UNIVERSITÀ DI MILANO BICOCCA

Dottorato di Ricerca in Matematica

Pura ed Applicata

Degree theory and quasilinear elliptic equations

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1 Notations

1 Notations

If D is a set, Y a normed linear space and $F : D \to Y$ a bounded map, we set

$$\|F\|_{\infty,D} = \sup\{\|F(u)\| : \ u \in D\}$$

(we agree that $\|F\|_{\infty,D} = 0$ if $D = \emptyset$).

If X, Y are finite dimensional normed linear spaces, A an open subset of X and $F : A \to Y$ a map of class C^1, we set

$$S_F = \{u \in A : \ dF(u) \text{ is not surjective}\} \ ,$$

where $dF(u) : X \to Y$ denotes the differential of F at u.

If X, Y are finite dimensional normed linear spaces, A an open subset of X and $F : A \to Y$ a map of class C^1, we set

$$S_F = \{u \in A : \ dF(u) \text{ is not surjective}\} ,$$

where $dF(u) : X \to Y$ denotes the differential of F at u.

The elements of S_F are called *singular points* or *critical points* of F, while S_F is the *singular set* or *critical set* of F.

If $Y = \mathbb{R}$, then $u \in A$ is critical if and only if $dF(u) = 0$.

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If $X = Y$, we also introduce the *Jacobian determinant* of F at u

$$J_F(u) = \det dF(u)\,.$$

Thus, in the case $X = Y$, $u \in A$ is critical if and only if $J_F(u) = 0$.

2 Recalls on finite dimensional degree theory

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See e.g.

K. Deimling, “Nonlinear functional analysis”, Springer-Verlag, Berlin, 1985.

M. A. Krasnosel′skiĭ and P. P. Zabreĭko, “Geometrical methods of nonlinear analysis”, *Grundlehren der Mathematischen Wissenschaften*, **263**, Springer-Verlag, Berlin, 1984.

Throughout this section, X will denote a finite dimensional normed linear space over $\mathbb{R}$ and A a bounded and open subset of X.

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If $F : \partial A \to X$ is a continuous map and $w \in X \backslash F(\partial A)$, then one can define the *topological degree*

$$\deg (F, A, w) \in \mathbb{Z}$$

which satisfies the following properties:

(a) if $\mathrm{B}\,(w,r) \cap F(\partial A) = \emptyset$ and

$G \in C(\partial A; X)$ and $z \in X$ satisfy

$$\|G - F\|_{\infty,\partial A} + \|z - w\| < r\,,$$

then $z \notin G(\partial A)$ and

$$\deg\,(G, A, z) = \deg\,(F, A, w)\ ;$$

(b) if $F \in C(\overline{A}; X) \cap C^1(A; X)$ and $w \in X \setminus F(\partial A)$ satisfies $F^{-1}(w) \cap S_F = \emptyset$, then $F^{-1}(w)$ is a finite set and

$$\deg(F, A, w) = \sum_{u \in F^{-1}(w)} \operatorname{sgn}(J_F(u))$$

(we agree that $\deg(F, A, w) = 0$ if $F^{-1}(w) = \emptyset$).

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Here $\operatorname{sgn}(s) = \frac{s}{|s|}$ denotes the sign of s.

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(we agree that $\deg(F, A, w) = 0$ if $F^{-1}(w) = \emptyset$).

Here $\operatorname{sgn}(s) = \frac{s}{|s|}$ denotes the sign of s.

From these two facts one can deduce all the properties of the topological degree.

Even if F is defined only on ∂A, the degree depends on A, not only on ∂A.

For instance, let $X = \mathbb{R}$,

$$A_0 =]0,1[\cup]2,3[\cup]3,4[\,,$$

$$A_1 =]0,1[\cup]1,2[\cup]3,4[\,.$$

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Then A_0 and A_1 are two bounded and open subsets of $\mathbb{R}$ with $\partial A_0 = \partial A_1 = \{0,1,2,3,4\}$.
If we consider $w = \frac{3}{2}$ and $F(u) = u$, then $w \in A_1 \setminus \overline{A_0}$, so that

$$\deg(F, A_0, w) = 0\,, \quad \deg(F, A_1, w) = 1\,.$$

(2.1) Theorem *Let $F : \partial A \to X$ be a continuous map and let $w \in X$.*

Then $F(u) \neq w$ for any $u \in \partial A$ if and only if $F(u) - w \neq 0$ for any $u \in \partial A$, and in this case

$$\deg(F, A, w) = \deg(F - w, A, 0) \ .$$

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$$\deg(F, A, w) = \deg(F - w, A, 0) \ .$$

(2.2) Theorem (Existence criterion) *Let $F : \overline{A} \to X$ be a continuous map and let $w \in X \setminus F(\partial A)$ with $\deg(F, A, w) \neq 0$.*

Then $w \in F(A)$.

