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Los Alamos
NATIONAL LABORATORY
Ideas That Change the World

Dynamics of Observables in Quantum Nonlinear Systems: quantum effects as a singular perturbation

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Lecture 1: Closed PDEs for observables for quantum nonlinear Hamiltonians:
- Coherent states for bosonic systems and expressions for observables.
- PDF for observables in a closed form.
- Simple example of PDE, and characteristic parameters.

Lecture 2: Quantum effects as a singular perturbations:
- Properties of equations for observables.
- A singular character of quantum effects, and characteristic time-scales for quantum dynamics.

Lecture 3: Quantum dynamics in the vicinity of elliptic and hyperbolic points:
- The peculiarities of dynamics for observables in the vicinities of elliptic and hyperbolic points.
- Analysis of solutions by using exact examples.

Lecture 4: Influence of the thermal bath: Quantum effects after decoherence and relaxation:
- Interaction of quantum system with the thermal bath.
- Effects of decoherence and relaxation.

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Some Motivations

Broad Agency Announcement (BAA 07-68) for Defense Sciences Office (DSO) DARPA/DSO SOL, DARPA Mathematical Challenges, BAA 07-68; BAA CLOSING DATE: **9/8/08;** TECHNICAL POC: Dr. Benjamin Mann, DARPA/DSO, Ph: (571) 218-4246, Email: BAA07-68@darpa.mil; CFDA#: 12.910; URL: http://www.darpa.mil/dso/solicitations/solicit.htm;
Website Submission: http://www.sainc.com/dsobaa/

I. Funding Opportunity Description

DARPA is soliciting innovative research proposals in the area of DARPA Mathematical Challenges, with the goal of dramatically revolutionizing mathematics and thereby strengthening the scientific and technological capabilities of DoD. To do so, the agency has identified twenty-three mathematical challenges, listed below, which were announced at DARPA Tech 2007.

DARPA seeks innovative proposals addressing these Mathematical Challenges. Proposals should offer high potential for major mathematical breakthroughs associated to one or more of these challenges. Responses to multiple challenges should be addressed individually in separate proposals. Submissions that merely promise incremental improvements over the existing state of the art will be deemed unresponsive.

DARPA Challenges

- *Mathematical Challenge One: The Mathematics of the Brain*

  Develop a mathematical theory to build a functional model of the brain that is mathematically consistent and predictive rather than merely biologically inspired.

- ***Mathematical Challenge Two: The Dynamics of Networks***

  Develop the high-dimensional mathematics needed to accurately model and predict behavior in large-scale distributed networks that evolve over time occurring in communication, biology, and the social sciences.

- *Mathematical Challenge Three:* ***Capture and Harness Stochasticity in Nature***

  Address Mumford's call for new mathematics for the 21st century. Develop methods that capture persistence in stochastic environments.

- *Mathematical Challenge Four:* ***21st Century Fluids***

  Classical fluid dynamics and the Navier-Stokes Equation were extraordinarily successful in obtaining quantitative understanding of shock waves, turbulence, and solitons, but new methods are needed to tackle complex fluids such as foams, suspensions, gels, and liquid crystals.

- *Mathematical Challenge Five:* ***Biological Quantum Field Theory***

  Quantum and statistical methods have had great success modeling virus evolution. Can such techniques be used to model more complex systems such as bacteria? Can these techniques be used to control pathogen evolution?

- *Mathematical Challenge Six:* ***Computational Duality***

  Duality in mathematics has been a profound tool for theoretical understanding. Can it be extended to develop principled computational techniques where duality and geometry are the basis for novel algorithms?

- *Mathematical Challenge Seven:* ***Occam's Razor in Many Dimensions***

  As data collection increases can we do more with less by finding lower bounds for sensing complexity in systems? This is related to questions about entropy maximization algorithms.

- *Mathematical Challenge Eight:* ***Beyond Convex Optimization***

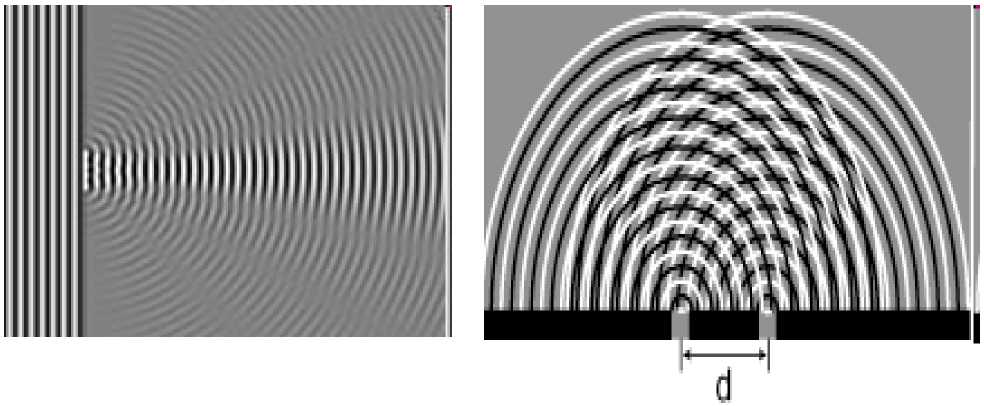
  Can linear algebra be replaced by algebraic geometry in a systematic way?

- *Mathematical Challenge Nine:*
  What are the Physical Consequences of Perelman's Proof of Thurston's Geometrization Theorem?
  Can profound theoretical advances in understanding three dimensions be applied to construct and manipulate structures across scales to fabricate novel materials?
- *Mathematical Challenge Ten:* ***Algorithmic Origami and Biology***
  Build a stronger mathematical theory for isometric and rigid embedding that can give insight into protein folding.
- *Mathematical Challenge Eleven:* ***Optimal Nanostructures***
  Develop new mathematics for constructing optimal globally symmetric structures by following simple local rules via the process of nanoscale self-assembly.
- *Mathematical Challenge Twelve:* ***The Mathematics of Quantum Computing, Algorithms, and Entanglement***
  In the last century we learned how quantum phenomena shape our world. In the coming century we need to develop the mathematics required to control the quantum world.
- *Mathematical Challenge Thirteen:* ***Creating a Game Theory that Scales***
  What new scalable mathematics is needed to replace the traditional Partial Differential Equations (PDE) approach to differential games?
- *Mathematical Challenge Fourteen:* ***An Information Theory for Virus Evolution***
  Can Shannon's theory shed light on this fundamental area of biology?
- *Mathematical Challenge Fifteen:* ***The Geometry of Genome Space***
  What notion of distance is needed to incorporate biological utility?
- *Mathematical Challenge Sixteen:* ***What are the Symmetries and Action Principles for Biology?***
  Extend our understanding of symmetries and action principles in biology along the lines of classical thermodynamics, to include important biological concepts such as robustness, modularity, evolvability, and variability.

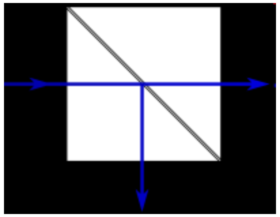
- *Mathematical Challenge Seventeen: Geometric Langlands and Quantum Physics*
  How does the Langlands program, which originated in number theory and representation theory, explain the fundamental symmetries of physics? And vice versa?
- *Mathematical Challenge Eighteen: Arithmetic Langlands, Topology, and Geometry*
  What is the role of homotopy theory in the classical, geometric, and quantum Langlands programs?
- *Mathematical Challenge Nineteen: Settle the Riemann Hypothesis*
  The Holy Grail of number theory.
- *Mathematical Challenge Twenty: Computation at Scale*
  How can we develop asymptotics for a world with massively many degrees of freedom?
- *Mathematical Challenge Twenty-one: Settle the Hodge Conjecture*
  This conjecture in algebraic geometry is a metaphor for transforming transcendental computations into algebraic ones.
- *Mathematical Challenge Twenty-two: Settle the Smooth Poincare Conjecture in Dimension 4*
  What are the implications for space-time and cosmology? And might the answer unlock the secret of "dark energy"?
- *Mathematical Challenge Twenty-three: What are the Fundamental Laws of Biology?*
  Dr. Tether's question will remain front and center in the next 100 years. I place this challenge last as finding these laws will undoubtedly require the mathematics developed in answering several of the questions listed above.

Quantum experiments exist independently of the theory

Double-slit experiment

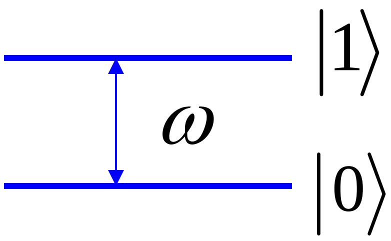


Randomness



Discrete energy levels: $E_n = -hcR_\infty \frac{Z^2}{n^2}$

Superpositions: $\psi = \frac{1}{\sqrt{2}}\left(|\text{ground state}\rangle + |\text{excited state}\rangle\right)$

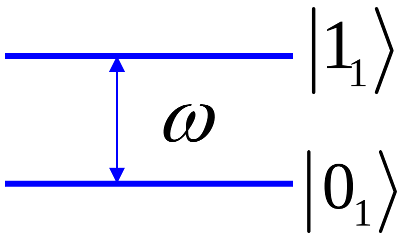


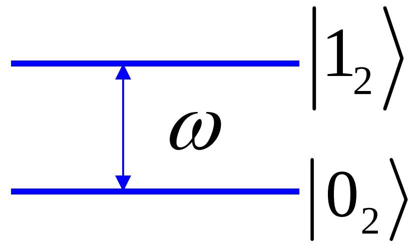
Entanglement: $\psi = \frac{1}{\sqrt{2}}\left(|H_1 H_2\rangle + |V_1 V_2\rangle\right)$

$B(t) = B_\perp \cos\omega t;\ \Omega_R = \gamma B_\perp;\ \Omega_R \tau = \frac{\pi}{2} - \text{pulse}$

Uncertainty principle in measurement: $\Delta x\, \Delta p \geq \frac{\hbar}{2}$

Superpositions and entanglement





$$\psi_1 = \alpha_1 |0_1\rangle + \beta_1 |1_1\rangle$$

$$\psi_2 = \alpha_2 |0_2\rangle + \beta_2 |1_2\rangle; \ |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Superpositional states

$$\psi \neq \psi_1 \otimes \psi_2; \qquad \rho = Tr_T \rho = \sum_n w_n \rho_n^{(1)} \otimes \rho_n^{(2)}$$

Entangled states

$$\psi = \frac{1}{\sqrt{2}}\left(|0_1 0_2\rangle + |1_1 1_2\rangle\right)$$ **-Example of entangled state—strong quantum correlations**

Classical correlations in a similar system (Andrey Kolmogorov's approach)

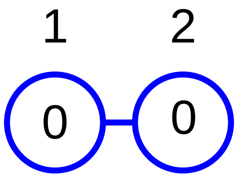
$$A_1 = [0_1 0_2], \ A_2 = [0_1 1_2], \ A_3 = [1_1 0_2], \ A_4 = [1_1 1_2]$$

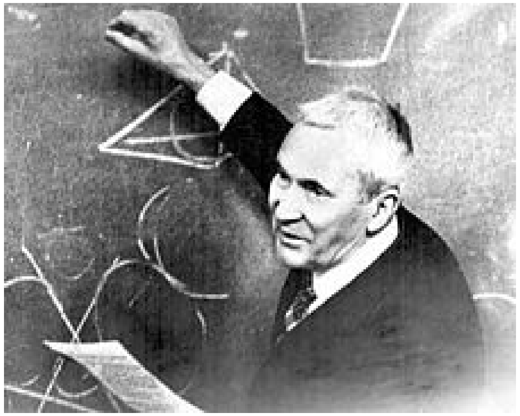
$$P(A_1) = \frac{1}{2}, \ P(A_2) = 0, \ P(A_3) = 0, \ P(A_4) = \frac{1}{2}$$

Corresponding quantum density matrix

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_2 + \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_1 \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_2,$$

$$\langle 0_1 0_2 | \rho | 0_1 0_2 \rangle = \frac{1}{2}, \ \langle 1_1 1_2 | \rho | 1_1 1_2 \rangle = \frac{1}{2}, \ \text{all other}=0$$





1903-1987

Problems with the Quasi-Classical Asymptotic

In quasi-classical region the wave function has the form: $\Psi(x,t) \sim e^{i\frac{S(x,t)}{\hbar}}$

where $\left|\frac{S(x,t)}{\hbar}\right| \sim \frac{I}{\hbar} \sim n \gg 1,$ I is the action of the system, n is a characteristic energy level

The wave function (density matrix, Wigner function) oscillates very fast, and it is difficult:

(i) to separate fast and slow variables,
(ii) to separate physical effects from mathematical corrections,
(iii) to get a convergence for asymptotic behavior, and
(iv) to derive the expressions for expectation values.

Our approach is based on PDEs for observable values

Heisenberg Uncertainty Relation and Coherent States

$A = \langle \Psi | \hat{A} | \Psi \rangle -$ average (expectation value, observable value);

$\sigma_{\mathrm{A}}^2 = \langle \Psi | \left(\hat{A} - A \right)^2 | \Psi \rangle -$ standard deviation (variance, valatility)

$| \Psi(t) \rangle -$ can depend on time (then: $A \Rightarrow A(t)$)

In what follows I will be mainly untested in the so-called quasi-classical regime. So, I will need quantum states which are closest to the classical ones in a certain sense. One choice of such states is based on minimizing the well-known **Heisenberg uncertainty relation**:

$$\sigma_A \sigma_B \geq \frac{1}{2} \left| \langle \Psi | \left[\hat{A}, \hat{B} \right] | \Psi \rangle \right|, \text{ where } \hat{A} \text{ and } \hat{B} \text{ are Hermitian operators,}$$

$$\left[\hat{A}, \hat{B} \right] \equiv \hat{A}\hat{B} - \hat{B}\hat{A} - \text{ commutator}$$

Proof of the Heisenberg Uncertainty Relation: $\sigma_A \sigma_B \geq \frac{1}{2}\left|\left\langle \Psi \left| \left[\hat{A}, \hat{B}\right] \right| \Psi \right\rangle\right|$

Define a state: $\left|\Phi\right\rangle = \hat{F}\left|\Psi\right\rangle$, where

$\hat{F} = \hat{A} + i\lambda\hat{B}$ is non-Hermitian, and λ is a real number

We have: $0 \leq \|\Phi\|^2 = \left\langle \Psi \right| \left(\hat{A} - i\lambda\hat{B}\right)\left(\hat{A} + i\lambda\hat{B}\right)\left|\Psi\right\rangle = \left\langle \hat{A}^2 \right\rangle + \lambda^2 \left\langle \hat{B}^2 \right\rangle + i\lambda\left\langle \left[\hat{A}, \hat{B}\right] \right\rangle.$

We have: (a) $\left\langle \left[\hat{A}, \hat{B}\right] \right\rangle$ is an imaginary value, and

(b) the right part has a minimum at $\lambda_0 = -\frac{i}{2}\frac{\left\langle \left[\hat{A}, \hat{B}\right] \right\rangle}{\left\langle \hat{B}^2 \right\rangle}$. Then,

$$\|\Phi\|^2_{\lambda_0} = \left\langle \hat{A}^2 \right\rangle + \frac{1}{4}\frac{\left(\left\langle \left[\hat{A}, \hat{B}\right] \right\rangle\right)^2}{\left\langle \hat{B}^2 \right\rangle} \geq 0, \text{ or}$$

$$\left\langle \hat{A}^2 \right\rangle \left\langle \hat{B}^2 \right\rangle \geq -\frac{1}{4}\left(\left\langle \left[\hat{A}, \hat{B}\right] \right\rangle\right)^2.$$

Substitute: $\hat{A} \Rightarrow \hat{A} - A$, $\hat{B} \Rightarrow \hat{B} - B$, $\left\langle \left[\hat{A}, \hat{B}\right] \right\rangle = -\left|\left\langle \left[\hat{A}, \hat{B}\right] \right\rangle\right|$, and use: $\left[\hat{A} - A, \hat{B} - B\right] = \left[\hat{A}, \hat{B}\right]$.

We have:

$$\sigma_A^2 \sigma_B^2 \equiv \left\langle \left(\hat{A} - A\right)^2 \right\rangle \left\langle \left(\hat{B} - B\right)^2 \right\rangle \geq \frac{1}{4}\left|\left\langle \left[\hat{A}, \hat{B}\right] \right\rangle\right|^2 - \text{ Heisenberg uncertaity relation}$$

Coherent States

Let $\hat{A}=\hat{x},\ \hat{B}=\hat{p}$ (in the coordinate representation: $\hat{p}=-i\hbar\partial/\partial x$),

then: $\left[\hat{x},\hat{p}\right]=i\hbar,$ and: $\sigma_x\sigma_p \geq \frac{\hbar}{2}.$

Let find a state $\left|\Psi\right\rangle$ which minimizes the uncertaity.

Additionally we requere a normalization condition: $\left\langle\Psi\middle|\Psi\right\rangle=1.$

Introduce a functional:

$$U=\left\langle\Psi\right|\left(\hat{x}-x\right)^2\left|\Psi\right\rangle\left\langle\Psi\right|\left(\hat{p}-p\right)^2\left|\Psi\right\rangle+\nu\left(\left\langle\Psi\middle|\Psi\right\rangle-1\right),$$

where: $x=\left\langle\hat{x}\right\rangle,\ p=\left\langle\hat{p}\right\rangle,\ \nu$ is a Lagrange multiplier.

The minimization problem for U reduces to the PDE:

$$\frac{\delta U}{\delta\left\langle\Psi\right|}=\left[\frac{\left(x-\left\langle\hat{x}\right\rangle\right)^2}{\sigma_x^2}+\frac{\left(-i\hbar\partial/\partial x-\left\langle\hat{p}\right\rangle\right)^2}{\sigma_p^2}-2\right]\left|\Psi\right\rangle=0.$$ Solution (coherent state):

$$\left|\Psi\left(x\right)\right\rangle=\frac{1}{\left(2\pi\sigma_x^2\right)^{1/4}}\exp\left(-\frac{\left(\left(x-\left\langle\hat{x}\right\rangle\right)^2\right)}{4\sigma_x^2}+i\frac{\left\langle\hat{p}\right\rangle\left(x-\left\langle\hat{x}\right\rangle\right)}{\hbar}+i\frac{\left\langle\hat{x}\right\rangle\left\langle\hat{p}\right\rangle}{2\hbar}\right),\quad\left(\sigma_x\sigma_p=\frac{\hbar}{2}\right)$$

Coherent States and Quantum Linear Oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{x}^2}{2}, \; \hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right), \; \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right), \; \left[\hat{a}, \hat{a}^\dagger\right] = 1,$$

$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$. *It can be shown that the coherent state,* $\left|\alpha\right\rangle$, *is a solution of equation*:

$$\hat{a}\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle; \; \left|\alpha\right\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\left(\sqrt{\frac{m\omega}{2\hbar}}x - \alpha\right)^2 + \frac{\alpha^2}{2} - \frac{\left|\alpha\right|^2}{2}\right],$$

$$\sigma_x^2 = \left\langle\alpha\right|\left(\hat{x} - x\right)^2\left|\alpha\right\rangle = \hbar/2m\omega, \; \sigma_p^2 = \left\langle\alpha\right|\left(\hat{p} - p\right)^2\left|\alpha\right\rangle = \hbar m\omega/2, \; \alpha = \sqrt{\frac{m\omega}{2\hbar}}\left(x + i\frac{p}{m\omega}\right).$$

Eigenvalue problem: $\hat{H}u_n\left(x\right) = E_n u_n\left(x\right);$

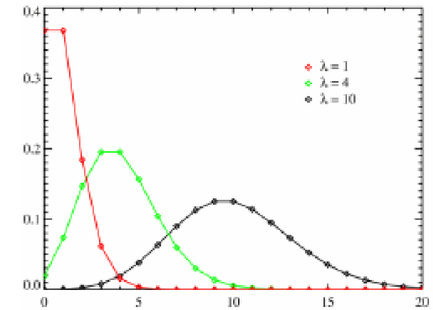
$$u_n\left(x\right) \equiv \left|n\right\rangle = \left(\frac{1}{2^n n!}\sqrt{\frac{m\omega}{\pi\hbar}}\right)^{1/2} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)\exp\left(-\frac{m\omega x^2}{2\hbar}\right), \; E_n = \hbar\omega\left(n + \frac{1}{2}\right), \; n = 1, 2, \ldots$$

$$\left|\alpha\right\rangle = e^{-\frac{\left|\alpha\right|^2}{2}}\sum_0^\infty \frac{\alpha^n}{\sqrt{n!}}\left|n\right\rangle - \textit{coherent state}$$

Some Properties of Coherent States

P_n

$\lambda = \overline{n} = |\alpha|^2$



n

1) In CS $|\alpha\rangle$ the uncertainty is minimal: $\sigma_x \sigma_p = \hbar/2$.

2) The CS is normalized: $\langle\alpha|\alpha\rangle = 1$.

3) The physical meaning of complex α is the following:

Introduce the probability: $P_n(\alpha) = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2}\dfrac{|\alpha|^{2n}}{n!} = \dfrac{e^{-\overline{n}}\overline{n}^n}{n!}$ $\left(\overline{n} = |\alpha|^2\right)$ – Poissonian distribution

4) The CSs with different α and β are not orthogonal: $\langle\alpha|\beta\rangle = \exp\left(-\dfrac{|\alpha|^2}{2} - \dfrac{|\beta|^2}{2} + \alpha^*\beta\right)$.

5) The CS $|\alpha\rangle$ can be constructed using a shift operator: $D_s(\alpha,\alpha^*) = \exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right)$.

Namely: $|\alpha\rangle = D_s(\alpha,\alpha^*)|0\rangle$. Proof:

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle,\ (\hat{a}|0\rangle = 0);\ \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle;\ \hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle;$$

$$D_s(\alpha,\alpha^*) = \exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right) = D_s(\alpha,\alpha^*) = e^{-\frac{|\alpha|^2}{2}} e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}} \text{ which follows from:}$$

$$e^{\hat{A}+\hat{B}} = e^{-\frac{[\hat{A},\hat{B}]}{2}} e^{\hat{A}} e^{\hat{B}} \quad \left(if\ \left[\hat{A},\hat{B}\right] \text{ is a c-number}\right)$$

$$D_s(\alpha,\alpha^*)|0\rangle = e^{\frac{-|\alpha|^2}{2}} e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}}|0\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha\hat{a}^\dagger}|0\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}\sqrt{n!}|n\rangle = |\alpha\rangle$$

6) The CSs form a complete system/basis

$$\frac{1}{\pi}\int d^2\alpha\left|\alpha\right\rangle\left\langle\alpha\right| = I,\ \left(d^2\alpha = d\,\mathrm{Re}\,\alpha\times d\,\mathrm{Im}\,\alpha\right)$$

7) For any quadratic Hamiltonians quantum and classical dynamics coincide (up to renormalization of coefficients)

$$\hat{H} = A(t)\hat{a} + A^*(t)\hat{a}^\dagger + B(t)\hat{a}^\dagger\hat{a} + C(t)\hat{a}^2 + C^*(t)\left(\hat{a}^\dagger\right)^2$$

as the operator equation for $\hat{a}$ is a linear one: $i\hbar\dot{\hat{a}} = \left[\hat{a},\hat{H}\right]$;

8) Consider an arbitrary operator function $\hat{f}(\hat{a}^\dagger,\hat{a})$

of bosonic operators that posesses a formal power series axpansion.

