Long-Time Decay Properties of Kepler Map.

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Summary. — Starting from the Kepler map approximation of highly excited H atoms in a microwave field, we construct a simple model to describe the decay of the survival probability. Taking into account the non-Markov character of this process, we derive an asymptotic $t^{-2/3}$ law for the survival probability, well confirmed by numerical experiments.

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1. – Introduction.

The problem of microwave ionization of highly excited hydrogen atoms is a striking example of the relevance of chaos in microphysics. The simplest classical model for this phenomenon describes an electron moving on a half-line under the combined action of the Coulomb field of the nucleus and of a monochromatic electric field [1]. At some critical value of the perturbation strength, a transition to chaotic motion occurs and the electron «diffuses» through the bound state region of the classical phase space until it eventually reaches the «continuum», *i.e.* ionizes.

The process of chaotic excitation—or, more properly, some quantum counterpart of this process [1-8]—appears to be responsible for the intense under-

(1) G. CASATI, B. V. CHIRIKOV, I. GUARNERI and D. L. SHEPELYANSKY: *Phys. Rep.*, **154**, 77 (1987).
(2) J. E. BAYFIELD and P. M. KOCH: *Phys. Rev. Lett.*, **33**, 258 (1974).
(3) R. V. JENSEN: *Phys. Rev. A*, **30**, 386 (1984).
(4) B. V. CHIRIKOV: *Proc. R. Soc. London, Ser. A*, **413**, 145 (1987).
(5) B. V. CHIRIKOV and D. L. SHEPELYANSKY: *The chaos border and statistical*

threshold ionization which was observed as early as 1974 (2). The complexity of motion in the stochastic regime calls for a statistical description, which is in fact accomplished by means of a suitable Fokker-Planck equation (3). The diffusion coefficient turns out to be the relevant quantity ruling the classical ionization (*). Numerical simulation showed that this Fokker-Planck equation provides a good local description of the classical excitation process (1). However, this equation should not be used to analyse important aspects of this process involving long-time behaviour such as, *e.g.*, the time decay of the survival probability, for two reasons at least. The first of these is well known from previous investigations of dynamical systems in which a chaos border is present. Near the last KAM curve, the geometry of the chaotic component of the phase space is astonishingly complicated by a structure of residual stable islands and remnants of broken tori. This «critical zone» acts as a trapping region for chaotic trajectories and produces long tails in the decay of correlations («statistical anomalies») (4,5).

Besides this, in the present problem there is still another reason why a Markovian approximation is invalid for long times. According to a simple resonance analysis, chaos is generated by the interaction of the external field frequency with the harmonics of the unperturbed motion. Thus the «diffusive time scale», *i.e.* the time scale on which the loss of memory due to local instability justifies a Markovian approximation, is of the order of the unperturbed period and therefore sharply increases as the electron diffuses upwards in energy towards the continuum. As a matter of fact, longer and longer pseudo-integrable segments appear in the orbit, as the orbit itself approaches the ionization border.

The present paper is devoted to the formulation of a statistical description for the chaotic excitation of 1-dimensional H atom in microwave fields, that circumvents the above difficulties and allows for the determination of the decay law.

2. – Mathematical model.

The essential ingredient of our model will be the «Kepler map» formulation of the H atom dynamics (7,8). The classical dynamics of a 1-*d* H atom with unperturbed action-angle variables (n, θ) in a monochromatic electric field of

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anomalies, Preprint INP 86-174 (Novosibirsk, 1986).

(6) C. F. F. KARNEY: *Physica D*, 8, 360 (1983).

(7) G. CASATI, I. GUARNERI and D. L. SHEPELYANSKY: *Phys. Rev. A*, **36**, 350 (1987).