(2.3) Theorem (Excision-additivity) *Let $\{U_j : j \in J\}$ be a family of pairwise disjoint open subsets of A, let*

$$F : \overline{A} \setminus \left(\bigcup_{j \in J} U_j \right) \to X$$

be a continuous map and let

$$w \in X \setminus F\left(\overline{A} \setminus \left(\bigcup_{j \in J} U_j \right) \right).$$

Then

$$\partial A \subseteq \overline{A} \setminus \left(\bigcup_{j \in J} U_j \right),$$

$$\partial U_j \subseteq \overline{A} \setminus \left(\bigcup_{j \in J} U_j \right) \qquad \text{for any } j \in J\,,$$

$$\deg(F, U_j, w) \neq 0 \qquad \text{for at most}$$

$$\text{finitely many } j \in J\,,$$

$$\deg(F, A, w) = \sum_{j \in J} \deg(F, U_j, w)\ .$$

(2.4) Theorem (Homotopy invariance) *Let* $\mathcal{H} : \partial A \times [0,1] \to X$ *be a continuous map and let* $w \in X \setminus \mathcal{H}(\partial A \times [0,1])$.

Then the function

$$\{t \longmapsto \deg(\mathcal{H}_t, A, w)\}$$

is constant on $[0,1]$

(we write $\mathcal{H}_t(u)$ *instead of* $\mathcal{H}(u,t)$*).*

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(we write $\mathcal{H}_t(u)$ instead of $\mathcal{H}(u,t)$).

A remarkable result of H. Hopf provides a form of converse.

(2.5) Theorem (Hopf)

Assume that A *and* $X \setminus \overline{A}$ *are both connected. Let* $F_0, F_1 : \partial A \to X$ *be two continuous maps and let* $w \in X \setminus (F_0(\partial A) \cup F_1(\partial A))$ *with*

$$\deg(F_0, A, w) = \deg(F_1, A, w) \ .$$

Then there exists a continuous map $\mathcal{H} : \partial A \times [0,1] \to X$, *with* $w \notin \mathcal{H}(\partial A \times [0,1])$, *such that* $\mathcal{H}_0(u) = F_0(u)$ *and* $\mathcal{H}_1(u) = F_1(u)$ *for any* $u \in \partial A$.

(2.6) Definition *Let $L : X \to X$ be linear. We say that $\lambda \in \mathbb{R}$ is an* eigenvalue *of L if there exists $u \in X \setminus \{0\}$ such that*

$$Lu = \lambda u\,.$$

(2.6) Definition *Let $L : X \to X$ be linear. We say that $\lambda \in \mathbb{R}$ is an* eigenvalue *of L if there exists $u \in X \setminus \{0\}$ such that*

$$Lu = \lambda u\,.$$

The linear space $\mathcal{N}(L - \lambda \mathrm{Id})$ is said to be the eigenspace *relative to λ, while any element of $\mathcal{N}(L - \lambda \mathrm{Id}) \setminus \{0\}$ is said to be an* eigenvector *relative to λ.*

The integer

$$\dim \mathcal{N}\left(L-\lambda \mathrm{Id}\right)$$

is called *geometric multiplicity* of λ and the integer

$$\lim_{m\to\infty}\left(\dim \mathcal{N}\left(\left(L-\lambda \mathrm{Id}\right)^m\right)\right)$$

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algebraic multiplicity of λ.

The limit does exist, as

$$\mathcal{N}\left(\left(L-\lambda \mathrm{Id}\right)^{m}\right) \subseteq \mathcal{N}\left(\left(L-\lambda \mathrm{Id}\right)^{m+1}\right),$$

$$\dim \mathcal{N}\left(\left(L-\lambda \mathrm{Id}\right)^{m}\right) \leq \dim \mathcal{N}\left(\left(L-\lambda \mathrm{Id}\right)^{m+1}\right).$$

(2.7) Theorem *Let $L : X \to X$ be linear and bijective.*

Then, for every $w \in L(A)$, we have

$$\deg(L, A, w) = \operatorname{sgn}(\det L) = (-1)^m ,$$

where m is the sum of the algebraic multiplicities of the eigenvalues λ of L with $\lambda < 0$ (we agree that $m = 0$ if there is no eigenvalue λ with $\lambda < 0$).