Then, we can prove: $\left\langle z\right|\hat{f}\hat{a}^\dagger\left|z\right\rangle = e^{-|z|^2}\frac{\partial}{\partial z}e^{|z|^2}f$, $\left\langle z\right|\hat{a}^\dagger\hat{f}\left|z\right\rangle = z^* f$, $\left\langle z\right|\hat{f}\hat{a}\left|z\right\rangle = zf$, $\left\langle z\right|\hat{a}\hat{f}\left|z\right\rangle = e^{-|z|^2}\frac{\partial}{\partial z^*}e^{|z|^2}f$,

$f = \left\langle z\right|\hat{f}\left|z\right\rangle$,

Example of proof:

$$Q \equiv \left\langle z\right|\hat{a}\hat{f}\left|z\right\rangle = e^{-|z|^2}\left\langle 0\right|e^{-z\hat{a}^\dagger}e^{z^*\hat{a}}\left(\hat{a}\hat{f}\right)e^{z\hat{a}^\dagger}e^{-z^*\hat{a}}\left|0\right\rangle,\ \left(e^{-z^*\hat{a}}\left|0\right\rangle = \left|0\right\rangle\right),$$

$$Q = e^{-|z|^2}\left\langle 0\right|e^{z^*\hat{a}}\left(\hat{a}\hat{f}\right)e^{z\hat{a}^\dagger}\left|0\right\rangle = e^{-|z|^2}\frac{\partial}{\partial z^*}\left\langle 0\right|e^{z^*\hat{a}}\hat{f}e^{z\hat{a}^\dagger}\left|0\right\rangle = e^{-|z|^2}\frac{\partial}{\partial z^*}e^{|z|^2}f$$

Why CSs are Useful:
CSs remain CSs for Linear Oscillator

$$|\Psi(t=0)>=|\alpha_0> \qquad i\hbar\frac{\partial|\Psi(t)>}{\partial t}=\mathbf{H}|\Psi(t)>$$

$$|\Psi(t)>=e^{-\frac{i}{\hbar}\mathbf{H}t}|\alpha_0>=e^{-i\omega(\mathbf{a}^+\mathbf{a}+1/2)t}|\alpha_0>$$

$$|\Psi(t)>=e^{-\frac{i\omega t}{2}}e^{-\frac{|\alpha_0|^2}{2}}\sum_{n=0}^{\infty}\frac{[\alpha_0\exp(-i\omega t)]^n}{\sqrt{n!}}|n>$$

$$=e^{-\frac{i\omega t}{2}}|\alpha_0\exp(-i\omega t)>\equiv e^{-i\frac{\omega t}{2}}|\alpha(t)>$$

$$\alpha(t)=\alpha_0\exp(-i\omega t)$$

$$x(t)=\langle\Psi(t)|\hat{x}|\Psi(t)\rangle, \qquad x(t)=x_0\cos(\omega t)+\frac{p_0}{m\omega}\sin(\omega t),$$

$$p(t)=\langle\Psi(t)|\hat{p}|\Psi(t)\rangle \qquad p(t)=p_0\cos(\omega t)-m\omega x_0\sin(\omega t)$$

$$x^2(t)+p^2(t)=const$$

Examples of Coherent States

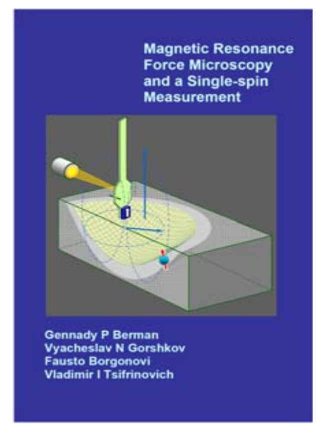
Laser radiation: He-Ne laser; Power: P=10mW, $\lambda = 0.63\mu m$,

$$P = \frac{\hbar\omega\bar{n}}{\tau} \Rightarrow$$

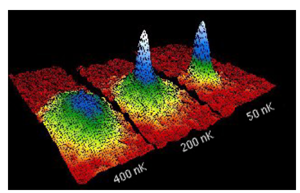
$$\bar{n} = \frac{P\tau}{\hbar\omega} = \frac{P\tau}{\hbar ck} = \frac{P\tau\lambda}{\hbar c 2\pi} = \frac{10\times10^{-3}\times10^{7}\,erg\times\sec^{-1}\times1\sec\times0.63\times10^{-6}\times10^{2}\,cm}{10^{-27}\,erg\times\sec\times3\times10^{10}\,cm\times\sec^{-1}\times6.28} \approx 3\times10^{16}\,\frac{photons}{\sec ond}$$

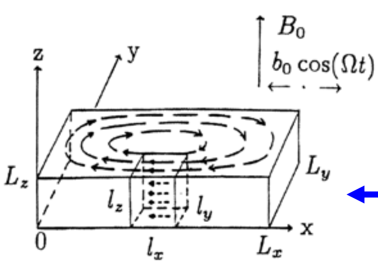
Cell phone station: Power: P=3W, $\omega/2\pi = 800MHz$; $\bar{n} \approx 6\times10^{24}\,\frac{photons}{\sec ond}$

Cantilever: quantum cooling and above



BEC: $\bar{n} \approx 10^{3} - 10^{6}$ atoms





Resonators

Radio waves: amplitude and phase are good defined

Exact C-Number Equation for Quantum Expectation Values

Time-independent Hamiltonian with one degree of freedom $H\left(a^{\dagger},a\right)$

Introduce the operator function: $f\left(a^{\dagger},a\right)$

We write the Heisenberg equation for: $f\left(a^{\dagger},a,t\right)\equiv e^{i\frac{H}{\hbar}t}f\left(a^{\dagger},a\right)e^{-i\frac{H}{\hbar}t}$

$\dot{f}=\frac{i}{\hbar}\left[H,f\right]$ We derive a closed PDE for the expectation value:

$f(\alpha^{*},\alpha,t)=\left\langle \alpha\right| f(a^{\dagger},a,t)\left|\alpha\right\rangle .$ We have from the Heisenberg equation:

$$\dot{f}=\frac{i}{\hbar}\left(\left\langle \alpha\right|Hf\left|\alpha\right\rangle-\left\langle \alpha\right|fH\left|\alpha\right\rangle\right)$$

Present the Hamiltonian and the function f in the normal-ordered form

$$H=\sum_{m,n}H_{m,n}\left(a^{\dagger}\right)^{m}a^{n},\ \ f\left(t\right)=\sum_{k,l}f_{k,l}\left(t\right)\left(a^{\dagger}\right)^{k}a^{l}.$$ Then, we have:

$$\left\langle \alpha\right|Hf\left|\alpha\right\rangle=\sum_{m,n}\sum_{k,l}H_{m,n}f_{k,l}\left(t\right)\left\langle \alpha\right|\left(a^{\dagger}\right)^{m}a^{n}\left(a^{\dagger}\right)^{k}a^{l}\left|\alpha\right\rangle=$$

$$\sum_{m,n}\sum_{k,l}H_{m,n}f_{k,l}\left(t\right)\left(\alpha^{*}\right)^{m}\alpha^{l}\left\langle \alpha\right|a^{n}\left(a^{\dagger}\right)^{k}\left|\alpha\right\rangle$$

Presentation of $\langle \alpha | a^n (a^\dagger)^k | \alpha \rangle$

Using: $Q = \langle z | af | z \rangle = e^{-|z|^2} \frac{\partial}{\partial z^*} e^{|z|^2} f,$ we have:

$$\langle \alpha | a^n (a^\dagger)^k | \alpha \rangle = e^{-|\alpha|^2} \frac{\partial^n}{\partial (\alpha^*)^n} (\alpha^*)^k e^{|\alpha|^2}$$

We derive the closed form expression:

$$\langle \alpha | Hf | \alpha \rangle = e^{-|\alpha|^2} \sum_{m,n} \sum_{k,l} H_{m,n} f_{k,l}(t) (\alpha^*)^m \alpha^l \frac{\partial^n}{\partial (\alpha^*)^n} (\alpha^*)^k e^{|\alpha|^2} =$$

$$e^{-|\alpha|^2} H\left(\alpha^*, \frac{\partial}{\partial \alpha^*}\right) f e^{|\alpha|^2}; \quad \left(a^\dagger \Rightarrow \alpha^*;\ a \Rightarrow \frac{\partial}{\partial \alpha^*}\right)$$

Analogously: $\langle \alpha | fH | \alpha \rangle = e^{-|\alpha|^2} H\left(\frac{\partial}{\partial \alpha}, \alpha\right) f e^{|\alpha|^2}$

Closed Equation for Observable Value

$$\dot{f} = \hat{K} f; \quad f = f\left(\alpha^*, \alpha, t\right);$$

$$f = f\left(\alpha^*, \alpha, t\right);$$

$$f\left(0\right) = f\left(\alpha^*, \alpha\right);$$

$$\hat{K} = \frac{i}{\hbar} e^{-|\alpha|^2} \left[H\left(\alpha^*, \frac{\partial}{\partial \alpha^*} \right) - H\left(\frac{\partial}{\partial \alpha}, \alpha \right) \right] e^{|\alpha|^2}$$

Quantum Nonlinear Oscillator

$$\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a} + \hbar^2\mu\left(\hat{a}^\dagger\hat{a}\right)^2; \;\; \left(\mu \geq 0\right)$$

$$\dot{f} = i\left(\omega + \mu\hbar + 2\mu\hbar\left|\alpha\right|^2\right)\left(\alpha^*\frac{\partial}{\partial\alpha^*} - \alpha\frac{\partial}{\partial\alpha}\right)f + i\mu\hbar\left(\left(\alpha^*\right)^2\frac{\partial^2}{\partial\left(\alpha^*\right)^2} - \alpha^2\frac{\partial^2}{\partial\alpha^2}\right)f,$$

$$\left(f\left(\alpha^*,\alpha,t\right) = \langle\alpha|\, f\left(\hat{a}^\dagger,\hat{a},t\right)|\alpha\rangle\right);$$

$$\alpha = \sqrt{J/\hbar}\,\exp\left(-i\theta\right); \qquad \left(\left|\frac{\partial\left(\sqrt{\hbar}\alpha,\sqrt{\hbar}\alpha^*\right)}{\partial\left(J,\theta\right)}\right| = 1;\; J = \hbar\left|\alpha\right|^2\right);$$

$$\alpha^*\frac{\partial}{\partial\alpha^*} - \alpha\frac{\partial}{\partial\alpha} = -i\frac{\partial}{\partial\theta}; \qquad \alpha^*\frac{\partial}{\partial\alpha^*} + \alpha\frac{\partial}{\partial\alpha} = 2J\frac{\partial}{\partial J};$$

$$\left(\alpha^*\right)^2\frac{\partial^2}{\partial\left(\alpha^*\right)^2} - \alpha^2\frac{\partial^2}{\partial\alpha^2} = i\frac{\partial}{\partial\theta} - 2J\frac{\partial^2}{\partial J\partial\theta};$$

Action-angle variables

Behavior of observables for quantum nonlinear oscillator

Initial condition: $f\left(\alpha^*,\alpha,0\right)=\left(\alpha^*\right)^m\alpha^q$

Solution: $f\left(\alpha^*,\alpha,t\right)=\left(\alpha^*\right)^m\alpha^q epx\left\{i\omega t\left(m-q\right)+i\mu\hbar t\left(m\left(m-1\right)-q\left(q-1\right)\right)\right\}\times$
$\exp\left\{\left(e^{2i\mu\hbar(m-q)t}-1\right)\left|\alpha\right|^2\right\}$

$$m=0,\ q=1;\ f\left(\alpha^*,\alpha,0\right)=\alpha;$$

$$f\left(\alpha^*,\alpha,t\right)=\alpha\left(\alpha^*,\alpha,t\right)=\left\langle\alpha\left|\hat{a}\left(t\right)\right|\alpha\right\rangle=\alpha e^{-i\omega t}\,\mathrm{e}^{\left\{\left(e^{-2i\mu\hbar t}-1\right)\left|\alpha\right|^2\right\}}$$

Classical limit: $\left(\hbar \Rightarrow 0,\ |\alpha|^2 \Rightarrow \infty,\ J \Rightarrow const\right)$

Introduce classical action of linear oscillator:

$$J = \hbar|\alpha|^2 = \hbar\bar{n},\ \bar{n} = \langle\alpha|a^+a|\alpha\rangle$$

Quasi-classical parameter: $\varepsilon = \dfrac{\hbar}{J} = \dfrac{1}{\bar{n}} << 1$

Classical Hamiltonian:

$$H_{cl}(J) = \omega J + \mu J^2; \quad \left(\omega_{cl}(J) = dH_{cl}(J)/dJ = \omega + 2\mu J\right)$$

Classical equations of motion:

$$\dot{f}_{cl} = i(\omega + 2\mu J)\left(\alpha^* \frac{\partial}{\partial\alpha^*} - \alpha\frac{\partial}{\partial\alpha}\right) f_{cl} = -i(\omega + 2\mu J)\frac{\partial}{\partial\theta} f_{cl};$$

$$\alpha_{cl}(t) = e^{-i(\omega + 2\mu J)t}\alpha; \quad (m = 0,\ q = 1)$$

Characteristic time scale at which quantum corrections become significant

Assuming: $\mu\hbar t \ll 1;\ m \neq q;$ we get:

$$f\left(\alpha^*,\alpha,t\right) = f_{cl}\left(\alpha^*,\alpha,t\right)\left[1 + i\mu\hbar t\left(m\left(m-1\right) - q\left(q-1\right) + O\left(\mu\hbar t\right)\right)\right]\times$$
$$\exp\left(-2\mu^2\hbar^2 t^2\left|\alpha\right|^2\left(m-q\right)^2 + O\left(\mu^3\hbar^3 t^3\left|\alpha\right|^2\right)\right)$$

In particular, by the time $\dfrac{1}{\mu|\alpha|\hbar}$ *quantum corrections are of the same order as classical solution*

Ehrenfest time scale:

$$t_E = \frac{1}{2\mu|\alpha|\hbar}$$

$$\alpha(t) \approx \alpha_{cl}(t)e^{-t^2/2t_E^2}$$

Characteristic parameters

Quasi-classical parameter: $\varepsilon = \frac{\hbar}{J} << 1 \left(J = \hbar |\alpha|^2 \right)$

Classical parameter of nonlinearity: $\mu_{cl} = \frac{\mu |\alpha|^4}{\omega |\alpha|^2} = \frac{\mu J}{\omega};$

Quantum parameter of nonlinearity: $\bar{\mu} = \frac{\mu\hbar}{\omega}; \quad \left(\frac{\bar{\mu}}{\mu_{cl}} = \varepsilon \right)$

Classical nonlinear frequency:

$$\omega_{cl} = dH_{cl} / dJ = \omega + 2\mu J; \; (H_{cl} = \omega J + \mu J^2)$$

Classical period: $\tau_{cl} = \frac{2\pi\omega}{\omega_{cl}} = \frac{2\pi}{1 + 2\mu_{cl}};$

Quantum recurrence time: $\tau_R = \frac{\pi}{\bar{\mu}}; \; (\tau = \omega t) \qquad \frac{\bar{\mu}}{\varepsilon} = \mu_{cl}$

Ehrenfest time: $\tau_E = \frac{1}{2\mu_{cl}\sqrt{\varepsilon}}$

$$\boxed{\alpha\left(\alpha^*, \alpha, \tau\right) = \alpha e^{-i\tau}\, \mathrm{e}^{\left\{\left(e^{-2i\bar{\mu}\tau} - 1\right)\frac{1}{\varepsilon}\right\}}}$$

General remarks on quasi-classical behavior of mesoscopic systems

Equation for quantum averages has the form

$$\frac{\partial f}{\partial \tau} = \left(\hat{K}_{cl} + \hbar \hat{K}_q \right) f$$

Includes high order derivatives

Includes only first order derivatives

For quantum nonlinear oscillator this equation has the form

$$\frac{\partial f}{\partial \tau} = \left(1 + 2\mu_{cl}\right) \frac{\partial f}{\partial \theta} + 2\varepsilon \mu_{cl} \frac{\partial^2 f}{\partial J \partial \theta};$$

$$\left(\alpha = \sqrt{J/\hbar}\, e^{-i\theta};\ \tau = \omega t \right)$$

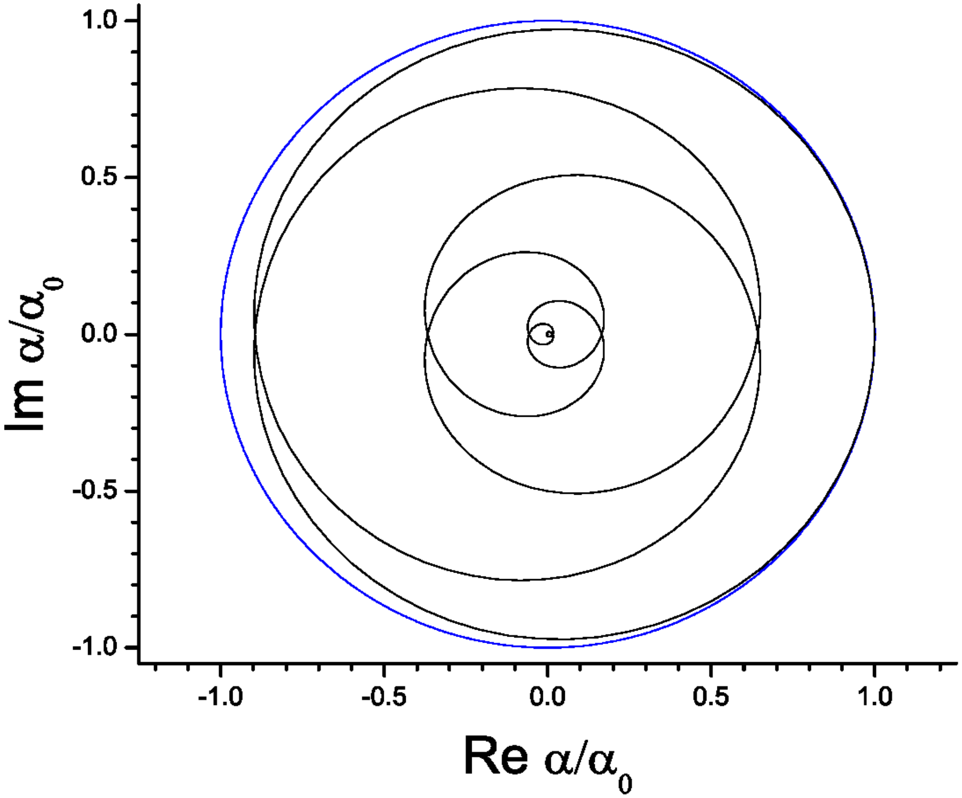
$$\left[\begin{array}{l} \varepsilon = \dfrac{\hbar}{J} << 1, \\ \mu_{cl} = \dfrac{\mu J}{\omega} \end{array} \right.$$

$$\hat{f} = \hat{a};\ \alpha(t) = e^{-i\tau} e^{\left(e^{-2i\mu_{cl}\varepsilon\tau} - 1\right)|\alpha|^2} \alpha$$

Solution

Two dimensionless parameters, quantum and classical

Demonstration of quantum dynamics



$$\frac{\alpha(\tau)}{\alpha_0} = e^{-i\tau} \exp\left[\left(e^{-2i\bar{\mu}\tau} - 1\right)/\varepsilon\right];$$
$$\left(\alpha = \alpha(0) \equiv \alpha_0\right)$$

$$\frac{\alpha_{cl}(\tau)}{\alpha_0} = e^{-i(1+2\mu_{cl})\tau}$$

$$\bar{\mu} = 0.05, \;\; \varepsilon = 0.05$$

$$\tau_R > \tau_E > \tau_{cl}$$

$$\tau_E = 2.27, \;\; \tau_R = 62.8, \;\; \tau_{cl} = 2.09$$

- Quantum dynamics is characterized by Ehrenfest time τ_E;
- Quantum recurrence time is very large in quasiclassical limit.

