Some Basic Tools of Critical Point Theory

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Chapter I

Some basic tools of critical point theory

1 The deformation theorem

Throughout this section, we will consider a real Banach space $X$ and a function $f : X \to \mathbb{R}$ of class $C^1$.

(1.1) Definition We say that $u \in X$ is a critical point of $f$, if $f'(u) = 0$. We say that $c \in \mathbb{R}$ is a critical value of $f$, if there exists a critical point $u$ of $f$ with $f(u) = c$. We say that $c \in \mathbb{R}$ is a regular value of $f$, if it is not a critical value of $f$.

(1.2) Definition Let $c \in \mathbb{R}$. We say that $(u_h)$ is a Cerami-Palais-Smale sequence at level $c$ (CPS$_c$-sequence, for short) for $f$, if $f(u_h) \to c$ and $(1 + \|u_h\|) f'(u_h) \to 0$.

We say that $f$ satisfies the Cerami-Palais-Smale condition at level $c$ (condition (CPS)$_c$, for short), if every (CPS)$_c$-sequence for $f$ admits a (strongly) convergent subsequence in $X$.

(1.3) Remark In the classical Palais-Smale condition, one considers sequences with $f'(u_h) \to 0$ instead of $(1 + \|u_h\|) f'(u_h) \to 0$. This useful variant, which is clearly a weaker condition, was introduced by Cerami [4].

For every $b \in \mathbb{R} \cup \{+\infty\}$ and $c \in \mathbb{R}$, we set

$$f^b := \{u \in X : f(u) \leq b\} ,$$

$$K_c := \{u \in X : f(u) = c, f'(u) = 0\} .$$

(1.4) Definition Given $u \in X$, we say that $v \in X$ is a pseudogradient vector for $f$ at $u$, if $\|v\| \leq 2\|f'(u)\|$ and $(f'(u), v) \geq \|f'(u)\|^2$.

We say that

$$V : \{u \in X : f'(u) \neq 0\} \to X$$

is a pseudogradient vector field for $f$, if $V$ is locally Lipschitz and $V(u)$ is a pseudogradient vector for $f$ at $u$ for any $u$ in the domain of $V$.

(1.5) Remark If $v$ is a pseudogradient vector for $f$ at $u$, we have

$$\|f'(u)\|^2 \leq (f'(u), v) \leq \|f'(u)\|\|v\| ,$$

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hence \( \|f'(u)\| \leq \|v\| \).

(1.6) **Lemma**  Let \( Y \) be a metric space, \( Z \) a normed space and for every \( y \in Y \) let \( \mathcal{F}(y) \) be a convex subset of \( Z \). Assume that for every \( y \in Y \) there exists a neighbourhood \( U \) of \( y \) such that

\[
\bigcap_{\xi \in U} \mathcal{F}(\xi) \neq \emptyset.
\]

Then there exists a locally Lipschitz map \( F : Y \to Z \) such that \( F(y) \in \mathcal{F}(y) \) for every \( y \in Y \).

**Proof.** For every \( y \in Y \) let \( U_y \) be an open neighbourhood of \( y \) such that

\[
\bigcap_{\xi \in U_y} \mathcal{F}(\xi) \neq \emptyset.
\]

Since \( \{U_y : y \in Y\} \) is an open cover of \( Y \) and \( Y \) is paracompact (see e.g. [8]), there exists a locally finite open cover \( \{W_j : j \in J\} \) of \( Y \) refining \( \{U_y : y \in Y\} \). Assume first that \( W_j \neq Y \) for any \( j \in J \). If we set

\[
\psi_j(y) = d(y, Y \setminus W_j), \quad \Psi(y) = \sum_{j \in J} \psi_j(y),
\]

then \( \psi_j \) is Lipschitz and \( \Psi \) is well defined and locally Lipschitz, as \( \{W_j : j \in J\} \) is locally finite. Since \( \{W_j : j \in J\} \) is an open cover, we also have \( \Psi(y) \neq 0 \) for every \( y \in Y \). Therefore, if we set

\[
\varphi_j(y) = \frac{\psi_j(y)}{\Psi(y)},
\]

it turns out that \( \{\varphi_j : j \in J\} \) is a locally Lipschitz partition of unity subordinated to \( \{W_j : j \in J\} \). If there exists \( j_0 \in J \) with \( W_{j_0} = Y \), set \( \varphi_{j_0} = 1 \) and \( \varphi_j = 0 \) for \( j \neq j_0 \). Then also in this case \( \{\varphi_j : j \in J\} \) is a locally Lipschitz partition of unity subordinated to \( \{W_j : j \in J\} \).

Since \( \{W_j : j \in J\} \) refines \( \{U_y : y \in Y\} \), for every \( j \in J \) we have

\[
\bigcap_{y \in W_j} \mathcal{F}(y) \neq \emptyset.
\]

If for every \( j \in J \) we choose a \( z_j \in \bigcap_{y \in W_j} \mathcal{F}(y) \), we can define a locally Lipschitz map \( F : Y \to Z \) by

\[
F(y) = \sum_{j \in J} \varphi_j(y) z_j.
\]

Given \( y \in Y \), there is only a finite number \( W_{j_1}, \ldots, W_{j_n} \) of \( W_j \)'s such that \( y \in W_j \). Then

\[
F(y) = \sum_{k=1}^{n} \varphi_{j_k}(y) z_{j_k}, \quad \sum_{k=1}^{n} \varphi_{j_k}(y) = 1.
\]

For every \( k = 1, \ldots, n \), from \( y \in W_{j_k} \) it follows \( z_{j_k} \in \mathcal{F}(y) \). Since \( \mathcal{F}(y) \) is convex, we conclude that \( F(y) \in \mathcal{F}(y) \).