(2.8) Corollary *Let $K : X \to X$ be linear with $\mathrm{Id} - K$ bijective.*

Then, for every $w \in (\mathrm{Id} - K)(A)$, we have

$$\deg(\mathrm{Id} - K, A, w) = (-1)^m,$$

where m is the sum of the algebraic multiplicities of the eigenvalues λ of K with $\lambda > 1$ (we agree that $m = 0$ if there is no eigenvalue λ with $\lambda > 1$).

(2.9) Theorem *Let $F : \partial A \to X$ be a continuous map, $L : X \to X$ linear and bijective and $w \in X \setminus (L \circ F)(\partial A)$.*

Then

$$\deg(L \circ F, A, w) = [\operatorname{sgn}(\det L)] \deg\left(F, A, L^{-1}w\right) .$$

(2.10) Theorem (Reduction) *Let* $F : \partial A \to X$ *be a continuous map and* $w \in X \setminus F(\partial A)$.

Assume there exists a linear subspace Y *of* X *such that* $w \in Y$ *and* $(\mathrm{Id} - F)(\partial A) \subseteq Y$.

Then

$$\partial_Y(A \cap Y) \subseteq Y \cap \partial A\,, \quad F(\partial_Y(A \cap Y)) \subseteq Y\,,$$

$$w \notin F(\partial_Y(A \cap Y))\,,$$

$$\deg(F, A, w) = \deg\left(F\big|_{\partial_Y(A\cap Y)}, A \cap Y, w\right)\,.$$

3 Linear spaces with scalar product

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Assume now that X is a finite dimensional linear space over $\mathbb{R}$ endowed with a scalar product $(\ |\ )$, while A is again a bounded and open subset of X.

(3.1) Theorem *Let* $0 \in A$ *and let* $F : \partial A \to X$ *be a continuous map with* $0 \notin F(\partial A)$. *Assume that*

$$(u|F(u)) \geq 0 \qquad \text{for any } u \in \partial A\,.$$

Then

$$\deg\,(F, A, 0) = 1\,.$$

(3.2) Theorem *Let $F : \partial A \to X$ be a continuous map. Assume there exists a linear subspace Y of X such that*

$$\Big\{ u \in \partial A : \ (u|F(u)) \leq 0 \, , \qquad (v|F(u)) = 0 \quad \forall v \in Y \Big\} = \emptyset \, .$$

Then

$$F(u) \neq 0 \quad \forall u \in \partial A\,,$$

$$\partial_Y(A \cap Y) \subseteq Y \cap \partial A\,,$$

$$(P_Y \circ F)(u) \neq 0 \quad \forall u \in \partial_Y(A \cap Y)\,,$$

$$\deg\,(F, A, 0) = \deg\Big((P_Y \circ F)\big|_{\partial_Y(A \cap Y)}, A \cap Y, 0\Big)\,.$$

Here $P_Y : X \to Y$ is the orthogonal projection.

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Throughout this section, X will denote a finite dimensional normed linear space over $\mathbb{R}$ and A a bounded and open subset of X.

We aim to consider a continuous map

$F : \partial A \to X'$ and $w \in X' \setminus F(\partial A)$.

(4.1) Proposition *The following facts hold:*

(a) *there exists a linear and bijective map* $J : X \to X'$ *such that*

$$\langle Ju, u\rangle \geq 0 \qquad \forall u \in X\,;$$

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(b) *if* $F : \partial A \to X'$ *is a continuous map,* $w \in X' \setminus F(\partial A)$ *and* $J_1,\ J_2$ *are as in* (a), *then*

$$\deg\left(J_1^{-1} \circ F, A, J_1^{-1}w\right) = \deg\left(J_2^{-1} \circ F, A, J_2^{-1}w\right)\ .$$

(4.2) Definition *If* $F : \partial A \to X'$ *is a continuous map and* $w \in X' \setminus F(\partial A)$*, we set*

$$\deg(F, A, w) = \deg\left(J^{-1} \circ F, A, J^{-1}w\right) ,$$

where $J : X \to X'$ *is any linear and bijective map such that*

$$\langle Ju, u\rangle \geq 0 \qquad \forall u \in X .$$

(4.3) Theorem *Let $F : \partial A \to X'$ be a continuous map and let $w \in X'$.*

Then $F(u) \neq w$ for any $u \in \partial A$ if and only if $F(u) - w \neq 0$ for any $u \in \partial A$, and in this case

$$\deg(F, A, w) = \deg(F - w, A, 0) \ .$$

(4.3) Theorem *Let* $F : \partial A \to X'$ *be a continuous map and let* $w \in X'$.