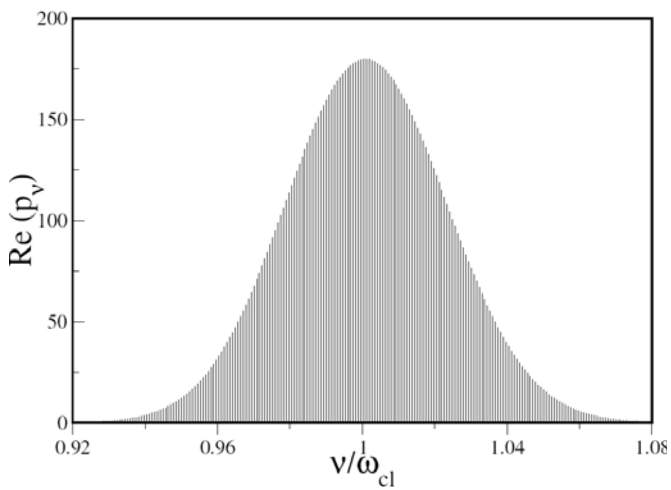
Fourier Spectrum

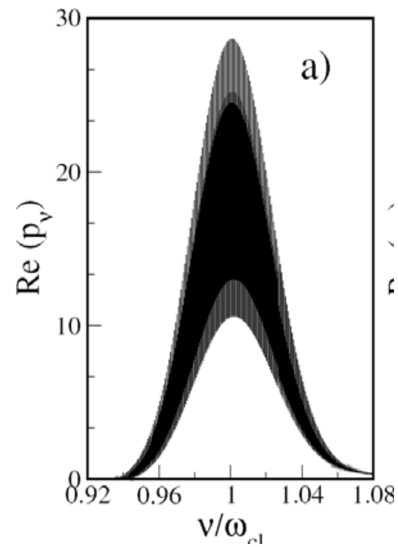
$$f\left(\alpha^*,\alpha,t\right)=\alpha\left(\alpha^*,\alpha,t\right)=$$

$$\left\langle\alpha\left|\hat{a}\left(t\right)\right|\alpha\right\rangle=\alpha e^{-i\omega t}\,\mathrm{e}^{\left\{\left(e^{-2i\mu\hbar t}-1\right)\left|\alpha\right|^2\right\}}=$$

$$\alpha e^{-i\omega t}e^{-2\left|\alpha\right|^2\sin^2\mu\hbar t}e^{-i\left|\alpha\right|^2\sin 2\mu\hbar t}\approx\overbrace{\alpha e^{-i\left(\omega+2\mu J\right)t}}^{classical}\;\overbrace{e^{-2\left|\alpha\right|^2\left(\mu\hbar t\right)^2}}^{quantum\ correction}=\alpha_{cl}\left(\tau\right)e^{-\frac{\tau^2}{2\tau_E^2}};$$

$$\alpha_\nu=\alpha\int_{-\infty}^{\infty}e^{i\left(\nu-\omega\left(J\right)\right)\tau}e^{-\frac{\tau^2}{2\tau_E^2}}d\tau=\sqrt{2\pi\tau_E^2}\,e^{-\frac{\tau_E^2\left(\nu-\omega\left(J\right)\right)^2}{2}}$$





$$\Delta\nu=\frac{2\sqrt{2}}{\tau_E}$$

Fourier Spectrum

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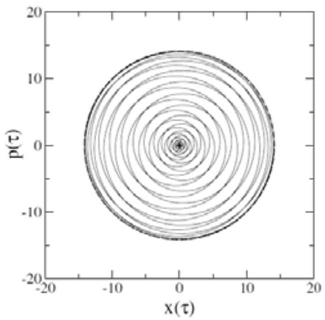


FIG. 1: Quasi-classical dynamics as described by the observable in Eq. (3). Parameters are $\epsilon = 0.02$, $\bar{\mu} = 0.01$, $\tau_{\rm E} = 5$, $\tau_{\rm R} = 314$, $\tau_{\rm cl} = 2.09$, $|\alpha|^2 = 100$, $\mu_{\rm cl} = 1$. Hence $\tau_{\rm cl} < \tau_{\rm E} < \tau_{\rm R}$.

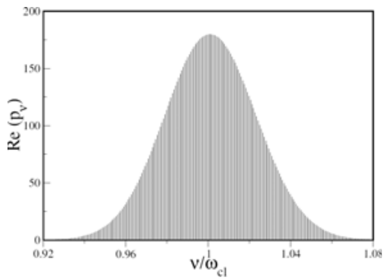


FIG. 2: Frequency spectrum of the effective momentum $p(\tau)$. Parameters are $\epsilon = 1/900$, $\bar{\mu} = 1/900$, $\tau_{\rm E} \approx 15$, $\tau_{\rm R} \approx 900\pi$, $\tau_{\rm cl} \approx 2\pi/3$, and $|\alpha|^2 = 900$. Hence $\tau_{\rm cl} < \tau_{\rm E} < \tau_{\rm R}$.

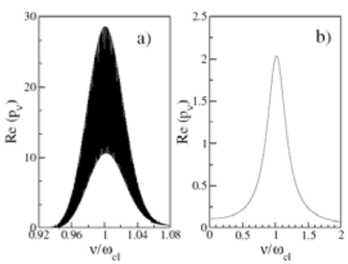


FIG. 3: Fourier frequency spectrum of the momentum $p(\tau)$ obtained from Eq. (7). Parameters are: a) $\gamma = 0.0005$, $\tau_\gamma = 2000 \gg \tau_{\rm E} \approx 15$; b) $\gamma = 0.5$, $\tau_\gamma = 2 < \tau_{\rm E} \approx 15$; all other parameters are the same as in Fig. 2.

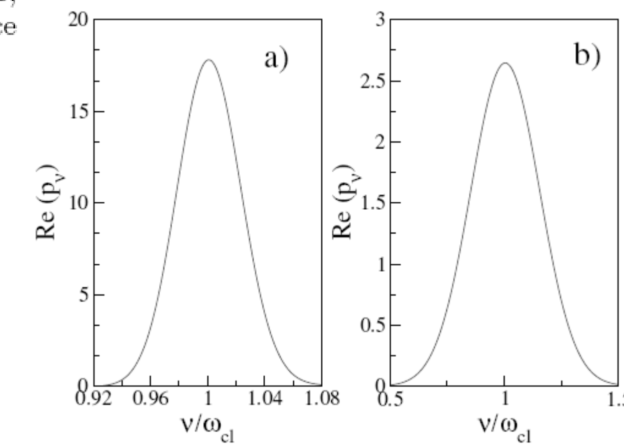


FIG. 4: Fourier frequency spectrum of the effective momentum $p(\tau)$ obtained from Eq. (10) for different values of $\gamma_{\mathcal{E}}$. All other parameters are the same as in Fig. 2. (a) $\gamma_{\mathcal{E}} = 5\times10^{-5}$. The characteristic relaxation time due to the interaction with the environment is $\tau_{\gamma_{\mathcal{E}}} = 200$. The characteristic width of the spectral line due to the interaction with the environment is $\Delta\nu_{\gamma_{\mathcal{E}}} \approx 2\sqrt{2\gamma_{\mathcal{E}}} = 0.02$. The Ehrenfest time scale is $\tau_{\rm E} \approx 15$, hence $\tau_{\gamma_{\mathcal{E}}} \gg \tau_{\rm E}$. The width of the spectral line due to the Ehrenfest time scale is $\Delta\nu_{\rm E} \approx 0.19$, and the numerical result give $\Delta\nu \approx 0.183$; (b) $\gamma_{\mathcal{E}} = 0.2$ ($\tau_{\gamma_{\mathcal{E}}} = 3.16$ and $\Delta\nu_{\gamma_{\mathcal{E}}} \approx 1.26$). The Ehrenfest time scale is $\tau_{\rm E} \approx 15$, hence $\tau_{\gamma_{\mathcal{E}}} < \tau_{\rm E}$. The numerical result for the width of the line is approximately 1.29, that corresponds to $\Delta\nu_{\gamma_{\mathcal{E}}}$.

Relation to the Navies Stokes Equation

$$\overbrace{\rho\Big(\underbrace{\frac{\partial \mathbf{v}}{\partial t}}_{\text{Unsteady acceleration}} + \underbrace{\mathbf{v}\cdot\nabla\mathbf{v}}_{\text{Convective acceleration}}\Big)}^{\text{Inertia}} = \underbrace{-\nabla p}_{\text{Pressure gradient}} + \underbrace{\mu\nabla^2\mathbf{v}}_{\text{Viscosity}} + \underbrace{\mathbf{f}}_{\text{Other forces}}$$

$$\mathrm{Re}_L = \frac{\rho V L}{\mu} = \frac{V L}{\nu}$$ ⟸ **Reynolds number**

$$\varepsilon = \frac{1}{\mathrm{Re}_L} \sim \frac{\nu}{Vk} \ll 1$$ ⟸ **Small parameter for large** $\mathrm{Re}_L \gg 1$

Term with viscosity is a singular perturbation

Geometrical Issues

The equation $\frac{\partial f}{\partial \tau} = \left(\hat{K}_{cl} + \hbar \hat{K}_q \right) f$ can be written in the form

$$df = \left(\hat{K}_{cl} + \hbar \hat{K}_q \right) f d\tau$$

Differential

First order derivatives

Higher order derivatives

Generalization for Many Degrees of Freedom

We consider a time-independent Hamiltonian

$$H(a_1^\dagger, \ldots, a_N^\dagger, a_1, \ldots, a_N) = \sum H_{\ell s} a^{\dagger \ell} a^s \equiv \sum H_{\ell s} a_1^{\dagger \ell_1} a_2^{\dagger \ell_2} \cdots a_N^{\dagger \ell_N} a_1^{s_1} a_2^{s_2} \cdots a_N^{s_N}$$

where

$$\ell = (\ell_1, \ldots, \ell_N) \in \mathbb{Z}_+^N, \qquad s = (s_1, \ldots, s_N) \in \mathbb{Z}_+^N$$

$$a^{\dagger \ell} = a_1^{\dagger \ell_1} \cdots a_N^{\dagger \ell_N}; \; a^s = a_1^{s_1} \cdots a_N^{s_N}$$

$$a_k^\dagger = \frac{1}{\sqrt{2}}\left(x_k - \hbar \frac{\partial}{\partial x_k}\right), \qquad a_k = \frac{1}{\sqrt{2}}\left(x_k + \hbar \frac{\partial}{\partial x_k}\right)$$

$$[a_k^\dagger, a_\ell] = -\delta_{k\ell}\hbar \text{ for } k, \ell = 1, \ldots, N$$

$$H_{\ell s} = H_{s\ell}^*$$

Coherent states

We introduce Poisson vectors Φ_α and coherent states $|\alpha\rangle$ as follows: for any $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{C}^N$

$$\Phi_\alpha = (\pi\hbar)^{-N/4} \exp\left\{-\frac{1}{2\hbar}\left(x^2 - 2\sqrt{2}\,x\cdot\alpha + \alpha^2\right)\right\},$$

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2\hbar}\right)|\Phi_\alpha\rangle.$$

Then, $a\Phi_\alpha = \alpha\Phi_\alpha$, i.e.,

$$a_k\Phi_\alpha = \alpha_k\Phi_\alpha, \quad k = 1, \ldots, N, \quad \text{also}$$

$$\hbar\frac{\partial}{\partial\alpha_k}\Phi_\alpha = a_k^\dagger\Phi_\alpha, \quad k = 1, \ldots, N.$$

$$|\alpha\rangle = e^{-|\alpha|^2/2\hbar}\sum_{n=0}^{\infty}\frac{\left(\alpha/\sqrt{\hbar}\right)^n}{\sqrt{n!}}|n\rangle$$

$$H_0|n\rangle = E_n|n\rangle, \; E_n = \hbar(n+1/2),$$

$$H_0 = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial x^2} + \frac{x^2}{2}, \; p = -i\hbar\frac{\partial}{\partial x}$$

Quantum observables for arbitrary operator

Heisenberg equations of motion for arbitrary operator

$$f(\alpha,t)=\langle\alpha|F(t)|\alpha\rangle,\ F(t)=e^{iHt/\hbar}Fe^{-iHt/\hbar}$$

$$|\alpha\rangle=|\alpha_1\rangle|\alpha_2\rangle\cdots|\alpha_N\rangle$$

The Heisenberg equation for an arbitrary operator valued function $F(t)$ is

$$\dot{F}=\frac{i}{\hbar}[H,F].$$

We want to derive a closed equations for

$$f(\alpha,t)=\langle\alpha|F(t)|\alpha\rangle$$

Let $F(t)=\sum_{m,q\in\mathbb{Z}_+^N}F_{mq}(t)a^{\dagger m}a^q$

Closed equations for quantum observables

$$\frac{d}{dt}\langle \Phi_\alpha | F \Phi_\alpha \rangle$$

$$= \frac{i}{\hbar}\left\langle \Phi_\alpha \middle| (HF - FH)\Phi_\alpha \right\rangle$$

$$= \frac{i}{\hbar}\sum_{\ell,s,m,q} H_{\ell s} F_{mq}(t)\left\{\left\langle \Phi_\alpha \middle| a^{\dagger\ell} a^s a^{\dagger m} a^q \Phi_\alpha \right\rangle - \left\langle \Phi_\alpha \middle| a^{\dagger m} a^q a^{\dagger\ell} a^s \Phi_\alpha \right\rangle\right\}$$

$$= \frac{i}{\hbar}\sum_{\ell,s,m,q} H_{\ell s} F_{mq}\left\{\alpha^{*\ell}\alpha^q \left\langle a^{\dagger s}\Phi_\alpha \middle| a^{\dagger m}\Phi_\alpha \right\rangle - \alpha^{*m}\alpha^s \left\langle a^{\dagger q}\Phi_\alpha \middle| a^{\dagger\ell}\Phi_\alpha \right\rangle\right\}$$

$$= \frac{i}{\hbar}\sum_{\ell,s,m,q} H_{\ell s} F_{mq}\left\{\alpha^{*\ell}\alpha^q \left(\hbar\frac{\partial}{\partial\alpha^*}\right)^s \left(\hbar\frac{\partial}{\partial\alpha}\right)^m - \alpha^{*m}\alpha^s \left(\hbar\frac{\partial}{\partial\alpha^*}\right)^q \left(\hbar\frac{\partial}{\partial\alpha}\right)^\ell\right\} \exp\frac{|\alpha|^2}{\hbar}$$

$$= \frac{i}{\hbar}\sum_{\ell,s,m,q} H_{\ell s} F_{mq}\left\{\alpha^{*\ell}\left(\hbar\frac{\partial}{\partial\alpha^*}\right)^s \alpha^{*m}\alpha^q - \alpha^s \left(\hbar\frac{\partial}{\partial\alpha}\right)^\ell \alpha^{*m}\alpha^q\right\} \exp\frac{|\alpha|^2}{\hbar}$$

$$= \frac{i}{\hbar}\sum_{\ell,s} H_{\ell s}\left\{\alpha^{*\ell}\left(\hbar\frac{\partial}{\partial\alpha^*}\right)^s - \alpha^s \left(\hbar\frac{\partial}{\partial\alpha}\right)^\ell\right\} \exp\frac{|\alpha|^2}{\hbar}\langle\alpha|F(t)|\alpha\rangle.$$

Introduce a polynomial

$$\mathcal{H}(z^*, z) = \sum_{\ell, s} H_{\ell s} z^{*\ell} z^s$$

A closed equation for quantum observable *F* has a form

$$\frac{d}{dt}\langle\alpha|F|\alpha\rangle = \frac{i}{\hbar}\sum_{r\in\mathbb{Z}_+^N}\frac{1}{r!}\left(\left(\frac{\partial}{\partial\alpha}\right)^r\mathcal{H}(\alpha^*,\alpha)\left(\hbar\frac{\partial}{\partial\alpha^*}\right)^r - \left(\frac{\partial}{\partial\alpha^*}\right)^r\mathcal{H}(\alpha^*,\alpha)\left(\hbar\frac{\partial}{\partial\alpha}\right)^r\right)\langle\alpha|F|\alpha\rangle.$$

Here, for $r = (r_1, \ldots, r_N) \in \mathbb{Z}_+^N$, $r! = r_1! \cdots r_N!$

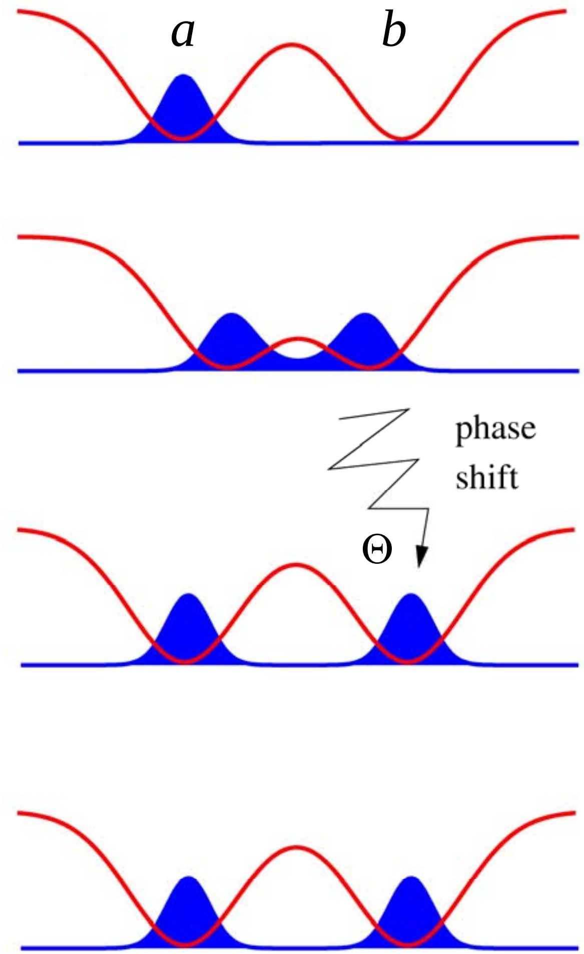
$$\left(\frac{\partial}{\partial\alpha_1}\right)^r = \left(\frac{\partial}{\partial\alpha_1}\right)^{r_1}\cdots\left(\frac{\partial}{\partial\alpha_N}\right)^{r_N}; \qquad \left(\frac{\partial}{\partial\alpha^*}\right)^r = \left(\frac{\partial}{\partial\alpha_1^*}\right)^{r_1}\cdots\left(\frac{\partial}{\partial\alpha_N^*}\right)^{r_N}$$

Solution: $f(\alpha^*, \alpha, t) = \langle\alpha|F(t)|\alpha\rangle, \quad \alpha \in \mathbb{C}^N$

Initial condition: $f(\alpha^*, \alpha, 0) = \alpha^{*m}\alpha^q; \quad m, q \in \mathbb{Z}_+^N$

Mathematical Model

$$\left|\psi_{inp}\right\rangle = \left|N\right\rangle_a \left|0\right\rangle_b$$



$$\begin{cases} \hat{H} = \hbar\omega\,\hat{a}^{+}\hat{a} + \hbar\omega\,\hat{b}^{+}\hat{b} + E_J(t)\left(\hat{a}^{+}\hat{b} + \hat{b}^{+}\hat{a}\right) \\ \left|\psi_{BS}\right\rangle = e^{-i\frac{\pi}{4}\left(\hat{a}^{+}\hat{b}+\hat{b}^{+}\hat{a}\right)}\left|N\right\rangle_a \left|0\right\rangle_b \end{cases}$$

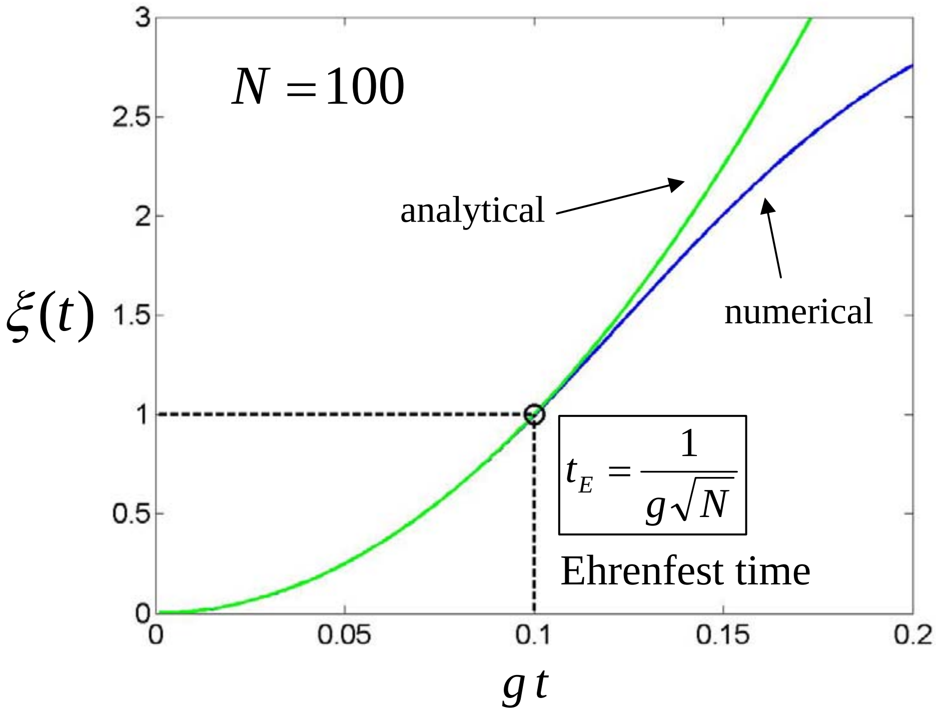
$$\int_0^{+\infty} E_J(t)\,\mathrm{dt} = \frac{\pi}{2}$$

$$\left|\psi_{PS}\right\rangle = e^{-i\frac{\Theta}{2}\left(\hat{a}^{+}\hat{a}-\hat{b}^{+}\hat{b}\right)}\left|\psi_{BS}\right\rangle = \left|\psi_{NL}(g=0)\right\rangle$$

We introduce a non-linear evolution inside the interferometer

$$\begin{cases} \hat{H} = \hbar\omega\,\hat{a}^{+}\hat{a} + \hbar^2 g\left(\hat{a}^{+}\hat{a}\right)^2 + \hbar\omega\,\hat{b}^{+}\hat{b} + \hbar^2 g\left(\hat{b}^{+}\hat{b}\right)^2 \\ \left|\psi_{NL}\right\rangle = e^{-i\frac{\hat{H}t}{\hbar}}\left|\psi_{PS}\right\rangle = \sum_{k=0}^{N} \frac{1}{2^{N/2}}\sqrt{\frac{N!}{k!(N-k)!}}\, e^{-i\Theta k} e^{-i\hbar g t (N/2-k)^2} \left|k\right\rangle_a \left|N-k\right\rangle_b \end{cases}$$

Results

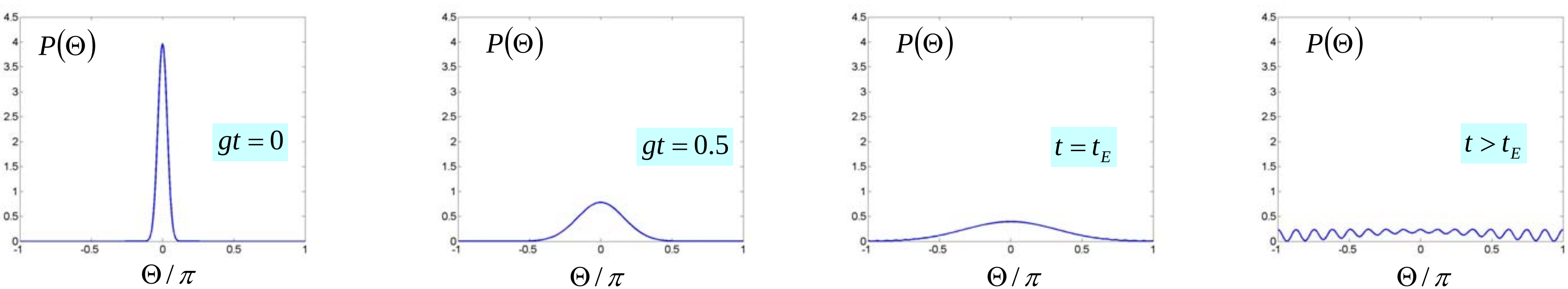


$$\xi(t) = [\Delta\Theta(t)]^2 - \frac{1}{N}$$

$$\xi_{cl} = 0$$

case g = 0 or $t = 0 : \xi(t = 0) = 0$

Green line: (analytical) $t < \tau_E : \xi(t) = Ng^2t^2$



There is a <u>recurrence time</u> when the system recovers the initial condition: $\xi(\tau_R) = 0 \; ; \; \tau_R = \frac{\pi}{g} >> \tau_E$

