) can also be given in terms of statistical properties. In Refs. [5, 6], on the basis of fluctuation theory [4, 18], it has been argued that metastable states relax to the most probable state on times that are proportional to $\exp(N\Delta s)$ where N is the number of degrees of freedom and Δs is the specific entropic barrier. In our case $\exp(N\Delta s)$ is nothing but P_{max}/P_0, see Fig. (1a). Empirically, for $B = 0$, we find a very good agreement with the reversal times setting

$$\tau = \frac{4}{J} \frac{P_{max}}{P_0}. \tag{6}$$

While the P_{max}/P_0 factor in this formula has a theoretical justification, because it represents the probability to cross the entropic barrier, the $1/J$ factor can be heuristically associated with the typical time scale of the system. A deeper theoretical justification of this formula should be obtained in view of its success in describing the numerical results for different N values, (Fig.(2)).

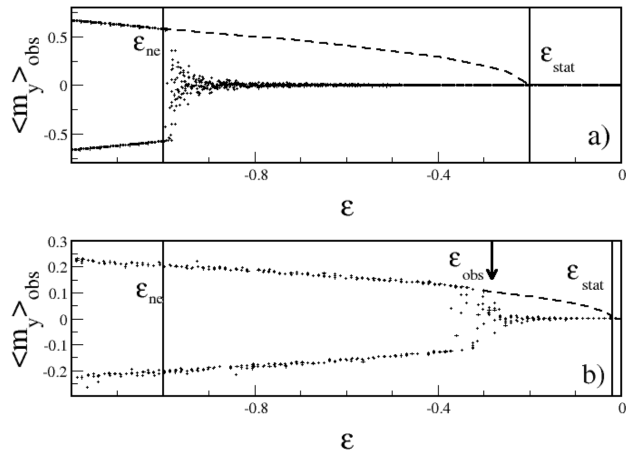

FIG. 3: Time average of m_y over the observational time τ_{obs} *vs* ϵ for different number of particles (a) $N = 5$, (b) $N = 50$, with fixed $J = B = 1$. Each single point has been obtained taking the time average over the time intervals $\tau_{obs} = 10^5$ (a) and $\tau_{obs} = 10^4$ (b). Dashed curves indicate the equilibrium value of m_y obtained from statistical mechanics. Vertical lines represent the non-ergodicity and the statistical threshold respectively. The arrow in (b) indicates the energy value ϵ_{obs} of the dynamical transition due to the finite observational time.

In the mean field model [11] one has $P_{max}/P_0 \sim 1/(\epsilon - \epsilon_{ne})^N$, which reproduces the main feature of Eq. (5) and gives $\alpha = 1$.

Summarizing, a simple spin model with anisotropic and long-range couplings has been studied as a paradigmatic example. We discuss the relevance of the ergodicity breaking [1, 2] occurring for any finite N with respect to the phase–transition. Two distinct energy thresholds are addressed: the non-ergodicity threshold, ϵ_{ne}, below which phase space is disconnected and magnetization cannot change sign in time, and the statistical threshold, ϵ_{stat}, at which a second order phase–transition occurs in the thermodynamic limit. The non-ergodicity threshold does not disappear in the thermodynamic limit and remains always distinct from the statistical threshold. In the highly chaotic regime, between ϵ_{ne} and ϵ_{stat}, the behavior of the system can be characterized by an average magnetization reversal time τ. Numerical simulations and statistical arguments allow us to give a characterization of the reversal time above the non–ergodicity threshold, pointing out a power law divergence at ϵ_{ne}. This behavior is likely to be valid beyond the toy model studied in this Letter and could be a characteristic signature of the non–ergodicity threshold. The dynamical magnetization reversal times are also found to be in good agreement with those obtained from simple fluctuation theory arguments. The infinite-time average of the magnetization is zero above the non-ergodicity threshold, and different from zero below. Therefore, the system, when chaotic, dynamically demagnetizes well below the statistical threshold. This is the reason why we could consider the non–ergodicity threshold as a *chaotic driven phase transition.*

I. ACKNOWLEDGMENTS

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