Then $F(u) \neq w$ *for any* $u \in \partial A$ *if and only if* $F(u) - w \neq 0$ *for any* $u \in \partial A$, *and in this case*

$$\deg(F, A, w) = \deg(F - w, A, 0) \ .$$

(4.4) Theorem (Existence criterion) *Let* $F : \overline{A} \to X'$ *be a continuous map and let* $w \in X' \setminus F(\partial A)$ *with* $\deg(F, A, w) \neq 0$.

Then $w \in F(A)$.

(4.5) Theorem (Excision-additivity) *Let $\{U_j : j \in J\}$ be a family of pairwise disjoint open subsets of A, let*

$$F : \overline{A} \setminus \left(\bigcup_{j \in J} U_j \right) \to X'$$

be a continuous map and let

$$w \in X' \setminus F\left(\overline{A} \setminus \left(\bigcup_{j \in J} U_j \right) \right).$$

Then

$$\partial A \subseteq \overline{A} \setminus \left(\bigcup_{j \in J} U_j \right),$$

$$\partial U_j \subseteq \overline{A} \setminus \left(\bigcup_{j \in J} U_j \right) \qquad \text{for any } j \in J\,,$$

$$\deg\left(F, U_j, w\right) \neq 0 \qquad \text{for at most finitely many } j \in J\,,$$

$$\deg\left(F, A, w\right) = \sum_{j \in J} \deg\left(F, U_j, w\right)\,.$$

(4.6) Theorem (Homotopy invariance) *Let* $\mathcal{H} : \partial A \times [0,1] \to X'$ *be a continuous map and let* $w \in X' \setminus \mathcal{H}(\partial A \times [0,1])$. *Then the function*

$$\{t \longmapsto \deg(\mathcal{H}_t, A, w)\}$$

is constant on $[0,1]$.

(4.7) Theorem (Hopf)

Assume that A and $X \setminus \overline{A}$ are both connected. Let $F_0, F_1 : \partial A \to X'$ be two continuous maps and let $w \in X' \setminus (F_0(\partial A) \cup F_1(\partial A))$ with

$$\deg(F_0, A, w) = \deg(F_1, A, w) \ .$$

Then there exists a continuous map $\mathcal{H} : \partial A \times [0,1] \to X'$, with $w \notin \mathcal{H}(\partial A \times [0,1])$, such that $\mathcal{H}_0(u) = F_0(u)$ and $\mathcal{H}_1(u) = F_1(u)$ for any $u \in \partial A$.

(4.8) Theorem *Let* $F : \partial A \to X'$ *be a continuous map,* $L : X' \to X'$ *linear and bijective and* $w \in X' \setminus (L \circ F)(\partial A)$.

Then

$$\deg (L \circ F, A, w)$$
$$= [\operatorname{sgn} (\det L)] \deg \left(F, A, L^{-1}w\right) \, .$$

(4.9) Theorem *Let* $0 \in A$ *and let* $F : \partial A \to X'$ *be a continuous map with* $0 \notin F(\partial A)$. *Assume that*

$$\langle F(u), u \rangle \geq 0 \qquad \text{for any } u \in \partial A\,.$$

Then

$$\deg(F, A, 0) = 1\,.$$

(4.10) Theorem *Let $F : \partial A \to X'$ be a continuous map. Assume there exists a linear subspace Y of X such that*

$$\Big\{ u \in \partial A : \ \langle F(u), u \rangle \leq 0 \,, \qquad \langle F(u), v \rangle = 0 \quad \forall v \in Y \Big\} = \emptyset \,.$$

Then

$$F(u) \neq 0 \quad \forall u \in \partial A\,,$$

$$\partial_Y(A \cap Y) \subseteq Y \cap \partial A\,,$$

$$(i' \circ F)(u) \neq 0 \quad \forall u \in \partial_Y(A \cap Y)\,,$$

$$\deg\,(F, A, 0) = \deg\left((i' \circ F)\big|_{\partial_Y(A\cap Y)}, A \cap Y, 0\right)\,.$$