Long time evolution of quantum averages for unstable classical dynamics (N=1; 1D case)

[G. Berman, M. Vishik, Phys Lett. A **319**, 351 (2003)]

Consider 1D Hamiltonian $H(a^\dagger, a) = i\omega(a^{\dagger 2} - a^2) + \mu(a^{\dagger 2} - a^2)^2$

Introduce operators of coordinate and momentum

$$\hat{x} = \frac{1}{\sqrt{2}}(a^\dagger + a), \qquad \hat{p} = \frac{i}{\sqrt{2}}(a^\dagger - a)$$

In $\hat{x}$ and $\hat{p}$ operators Hamiltonian has the form

$$H = 2\omega\hat{x}\hat{p} - i\omega\hbar + \mu(2i\hat{x}\hat{p} + \hbar)^2$$

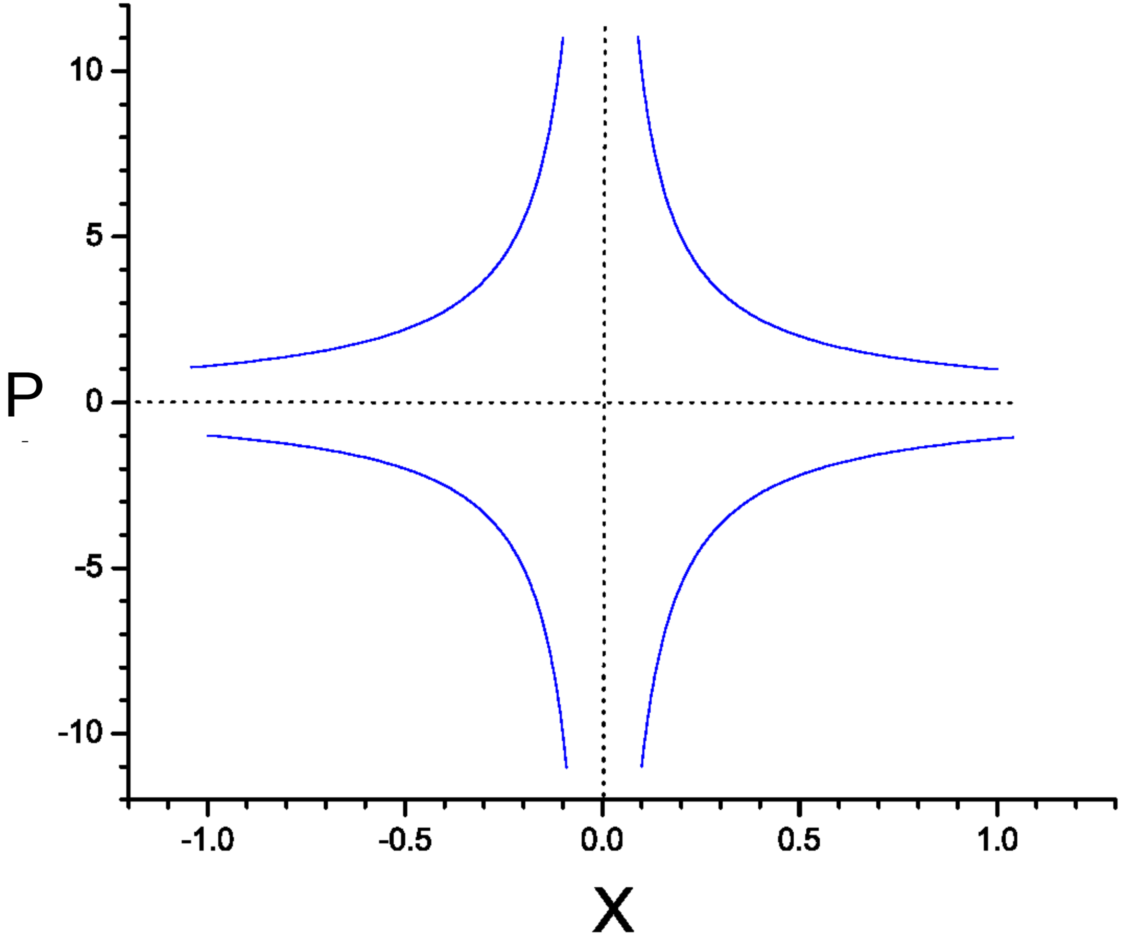
Evolution of operators $\hat{x}$ and $\hat{p}$:

$$X(t) = e^{\frac{i}{\hbar}Ht}\hat{x}e^{-\frac{i}{\hbar}Ht}$$

$$P(t) = e^{\frac{i}{\hbar}Ht}\hat{p}e^{-\frac{i}{\hbar}Ht}$$

$$X(0) = \hat{x},\ P(0) = \hat{p},\ [X(t), P(t)] = i\hbar$$

Unstable classical dynamics



$$X(t) = x_0 e^{(2-8\mu_{cl} x_0 p_0)\tau}$$

$$P(t) = p_0 e^{-(2-8\mu_{cl} x_0 p_0)\tau}$$

$$(\mu_{cl} = 1)$$

Equation for quantum observables

$$\frac{\partial}{\partial t} f = i[-2i\omega\alpha - 4\mu(\alpha^{*2} - \alpha^2)\alpha - 4\mu\hbar\alpha^*]\frac{\partial}{\partial\alpha^*} f - i[2i\omega\alpha^* + 4\mu(\alpha^{*2} - \alpha^2)\alpha^* - 4\mu\hbar\alpha]\frac{\partial}{\partial\alpha} f$$
$$+ i\hbar[-i\omega - 2\mu(\alpha^{*2} - 3\alpha^2)]\left(\frac{\partial}{\partial\alpha^*}\right)^2 f - i\hbar[i\omega + 2\mu(3\alpha^{*2} - \alpha^2)]\left(\frac{\partial}{\partial\alpha}\right)^2 f$$
$$+ 4i\hbar^2\mu\alpha\left(\frac{\partial}{\partial\alpha^*}\right)^3 f - 4i\hbar^2\mu\alpha^*\left(\frac{\partial}{\partial\alpha}\right)^3 f + i\hbar^3\mu\left(\frac{\partial}{\partial\alpha^*}\right)^4 f - i\hbar^3\mu\left(\frac{\partial}{\partial\alpha}\right)^4 f.$$

Solution:

$$f(\alpha^*, \alpha, t) = \frac{\hbar^{n/2}}{\sqrt{\pi}} \exp\left\{-\frac{(\alpha+\alpha^*)^2}{2\hbar}\right\} \frac{e^{2\omega n t}}{(\cos 8\mu n\hbar t)^{\frac{n+1}{2}}} \exp\left\{\frac{(\alpha^* e^{-4in\mu\hbar t} + \alpha e^{4in\mu\hbar t})^2}{2\hbar\cos 8n\mu\hbar t}\right\}$$
$$\times \sum_{k=0}^{[n/2]} \frac{n!}{(2k)!(n-2k)!} \frac{(\alpha^* e^{-4in\mu\hbar t} + \alpha e^{4in\mu\hbar t})^{n-2k}}{\hbar^{\frac{n}{2}-k}(2\cos 8n\mu\hbar t)^{\frac{n}{2}-k}} \int_{-\infty}^{\infty} x^{2k} e^{-x^2}\, dx$$
$$= \exp\left\{-\frac{(\alpha+\alpha^*)^2}{2\hbar}\right\} \frac{e^{2\omega n t}}{(\cos 8\mu n\hbar t)^{\frac{n+1}{2}}} \exp\left\{\frac{(\alpha^* e^{-4in\mu\hbar t} + \alpha e^{4in\mu\hbar t})^2}{2\hbar\cos 8n\mu\hbar t}\right\}$$
$$\times 2^{-n/2} \sum_{k=0}^{[n/2]} \frac{n!(2k-1)!!}{(2k)!(n-2k)!2^k} 2^k \hbar^k \frac{(\alpha^* e^{-4in\mu\hbar t} + \alpha e^{4in\mu\hbar t})^{n-2k}}{(\cos 8n\mu\hbar t)^{\frac{n}{2}-k}}$$
$$= \exp\left\{-\frac{(\alpha+\alpha^*)^2}{2\hbar}\right\} \frac{e^{2\omega n t}}{(\cos 8\mu n\hbar t)^{\frac{n+1}{2}}} \exp\left\{\frac{(\alpha^* e^{-4in\mu\hbar t} + \alpha e^{4in\mu\hbar t})^2}{2\hbar\cos 8n\mu\hbar t}\right\}$$
$$\times 2^{-n/2} \sum_{k=0}^{[n/2]} \frac{n!}{2^k k!(n-2n)!} \hbar^k \frac{(\alpha^* e^{-4in\mu\hbar t} + \alpha e^{4in\mu\hbar t})^{n-2k}}{(\cos 8\mu n\hbar t)^{\frac{n}{2}-k}}.$$

Collapse of quantum averages

$$f(\alpha^*, \alpha, t) = \langle \alpha | X^n(t) | \alpha \rangle$$

Solution for n=2:

$$f(\alpha^*, \alpha, t) = \frac{1}{2} \exp\left\{ -\frac{(\alpha + \alpha^*)^2}{2\hbar} \right\} \frac{e^{4\omega t}}{(\cos 16\mu\hbar t)^{3/2}} \exp\left\{ \frac{(\alpha^* e^{-8i\mu\hbar t} + \alpha e^{8i\mu\hbar t})^2}{2\hbar \cos 16\mu\hbar t} \right\}$$
$$\times \left[\frac{(\alpha^* e^{-8i\mu\hbar t} + \alpha e^{8i\mu\hbar t})^2}{\cos 16\mu\hbar t} + \hbar \right].$$

We discuss the limit of this expression as $t \to \frac{\pi}{32\mu\hbar} - 0$. For simplicity, let $\alpha = i$. Since

$$(\alpha^* e^{-8i\mu\hbar t} + \alpha e^{8i\mu\hbar t})^2 = \left(-i\frac{1-i}{\sqrt{2}} + i\frac{1+i}{\sqrt{2}} \right)^2 = 2 \quad \text{at } t = \frac{\pi}{32\mu\hbar},$$

we have

$$f(-i, i, t) \to \infty \quad \text{as } t \to \frac{\pi}{32\mu\hbar} - 0.$$

Times of collapses:

$$t_\ell = \frac{\pi}{32\mu\hbar} + \ell \frac{\pi}{16\mu\hbar}, \qquad \ell = 0, \pm 1, \pm 2, \ldots .$$

Time dependence of quantum corrections

Average quantum coordinate:

$$f(\alpha^*,\alpha,t) = \langle\alpha|X(t)|\alpha\rangle = \frac{1}{\sqrt{2}}\exp\left\{-\frac{(\alpha+\alpha^*)^2}{2\hbar}\right\}\frac{e^{2\omega t}}{(\cos 8\mu\hbar t)^{3/2}}\exp\left\{\frac{(\alpha^* e^{-4i\mu\hbar t}+\alpha e^{4i\mu\hbar t})^2}{2\hbar\cos 8\mu\hbar t}\right\} \times \left(\alpha^* e^{-4i\mu\hbar t}+\alpha e^{4i\mu\hbar t}\right).$$

Under conditions: $|\mu\hbar t| \ll 1, \quad |\alpha|^2 \gg \hbar, \quad \mu^2\hbar t^2|\alpha|^2 \ll 1$

$$f(\alpha^*,\alpha,t) = \frac{1}{\sqrt{2}}e^{2\omega t+4i\mu t(\alpha^2-\alpha^{*2})}\left(1+\mathcal{O}\left(\mu^2\hbar^2 t^2\right)\right)\left((\alpha^*+\alpha)+4i\mu\hbar t(\alpha-\alpha^*)+\mathcal{O}\left(|\alpha|\mu^2\hbar^2 t^2\right)\right) \times \exp\left(16\mu^2\hbar t^2|\alpha|^2+\mathcal{O}\left(|\alpha|^2\left|\mu^3\hbar^2 t^3\right|\right)\right).$$

Classical coordinate: $f_{\mathrm{cl}}(\alpha^*,\alpha,t) = \frac{1}{\sqrt{2}}\exp\left\{2\omega t+4i\mu t\left(\alpha^2-\alpha^{*2}\right)\right\}(\alpha^*+\alpha)$

At logarithmically small time $C\log\frac{1}{\varepsilon\mu_{cl}}$ the quantum corrections become at least of order of 1.

Comparison of quantum and classical dispersion

$$D(\alpha^*,\alpha,t) = \langle\alpha|X^2(t)|\alpha\rangle - \langle\alpha|X(t)|\alpha\rangle^2$$

$$D(\alpha^*,\alpha,t) \approx \exp\{4\omega t - 8i\mu\hbar t(\alpha^2-\alpha^{*2})\}\left(\frac{1}{2}\hbar + 4i\mu\hbar t(\alpha^2-\alpha^{*2}) + 16(\alpha^*+\alpha)^2\mu^2\hbar t^2|\alpha|^2\right)$$

Quantum dispersion vanishes when $\varepsilon=0$ (classical limit)

Conclusion

- For nonlinear oscillator quantum dynamics is characterized by three time scales: (i) classical period τ_{cl}, (ii) Ehrenfest time τ_E, (iii) quantum recurrence time τ_R.
- In the quasi-classical region τ_R is usually large: $\tau_R > \tau_E > \tau_{cl}$.
- To observe quantum effects one should measure the quantum features related to Ehrenfest time scale τ_E.
- For stable classical dynamics $\tau_E = 1/\mu_{cl}\sqrt{\varepsilon}$; for unstable classical dynamics $C\log(1/\varepsilon\mu_{cl})$.
- Time scale τ_E can be extracted from the frequency spectrum.
- Quantum collapses should be investigated in more detail.

Asymptotic theory for quantum Bose systems with many degrees of freedom

[M. Vishik, G. Berman, Phys Lett. A **313**, 37 (2003)]

We start with the closed equation for quantum averages

$$\frac{d}{dt}\langle\alpha|F|\alpha\rangle = \frac{i}{\hbar}\sum_{r\in\mathbb{Z}_+^N}\frac{1}{r!}\left(\left(\frac{\partial}{\partial\alpha}\right)^r \mathcal{H}(\alpha^*,\alpha)\left(\hbar\frac{\partial}{\partial\alpha^*}\right)^r - \left(\frac{\partial}{\partial\alpha^*}\right)^r \mathcal{H}(\alpha^*,\alpha)\left(\hbar\frac{\partial}{\partial\alpha}\right)^r\right)\langle\alpha|F|\alpha\rangle.$$

Here, for $r=(r_1,\ldots,r_N)\in\mathbb{Z}_+^N$, $r! = r_1!\cdots r_N!$

Initial condition: $\langle\alpha|F(0)|\alpha\rangle = \alpha^{*m}\alpha^q \exp\left(-\frac{|\alpha-\alpha_0|^2}{\hbar}\right), \quad \alpha_0\in\mathbb{C}^N$

$$F(0) = a^{\dagger m}|\alpha_0\rangle\langle\alpha_0|a^q.$$

Indeed, for $\alpha\in\mathbb{C}^N$,

$$\langle\alpha|a^{\dagger m}|\alpha_0\rangle\langle\alpha_0|a^q|\alpha\rangle = \alpha^{*m}\alpha^q\left|\langle\alpha_0,\alpha\rangle\right|^2 = \alpha^{*m}\alpha^q\left|\exp\left(\frac{\alpha_0^*\alpha}{\hbar} - \frac{|\alpha_0|^2}{2\hbar} - \frac{|\alpha|^2}{2\hbar}\right)\right|^2$$

$$= \alpha^{*m}\alpha^q \exp\left(-\frac{|\alpha-\alpha_0|^2}{\hbar}\right).$$

. .

Quasi-classical asymptotic theory for quantum averages

Assume formally

$$f(\alpha^*, \alpha, t) = e^{\frac{S(\alpha^*, \alpha, t)}{\hbar}} \sum_{j=0}^{\infty} b_j(\alpha^*, \alpha, t)\hbar^j \quad (*)$$

where the phase $S(\alpha^*, \alpha, t)$ and the coefficients b_j, $j = 0, 1, 2, \ldots$ are to be determined.

The initial conditions are

$$S(\alpha^*, \alpha, 0) = -|\alpha - \alpha_0|^2,$$

$$b_0(\alpha^*, \alpha, 0) = \alpha^{*m} \alpha^q,$$

$$b_j(\alpha^*, \alpha, 0) = 0 \quad \text{for } j \geqslant 1.$$

For the phase S we have from Taylor's formula:

$$\dot{S} = i\left\{\mathcal{H}\left(\alpha^*, \alpha + \frac{\partial S}{\partial \alpha^*}\right) - \mathcal{H}\left(\alpha^* + \frac{\partial S}{\partial \alpha}, \alpha\right)\right\}$$

This Hamilton-Jacobi equation is real and therefore the classical Hamilton-Jacobi theory applies.

- The expansion (*) describes Laplace asymptotics unlike WKB asymptotics for Schrödinger equation describing the evolution of wave function

Effective Hamiltonian

We introduce momenta p and p* and the effective Hamiltonian associated with $\mathcal{H}(z^*, z) = \sum_{\ell,s} H_{\ell s} z^{*\ell} z^s$

$$W(\alpha^*, \alpha, p^*, p) = -i\{\mathcal{H}(\alpha^*, \alpha + p^*) - \mathcal{H}(\alpha^* + p, \alpha)\}.$$

The Hamiltonian dynamics in these variables is

$$\begin{cases} \dot{\alpha}_k = i\dfrac{\partial \mathcal{H}(\alpha^* + p, \alpha)}{\partial \alpha_k^*}, \\ \dot{p}_k = i\left(\dfrac{\partial \mathcal{H}(\alpha^*, \alpha + p^*)}{\partial \alpha_k} - \dfrac{\partial \mathcal{H}(\alpha^* + p, \alpha)}{\partial \alpha_k}\right), \\ \dot{\alpha}_k^* = -i\dfrac{\partial \mathcal{H}(\alpha^*, \alpha + p^*)}{\partial \alpha_k}, \\ \dot{p}_k^* = i\left(\dfrac{\partial \mathcal{H}(\alpha^*, \alpha + p^*)}{\partial \alpha_k^*} - \dfrac{\partial \mathcal{H}(\alpha^* + p, \alpha)}{\partial \alpha_k^*}\right). \end{cases}$$

Example for N=1, $\mathcal{H}(\alpha^*, \alpha) = \omega\alpha^*\alpha + \mu\alpha^{*2}\alpha^2$

$$\dot{S} = i\left(\omega + 2\mu|\alpha|^2\right)\left(\alpha^* \frac{\partial S}{\partial \alpha^*} - \alpha \frac{\partial S}{\partial \alpha}\right) + i\mu\left(\alpha^{*2}\left(\frac{\partial S}{\partial \alpha^*}\right)^2 - \alpha^2\left(\frac{\partial S}{\partial \alpha}\right)^2\right).$$

The first transport equation is

$$\begin{aligned}
\dot{b}_0 &= i\left(\omega + 2\mu|\alpha|^2\right)\left(\alpha^* \frac{\partial}{\partial \alpha^*} - \alpha \frac{\partial}{\partial \alpha}\right) b_0 + 2i\mu\left(\alpha^{*2} \frac{\partial S}{\partial \alpha^*} \frac{\partial}{\partial \alpha^*} - \alpha^2 \frac{\partial S}{\partial \alpha} \frac{\partial}{\partial \alpha}\right) b_0 \\
&\quad + i\mu\left(\alpha^{*2} \frac{\partial^2 S}{\partial \alpha^{*2}} - \alpha^2 \frac{\partial^2 S}{\partial \alpha^2}\right) b_0 \\
&\equiv L_0 b_0.
\end{aligned}$$

Solution

A simple computation yields:

$$S(\alpha^*,\alpha,t) = -\left|\alpha(0)-\alpha_0\right|^2 - i\mu t\left(\alpha^*(0)^2\alpha_0^2 - \alpha(0)^2\alpha_0^{*2}\right) + \left|\alpha(0)\right|^2\left(1 - e^{-2i\mu t(\alpha^*(0)\alpha_0 - \alpha(0)\alpha_0^*)}\right),$$

$$b_0(\alpha^*,\alpha,t) = \alpha(0)^{*m}\alpha(0)^q \frac{1}{\sqrt{1+4\mu^2 t^2|\alpha_0|^2|\alpha(0)|^2}} \exp\left\{\frac{i(\alpha(0)^*\alpha_0 - \alpha(0)\alpha_0^*)}{2|\alpha(0)||\alpha_0|}\arctan\left(2\mu t|\alpha_0||\alpha(0)|\right)\right\}$$

where

$$\alpha = \alpha(0)\exp\left\{i\omega t + 2\mu t\alpha_0^*\alpha(0)\right\}.$$

To solve the problem with the initial conditions

$$\langle\alpha|F(0)|\alpha\rangle = \alpha^{*m}\alpha^q$$

we use the completeness relation

$$\frac{1}{(\pi\hbar)^N}\int |\alpha_0\rangle\langle\alpha_0|\, d^{2N}\alpha_0 = \mathrm{id}$$

which leads to the expansion

$$\langle\alpha|F(t)|\alpha\rangle = \frac{1}{(\pi\hbar)^N}\int d^{2N}\alpha_0 e^{\frac{S(\alpha^*,\alpha,t)}{\hbar}}\sum_{j=0}^{\infty} b_j(\alpha^*,\alpha,t)\hbar^j.$$

Here in the right-hand side the phase S and the coefficients b_j, $j \geqslant 0$ depend on a parameter $\alpha_0 \in \mathbb{C}^N$.