(8) G. CASATI, I. GUARNERI and D. L. SHEPELYANSKY, to appear in the July 1988 Special Issue of the *IEEE J. Quantum Electron.*

(*) Here we consider the case when the ratio of field frequency to the Kepler frequency is larger than 1.

strength ε and frequency ω can be viewed as a time-independent Hamiltonian problem in the extended 4-dimensional phase space. Subsequent crossings of the plane $\theta = \pi$ («aphelion») define a canonical Poincaré map, that for $\omega_0 \equiv \omega n_0^3 \gg 1$ has the form

$$\bar{x} = x + k \sin\phi\,, \qquad \bar{\phi} = \phi + 2\pi\omega(-2\omega\bar{x})^{-3/2}\,, \tag{2.1}$$

where $x = -1/2n^2\omega$ is the unperturbed electron energy divided by ω, ϕ is the field phase at perihelion, and $k = 2.58\,\varepsilon\omega^{-5/3}$. The «Kepler map» (2.1) is area-preserving and is defined for all bound states $\bar{x} < 0$ but carries some of them into the region $\bar{x} > 0$; this accounts for ionization. Orbits leaving with a value x_0 undergo a stochastic transition for

$$k > k_0 \sim \frac{1}{6\pi\omega^2(-2x_0\,\omega)^{5/2}}\,.$$

We mention here that a map with a similar structure has been recently proposed in order to describe the Halley comet dynamics (9). By iterating (2.1) we get a simplified description of the dynamics in the discrete time n defined by the number of passages at perihelion. In the fully chaotic regime the phase ϕ can be assumed to change at random and this leads to a statistical picture in which the electron performs a random walk in x, with an absorbing boundary at $x = 0$ and a zero flux boundary at the bottom of the chaotic region. However, this picture accounts for the decay law only in its initial stage (fig. 2). Indeed, due to critical effects, the long time decay is algebraic and not exponential (as would be predicted by the pure random-walk model) (4).

Anyway, we wish to describe how the chaotic excitation develops in real time; for this purpose we need to modify the basic model (2.1). This map was obtained by a stationary phase analysis of the change in x between subsequent passages at the aphelion (8). That analysis shows that for $\omega_0 > 1$ the effect of the external field is concentrated in a neighbourhood of the perihelion and this suggests a picture in which the electron, away from perihelion moves along an unperturbed Kepler orbit; at perihelion it undergoes a «kick», described by (2.1), that throws it on a new orbit, or directly into the continuum. In this picture the small field-induced oscillations around the unperturbed Kepler motion are neglected, so that its limits of validity are defined by $\varepsilon \ll 5\omega^{4/3}$ (8). With this picture in mind, we shall investigate the continuous time motion of a point described by 3 coordinates, $-\infty < x < 0$, $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant \phi \leqslant 2\pi$. The motion will be described by

$$\dot{x} = 0\,, \qquad \dot{\phi} = 0\,, \qquad \dot{\theta} = (-2\omega x)^{3/2} \tag{2.2}$$

(9) B. V. Chirikov and V. V. Vecheslavov: *Chaotic dynamics of the Comet Halley*, Preprint INP 86-184 (Novosibirsk, 1986).

as long as $0<\theta<2\pi$. Instead, when an orbit defined by (2.2) impinges in $\theta=2\pi$, x and ϕ change according to (2.1) and θ becomes 0. If the orbit approaches $x=0$, then longer and longer «integrable segments» defined by (2.2) appear, so that we cannot assume the changes of x in time to have a Markov character, even when the map (2.1) is chaotic. Notice that the invariant measure for this model in the (x, θ, ϕ) phase space is defined by the volume element $x^{-3/2}\,dx\,d\theta\,d\phi$. This means that any region bounded on the right by $x=0$ has an infinite measure; this remark will be important below. A statistical model for excitation in the fully chaotic region will now be derived under the assumption that subsequent kicks are uncorrelated. We shall then study the evolution of an ensemble of points, distributed in x and θ with a density $w(x, \theta, t)$, which move along orbits (2.2) and in $\theta=2\pi$ change their energy at random, according to some transition kernel $k(x, x')$.

The region of interest will be bounded below by a zero-flux boundary at $x=\overline{x}$, corresponding to the last KAM curve for mapping (2.1). The density $w(x, \theta, t)$ must then satisfy

$$\frac{\partial w(x, \theta, t)}{\partial t} = -x^{3/2}\frac{\partial w(x, \theta, t)}{\partial \theta} \tag{2.3}$$

for $0<\theta<2\pi$, which is the Liouville equation associated with (2.2). (For convenience we assumed $2\omega=1$ and changed variable to $-x$.)