Here $i : Y \to X$ is the inclusion map and $i' : X' \to Y'$ the dual map to i, so that

$$\langle i'w, v\rangle = \langle w, iv\rangle \qquad \forall w \in X'\,,\ \forall v \in Y\,.$$

(4.11) Theorem *Let $L, J : X \to X'$ be linear and bijective with*

$$\langle Ju, u \rangle \geq 0 \qquad \forall u \in X\,.$$

Then, for every $w \in L(A)$, we have

$$\deg\left(L, A, w\right) = (-1)^m\,,$$

where m is the sum of the algebraic multiplicities of the eigenvalues λ of $(J^{-1} \circ L)$ with $\lambda < 0$ (we agree that $m = 0$ if there is no eigenvalue λ with $\lambda < 0$).

(4.12) Definition *Let $L, K : X \to X'$ be two linear maps. We say that $\lambda \in \mathbb{R}$ is an* eigenvalue *of (L, K) if there exists $u \in X \setminus \{0\}$ such that*

$$Lu = \lambda K u \,.$$

(4.12) Definition *Let* $L, K : X \to X'$ *be two linear maps. We say that* $\lambda \in \mathbb{R}$ *is an* eigenvalue *of* (L, K) *if there exists* $u \in X \setminus \{0\}$ *such that*

$$Lu = \lambda K u\,.$$

The linear space $\mathcal{N}(L - \lambda K)$ *is said to be the* eigenspace *relative to* λ, *while any element of* $\mathcal{N}(L - \lambda K) \setminus \{0\}$ *is said to be an* eigenvector *relative to* λ.

The integer

$$\dim \mathcal{N}\left(L - \lambda K\right)$$

is called *geometric multiplicity* of λ.

(4.13) Proposition *Assume that*

$$\langle Ku, u\rangle \geq 0 \qquad \forall u \in X$$

and that there exists $\overline{\lambda} \in \mathbb{R}$ *such that*

$$\langle (L + \overline{\lambda} K)u, u\rangle > 0 \qquad \forall u \in X \setminus \{0\}\,. \tag{4.14}$$

(4.13) Proposition *Assume that*

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and that there exists $\overline{\lambda} \in \mathbb{R}$ *such that*

$$\langle (L + \overline{\lambda} K)u, u \rangle > 0 \qquad \forall u \in X \setminus \{0\} \,. \quad (4.14)$$

Then the following facts hold:

(a) $(L + \overline{\lambda} K)$ *is bijective and any eigenvalue* λ *of* (L, K) *satisfies* $\lambda > -\overline{\lambda}$*;*

(b) *for every* $\lambda > -\overline{\lambda}$, *we have*

$$\mathcal{N}(L - \lambda K) = \mathcal{N}\left((L + \overline{\lambda} K)^{-1} \circ K - \frac{1}{\lambda + \overline{\lambda}} \mathrm{Id} \right) ;$$

in particular, λ *is an eigenvalue of* (L, K) *if and only if* $\frac{1}{\lambda+\overline{\lambda}}$ *is an eigenvalue of* $(L + \overline{\lambda} K)^{-1} \circ K$ *and the geometric multiplicity is the same;*

(c) *if also* $\hat{\lambda} \in \mathbb{R}$ *satisfies* (4.14), *then the algebraic multiplicity of* $\frac{1}{\lambda+\overline{\lambda}}$ *as an eigenvalue of* $(L+\overline{\lambda}K)^{-1} \circ K$ *is equal to that of* $\frac{1}{\lambda+\hat{\lambda}}$ *as an eigenvalue of* $(L+\hat{\lambda}K)^{-1} \circ K$.

(4.15) Definition *Assume that*

$$\langle Ku, u\rangle \geq 0 \qquad \forall u \in X\,,$$

$$\langle (L+\mu K)u, u\rangle > 0 \qquad \forall u \in X \setminus \{0\}\,,$$

for some $\mu \in \mathbb{R}$.

(4.15) Definition *Assume that*

$$\langle Ku, u\rangle \geq 0 \qquad \forall u \in X\,,$$

$$\langle (L+\mu K)u, u\rangle > 0 \qquad \forall u \in X\setminus\{0\}\,,$$

for some $\mu \in \mathbb{R}$.