Additional study requires to apply the theory to concrete systems

Quantum nonlinear oscillator interacting with environment (exact solution)

Hamiltonian of quantum nonlinear oscillator: $H_S = \hbar\omega a^\dagger a + \mu\hbar^2 (a^\dagger a)^2$

Hamiltonian of thermal bath: $H_E = \sum_j \hbar\omega_j b_j^\dagger b_j$

Interaction Hamiltonian: $H_{\text{int}} = \sum_j g_j a^\dagger a \, b_j^\dagger b_j$

Joint observable of system+bath: $\hat{f} = \hat{f}(\hat{a}^\dagger(t), \hat{a}(t), \{\hat{b}_j^\dagger(t)\}, \{\hat{b}_j(t)\})$

Exact PDE for joint evolution: $$\frac{\partial f}{\partial t} = (\hat{K}_\alpha + \hat{K}_\beta + \hat{K}_{\text{int}}) f$$

$$\hat{K}_\alpha f = i(\omega + \mu\hbar + 2\mu\hbar|\alpha|^2)\left(\alpha^* \frac{\partial}{\partial \alpha^*} - \alpha \frac{\partial}{\partial \alpha}\right) f + \quad i\mu\hbar \left[(\alpha^*)^2 \frac{\partial^2}{\partial(\alpha^*)^2} - \alpha^2 \frac{\partial^2}{\partial \alpha^2}\right] f$$

$$\hat{K}_\beta f = i \sum_j \omega_j \left(\beta_j^* \frac{\partial}{\partial \beta_j^*} - \beta_j \frac{\partial}{\partial \beta_j}\right) f$$

$$\hat{K}_{\text{int}} f = (\hat{K}_{\text{int}}^{(1)} + \hat{K}_{\text{int}}^{(2)} + \hat{K}_{\text{int}}^{(3)}) f$$

$$\hat{K}_{\text{int}}^{(1)} f = \frac{i}{\hbar}\left(\sum_j g_j |\beta_j|^2\right)\left(\alpha^* \frac{\partial}{\partial \alpha^*} - \alpha \frac{\partial}{\partial \alpha}\right) f$$

$$\hat{K}_{\text{int}}^{(2)} f = \frac{i}{\hbar}|\alpha|^2 \sum_j g_j \left(\beta_j^* \frac{\partial}{\partial \beta_j^*} - \beta_j \frac{\partial}{\partial \beta_j}\right) f$$

$$\hat{K}_{\text{int}}^{(3)} f = \frac{i}{\hbar} \sum_j g_j \left(\alpha^* \frac{\partial}{\partial \alpha^*} \beta_j^* \frac{\partial}{\partial \beta_j^*} - \alpha \frac{\partial}{\partial \alpha} \beta_j \frac{\partial}{\partial \beta_j}\right) f$$

Initial state: System (coherent state) $\otimes$ Environment (thermal state)

$$\rho_S = |\alpha\rangle\langle\alpha| \qquad \rho_E = \prod_j \int d^2\beta_j \, P(\beta_j^*, \beta_j) \, |\beta_j\rangle\langle\beta_j| \qquad P(\beta_j^*, \beta_j) = \frac{1}{\pi \bar{n}_j} \, e^{-|\beta_j|^2/\bar{n}_j}$$

Exact solution:

$$f(\alpha, \alpha^*, \{\beta_j\}, \{\beta_j^*\}; t) = f_\alpha(\alpha, \alpha^*; t) \times f_\beta(\{\beta_j\}, \{\beta_j^*\}; t)$$

$$f_\alpha(t) = \alpha(t) \qquad f_\beta(t) = \prod_j f_\beta^{(j)}(t) = \prod_j \exp\left[-|\beta_j|^2 (1 - e^{-i g_j t/\hbar})\right]$$

Reduced observable:

$$f_S(t) = \alpha(t) \times \int \prod_j d^2\beta_j P(\beta_j^*, \beta_j) f_\beta(t)$$

$$= \alpha(t) R(t)$$

$R(t)$ → Decoherence factor

$$f_S(t) = e^{-\frac{\gamma t^2}{2}} \, e^{-i\delta\omega t} \, \alpha(t) \qquad \alpha(t) = \alpha \, e^{-i(\omega + \mu\hbar)t} \, \exp[|\alpha|^2 (e^{-2i\mu\hbar t} - 1)]$$

D.A.R. Dalvit, G.P. Berman, M. Vishik, Phys. Rev. A.; 2006; v.73, p.013803

<u>Open system:</u> <u>Nonlinear</u> quantum oscillator- bath: $H_{\rm total} = H_{\rm BEC} + H_{\rm bath} + H_{\rm int}$

$$H_{\rm BEC} = \hbar\omega_{\rm clas}\, a^\dagger a + \frac{U}{2}\, a^\dagger a\,(a^\dagger a - 1) \longrightarrow$$ **BEC as a non-linear oscillator**

$$H_{\rm bath} = \sum_j \hbar\omega_j\, b_j^\dagger\, b_j \longrightarrow$$ **set of harmonic oscillators** $\quad x = \frac{1}{\sqrt{2}}(a^+ + a)$

$$H_{\rm int} = \sum_j \lambda_j\, x\, q_j \longrightarrow$$ **position-position coupling** $\quad q_j = \frac{1}{\sqrt{2}}(b_j^+ + b_j)$

Five time-scales:

$\tau_{\rm clas} = 1$ Period of classical linear oscillator

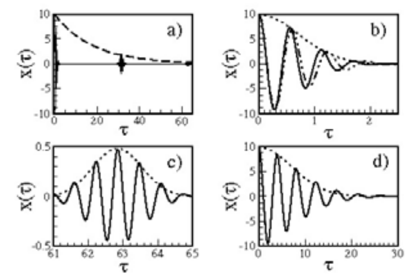
$\tau_R = \frac{\pi}{\bar{\mu}}$ Revival time

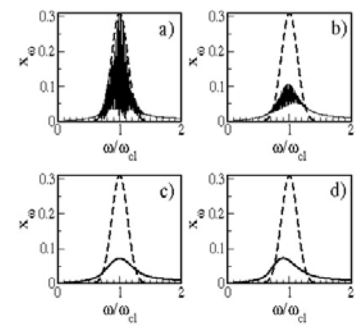
$\tau_E = \frac{1}{2\bar{\mu}|\alpha|}$ Ehrenfest time

$\tau_\gamma = \gamma^{-1}$ Relaxation time

$\tau_D = \frac{\tanh(\bar{\beta}\bar{\Omega}/2)}{\gamma|\alpha|^2\bar{\Omega}}$ Decoherence time

$\bar{\Omega} = 1 + \bar{\mu}(1 + 2|\alpha|^2)$





$|\alpha|^2 = 50 \quad \bar{\beta} = 1$

$(a-c)\ \bar{\mu} = 0.1\ ;\ \gamma = 10^{-4} \rightarrow \tau_D \gg \tau_E$

$(d)\ \bar{\mu} = \gamma = 10^{-2} \rightarrow \tau_D \ll \tau_E$

We showed that when $\tau_D \ll \tau_E < \tau_\gamma$ **one can still observe quantum effects for observables (width of revival bumps given by** τ_E**). Decoherence is insufficient for recovering quantum-classical correspondence in nonlinear systems**

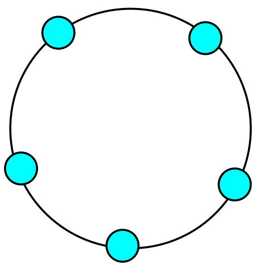
"*Survival of quantum effects for observables after decoherence*", G.P. Berman, A.R. Bishop, F. Borgonovi, and D.A.R. Dalvit, Phys. Rev. A 69, 062110 (2004).

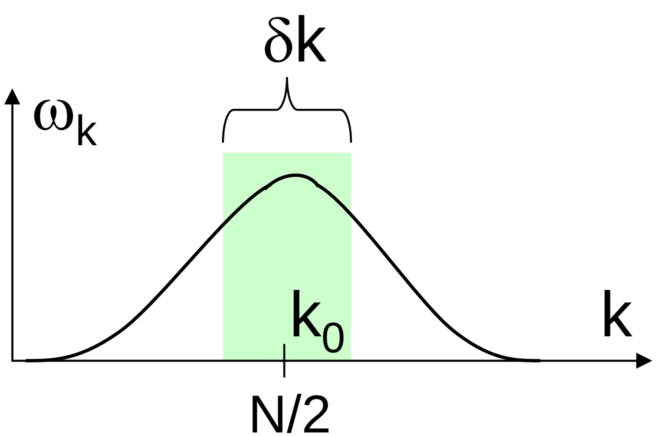
Quantum dynamics in Fermi Pasta Ulam Problem

G.P. Berman (LANL), N. Tarkhanov (U. of Potsdam)

Hamiltonian: $H = \sum_{n=1}^{N} \left(\frac{p_n^2}{2m} + \frac{\epsilon}{2} (u_{n+1} - u_n)^2 + \frac{\nu}{4} (u_{n+1} - u_n)^4 \right)$

Periodic boundary conditions: $p_{n+N} = p_n$ and $u_{n+N} = u_n$





Classical limit: new canonical variables: a_k, a_k*

$$a_k = \frac{1}{\sqrt{2m\hbar\omega_k}} (P_k - i\omega_k U_k^*),$$

$$P_k = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} p_n \, e^{-2\pi \frac{k}{N} n},$$

$$U_k = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} u_n \, e^{2\pi \frac{k}{N} n},$$

$$\omega_k = 2\sqrt{\frac{\epsilon}{m}} \sin \pi \frac{k}{N}.$$

Narrow packet approximation (NPA): Classical limit

$$\delta k / k_0 << 1,\ \delta k = |k - k_0|,\ k_0 \approx N/2$$

Hamiltonian in NPA:

$$H = \sum_{k=1}^{N} \hbar\omega_k a_k^* a_k + \frac{\hbar^2}{2} \sum_{k_1,k_2,k_3,k_4} V_{k_1k_2k_3k_4}\, a_{k_1}^* a_{k_2}^* a_{k_3} a_{k_4}\, \delta_{k_1+k_2-k_3-k_4,0} + O(1),$$

where

$$V_{k_1k_2k_3k_4} = \frac{3\nu}{\varepsilon m N}\left(\sin\pi\frac{k_1}{N}\ \sin\pi\frac{k_2}{N}\ \sin\pi\frac{k_3}{N}\ \sin\pi\frac{k_4}{N}\right)^{1/2}.$$

$$\begin{aligned} \omega_k &\approx \omega_{k_0} + c(k-k_0) - \Omega(k-k_0)^2, \\ V_{k_1k_2k_3k_4} &\approx V_0 \end{aligned}$$

where

$$c = 2\sqrt{\frac{\epsilon}{m}}\frac{\pi}{N}\cos\pi\frac{k_0}{N}, \quad \Omega = \sqrt{\frac{\epsilon}{m}}\left(\frac{\pi}{N}\right)^2 \sin\pi\frac{k_0}{N}, \quad V_0 = \frac{3\nu}{\varepsilon m N}\left(\sin\pi\frac{k_0}{N}\right)^2$$

Narrow packet approximation (NPA): Classical limit

Classical equations of motion for four wave interactions:

$$\imath \dot{a}_k = \frac{\partial H}{\partial a_k^*}$$

$$\imath \dot{A}_j = -j^2 \Omega \, A_j + \hbar V_0 \sum_{j_2, j_3, j_4} A_{j_2}^* A_{j_3} A_{j_4} \, \delta_{j+j_2-j_3-j_4,0},$$

$$A_j = \exp\Big((\omega_{k_0} + cj) t \, \imath \Big) a_{j+k_0}.$$

Introduce: $\Phi(\theta,t) = \sum_j A_j(t) e^{iJ\theta} = \Phi(\theta + 2\pi, t)$ - envelope

Equation for $\Phi(\theta,t)$: $i\frac{\partial \Phi}{\partial t} = \frac{\partial^2 \Phi}{\partial \theta^2} + \hbar V_0 |\Phi|^2 \Phi$ - NS or GP equation

Solution for finite amplitude wave:

$$\begin{aligned} A_k(t) &= \exp\Big((\Omega_k - \hbar V_0 |A_k|^2) t \, \imath \Big) A_k, \\ A_j(t) &= 0, \qquad \text{if} \quad j \neq k \end{aligned}$$

$$\Omega_k = k^2 \Omega$$

Stability of the finite amplitude wave with respect to the decay in the neighboring modes

$$2k \mapsto (k-l) + (k+l)$$

Assume that $|A_j| \ll |A_k|$

The linearized system of equations:

$$\begin{aligned}
\imath \dot{A}_k &= -(\Omega_k A_k + \hbar V_0 |A_k|^2) A_k, \\
\imath \dot{A}_{k-l} &= -\Omega_{k-l} A_{k-l} + 2\hbar V_0 |A_k|^2 A_{k-l} + \hbar V_0 A_k^2 A_{k+l}^* \\
\imath \dot{A}_{k+l} &= -\Omega_{k+l} A_{k+l} + 2\hbar V_0 |A_k|^2 A_{k+l} + \hbar V_0 A_k^2 A_{k-l}^*
\end{aligned}$$

The amplitudes of small waves grow exponentially with the increment:

$$\lambda_l = \sqrt{V_0 I(\Delta\Omega) - \left(\frac{\Delta\Omega}{2}\right)^2} = l\Omega\sqrt{\frac{2V_0 I}{\Omega} - l^2}$$

$$\Delta\Omega = \Omega_{k-l} + \Omega_{k+l} - 2\Omega_k = 2l^2\Omega \qquad I = \hbar |A_k|^2$$

Condition for instability: $2V_0 I/\Omega > 1$ or $\nu > \frac{\pi^2}{3NE_{k_0}} \sim \frac{1}{NE}$

Quantum equations of decays $[u_j, p_k] = \imath\hbar\,\delta_{jk}$

$$\imath \dot{A}_j = -j^2(1+q)\Omega\, A_j + \hbar V_0 \sum_{j_2, j_3, j_4} A^\dagger_{j_2} A_{j_3} A_{j_4}\, \delta_{j+j_2-j_3-j_4,0},$$

$$\begin{aligned} [A_j, A^\dagger_k] &= \delta_{jk}, \\ q &= \hbar \frac{\nu \cot \dfrac{\pi}{2N}}{32N\sqrt{m\epsilon^3}}, \end{aligned}$$

Quantum observables: $$\begin{aligned} \alpha_j(t) &= \langle \vec{\alpha} |\, A_j(t)\, | \vec{\alpha} \rangle \\ &= \alpha_j(t, \vec{\alpha}, \vec{\alpha}^*), \end{aligned}$$

Heisenberg equations: $\imath\hbar\, \dot{A}_j = [A_j(t), H_{\rm eff}]$

Effective quantum Hamiltonian:

$$H_{\rm eff} = -\hbar \sum_k \Omega_k A^\dagger_k A_k + \frac{1}{2}\hbar^2 V_0 \sum_{k_1, k_2, k_3, k_4} A^\dagger_{k_1} A^\dagger_{k_2} A_{k_3} A_{k_4}\, \delta_{k_1+k_2-k_3-k_4,0}$$

Equations for quantum observables

$$\begin{aligned} \imath\dot{\alpha}_j(t) &= \hat{T}\,\alpha_j(t), \\ \alpha_j(0) &= \alpha_j, \end{aligned}$$

$$\begin{aligned} \hat{T} &= -\sum_k \Omega_k \Big(\alpha_k \frac{\partial}{\partial \alpha_k} - C.C.\Big) \\ &+ \hbar V_0 \sum_{k_1,k_2,k_3,k_4} \Big(\alpha^*_{k_1}\alpha_{k_2}\alpha_{k_3}\frac{\partial}{\partial \alpha_{k_4}} - C.C.\Big)\delta_{k_1+k_2-k_3-k_4,0} \\ &+ \frac{1}{2}\hbar V_0 \sum_{k_1,k_2,k_3,k_4} \Big(\alpha_{k_1}\alpha_{k_2}\frac{\partial}{\partial \alpha_{k_3}}\frac{\partial}{\partial \alpha_{k_4}} - \ldots\Big)\delta_{k_1+k_2-k_3-k_4,0} \end{aligned}$$

Exact solution in the form of finite amplitude wave:

$$\begin{aligned} \alpha_k(t) &= \exp\Big(\Omega_k t\imath - (1-\exp(-\hbar V_0 t\imath))|\alpha_k|^2\Big)\,\alpha_k, \\ \alpha_j(t) &= 0, \qquad\qquad \text{if} \quad j \neq k \end{aligned}$$

$$\Omega_k = k^2(1+q)\Omega$$

Solutions for quantum observables in the form of expansion in α_j

$$\begin{aligned}
\alpha_{k+l}(t,\vec{\alpha},\vec{\alpha}^*) &= c_{l,0}(t,\alpha_k,\alpha_k^*) \\
&+ \sum_{j\neq 0}\left(c_{l,j}^{(1,0)}(t,\alpha_k,\alpha_k^*)\alpha_{k+j} + c_{l,j}^{(0,1)}(t,\alpha_k,\alpha_k^*)\alpha_{k+j}^*\right) \\
&+ \ldots,
\end{aligned}$$

Initial conditions:

$$\begin{aligned}
c_{0,0}(0,\alpha_k,\alpha_k^*) &= \alpha_k, & c_{l,0}(0,\alpha_k,\alpha_k^*) &= 0 \\
c_{l,j}^{(1,0)}(0,\alpha_k,\alpha_k^*) &= \delta_{lj}, & c_{l,j}^{(0,1)}(0,\alpha_k,\alpha_k^*) &= 0
\end{aligned}$$

Equations for observables:

$$\imath\,\dot{c}_{l,0} = \hat{M}c_{l,0},$$

$$\imath\,\dot{c}_{l,j}^{(1,0)} = \hat{M}c_{l,j}^{(1,0)} - (\Omega_{k+j} - 2\hbar V_0|\alpha_k|^2)c_{l,j}^{(1,0)} + 2\hbar V_0\alpha_k\frac{\partial}{\partial\alpha_k}c_{l,j}^{(1,0)} - \hbar V_0{\alpha_k^*}^2 c_{l,-j}^{(0,1)},$$

$$\imath\,\dot{c}_{l,-j}^{(0,1)} = \hat{M}c_{l,-j}^{(0,1)} - (\Omega_{k-j} - 2\hbar V_0|\alpha_k|^2)c_{l,-j}^{(0,1)} - 2\hbar V_0\alpha_k\frac{\partial}{\partial\alpha_k}c_{l,-j}^{(0,1)} + \hbar V_0\alpha_k^2 c_{l,j}^{(1,0)},$$

Equations for observables for small waves

Transformation to new functions f and g:

$$\begin{aligned} c_{l,l}^{(1,0)} &= \exp\Big(-(\Omega_{k-l} - 2\Omega_k)t\imath\Big) f, \\ c_{l,-l}^{(0,1)} &= \frac{\alpha_k}{\alpha_k^*} \exp\Big(-(\Omega_{k-l} - 2\Omega_k)t\imath\Big) g \end{aligned}$$

Equations for f and g:

$$\begin{aligned} \imath\,\dot{f} &= (2V_0 I - (\Omega_{k-l} + \Omega_{k+l} - 2\Omega_k))f + 2\hbar V_0\, I\frac{\partial f}{\partial I} - V_0 I g \\ \imath\,\dot{g} &= V_0 I\, f + \hbar V_0\, g \end{aligned}$$

with initial data

$$\begin{aligned} f(0) &= 1, \\ g(0) &= 0, \end{aligned}$$

Equations for f and g

$$\begin{aligned} f &\mapsto \exp(-\hbar V_0 t\,\imath) f \\ g &\mapsto \exp(-\hbar V_0 t\,\imath) g \end{aligned}$$

$$t \mapsto t\Omega \qquad x = V_0 I/\Omega \qquad l = 1$$

$$\begin{aligned} \imath \dot{f} &= 2(x-1)f + 2\varepsilon x \frac{\partial f}{\partial x} - xg \\ \imath \dot{g} &= x f, \end{aligned}$$

Quantum parameter: $\varepsilon = \hbar \dfrac{V_0}{\Omega},$

Numerical simulations for BEC are described in

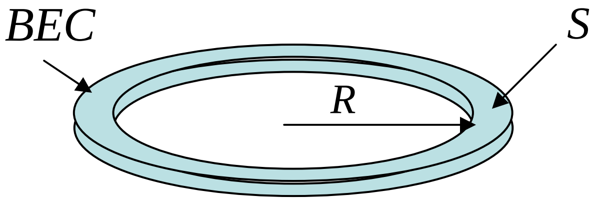
G.P. Berman, A. Smerzi, A.R. Bishop, Phys. Rev. Lett. **88**, 120402 (2002)

Mathematical theory was developed by N. Tarkhanov (2004)

Stability of the Quantum Dynamics of a Bose-Einstein Condensate Trapped in a One-Dimensional Toroidal Geometry

(G.P. Berman, A.R. Bishop, D.A.R. Dalvit, G.V. Shlyapnikov, N. Tarkhanov, E.M. Timmermans, Int. J. Theor. Phys., 2008)

Ring geometry



Equation for quantum field operator

$$i\frac{\partial\hat{\Psi}}{\partial\tau}=\left(-\frac{\partial^2}{\partial\theta^2}+2\pi\varepsilon\hat{\Psi}^\dagger\hat{\Psi}\right)\hat{\Psi},\quad \hat{\Psi}(\theta,\tau)=\frac{1}{\sqrt{2\pi}}\sum_{j=-\infty}^{\infty}\hat{a}_j(\tau)e^{ij\theta}$$

$$\varepsilon=4a\frac{R}{S}\rightarrow 0$$

Quantum Hamiltonian

$$\hat{H}_{eff}=\sum_p k^2\,\hat{a}_k^+\,\hat{a}_k+\frac{\varepsilon}{2}\sum_{k_1,k_2,k_3,k_4=-\infty}^{\infty}\hat{a}_{k_1}^+\,\hat{a}_{k_2}^+\hat{a}_{k_3}\,\hat{a}_{k_4}\delta_{k_1+k_2-k_3-k_4,0}$$

Quantum Perturbation Theory

$$\imath\, \dot{\hat{a}}_j = j^2 \hat{a}_j + \varepsilon \sum_{k_1,k_2,k_3=-\infty}^{\infty} \hat{a}^{\dagger}_{k_1} \hat{a}_{k_2} \hat{a}_{k_3} \delta_{j+k_1-k_2-k_3,0} \qquad |\vec{\alpha}\rangle = \prod_{j=-\infty}^{\infty} |\alpha_j\rangle$$

$$\imath\, \dot{\hat{a}}_j = [\hat{a}_j(\tau), \hat{H}_{\mathrm{eff}}]$$

$$\imath\, \dot{\alpha}_j(\tau) = \hat{T} \alpha_j(\tau),$$

$$\alpha_j(0) = \alpha_j,$$

$$\hat{T} = \sum_{k=-\infty}^{\infty} k^2 \left(\alpha_k \frac{\partial}{\partial \alpha_k} - C.C. \right)$$

$$+ \varepsilon \sum_{k_1,k_2,k_3,k_4=-\infty}^{\infty} \left(\alpha_{k_1} \alpha_{k_2} \alpha^*_{k_3} \frac{\partial}{\partial \alpha_{k_4}} - C.C. \right) \delta_{k_1+k_2-k_3-k_4,0}$$

$$+ \frac{1}{2} \varepsilon \sum_{k_1,k_2,k_3,k_4=-\infty}^{\infty} \left(\alpha_{k_1} \alpha_{k_2} \frac{\partial}{\partial \alpha_{k_3}} \frac{\partial}{\partial \alpha_{k_4}} - C.C. \right) \delta_{k_1+k_2-k_3-k_4,0},$$