The effect of kicks at $\theta=2\pi$ is described by

$$w(x, 0, t) = x^{-3/2}\int_0^{\overline{x}} dx'\; k(x, x')\, w(x', 2\pi, t)\, x'^{3/2}, \tag{2.4}$$

where the factors $x^{-3/2}$, $x'^{3/2}$ account for the change in phase volume. There will be no need to give the specific form of $k(x, x')$; anyway the boundary condition at $x=\overline{x}$ should be taken into account when specifying k. We wish to find how the distribution in energy

$$P(x, t) = \int_0^{2\pi} d\theta\, w(x, \theta, t) \tag{2.5}$$

evolves in time. By using (2.3) and (2.4) we get

$$\frac{\partial P}{\partial t} = \int_0^{2\pi} dx'\, x'^{3/2}[k(x, x') - \delta(x-x')]\, w(x', 2\pi, t), \tag{2.6}$$

so we need an equation for $w(x, 2\pi, t)$ and this is

$$w(x, 2\pi, t) = \theta(2\pi x^{-3/2} - t)\, w(x, 2\pi - x^{3/2}t, 0) + \\ + \theta(t - 2\pi x^{-3/2})\, w(x, 0, t - 2\pi x^{-3/2}), \tag{2.7}$$

where θ is the unit step function.

(2.7) just says that particles at $\theta = 2\pi$ at time t are either incoming there for the first time after $t = 0$ or have undergone a kick at a previous time $t - 2\pi x^{-3/2}$. Assuming the initial condition $w(x, \theta, 0) \equiv g(x)$, sharply peaked around some value x_0 $(0 < x_0 < \bar{x})$ and, using (2.4), we get

$$w(x, 2\pi, t) = \theta(2\pi x^{-3/2} - t)\, g(x) + \theta(t - 2\pi x^{-3/2})\, x^{-3/2} \int_0^{\bar{x}} \mathrm{d}x'\, k(x, x')\, x'^{3/2}\, w(x', 2\pi, t - 2\pi x^{-3/2})\,. \tag{2.8}$$

Let us put $\Phi(x, t) = w(x, 2\pi, t)$. By taking Laplace transforms (in time) of (2.6), (2.8) we get two coupled equations for the transforms $\widetilde{\Phi}(x, \lambda)$ and $\widetilde{P}(x, \lambda)$. In operator form these equations read

$$\lambda \widetilde{P} = 2\pi g + (\boldsymbol{K} - \boldsymbol{1})\, \psi\,, \qquad \psi = h + \boldsymbol{E}\boldsymbol{K}\psi\,, \tag{2.9}$$

where $\psi(x, \lambda) = x^{3/2}\, \widetilde{\Phi}(x, \lambda)$, the operator $\boldsymbol{E}$ is multiplication by $\exp[-2\pi\lambda x^{-3/2}]$, $\boldsymbol{K}$ is the integral operator with kernel $k(x, x')$ and

$$h(x) = \frac{x^{3/2}}{\lambda} \left(\frac{1 - \exp[-2\pi\lambda x^{-3/2}]}{\lambda} \right) g(x)\,.$$

Being interested in the long time behaviour of P we shall analyse $\widetilde{P}$ near $\lambda = 0$. For small λ, $(\boldsymbol{1} - \boldsymbol{E})$ is «small», and this motivates the expansion

$$\psi = (\boldsymbol{1} - \boldsymbol{E}\boldsymbol{K})^{-1} h = (\boldsymbol{1} - \boldsymbol{K})^{-1} [\boldsymbol{1} - (\boldsymbol{E} - \boldsymbol{1}) \boldsymbol{K} (\boldsymbol{1} - \boldsymbol{K})^{-1}]^{-1} h = (\boldsymbol{1} - \boldsymbol{K})^{-1} [\boldsymbol{1} + (\boldsymbol{E} - \boldsymbol{1}) \boldsymbol{K} (\boldsymbol{1} - \boldsymbol{K})^{-1} + \ldots]\, h\,.$$

Upon substituting this into the 1st equation (2.9) we finally get

$$\lambda \widetilde{P}(x, \lambda) \simeq 2\pi g(x) - [\boldsymbol{1} + (\boldsymbol{E} - \boldsymbol{1}) \boldsymbol{K} (\boldsymbol{1} - \boldsymbol{K})^{-1}]\, h(x)\,.$$