If λ *is an eigenvalue of* (L, K), *the* algebraic multiplicity *of* λ *is defined as the algebraic multiplicity of* $\frac{1}{\lambda+\overline{\lambda}}$ *as an eigenvalue of* $(L+\overline{\lambda}K)^{-1}\circ K$, *where* $\overline{\lambda}$ *is any real number satisfying*

$$\langle (L+\overline{\lambda}K)u, u\rangle > 0 \qquad \forall u \in X\setminus\{0\}\,.$$

(4.16) Theorem *Let $L, K : X \to X'$ be two linear maps such that L is bijective and*

$$\langle Ku, u \rangle \geq 0 \qquad \forall u \in X\,.$$

Assume there exists $\overline{\lambda} \in \mathbb{R}$ such that

$$\langle (L + \overline{\lambda} K)u, u \rangle > 0 \qquad \forall u \in X \setminus \{0\}\,.$$

Then, for every $w \in L(A)$, we have

$$\deg(L, A, w) = (-1)^m\,,$$

where m is the sum of the algebraic multiplicities of the eigenvalues λ of (L, K) with $\lambda < 0$.

(4.17) Theorem *Let* $0 \in A$ *and let* $L, K : X \to X'$ *be two linear maps such that*

$$\langle Ku, u \rangle \geq 0 \qquad \forall u \in X\,.$$

Assume there exists $\overline{\lambda} \in \mathbb{R}$ *such that*

$$\langle (L + \overline{\lambda} K)u, u \rangle > 0 \qquad \forall u \in X \setminus \{0\}$$

and let μ *be an eigenvalue of* (L, K) *of odd algebraic multiplicity.*

Then we have

$$\lim_{\lambda \to \mu^-} \deg\left(L - \lambda K, A, 0\right) \neq \lim_{\lambda \to \mu^+} \deg\left(L - \lambda K, A, 0\right) .$$

5 Infinite dimensional spaces

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See e.g.

H. BREZIS, “Functional analysis, Sobolev spaces and partial differential equations”, Universitext, Springer, New York, 2011.

(5.1) Theorem *Let X be a normed space, E a bounded subset of X and $u \in X$ belonging to the weak closure of E.*

(5.1) Theorem *Let X be a normed space, E a bounded subset of X and $u \in X$ belonging to the weak closure of E.*
Then there exists a separable and closed linear subspace X_0 of X such that $u \in X_0$ and u belongs to the weak closure of $E \cap X_0$ in X_0.

(5.2) Theorem *Let X be a reflexive Banach space, E a bounded subset of X and $u \in X$ belonging to the weak closure of E.*

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Then there exists a sequence (u_k) in E weakly convergent to u.

6 Maps of class $(S)_+$

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In the following, X will denote a reflexive Banach space over $\mathbb{R}$.

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In the following, X will denote a reflexive Banach space over $\mathbb{R}$.

(6.1) Definition *A map $F : D \to X'$, with $D \subseteq X$, is said to be* demicontinuous *if, for every sequence (u_k) convergent to some u in D, we have*

$$\lim_k \langle F(u_k), v \rangle = \langle F(u), v \rangle \qquad \forall v \in X\,.$$

(6.2) Definition *A map* $F : D \to X'$, *with* $D \subseteq X$, *is said to be* of class $(S)_+$ *if, for every sequence* (u_k) *in* D *weakly convergent to some* u *in* X *with*

$$\limsup_k \langle F(u_k), u_k - u \rangle \leq 0 \,,$$

it holds $\|u_k - u\| \to 0$.

(6.3) Definition *A map* $F : D \to X'$, *with* $D \subseteq X$, *is said to be* completely continuous *if it is continuous and, for every bounded sequence* (u_k) *in* D, *the sequence* $(F(u_k))$ *admits a convergent subsequence in* X'.

(6.4) Proposition *The following facts hold:*

(a) *if* $F_1 : D_1 \to X'$ *and* $F_2 : D_2 \to X'$ *are of class* $(S)_+$, *then* $(F_1 + F_2) : D_1 \cap D_2 \to X'$ *is of class* $(S)_+$;

(b) *if* $F : D \to X'$ *is of class* $(S)_+$ *and* $t > 0$, *then* $tF : D \to X'$ *is of class* $(S)_+$;

(c) *if* $F_1 : D_1 \to X'$ *is of class* $(S)_+$, *and* $F_2 : D_2 \to X'$ *is completely continuous, then* $(F_1 + F_2) : D_1 \cap D_2 \to X'$ *is of class* $(S)_+$;

(d) *if* $F : D \to X'$ *is of class* $(S)_+$ *and* $w \in X'$, *then* $(F + w) : D \to X'$ *is of class* $(S)_+$.