Periodic Quantum Wave and Small Perturbations

$$\alpha_j(\tau) = \exp(-\iota\, j^2\tau - (1-\exp(-\iota\,\varepsilon\tau))|\alpha_j|^2)\alpha_j,$$ **Quantum wave**

$$\alpha_{j'}(\tau) = 0, \quad \text{if } j' \neq j.$$

$$\alpha_{\mathrm{cl},j}(\tau) = \exp(-\iota(j^2 + \varepsilon|\alpha_j|^2)\tau)\alpha_{\mathrm{cl},j},$$ **Classical wave**

$$\tau_\hbar = \frac{\sqrt{2}}{|\alpha_j||\varepsilon|}$$ **Validity of classical (GP) solution**

$$\tau_r = \frac{2\pi}{|\varepsilon|}.$$ **Quantum revivals**

Equations for quantum dynamics

$$\alpha_{k+\ell}(\tau, \vec{\alpha}, \vec{\alpha}^*) = c_{\ell,0}(\tau, \alpha_k, \alpha_k^*) + \sum_{j\neq 0}\left(c_{\ell,j}^{(1,0)}(\tau, \alpha_k, \alpha_k^*)\alpha_{k+j} + c_{\ell,j}^{(0,1)}(\tau, \alpha_k, \alpha_k^*)\alpha_{k+j}^*\right) + \dots,$$

$$\imath\, \dot{c}_{\ell,0} = \hat{M} c_{\ell,0},$$

$$\imath\, \dot{c}_{\ell,j}^{(1,0)} = \hat{M} c_{\ell,j}^{(1,0)} + ((k+j)^2 + 2\varepsilon|\alpha_k|^2) c_{\ell,j}^{(1,0)} + 2\varepsilon\alpha_k \frac{\partial}{\partial\alpha_k} c_{\ell,j}^{(1,0)} - \varepsilon\alpha_k^{*2} c_{\ell,-j}^{(0,1)},$$

$$\imath\, \dot{c}_{\ell,-j}^{(0,1)} = \hat{M} c_{\ell,-j}^{(0,1)} - ((k-j)^2 + 2\varepsilon|\alpha_k|^2) c_{\ell,-j}^{(0,1)} - 2\varepsilon\alpha_k \frac{\partial}{\partial\alpha_k} c_{\ell,-j}^{(0,1)} + \varepsilon\alpha_k^2 c_{\ell,j}^{(1,0)},$$

$$\hat{M} = (k^2 + \varepsilon|\alpha_k|^2)\alpha_k \frac{\partial}{\partial\alpha_k} + \frac{1}{2}\varepsilon\alpha_k^2 \frac{\partial^2}{\partial\alpha_k^2} - C.C.,$$

Results on Convergence of Quantum Solution to GP

Stable case (Repulsive interaction)

$$T \sim \frac{1}{a_k(x)} \frac{1}{\sqrt{|\varepsilon|}} \quad < \quad \tau_\hbar = \frac{\sqrt{2}}{|\alpha_j||\varepsilon|}$$

Due to quantum tunneling

Unstable case (Attractive interaction)

$$[0,T] \times \{x \in \mathbb{C} : |x| \le R\} \times \left\{\varepsilon \in \mathbb{R} : |\varepsilon| \le \frac{1}{2Te^{3TR}}\right\}$$

Some Properties of Spin Coherent States

$$\mathbf{S}^x = \frac{1}{2}\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \qquad \mathbf{S}^y = \frac{1}{2}\begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}, \qquad \mathbf{S}^z = \frac{1}{2}\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}.$$

Spin ½ operators

$$[\mathbf{S}^+, \mathbf{S}^-] = 2\mathbf{S}^z, \qquad [\mathbf{S}^\pm, \mathbf{S}^z] = \mp \mathbf{S}^\pm,$$

$$[\mathbf{S}^\pm, \mathbf{S}^\pm] = 0, \qquad \mathbf{S}^\pm = \mathbf{S}^x \pm i\mathbf{S}^y.$$

Commutation relations

Spin coherent state is a superposition $|\xi >= (1 + |\xi|^2)^{-S} \exp(\xi \mathbf{S}^+)|S, -S >$ of spin states $|S, M\rangle$ with $S^z = M; -S \leq M \leq S$

generated from the ground state

$$\mathbf{S}^z|S, -S >= -S|S, -S >,$$

$$\mathbf{S}^2|S, -S >= S(S+1)|S, -S >,$$

$$\mathbf{S}^2 = (\mathbf{S}^x)^2 + (\mathbf{S}^y)^2 + (\mathbf{S}^z)^2,$$

Some properties of spin coherent states

$$|\xi> = \sum_{M=-S}^{S} U_M(\xi)|S, M>,$$ expansion of the CS in the basis of states $\left|S, M\right\rangle$

$$U_M(\xi) = (1+|\xi|^2)^{-S}\left[\frac{2S}{(S+M)!(S-M)!}\right]^{1/2}\xi^{S+M}$$ coefficients of expansion

$$<\xi|\mu> = [(1+|\xi|^2)(1+|\mu|^2]^{-S}(1+\xi^*\mu)^{2S}$$ two CSs are non-orthogonal

$$P_M(\xi) = \left|\left\langle S, M\,|\,\xi\right\rangle\right|^2 = \frac{(2S)!|\xi|^{2(S+M)}}{\left(1+|\xi|^2\right)^{2S}(S+M)!(S-M)!}$$ probability to find the projection *M* in CS

$$S^z \equiv <\xi|\mathbf{S}^z|\xi> = -S\frac{1-|\xi|^2}{1+|\xi|^2}$$ expectation value of $\hat{S}^z$

$$p = S - M, \quad \bar{p} = S - S^z, \quad (0 \le p, \bar{p} \le 2S)$$ two parameters

We are interested in the distribution function of *p under the condition* $|\xi|^2 \approx \frac{2S}{\bar{p}} \gg 1$

$$P_{\bar{p}}(p) = \frac{e^{-\bar{p}}\bar{p}^p}{p!}, \quad (\bar{p}/2S \ll 1)$$ Poisson distribution

Single spin in coherent state

$$|\xi> = \sum_{M=-S}^{S} U_M(\xi)|S, M>, \quad (S = 1/2)$$

$$\left|\xi, \frac{1}{2}\right\rangle = \sum_{M=-\frac{1}{2}}^{\frac{1}{2}} U_M(\xi)\left|M, \frac{1}{2}\right\rangle = U_{-\frac{1}{2}}(\xi)\left|-\frac{1}{2}, \frac{1}{2}\right\rangle + U_{\frac{1}{2}}(\xi)\left|\frac{1}{2}, \frac{1}{2}\right\rangle =$$

$$\frac{1}{\sqrt{1+|\xi|^2}}|\downarrow\rangle + \frac{\xi}{\sqrt{1+|\xi|^2}}|\uparrow\rangle$$

Length of a single spin

$$\vec{S}^2 = \left(S^x\right)^2 + \left(S^y\right)^2 + \left(S^z\right)^2 = \frac{3}{4} = S\left(S+1\right);$$

$$S^{x,y,z} = \frac{1}{2}\sigma^{x,y,z}; \ \left(\sigma^{x,y,z}\right)^2 = 1;$$

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma^y = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \left|\uparrow\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \left|\downarrow\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

Say, we have: $\psi_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \ S^z\psi_z = \frac{1}{2}\psi_z; \ \text{But: } \vec{S}^2\psi_z = \frac{3}{4}\psi_z;$

In classical case it would be: $\vec{S}_{cl}^2 = \frac{1}{4},$ **because in classical case:**

$S_{cl}^{x,y} = 0,$ **but in quantum case:**

$$\psi_z^\dagger S^x \psi_z = 0; \ S^x\psi_z = \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \neq 0, \ \psi_x = \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right); \ \left\langle \psi_z \middle| \psi_x \right\rangle = \frac{1}{\sqrt{2}}; \ \left|\left\langle \psi_z \middle| \psi_x \right\rangle\right|^2 = \frac{1}{2} \neq 0$$

Two spins in coherent state

$$\left|\xi,1\right\rangle = \sum_{M=-1}^{1} U_M\left(\xi\right)\left|M,1\right\rangle = U_{-1}\left(\xi\right)\left|-1,1\right\rangle + U_0\left(\xi\right)\left|0,1\right\rangle + U_1\left(\xi\right)\left|1,1\right\rangle =$$

$$\frac{1}{\left(1+\left|\xi\right|^2\right)}\left|-1,1\right\rangle + \frac{\sqrt{2}\xi}{\left(1+\left|\xi\right|^2\right)}\left|0,1\right\rangle + \frac{\xi^2}{\left(1+\left|\xi\right|^2\right)}\left|1,1\right\rangle =$$

$$\frac{1}{\left(1+\left|\xi\right|^2\right)}\left|\downarrow_1\downarrow_2\right\rangle + \frac{\sqrt{2}\xi}{\left(1+\left|\xi\right|^2\right)}\left[\frac{1}{\sqrt{2}}\left(\left|\downarrow_1\uparrow_2\right\rangle + \left|\uparrow_1\downarrow_2\right\rangle\right)\right] + \frac{\xi^2}{\left(1+\left|\xi\right|^2\right)}\left|\uparrow_1\uparrow_2\right\rangle = \left|\xi,\frac{1}{2}\right\rangle_1\left|\xi,\frac{1}{2}\right\rangle_2$$

Prove this for *N* spins!

Some expressions

$$\vec{S}^2 = \left(\vec{S}_1 + \vec{S}_2\right)^2 = \vec{S}_1^2 + \vec{S}_2^2 + 2\vec{S}_1\vec{S}_2 =$$

$$\left(S_1^x\right)^2 + \left(S_1^y\right)^2 + \left(S_1^z\right)^2 + \left(S_2^x\right)^2 + \left(S_2^y\right)^2 + \left(S_2^z\right)^2 + \frac{1}{2}\vec{\sigma}_1\vec{\sigma}_2 = \frac{3}{2} + \frac{1}{2}\vec{\sigma}_1\vec{\sigma}_2;$$

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma^y = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \left|\uparrow\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \left|\downarrow\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

$$\vec{S}^2\left|\uparrow_1\downarrow_2\right\rangle = \left|\uparrow_1\downarrow_2\right\rangle + \left|\downarrow_1\uparrow_2\right\rangle;\ \vec{S}^2\left|\downarrow_1\uparrow_2\right\rangle = \left|\uparrow_1\downarrow_2\right\rangle + \left|\downarrow_1\uparrow_2\right\rangle;$$

$$\vec{S}^2\frac{1}{\sqrt{2}}\left(\left|\uparrow_1\downarrow_2\right\rangle + \left|\downarrow_1\uparrow_2\right\rangle\right) = 2\frac{1}{\sqrt{2}}\left(\left|\uparrow_1\downarrow_2\right\rangle + \left|\downarrow_1\uparrow_2\right\rangle\right) = S\left(S+1\right)\frac{1}{\sqrt{2}}\left(\left|\uparrow_1\downarrow_2\right\rangle + \left|\downarrow_1\uparrow_2\right\rangle\right);$$

$$S^z = S_1^z + S_2^z;\ S^z\frac{1}{\sqrt{2}}\left(\left|\uparrow_1\downarrow_2\right\rangle + \left|\downarrow_1\uparrow_2\right\rangle\right) = 0\frac{1}{\sqrt{2}}\left(\left|\uparrow_1\downarrow_2\right\rangle + \left|\downarrow_1\uparrow_2\right\rangle\right)$$

Some properties of spin coherent states

$$< (\mathbf{S}^x)^2 >< (\mathbf{S}^y)^2 > \quad \geq \quad \frac{1}{4} < \mathbf{S}^z >^2$$

Spin CS minimize the Heisenberg uncertainty condition

$$\int |\xi >< \xi| d\mu_s(\xi) = \mathbf{I},$$

$$d\mu_s(\xi) = \frac{(2S+1)}{\pi} \frac{d^2\xi}{(1+|\xi|^2)^2}$$

Spin CSs represent a complete set

The following relations take place similar to bosonic CSs

$$\mathbf{f} = \mathbf{f}(\mathbf{S}^z, \mathbf{S}^+, \mathbf{S}^-)$$

$$< \xi|\mathbf{S}^+\mathbf{f}|\xi >= (1+|\xi|^2)^{-2S} \left(2S\xi^* - \xi^{*2}\frac{\partial}{\partial \xi^*}\right)(1+|\xi|^2)^{2S} f,$$

$$< \xi|\mathbf{S}^-\mathbf{f}|\xi >= (1+|\xi|^2)^{-2S} \frac{\partial}{\partial \xi^*}(1+|\xi|^2)^{2S} f,$$

$$< \xi|\mathbf{S}^z\mathbf{f}|\xi >= (1+|\xi|^2)^{-2S} \left(\xi^*\frac{\partial}{\partial \xi^*} - S\right)(1+|\xi|^2)^{2S} f,$$

$$< \xi|\mathbf{f}\mathbf{S}^+|\xi >= (1+|\xi|^2)^{-2S} \frac{\partial}{\partial \xi}(1+|\xi|^2)^{2S} f,$$

$$< \xi|\mathbf{f}\mathbf{S}^-|\xi >= (1+|\xi|^2)^{-2S} \left(2S\xi - \xi^2\frac{\partial}{\partial \xi}\right)(1+|\xi|^2)^{2S} f,$$

$$< \xi|\mathbf{f}\mathbf{S}^z|\xi >= (1+|\xi|^2)^{-2S} \left(\xi\frac{\partial}{\partial \xi} - S\right)(1+|\xi|^2)^{2S} f.$$

Closed PDE for Spin Observables in CSs

$$\mathbf{H}(\mathbf{S}^z, \mathbf{S}^+, \mathbf{S}^-) \qquad \dot{\mathbf{f}} = \frac{i}{\hbar}[\mathbf{H}, \mathbf{f}], \quad \mathbf{f} = \mathbf{f}\left(\mathbf{S}^z(t), \mathbf{S}^+(t), \mathbf{S}^-(t)\right)$$

$$f(t) \equiv f(\xi^*, \xi, t) = <\xi|\mathbf{f}\left(\mathbf{S}^z(t), \mathbf{S}^+(t), \mathbf{S}^-(t)\right)|\xi> \quad \dot{f} = \frac{i}{\hbar}(<\xi|\mathbf{Hf}|\xi> - <\xi|\mathbf{fH}|\xi>)$$

$$\mathbf{H} = \sum_{m,l,n} A_{m,l,n}(\mathbf{S}^z)^m(\mathbf{S}^+)^l(\mathbf{S}^-)^n, \qquad <\xi|\mathbf{Hf}|\xi> = \sum_{m,l,n} A_{m,l,n} <\xi|(\mathbf{S}^z)^m(\mathbf{S}^+)^l(\mathbf{S}^-)^n\mathbf{f}|\xi>$$

$$<\xi|(\mathbf{S}^z)^m(\mathbf{S}^+)^l(\mathbf{S}^-)^n\mathbf{f}|\xi> = (1+|\xi|^2)^{-2S}(\xi^*\frac{\partial}{\partial\xi^*} - S)^m(1+|\xi|^2)^{2S} <\xi|(\mathbf{S}^+)^l(\mathbf{S}^-)^n\mathbf{f}|\xi>$$

$$<\xi|(\mathbf{S}^z)^m(\mathbf{S}^+)^l(\mathbf{S}^-)^n\mathbf{f}|\xi> = (1+|\xi|^2)^{-2S}(\xi^*\frac{\partial}{\partial\xi^*} - S)^m(2S\xi^* - \xi^{*2}\frac{\partial}{\partial\xi^*})^l(\frac{\partial}{\partial\xi^*})^n(1+|\xi|^2)^{2S} <\xi|\mathbf{f}|\xi>$$

$$<\xi|\mathbf{Hf}|\xi>$$

$$= (1+|\xi|^2)^{-2S} H\left[(\xi^*\frac{\partial}{\partial\xi^*} - S), (2S\xi^* - \xi^{*2}\frac{\partial}{\partial\xi^*}), (\frac{\partial}{\partial\xi^*})\right](1+|\xi|^2)^{2S} <\xi|\mathbf{f}|\xi>$$

Closed PDE for Spin Observables in CSs

$$< \xi|\mathbf{fH}|\xi >= \sum_{m,l,n} A_{m,l,n} < \xi|\mathbf{f}(\mathbf{S}^z)^m(\mathbf{S}^+)^l(\mathbf{S}^-)^n|\xi >$$

$$\times(1+|\xi|^2)^{-2S} H\left[(\xi\frac{\partial}{\partial\xi}-S),(\frac{\partial}{\partial\xi}),(2S\xi-\xi^2\frac{\partial}{\partial\xi})\right]$$

$$(1+|\xi|^2)^{2S} < \xi|\mathbf{f}|\xi > .$$

$$\dot{f} = \hat{K}f, \qquad f \equiv f(\xi^*,\xi,t), \qquad f(0) = f(\xi^*,\xi),$$

$$\hat{K} = \frac{i}{\hbar}(1+|\xi|^2)^{-2S}\{H\left[(\xi^*\frac{\partial}{\partial\xi^*}-S),(2S\xi^*-\xi^{*2}\frac{\partial}{\partial\xi^*}),(\frac{\partial}{\partial\xi^*})\right]$$

$$-H\left[(\xi\frac{\partial}{\partial\xi}-S),(\frac{\partial}{\partial\xi}),(2S\xi-\xi^2\frac{\partial}{\partial\xi})\right]\}(1+|\xi|^2)^{2S}.$$

Example of Spin CS PDF

$$\mathbf{H}_0 = D\hbar^2(\mathbf{S}^z)^2$$

$$\hat{K} = iD\hbar\left(1+|\xi|^2\right)^{-2S}\left[\left(\xi^*\frac{\partial}{\partial\xi^*}-S\right)^2-\left(\xi\frac{\partial}{\partial\xi}-S\right)^2\right]\left(1+|\xi|^2\right)^{2S}$$
$$\equiv \hat{K}_{cl}+\hbar\hat{K}_q,$$

$$\hat{K}_{cl} = iD\left[\frac{4S_0|\xi|^2}{(1+|\xi|^2)}-\hbar(2S-1)\right]\left(\xi^*\frac{\partial}{\partial\xi^*}-\xi\frac{\partial}{\partial\xi}\right),$$

$$\hat{K}_q = iD\left(\xi^{*2}\frac{\partial^2}{\partial\xi^{*2}}-\xi^2\frac{\partial^2}{\partial\xi^2}\right). \quad S_0=\hbar S$$

$$\mathbf{J}^+(t)=\mathbf{S}^+(t)/S,\quad \mathbf{J}^-(t)=\mathbf{S}^-(t)/S,\quad \mathbf{J}^z(t)=\mathbf{S}^z(t)/S.$$

$$J^+(t) = <\xi|\mathbf{J}^+(t)|\xi>,\quad J^-(t) = <\xi|\mathbf{J}^-(t)|\xi>,$$
$$J^z(t) = <\xi|\mathbf{J}^z(t)|\xi>,$$

$$\frac{\partial J^+(t)}{\partial t}=\hat{K}J^+(t),\quad \frac{\partial J^-(t)}{\partial t}=\hat{K}J^-(t),\quad \frac{\partial J^z(t)}{\partial t}=\hat{K}J^z(t)$$

$$J^+(0)=\frac{2\xi^*}{1+|\xi|^2},$$

$$J^-(0)=\frac{2\xi}{1+|\xi|^2},$$

$$J^z(0)=-\frac{1-|\xi|^2}{1+|\xi|^2}.$$

Example of Spin CS PDF

$$J^+(t) \equiv < \xi|\mathbf{J}^+(t)|\xi >$$

$$= \frac{2\xi^* \exp\left[-i(2S-1)\hbar D t\right]}{\left[1+|\xi|^2 \exp(2i\hbar D t)\right]} \left[\frac{1+|\xi|^2 \exp(2i\hbar D t)}{1+|\xi|^2}\right]^{2S},$$

$$J^-(t) = \left(J^+(t)\right)^*, \quad J^z(t) = J^z(0) = -\frac{1-|\xi|^2}{1+|\xi|^2}.$$

In the classical limit $\omega(J) = 2S_0 D J, \quad J\epsilon[-1,1].$

$$S \to \infty, \quad \hbar \to 0, \quad \hbar S = S_0 = constant,$$

$$J_{cl}^+(t) = \frac{2\xi}{1+|\xi|^2} \exp\left[-2iS_0 D t \frac{(1-|\xi|^2)}{(1+|\xi|^2)}\right],$$

$$J_{cl}^-(t) = \left(J_{cl}^+(t)\right)^*, \quad J_{cl}^z(t) = J_{cl}^z(0) == -\frac{1-|\xi|^2}{1+|\xi|^2} \equiv J.$$

$$\delta(\tau) = \frac{|J^+(\tau) - J_{(cl)}^+(\tau)|}{|J_{(cl)}^+(\tau)|} \approx \left|\frac{i\tau}{2S} + \frac{\tau^2(1-J^2)}{4SJ^2}\right|$$

$$J_{qc}^+(t) = \frac{2\xi^*}{(1+|\xi|^2)} e^{i\omega(J)t}$$

Quasi-classical solution $\left(\tau = \omega(J)t\right)$

$$\times \left[1 + iD\hbar t - \frac{2i\hbar D t|\xi|^2}{(1+|\xi|^2)} - \frac{4\hbar S_0 D^2 t^2 |\xi|^2}{(1+|\xi|^2)^2}\right] = \frac{2\xi^*}{(1+|\xi|^2)} e^{i\tau} \left[1 - \frac{i\tau}{2S} - \frac{\tau^2(1-J^2)}{4SJ^2}\right]$$

Additional Averaging over the Distribution Function

$$\hat{\rho}_0 = \int d\mu_s(\xi) P(\xi^*, \xi) |\xi >< \xi| \qquad \int d\mu_s(\xi) P(\xi^*, \xi) = 1$$

$$\bar{f}(t) = Tr(\hat{\rho}_0 \mathbf{f}(t)) = \int d\mu_s(\xi) P(\xi^*, \xi) f(\xi^*, \xi, t)$$

$$P(\xi^*, \xi) = \frac{\nu}{(2S+1)} (1 + |\xi|^2)^2 e^{-\nu|\xi - \xi_0|^2} \qquad \lim_{\nu \to 0} \hat{\rho}_0 = |\xi_0 >< \xi_0|$$

$$\hat{\rho}_0 = \frac{1}{\pi} \int d^2\alpha P(\alpha^*, \alpha) |\alpha >< \alpha| \qquad Tr(\hat{\rho}_0) = \frac{1}{\pi} \int d^2\alpha P(\alpha^*, \alpha) = 1$$

$$\bar{f}(t) = Tr\left(\hat{\rho}_0 \mathbf{f}(t)\right) = \frac{1}{\pi} \int d^2\alpha P(\alpha^*, \alpha) f(\alpha^*, \alpha, t) \qquad f(\alpha^*, \alpha, t) \equiv < \alpha | \mathbf{f}(t) | \alpha >$$

$$P(\alpha^*, \alpha) = \nu \exp(-\nu |\alpha - \alpha_0|^2) \qquad \sigma_p \sigma_q \equiv \sqrt{\overline{(\mathbf{p} - \bar{p})^2} \cdot \overline{(\mathbf{q} - \bar{q})^2}} = \frac{\hbar}{2}(1 + \frac{2}{\nu})$$

$$\lim_{\hbar \to 0} \frac{\nu}{\hbar} = \nu_0 = constant. \qquad \lim_{\hbar \to 0} \sigma_p \sigma_q = \frac{1}{\nu_0}$$

← classical limit

Time-dependent Hamiltonian

Suppose that we have a time-periodic Hamiltonian

$$H = H_0 - \varepsilon\hbar S^x \cos\left(\nu t + \varphi_0\right) + H_T + H_{ST}, \qquad (1)$$

$$H_0 = -\hbar\omega S^z,$$

and as before H_T *and* H_{ST} are the thermal bath and the interaction .