Under not very restrictive smoothness assumptions on $k(x, x')$ the integral operator $\boldsymbol{K}$ has a complete orthonormal set of eigenfunctions $u_n(x)$ with related eigenvalues ξ_n $(0 < \xi_n < 1)$. If we assume $g(x) = \delta(x - x_0)$, and expand $\boldsymbol{K}(\boldsymbol{1} - \boldsymbol{K})^{-1}$ on the u_n's, we get

$$\widetilde{P}(x, \lambda) \simeq \frac{2\pi}{\lambda} \delta(x - x_0) + \frac{x_0^{3/2}}{\lambda} \left(\frac{1 - \exp[-2\pi\lambda x_0^{-3/2}]}{\lambda} \right) \delta(x - x_0) + x_0^{3/2} \frac{1 - \exp[-2\pi\lambda x_0^{-3/2}]}{\lambda} \frac{1 - \exp[-2\pi\lambda x^{-3/2}]}{\lambda} \sum_n \frac{\xi_n}{1 - \xi_n} u_n(x_0)\, u_n(x)\,. \tag{2.10}$$

The first two terms describe the decay of the initial δ-like distribution which is completely effaced as soon as all the particles in the ensemble undergo their first kick. Integrating in x, we obtain that the Laplace transform $\widetilde{P}(\lambda)$ of the total survival probability $P(t)$ behaves as $\lambda^{-1/3}$ near $\lambda = 0$. According to well-known estimates, this entails that $P(t)$ decays as $t^{-2/3}$.

In the same way it can be seen that the following terms of the expansion in $(\boldsymbol{E} - 1)$ have an asymptotic behaviour which acquires a $-2/3$ exponent each time, *i.e.* the second order $\sim t^{-4/3}$ and so on; then survival probability decays as $t^{-2/3}$ when t becomes large.

This is the asymptotic form of the decay law predicted by the stochastic model.

3. – Numerical results.

Unlike the «stochastic» model (2.3), (2.4) our model (2.1) (2.2) for chaotic excitation is completely deterministic.

Nevertheless, the above predictions about the decay law hold for that model as well, above the chaotic threshold. This is confirmed by numerical simulations.

In fig. 1 we show the results of one such simulation. Here we followed the evolution of 10^5 initial points according to the deterministic model with an initial

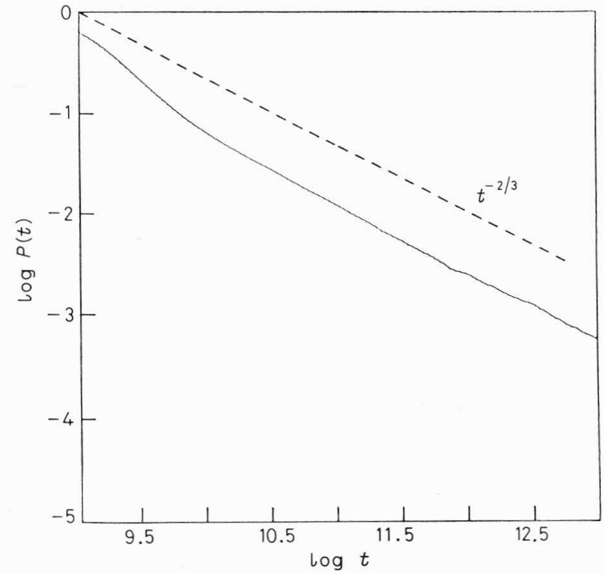

Fig. 1. – Plot in log-log scale of the survival probability $P(t)$ against time t—full line—(only the tail is shown). The dashed line is the predicted $t^{-2/3}$ law. Here $\omega = 1.2 \cdot 10^{-5}$, $k = 1.1$, with an ensemble of 100 000 points with initial energy $x_0 = -7$ and uniformly distributed phase.

value $x_0 = -7$ and θ homogeneously distributed in $(0, 2\pi)$. Particles with $x > 0$ were absorbed.

We took $k = 1.1$, $\omega = 1.2 \cdot 10^{-5}$, which correspond to an initial hydrogenic level $n_0 = 66$ and rescaled field and frequency $\varepsilon_0 = \varepsilon n_0^4 = 5.088 \cdot 10^{-2}$ and $\omega_0 = \omega n_0^3 = 3.45$.

ε_0 was therefore ~ 4 times the chaotic threshold; $\bar{x}$ could be roughly estimated by the resonance overlap criterion as $\bar{x} \sim -15$, in agreement with numerical results.