Let’s write instead of (1) the following effective time-independent Hamiltonian

.

$$H_{eff} = H_0 + \hbar\nu b^\dagger b - \frac{\varepsilon\hbar}{2\left|\beta_0\right|}\left(b^\dagger + b\right)S^x + H_T + H_{ST}, \qquad (2)$$

$$\left[b, b^\dagger\right] = 1.$$

Suppose that initially the b -field in in the coherent state

$$\left|\beta_0\right\rangle = e^{-\left|\beta_0\right|^2/2}\sum_{n=1}^{\infty}\frac{\beta_0^n}{n!}\left|n\right\rangle, \quad \left(b(0)\left|\beta_0\right\rangle = \beta_0\left|\beta_0\right\rangle\right). \qquad (3)$$

Suppose that the initial energy of the b -field is large enough

$$\nu\left|\beta_0\right|^2 \gg \max\{\varepsilon,\omega\} \qquad (4)$$

The equation for b has the form

$$i\hbar\dot{b} = \left[b, H_{eff}\right] = \hbar\nu b - \frac{\varepsilon\hbar}{2\left|\beta_0\right|}S^x. \qquad (5)$$

Under the condition (4) we can neglect the last term in (5). So, the solution for the operator *b* Is

$$b(t) \approx b(0)e^{-i\nu t}. \qquad (6)$$

So, using (3) we have

$$\langle\beta_0\left|b(t)\right|\beta_0\rangle \approx \langle\beta_0\left|b(0)\right|\beta_0\rangle e^{-i\nu t} = \beta_0 e^{-i\nu t} = \left|\beta_0\right| e^{-i\varphi_0} e^{-i\nu t}.$$

Let's substitute (6) into (2). We have

$$H_{eff} \approx H_0 + \hbar \nu b^{\dagger}(0) b(0) - \frac{\varepsilon \hbar}{2|\beta_0|}\left(b^{\dagger}(0) e^{i\nu t} + b(0) e^{-i\nu t}\right) S^x + H_T + H_{ST}. \quad (7)$$

If we average (7) over , $\langle \beta_0 | \ldots | \beta_0 \rangle$**, we get the Hamiltonian (1).**

Time-Dependent Hamiltonians

$$\mathbf{H} = \mathbf{H}_0 - \varepsilon\hbar \mathbf{S}^x \cos(\omega t), \quad \mathbf{H}_0 = D\hbar^2 (\mathbf{S}^z)^2$$

$$\mathbf{H}_{eff} = \hbar\omega \mathbf{b}^+\mathbf{b} + D\hbar^2(\mathbf{S}^z)^2 - \frac{\hbar\varepsilon}{2\beta_0}\left(\mathbf{b}^+ + \mathbf{b}\right)\mathbf{S}^x, \quad \left([\mathbf{b}, \mathbf{b}^+] = 1\right)$$

$$\hbar\omega|\beta_0|^2 \gg \left\{DS_0^2, \varepsilon S_0\right\}, \quad (S_0 = \hbar S) \quad \mathbf{b}(t) \approx e^{-i\omega t}\mathbf{b}(0)$$

$$f(t) = < \xi, \beta|\mathbf{f}(t)|\beta, \xi > = f(\xi^*, \xi; \beta^*, \beta, t) \quad |\beta, \xi > \equiv |\beta > |\xi >$$

$$\mathbf{f}(t) \equiv \mathbf{f}\left[\mathbf{S}^z(t), \mathbf{S}^+(t), \mathbf{S}^-(t), \mathbf{b}^+(t), \mathbf{b}(t)\right] \quad \dot{f} = \hat{K}f, \quad f(0) = f(\xi^*, \xi; \beta^*, \beta)$$

$$\hat{K} = \frac{i}{\hbar}e^{-|\beta|^2}\left(1 + |\xi|^2\right)^{-2S}\{\mathbf{H}_{eff}(\mathbf{S}^z = \xi^*\frac{\partial}{\partial\xi^*} - S, \mathbf{S}^+ = 2S\xi^* - \xi^{*2}\frac{\partial}{\partial\xi^*}, \mathbf{S}^- = \frac{\partial}{\partial\xi^*}, \mathbf{b}^+ = \beta^*, \mathbf{b} = \frac{\partial}{\partial\beta^*})$$

$$-\mathbf{H}_{eff}(\mathbf{S}^z = \xi\frac{\partial}{\partial\xi} - S, \ \mathbf{S}^+ = \frac{\partial}{\partial\xi}, \mathbf{S}^- = 2S\xi - \xi^2\frac{\partial}{\partial\xi}, \mathbf{b}^+ = \frac{\partial}{\partial\beta}, \mathbf{b} = \beta)\}$$

$$\times (1 + |\xi|)^{2S} e^{|\beta|^2}.$$

Quantum trajectories in "action-angle" representation

Gennady Berman, T-4, LANL

As a rule, the process of evolution of a quantum system in the Wigner representation is described in the "coordinate-momentum" variables. However, in the classical limit there are some advantages to describe the dynamics of the nonlinear system in the "action-angle" variables ensuring the separation of the motion into the fast (phase) and slow (e.g. diffusion in action). Therefore in the quantum analysis of such systems in Wigner representation it is useful to generalize it for the case of "action-angle" variables. We demonstrate this approach using a simple example of a quantum kicked rotator. We show that the quantum dynamics is reduced to a classical discrete map with a quasi-random force. In the quasi-classical region of parameters the influence of the quasi-random force is small, and the dynamics of observables can be described by the discrete trajectories.

Schrödinger Equation for Kicked Rotator

$$\hat{H} = \frac{\gamma\hbar T\hat{n}^2}{2} + \varepsilon f(\theta)\sum_{t=-\infty}^{\infty}\delta(\tau - Tt), \quad \hat{n} = -i\frac{\partial}{\partial\theta}$$

For kicked rotator: $f(\theta) = \cos\theta$

For rational: $\xi = \frac{\gamma\hbar T}{2\pi} = \frac{r}{q}$ the quantum resonances exist with

$\langle E(t)\rangle \sim t^2$ when $t \to \infty$

Wigner Representation for Quantum Rotator

The arbitrary operator A of the initial representation may be represented in the "angle–action" representation

$$\hat{A} \equiv A(\theta, \hat{n}) = A(\theta + 2\pi, \hat{n}), \quad \hat{n} = -\mathrm{i}\frac{\partial}{\partial\theta}. \qquad (2.4)$$

Define the Weyl transformation of the arbitrary operator $A(\theta, \hat{n})$ (2.4) in the C-number function $a(\varphi, p)$ in the following way:

$$a(\varphi, p) = \mathrm{Tr}\left[A(\theta, \hat{n})\hat{\Delta}(\varphi, p)\right], \quad 0 \le \varphi < 2\pi, \quad p = 0, \pm 1, \ldots, \qquad (2.5)$$

where the symbol Tr denotes the trace of the operator and

$$\hat{\Delta}(\varphi, p) = \sum_{m=-\infty}^{\infty} \frac{1}{2\pi} \times \int_{-\pi}^{\pi} \mathrm{d}\xi \exp\left[\mathrm{i}(\varphi - \theta)m + \mathrm{i}(p - \hat{n})\xi\right]. \qquad (2.6)$$

Then, the reverse transformation has the form

$$A(\theta, \hat{n}) = \sum_{p=-\infty}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\varphi\, a(\varphi, p)\hat{\Delta}(\varphi, p). \qquad (2.7)$$

Classical Limit

In the classical case the Hamiltonian of the kicked rotator has the form

$$H_{cl}(\varphi, I, t) = \frac{\gamma I^2}{2} + \varepsilon f(\varphi) \sum_{n=-\infty}^{\infty} \delta(t - nT), \tag{3.1}$$

where $f(\varphi + 2\pi) = f(\varphi)$, ε, γ are the parameters of perturbation and nonlinearity, respectively.

The classical mappings according to (3.1) have the form

$$I_{t+1} = I_t - \varepsilon \left. \frac{\mathrm{d}f(\varphi)}{\mathrm{d}\varphi} \right|_{\varphi=\varphi_t}, \qquad \varphi_{t+1} = \varphi_t + \gamma T I_{t+1}. \tag{3.2}$$

In (3.2) t is the discrete time; $\varphi_t \equiv \varphi(t-0)$, $I_t \equiv I(t-0)$ are the values of the functions $\varphi(t)$ and $I(t)$ immediately before the tth pulse.

The equation for the classical distribution function is, according to (3.2) of the form

$$\rho_{t+1}(\varphi, I) = \int_0^{2\pi} \mathrm{d}\varphi' \int_{-\infty}^{\infty} \mathrm{d}I' \times K_{cl}(\varphi, I|\varphi', I')\rho_t(\varphi', I'), \tag{3.3}$$

where K_{cl} is the classical Green function

$$K_{cl}(\varphi, I|\varphi', I') = \tilde{\delta}(\varphi' + \gamma T I - \varphi) \times \delta\left(I' - \varepsilon \left. \frac{\mathrm{d}f}{\mathrm{d}\varphi} \right|_{\varphi=\varphi'} - I \right). \tag{3.4}$$

Henceforth $\tilde{\delta}$ denotes a periodic δ-function with period 2π. The operator (3.4) preserves the area bound by the closed contour on the plane (φ, I). However, when increasing t the initial contour distorts getting a rather complicated structure. This circumstance leads to fast decay of the phase correlation functions enabling us to describe the system (3.1) by a statistical technique [1–4].

Quantum Case

Describe the evolution of the rotator in the quantum case by the Wigner function which is the image of the density matrix operator,

$$\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)|, \tag{3.5}$$

where the wave function $|\psi(t)\rangle \equiv \psi(\theta, t)$ satisfies the Schrödinger equation

$$i\hbar \frac{\partial\psi(\theta,t)}{\partial t} = \left[-\frac{\gamma\hbar^2}{2}\frac{\partial^2}{\partial\theta^2} + \varepsilon f(\theta)\sum_n \delta(t-nT) \right]\psi(\theta,t). \tag{3.6}$$

From (3.5), (3.6) we have

$$\hat{\rho}_{t+1} = \hat{U}\hat{\rho}_t\hat{U}^\dagger, \tag{3.7}$$

where the evolution operator $\hat{U}$ has the form

$$\hat{U} = \exp\left(i\pi\zeta \frac{\partial^2}{\partial\theta^2} \right) \exp\left[-i\kappa f(\theta) \right],$$

$$\zeta = \frac{\gamma\hbar T}{2\pi}, \quad \kappa = \frac{\varepsilon}{\hbar}. \tag{3.8}$$

In (3.7) t is the discrete time: $t = 0, 1, \ldots;$ $\hat{\rho}_t = \hat{\rho}(t-0)$. Using the results of section 2 obtain analogously to the case of (x, p)-representation [20, 21] an equation for the evolution of the Wigner function (appendix B)

$$\rho_{t+1}(\varphi, p) = \int_0^{2\pi} d\varphi' \times \sum_{p'=-\infty}^{\infty} K_Q(\varphi, p|\varphi', p')\rho_t(\varphi', p'), \tag{3.9}$$

$$K_Q(\varphi, p|\varphi', p') = \frac{1}{(2\pi)^2}\sum_{m=-\infty}^{\infty}\int_{-\pi}^{\pi} d\xi \times \exp\Big\{ im(\varphi' - \varphi + 2\pi\zeta p') + i\xi(p - p') + i\kappa\Big[f\Big(\varphi' - \frac{2\pi m\zeta + \xi}{2}\Big) - f\Big(\varphi' + \frac{2\pi m\zeta + \xi}{2}\Big)\Big]\Big\}. \tag{3.10}$$

The formulas (3.9), (3.10) are easily shown to transform at the formal transition $\hbar \to 0$, $p\hbar = I$ into the classical analogous (3.3), (3.4).

To simplify the form of analytical expressions restrict further to the case $f(\varphi) = \cos 2\varphi$. Then, from (3.10) get

$$K_Q(\varphi, p|\varphi', p') = \tilde{\delta}(\varphi' + 2\pi\zeta p - \varphi) \times J_{p-p'}(2\kappa \sin 2\varphi'), \tag{3.11}$$

where $J_p(z)$ is the Bessel function.

Note, that in the quantum case, as follows from (3.11), the map of the φ, p variables per one transformation step occurs locally in phase and nonlocally in action. Such a situation is typical of the systems kicked by δ-pulses [20, 21].

Invariant Sets in the Phase Space

We show that due to the discreteness of the phase space in the action I in the quantum case one can point out such countable $\{\varphi_n\}$ sets that the evolution of the Wigner function on these sets does not depend on the other values of the phase

φ. We introduce a succession of numbers

$$\varphi_n = \{\varphi_0 + 2\pi\zeta n\}_{2\pi}, \quad n = 0 \pm 1, \ldots, \tag{4.1}$$

where $\{\ldots\}_{2\pi}$ denotes the fractional part by the modulus 2π; φ_0 is the arbitrary initial phase. Then, from (3.9)–(3.11) we obtain

$$\rho_{t+1}(\varphi_n, p) = \sum_{p'=-\infty}^{\infty} J_{p-p'}(2\kappa \sin 2\varphi_{n-p})\rho_t(\varphi_{n-p}, p'). \tag{4.2}$$

Hence, the set $\{\varphi_n\}$ (4.1) is invariant and the evolution of the Wigner function $\rho_t(\varphi, p)$ over $\{\varphi_n\}$ does not depend on the values $\rho_t(\varphi, p)$ over the other region of the phase space (whose measure equals unity) (see fig. 1).

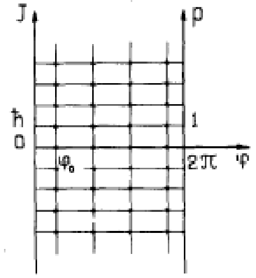


Fig. 1. The phase space and invariant set (denoted by the dots at $\zeta = 1/4$.

In the case of rational $\zeta = r/q$ (r, q are integers), the succession of the phases $\{\varphi_n\}$ in (4.1) is finite ($0 \le n < q$). In the case when ζ is irrational the succession $\{\varphi_n\}$ is infinite and densely covers the segment $0 \le \varphi < 2\pi$. The noted difference leads to different expressions for the definition of expectation values for the arbitrary operator $\hat{A}$, $\langle A(t)\rangle = \mathrm{Tr}[\hat{A}\hat{\rho}_t]$:

$$\langle A(t)\rangle = \sum_{p=-\infty}^{\infty} \int_0^{2\pi/q} d\varphi_0 \times \sum_{n=0}^{q-1} a(\varphi_n, p)\rho_t^{(\varphi_0)}(\varphi_n, p), \quad \zeta = \frac{r}{q}, \tag{4.3}$$

$$\langle A(t)\rangle = \sum_{p=-\infty}^{\infty} \lim_{N\to\infty} \frac{1}{N} \times \sum_{n=0}^{N} a(\varphi_n, p)\rho_t^{(\varphi_0)}(\varphi_n, p), \quad \zeta \ne \frac{r}{q}. \tag{4.4}$$

Here $a(\varphi, p)$ is the image of the operator $A(\theta, \hat{n})$ (see (2.5)); the upper index (φ_0) shows explicit dependence of the Wigner function on the phase φ_0. In (4.4) the additional condition of continuity in φ is imposed on the initial function $\rho_0(\varphi; p)$ and its first derivative. Under this condition the result of the calculations in (4.4) does not depend on the phase φ_0 (the phase φ_0 may be chosen arbitrarily).

Explicit Form of Wigner Function

From (5.4) analogous to (3.7), (3.9) obtain an equation for the Wigner function

$$\rho(\varphi, p) = \sum_{p'=-\infty}^{\infty} J_{p-p'}[2\kappa \sin 2(\varphi - 2\pi\zeta p)] \times \rho(\varphi - 2\pi\zeta p, p'). \quad (5.5)$$

Now, assume the eigenfunction in (5.1) to belong to the continuous spectrum and have the form (5.3). In this case we can write down the explicit form of the function $\rho(\varphi, p)$. According to (2.5), (3.5), (5.3) we get

$$\rho(\varphi, p) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\xi \, \mathrm{e}^{\mathrm{i}p\xi} \psi^*\left(\varphi + \frac{\xi}{2}\right) \psi\left(\varphi - \frac{\xi}{2}\right)$$

Quantum Equations of Motion

In the classical case the equations of motion connecting the action I and the phase φ over a single kick for $f(\varphi)=\cos 2\varphi$ have the form (see (3.2))

$$I_{t+1} = I_t + 2\varepsilon \sin 2\varphi_t, \qquad \varphi_{t+1} = \varphi_t + \gamma T I_{t+1}, \tag{6.1}$$

with the phase and the action changing continuously.

We introduce formally the following transformations;

$$p_{t+1} = p_t + \left\{ \frac{2\varepsilon}{\hbar} \sin 2\varphi_t \right\}_{\text{int}} + \Delta p_t, \qquad \varphi_{t+1} = \varphi_t + \gamma T \hbar p_{t+1} \tag{6.2}$$

where $\{\ldots\}_{\text{int}}$ denotes an integer part; p_t changes discretely ($p = 0, \pm 1, \ldots$); Δp_t is a quasirandom function taking discrete values and distributed by the law $W(\Delta p_t)$. It is easily seen that when we choose the function $W(\Delta p_t)$ in the form

$$W(\Delta p_t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\xi \times \exp\left[-\mathrm{i}\xi \Delta p_t - \mathrm{i}\xi \left\{ \frac{2\varepsilon}{\hbar} \sin 2\varphi_t \right\}_{\text{int}} + \mathrm{i} \frac{2\varepsilon}{\hbar} \sin 2\varphi_t \sin \xi \right], \tag{6.3}$$

eqs. (6.2) lead to the law of the evolution of the distribution function coinciding with the equation for the evolution of the Wigner function $\rho_t(\varphi, p)$ for the quantum rotator (3.9), (3.11). In fact, using the transformations (6.2) the equation for the distribution function $D_t(\varphi, p)$ in the general case may be put down

$$D_{t+1}(\varphi, p) = \sum_{\Delta p' = -\infty}^{\infty} W(\Delta p') \times \left[\sum_{p'=-\infty}^{\infty} \int_0^{2\pi} \mathrm{d}\varphi' \, \delta_{p' + \left\{ \frac{2\varepsilon}{\hbar} \sin 2\varphi' \right\}_{\text{int}} + \Delta p', p} \times \tilde{\delta}(\varphi' + \gamma T \hbar p - \varphi) D_t(\varphi', p') \right]$$

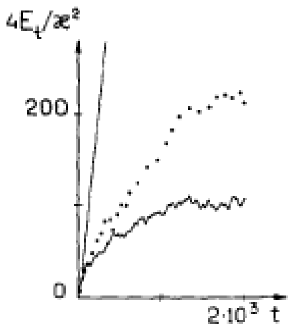
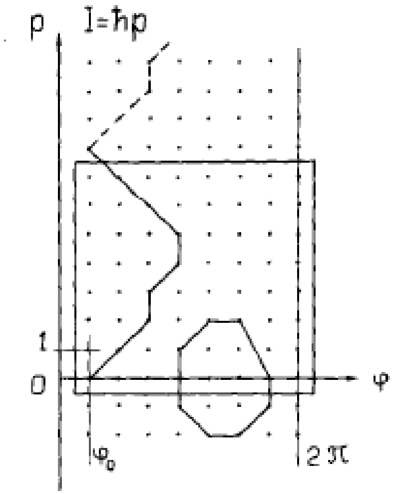


Fig. 2. The diffusion law for transformations (7.12).

Quantum Trajectories



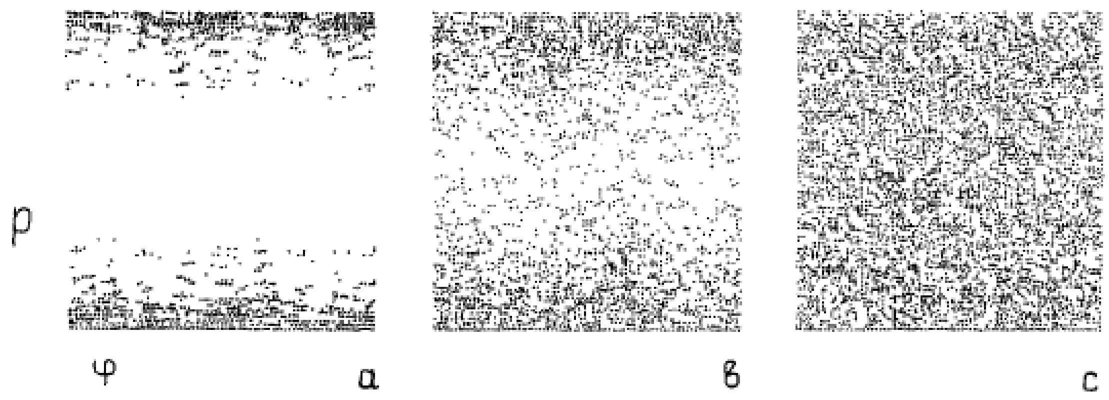


Fig. 2. Numerical data for the mapping (3.1). Figures show trajectories on a torus of $q = 101$ particles "starting" from the bottom line. Values of parameters $\zeta = 10/101$, $\varphi_0 = 0.0$, (a) $2\kappa = 5.0$, (b) $2\kappa = 10.0$, (c) $2\kappa = 20.0$. In (a) all trajectories have the value $J = 0$ (J is the number of rotations around torus along the axis of action); in (b) six particles have $J = \pm 1$; in (c) four have $J = \pm 1$, fourteen $J = \pm 2$, two $J = \pm 3$.

Numerical Results on Quantum Diffusion

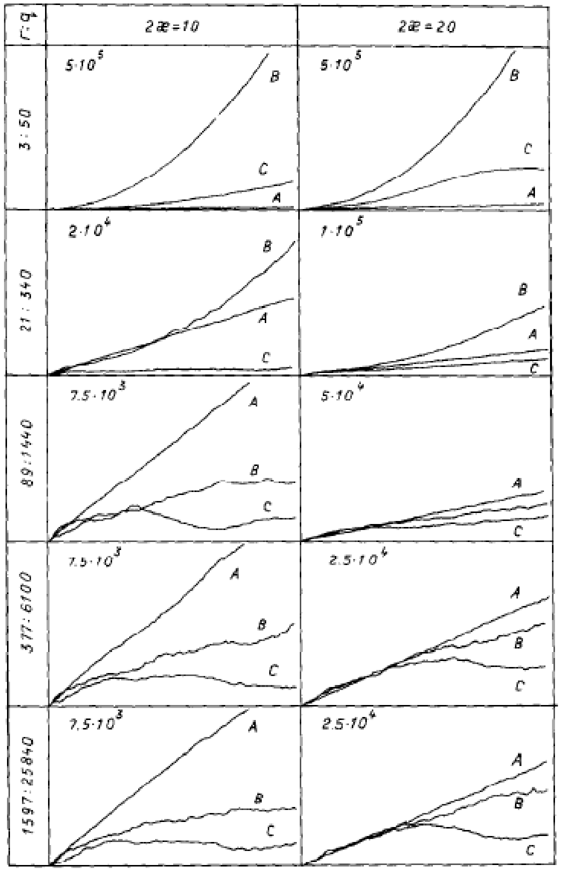


Fig. 3. Increase of the mean energy up to time $t = 200$ for classical rotator (curves A), classical model of quantum stochasticity (curves B) and quantum rotator (curves C). Numbers in figure show the scale of E_t along the vertical axis.

1. **Quantum chaos and peculiarities of diffusion in Wigner representation.** BERMAN, GP; KOLOVSKY, AR; IZRAILEV, FM; Physica A; 1988; vol.152, no.1-2, p.273-86.

2. **Dynamics of classically chaotic quantum systems in Wigner representation.** Berman, GP; Kolovsky, AR; Physica D, 1985; vol.17D, no.2, p.183-97.
3. **Quantum chaos in the Wigner representation.** Berman, GP; Kolovskii, AR; Izrailev, FM; Iomin, AM Source: Chaos; 1991; vol.1, no.2, p.220-3

Decoherence in Spin Systems (by G.M. Palma *et al.* Proc. R. Soc. Lond. A 452 (1996) 567)

MODEL

$$H = -\hbar\omega_L I_z + \sum_q \hbar\omega_q (a_q^\dagger a_q + 1/2) + \hbar I_z \sum_q (g_q a_q^\dagger + g_q^* a_q)$$

Commutation relations

$$I_z|\uparrow\rangle = \frac{1}{2}|\uparrow\rangle, \; I_z|\downarrow\rangle = -\frac{1}{2}|\downarrow\rangle$$

$$[a_q, a_{q'}^\dagger] = \delta_{q,q'}, \; [I_i, I_j] = \frac{i}{2}\epsilon_{ijk} I_k$$

Transformation to Interaction Representation with H_0

$$U_{int} = e^{i\omega_L I_z t - i\sum_q \hbar\omega_q (a_q^\dagger a_q + 1/2)}$$

Schrödinger equation in Interaction Representation

$$\Psi_{lab} = U_{int}\Psi_{int}$$

$$i\hbar\frac{\partial\Psi_{int}}{\partial t} = U_{int}^{\dagger}\hbar I_z \sum_q (g_q a_q^{\dagger} + g_q^* a_q) U_{int}\Psi_{int} =$$

$$\hbar I_z \sum_q (g_q a_q^{\dagger} e^{i\omega_q t} + g_q^* a_q e^{-i\omega_q t})\Psi_{int}$$

The relations were used

$$e^{\xi a_q^{\dagger} a_q} a_q^{\dagger} e^{-\xi a_q^{\dagger} a_q} = a_q^{\dagger} e^{\xi}$$

$$e^{\xi a_q^{\dagger} a_q} a_q e^{-\xi a_q^{\dagger} a_q} = a_q^{\dagger} e^{-\xi}$$

Solution of the Schrödinger equation

$$\Psi_{int}(t) = U(t)\Psi_{int}(0) = \hat{T}e^{-\frac{i}{\hbar}\int_0^t H_{int}(\tau)d\tau} =$$

$$e^{-\frac{i}{\hbar}H_{int}(t_n)\Delta t}e^{-\frac{i}{\hbar}H_{int}(t_{n-1})\Delta t}\ldots e^{-\frac{i}{\hbar}H_{int}(t_1)\Delta t}\Psi_{int}(0)$$

$$t_i = i\Delta t, \;\; i = 1, \ldots, n$$

Commutator

$$[H_{int}(t_m), H_{int}(t_k)] = -\frac{i\hbar^2}{2}\sum_q |g_q|^2 \sin\omega_q(t_m - t_k)$$

Two exponents can be combined

$$e^{-\frac{i}{\hbar}H_{int}(t_m)\Delta t}e^{-\frac{i}{\hbar}H_{int}(t_k)\Delta t} =$$

$$e^{-\frac{i\Delta t}{4}\sum_q |g_q|^2 \sin\omega_q(t_m - t_k)}e^{-\frac{i}{\hbar}H_{int}(t_m)\Delta t - \frac{i}{\hbar}H_{int}(t_k)\Delta t}$$

The relation was used

$$e^{A+B} = e^{-\frac{1}{2}[A,B]} e^A e^B$$

Evolution operator

$$U(t) = e^{i\varphi(t)} e^{2I_z \sum_q (g_q a_q^\dagger \eta_q(t) - g_q^* a_q \eta_q^*(t))}$$

$$\eta(t) = \frac{1 - e^{i\omega_q t}}{2\omega_q}$$

Initial wave function of "spin+thermal bath"

$$|\Psi(0)\rangle = |\Psi_T^{(1)}(0)\rangle \otimes |\uparrow\rangle + |\Psi_T^{(2)}(0)\rangle \otimes |\downarrow\rangle$$

Solution of the Schrödinger equation

$$|\Psi(t)\rangle = U(t)|\Psi(0)\rangle =$$

$$e^{\sum_q (g_q a_q^\dagger \eta_q(t) - g_q^* a_q \eta_q^*(t))} |\Psi_T^{(1)}(0)\rangle \otimes |\uparrow\rangle +$$

$$e^{-\sum_q (g_q a_q^\dagger \eta_q(t) - g_q^* a_q \eta_q^*(t))} |\Psi_T^{(2)}(0)\rangle \otimes |\downarrow\rangle$$

Density matrix

$$\rho(t) = |\Psi(t)\rangle\langle\Psi(t)| =$$

$$e^{\sum_q (g_q a_q^\dagger \eta_q(t) - g_q^* a_q \eta_q^*(t))} \rho_T^{(1,1)}(0) \otimes |\uparrow\rangle\langle\uparrow| e^{-\sum_q (g_q a_q^\dagger \eta_q(t) - g_q^* a_q \eta_q^*(t))} +$$

$$e^{-\sum_q (g_q a_q^\dagger \eta_q(t) - g_q^* a_q \eta_q^*(t))} \rho_T^{(2,2)}(0) \otimes |\downarrow\rangle\langle\downarrow| e^{\sum_q (g_q a_q^\dagger \eta_q(t) - g_q^* a_q \eta_q^*(t))} +$$

$$e^{\sum_q (g_q a_q^\dagger \eta_q(t) - g_q^* a_q \eta_q^*(t))} \rho_T^{(1,2)}(0) \otimes |\uparrow\rangle\langle\downarrow| e^{\sum_q (g_q a_q^\dagger \eta_q(t) - g_q^* a_q \eta_q^*(t))} +$$

$$e^{-\sum_q (g_q a_q^\dagger \eta_q(t) - g_q^* a_q \eta_q^*(t))} \rho_T^{(2,1)}(0) \otimes |\downarrow\rangle\langle\uparrow| e^{-\sum_q (g_q a_q^\dagger \eta_q(t) - g_q^* a_q \eta_q^*(t))}$$

$$\rho_T^{(i,j)}(0) = |\Psi_T^{(i)}(0)\rangle\langle\Psi_T^{(j)}(0)|$$

Time-evolution of non-diagonal density matrix component

$$\rho_{\uparrow\downarrow(t)} = Tr\left\{e^{\sum_q (g_q a_q^\dagger \eta_q(t) - g_q^* a_q \eta_q^*(t))} \rho_T^{(1,2)}(0) e^{\sum_q (g_q a_q^\dagger \eta_q(t) - g_q^* a_q \eta_q^*(t))}\right\} =$$

$$Tr\left\{\rho_T^{(1,2)} e^{2\sum_q (g_q a_q^\dagger \eta_q(t) - g_q^* a_q \eta_q^*(t))}\right\}$$

We use the relation

$$e^{2\sum_q (g_q a_q^\dagger \eta_q(t) - g_q^* a_q \eta_q^*(t))} = e^{2|g_q \eta_q(t)|^2} e^{-2g_q^* a_q \eta_q^*(t)} e^{2g_q a_q^\dagger \eta_q(t)}$$

Non-diagonal density matrix

$$\rho_{\uparrow\downarrow(t)} = e^{2|g_q \eta_q(t)|^2} Tr\left\{e^{2\sum_q g_q a_q^\dagger \eta_q(t)} \rho_T^{(1,2)}(0) e^{-2\sum_q (g_q^* a_q \eta_q^*(t)}\right\}$$

Calculation of the trace in coherent states

$$a_q|\alpha_q\rangle = \alpha_q|\alpha_q\rangle$$

Initial condition for thermal bath

$$\rho_T^{(i,j)}(0) = \rho_T(0) = \prod_q \left(1 - e^{-\frac{\hbar\omega_q}{k_B T}}\right) e^{-\frac{\hbar\omega_q a_q^\dagger a_q}{k_B T}}$$

Calculation of trace

$$\rho_{\uparrow\downarrow(t)} = e^{2|g_q\eta_q(t)|^2} \prod_q \left(1 - e^{-\frac{\hbar\omega_q}{k_B T}}\right) \times J_q$$

$$J_q = \frac{1}{\pi} \int d^2\alpha_q \langle\alpha_q| e^{2g_q a_q^\dagger \eta_q(t)} e^{-\frac{\hbar\omega_q a_q^\dagger a_q}{k_B T}} e^{-2g_q^* a_q \eta_q^*(t)} |\alpha_q\rangle$$

Calculation of integral J

$$J = \frac{1}{\pi} \int d^2\alpha e^{2g\alpha^*\eta(t) - 2g^*\alpha\eta^*(t)} \langle\alpha| e^{-\frac{\hbar\omega_q a_q^\dagger a_q}{k_B T}} |\alpha\rangle$$

We use relations

$$a^\dagger a|n\rangle = n|n\rangle$$

$$\langle\alpha|e^{-\frac{\hbar\omega a^\dagger a}{k_B T}}|\alpha\rangle = \sum_{n=0}^{\infty}\alpha|e^{-\frac{\hbar\omega a^\dagger a}{k_B T}}|n\rangle\langle n|e^{-|\alpha|^2/2}\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}|m\rangle =$$

$$\sum_{n=0}^{\infty}e^{-\frac{\hbar\omega n}{k_B T}}\langle\alpha|n\rangle e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}} = \sum_{n=0}^{\infty}e^{-\frac{\hbar\omega n}{k_B T}}e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{\sqrt{n!}} =$$

$$e^{-|\alpha|^2\left(1-e^{-\hbar\omega/K_B T}\right)}$$

Integral J takes the form

$$J = \frac{1}{\pi}\int d^2\alpha e^{2g\alpha^*\eta(t)-2g^*\alpha\eta^*(t)-|\alpha|^2\left(1-e^{-\hbar\omega/K_B T}\right)}$$

where

$$\Gamma(t) = \sum_q \frac{|g_q|^2}{\omega_q^2}(1-\cos\omega_q t)\coth(\hbar\omega_q/k_BT) =$$

$$\int d\vec{k}\frac{|g_{\vec{k}}|^2}{\omega_{\vec{k}}^2}(1-\cos\omega_k t)\coth(\hbar\omega_k/k_BT) =$$

$$\int d\omega\frac{dk}{d\omega}G(\omega)|g(\omega)|^2\frac{(1-\cos\omega t)}{\omega^2}\coth(\hbar\omega/k_BT)$$

where

$$d\vec{k} = \frac{dk}{d\omega}G(\omega)d\omega$$

$G(\omega)$ is the density of modes at frequency ω

Usually made assumption

$$\frac{dk}{d\omega}G(\omega)|g(\omega)|^2 \sim \omega^n e^{-\omega/\omega_c}$$

Assumption made in G. Massimo et al.

$$g(\omega) \sim \sqrt{\omega}\ \textit{(as in quantum optical systems)}$$

One dimensional case

$$G(\omega) = const,\ n = 1,\ \Gamma \sim \int d\omega e^{-\omega/\omega_c}\frac{(1-\cos\omega t)}{\omega}\coth\frac{\hbar\omega}{2k_BT}$$

Three dimensional case

$$G(\omega) \sim \omega^2,\ n = 3,\ \Gamma \sim \int d\omega\omega e^{-\omega/\omega_c}(1-\cos\omega t)\coth\frac{\hbar\omega}{2k_BT}$$

Change of variables

$$\beta = \sqrt{1 - e^{-\hbar\omega/k_BT}}\alpha, \ \ \beta^* = \sqrt{1 - e^{-\hbar\omega/k_BT}}\alpha^*$$

Integral takes the form

$$J = \frac{1}{\left(1 - e^{-\hbar\omega/k_BT}\right)} \times$$

$$\frac{1}{\pi}\int d^2\beta e^{2g\eta(t)\beta^*/\sqrt{1-e^{-\hbar\omega/k_BT}} - 2g^*\eta^*(t)\beta/\sqrt{1-e^{-\hbar\omega/k_BT}} - |\beta|^2}$$

We use (P.Caruthers and M. Nieto, Rev. Mod Phys., 40 (1968) 411)

$$\frac{1}{\pi}\int d^2\beta e^{\xi^*\beta - |\beta|^2} f(\beta^*) = f(\xi^*)$$

where

$$\xi = -\frac{2g\eta(t)}{\sqrt{1 - e^{-\hbar\omega/k_BT}}}, \ \ f(\beta^*) = e^{2g\eta(t)\beta^*/\sqrt{1-e^{-\hbar\omega/k_BT}}}$$

Finally

$$J = \frac{1}{\left(1 - e^{-\hbar\omega/k_BT}\right)} e^{-4|g\eta(t)|^2/(1-e^{-\hbar\omega/k_BT})}$$

Non-diagonal density matrix component

$$\rho_{\uparrow\downarrow}(t) = e^{2\sum_q |g_q\eta_q(t)|^2 - \sum_q 4|g_q\eta_q(t)|^2/\left(1-e^{-\hbar\omega/k_BT}\right)} =$$

$$e^{-2\sum_q |g_q\eta_q(t)|^2 \coth(\hbar\omega_q/2k_BT)} \equiv e^{-\Gamma(t)}$$

Problems with spin quantum computer

$$\mathcal{H} = \sum_q \omega_q (a_q^\dagger a_q + 1/2) + \sum_j I_{jz} \sum_q (\lambda_q a_q^\dagger + h.c.) + V.$$

$$\mathcal{H}_{int}(t) = \exp(i\mathcal{H}_0 t)\mathcal{H}_1 \exp(-i\mathcal{H}_0 t),$$

$$\exp(\xi a^\dagger a) a^\dagger \exp(-\xi a^\dagger a) = a^\dagger e^{\xi}, \;\; \exp(\xi a^\dagger a) a \exp(-\xi a^\dagger a) = a e^{-\xi},$$

$$U = \hat{T} \exp\left\{ -i \int_0^t \mathcal{H}_{int}(\tau) d\tau \right\}. \qquad e^{A+B} = e^{-[A,B]/2} e^A e^B,$$

$$U = \prod_{m>k} \exp\left\{ -\frac{\Delta\tau^2}{2} [\mathcal{H}_{int}(t_m), \mathcal{H}_{int}(t_k)] \right\} \exp\left\{ -i \int_0^t \mathcal{H}_{int}(\tau) d\tau \right\},$$

$$[\mathcal{H}_{int}(t_m), \mathcal{H}_{int}(t_k)] = -2i \left(\sum_j I_{jz} \right)^2 \sum_q |\lambda_q|^2 \sin \omega_q (t_m - t_k).$$

$$U = \prod_{m>k} \exp\{ i\Delta\tau^2 (\sum_j I_{jz})^2 \sum_q |\lambda_q|^2 \sin \omega_q (t_m - t_k) \} \times \exp\{ 2 \sum_j I_{jz} \sum_q (\lambda_q \eta_q(t) a_q^\dagger - h.c.),$$

$$\eta_q(t) = \frac{1 - \exp(i\omega_q t)}{2\omega_q}.$$

$$\rho(t) = (T_0/D){\sum_{n,n'}}' U\rho_e(0)|n\rangle\langle n'|U^\dagger. \qquad \rho_e(0) = \prod_q [1-\exp(-\omega_q/T)]\exp\left[-(\omega_q/T)a_q^\dagger a_q\right],$$

$$(T_0/D){\sum_{n,n'}}' \rho_e^{nn'}|n\rangle\langle n'|,$$

$$\rho_e^{nn'} = Tr\{\prod_{m>k}\exp[-i\Delta\tau^2 I_z^2(n)\sum_q|\lambda_q|^2\sin\omega_q(t_m-t_k)]\times\exp[2I_z(n)\sum_q(\lambda_q\eta_q(t)a_q^\dagger-h.c.)]\rho_e(0) \tag{22}$$

$$\times\prod_{m>k}\exp[i\Delta\tau^2 I_z^2(n')\sum_q|\lambda_q|^2\sin\omega_q(t_m-t_k)]\times\exp[-2I_z(n')\sum_q(\lambda_q\eta_q(t)a_q^\dagger-h.c.)]\}.$$

$$\rho_e^{nn'} = \exp\{i\alpha(t)[I_z^2(n)-I_z^2(n')]\}\times\ \exp\{-\beta(t)[I_z(n)-I_z(n')]^2\},$$

where

$$\alpha(t) = \Delta\tau^2\sum_{m>k}\sum_q|\lambda_q|^2\sin\omega_q(t_m-t_k) = \sum_q|\lambda_q|^2\int_0^t dt'\int_0^{t'}dt''\sin\omega_q(t'-t'') =$$

$$\sum_q\frac{|\lambda_q|^2}{\omega_q^2}(\omega_q t-\sin\omega_q t),$$

$$\beta(t) = 2\sum_q|\lambda_q\eta_q(t)|^2\coth(\omega_q/2T).$$

$$\alpha(t) = \int d\omega \kappa^2(\omega) \frac{\omega t - \sin(\omega t)}{\omega^2},$$

$$\beta(t) = 2 \int d\omega \kappa^2(\omega) \frac{\sin^2(\omega t/2)}{\omega^2} \coth \left(\frac{\omega}{2T} \right),$$

$$\kappa^2(\omega) = |\lambda(\omega)|^2 G(\omega) \frac{dq}{d\omega}.$$

Decoherence of a single qubit

Palma, Suominen & Ekert, P.Roy.Soc.Lond., A, 452, 567, 1996

Hamiltonian:

$$\hat{H} = \frac{1}{2}\omega_0 \hat{\sigma}_z + \sum_k \omega_k \hat{b}_k^+ \hat{b}_k^+ + \hat{\sigma}_z \sum_k (g_k \hat{b}_k^+ + g_k^* \hat{b}_k)$$

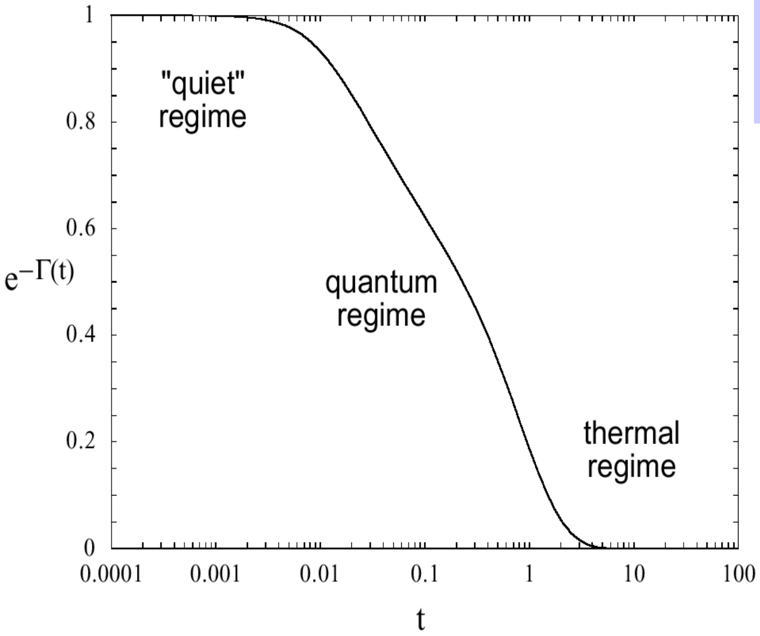
Initial condition:

$$\rho(0) = \rho^S(0) \prod_k R_k^T(0)$$

Solution:

$$\rho_{10}^S(t) = e^{-\Gamma(t)} \rho_{10}^S(0)$$

$$\Gamma(t) \propto \int d\omega \underbrace{\frac{dk}{d\omega} G(\omega) |g(\omega)|^2}_{\propto \omega^n \exp(-\omega/\omega_C)} (1 + 2\langle n(\omega)\rangle_T) \frac{1 - \cos\omega t}{\omega^2}$$



Time is in units T^{-1}.

n=1 and $\omega_C / T = 100$