In developing our statistical model we took no special steps to describe the critical region close to $x = \bar{x}$. An interesting question is then what the role of this critical region would be in the decay process. Indeed, since chaotic trajectories explore the whole chaotic component of phase space, particles diffusing downwards in energy must sooner or later enter this region, where they spend a long time as their orbits wind around residual stability islands in a pseudo-integrable way. This «trapping» produces a delay in the ionization process and one may expect that even the decay law would be changed.

The effect of the critical region on the decay of correlations and on the survival probability has been studied on different models. It was found that the survival probability in the critical region decays according to a $t^{-1/2}$ law. Since in our stochastic model no special care was taken to describe the trapping in the

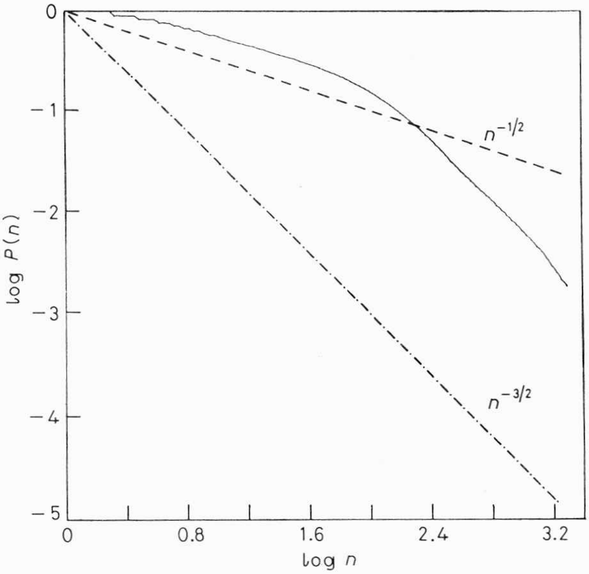

Fig. 2. – Plot in log-log scale of the survival probability $P(n)$ against the number n of iterations (full line) of the Kepler map. The dashed-line represents the $n^{-1/2}$ law predicted by the initial free diffusion; the dashed and dotted line shows the $n^{-3/2}$ law observed in other numerical experiments and discussed in ref. (5). Here $\omega = 1.2 \cdot 10^{-5}$, $k = 2.2$, with an ensemble of 75 000 points with initial energy $x_0 = -2$ and uniformly distributed phase.

region near $\overline{x}$, one might then expect the decay of the deterministic model to deviate from the $t^{-2/3}$ law, due to critical effects not accounted for in the model. One might even guess that the $t^{-1/2}$ critical decay, being slower than the $t^{-2/3}$ one, would dominate for very long times. This expectation is contradicted by numerical results which demonstrate the $t^{-2/3}$ decay in real time, in spite of the anomalies that appear when the survival probability is plotted against the fictitious time defined by the number of iterations (fig. 2). Our explanation is that the decay law is determined by the sojourn inside the phase space region bounded by $x = 0$ on the right which acts as a trapping region itself. Since this region has an infinite measure, it overwhelms the effect of the critical region, which has instead a finite measure.

In other words, the long time decay is determined by the waiting time before the last, ionizing kick, and the average of this time is infinite.

The above-described stochastic model appears to provide a statistical picture of the excitation process that, unlike the Fokker-Planck equation, accounts for the non-Markov character of the chaotic motion near the ionization border. It should then be possible to extract from the model other information about the statistics of chaotic excitation, besides the decay law; this will be the subject of future investigations.

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● RIASSUNTO

Partendo dalla mappa di Keplero, quale approssimazione della dinamica di atomi di idrogeno altamente eccitati in un campo a microonde, viene sviluppato un modello per descrivere il decadimento della probabilità di sopravvivenza. Il carattere non Markoviano di questo processo permette di ricavare il comportamento asintotico $t^{-2/3}$ per la probabilità di sopravvivenza, ben confermato dagli esperimenti numerici.

Свойства длинновременного затухания отображения Кеплера.

Резюме (*). — Исходя из приближения отображения Кеплера для сильно возбужденных атомов водорода в микроволновом поле, мы конструируем простую модель для описания затухания вероятности выживания. Учитывая немарковский характер этого процесса, мы выводим асимптотический закон $t^{-3/2}$ для вероятности выживания, что хорошо подтверждается численными экспериментами.

(*) *Переведено редакцией.*