(1.7) **Theorem**  There exists a pseudogradient vector field for \( f \).

**Proof.** Let

\[
Y = \{u \in X : f'(u) \neq 0\}.
\]
For every $u \in Y$, denote by $\mathcal{V}(u)$ the set of pseudogradient vectors for $f$ at $u$. It is readily seen that $\mathcal{V}(u)$ is a convex subset of $X$. Moreover, for every $u \in Y$ there exists $w \in X$ such that $\|w\| \leq 1$ and $\langle f'(u), w \rangle \geq \frac{\varepsilon}{3} \|f'(u)\|$. Then $v = \frac{\varepsilon}{3} \|f'(u)\| w$ satisfies $\|v\| \leq \frac{\varepsilon}{3} \|f'(u)\|$ and $\langle f'(u), v \rangle \geq \frac{\varepsilon}{3} \|f'(u)\|^2$. Since $f$ is of class $C^1$, there exists a neighborhood $U$ of $u$ such that $\|v\| < 2\|f'(u)\|$ and $\langle f'(u), v \rangle > \|f'(u)\|^2$ for every $\xi \in U$, so that

$$v \in \bigcap_{\xi \in U} \mathcal{V}(\xi).$$

From Lemma (1.6) we deduce that there exists a locally Lipschitz map $V : Y \to X$ with $V(u) \in \mathcal{V}(u)$ and the assertion follows.

Now we can prove the main result of this section.

**1.8 Theorem (Deformation Theorem)** Let $c \in \mathbb{R}$ be such that $f$ satisfies (CPS)$_c$. Then, for every $\varepsilon > 0$, every neighborhood $U$ of $K_c$ (if $K_c = \emptyset$, we allow $U = \emptyset$) and every $\lambda > 0$, there exist $\varepsilon \in ]0, \varepsilon[\; \text{ and a continuous map } \eta : X \times [0, 1] \to X$ such that for every $(u, t) \in X \times [0, 1]$ we have:

(a) $\|\eta(u, t) - u\| \leq \lambda (1 + \|u\|) t$;

(b) $f(\eta(u, t)) \leq f(u)$;

(c) $\eta(u, t) \neq u \implies f(\eta(u, t)) < f(u)$;

(d) $|f(u) - c| \geq \varepsilon \implies \eta(u, t) = u$;

(e) $f^{c+\varepsilon} \times \{1\}) \subseteq f^{c-\varepsilon} \cup U$.

**Proof.** From condition (CPS)$_c$ it easily follows that $K_c$ is compact. Therefore there exists $\rho > 0$ such that $B_\rho(K_c) \subseteq U$.

We claim there exist $\tilde{\varepsilon} \in ]0, \frac{\varepsilon}{2}[\; \text{ and } \sigma > 0$ such that

$$c - 2\tilde{\varepsilon} \leq f(u) \leq c + 2\tilde{\varepsilon}, u \notin B_\rho(K_c) \implies (1 + \|u\|) \|f'(u)\| \geq \sigma.$$

Actually, assume for a contradiction that $(u_n)$ is a sequence in $X$ with $f(u_n) \to c$, $u_n \notin B_\rho(K_c)$ and $(1 + \|u_n\|) \|f'(u_n)\| \to 0$. Then, up to a subsequence, $(u_n)$ is convergent to some $u$ with $f(u) = c$, $u \notin B_\rho(K_c)$ and $f'(u) = 0$, which is clearly impossible.

Let $\chi : X \to [0, 1]$ be a locally Lipschitz function such that

$$\left(|f(u) - c| \geq 2\tilde{\varepsilon} \text{ or } u \in B_\rho(K_c)\right) \implies \chi(u) = 0,$$

$$\left(|f(u) - c| \leq \tilde{\varepsilon} \text{ and } u \notin B_{2\rho}(K_c)\right) \implies \chi(u) = 1,$$

let $\mu > 0$ with

$$\exp \mu - 1 \leq \lambda$$

and let

$$W(u) = \begin{cases} \frac{\sigma \chi(u)}{\|V(u)\|^2} & \text{if } |f(u) - c| \leq 2\tilde{\varepsilon} \text{ and } u \notin B_\rho(K_c), \\ 0 & \text{otherwise}, \end{cases}$$

$$\begin{aligned}
\langle f'(u), v \rangle &= \frac{\varepsilon}{3} \|f'(u)\|, \\
\|v\| &= \frac{\varepsilon}{3} \|f'(u)\|, \\
\langle f'(u), v \rangle &= \frac{\varepsilon}{3} \|f'(u)\|^2.
\end{aligned}$$
where $V$ is a pseudogradient vector field for $f$. Then $W : X \to X$ is locally Lipschitz. Moreover, if $|f(u) - c| \leq 2 \hat{\varepsilon}$ and $u \notin B_{\hat{\varepsilon}}(K_c)$, we deduce from (1.9) and the definition of pseudogradient vector that
\[ \|W(u)\| \leq \sigma \mu \frac{1}{\|f(u)\|} \leq \sigma \mu \frac{1}{\|f'(u)\|} \leq \mu (1 + \|u\|), \]
\[ \langle f'(u), W(u) \rangle = -\sigma \mu \chi(u) \frac{\|f'(u), V(u)\|}{\|V(u)\|^2} \leq -\sigma \mu \chi(u) \frac{\|f'(u)\|^2}{\|V(u)\|^2} \leq -\frac{1}{4} \sigma \mu \chi(u). \]
It follows
\begin{equation}
\forall u \in X : \|W(u)\| \leq \mu(1 + \|u\|),
\end{equation}
\begin{equation}
\forall u \in X : \langle f'(u), W(u) \rangle \leq -\frac{1}{4} \sigma \mu \chi(u).
\end{equation}
Therefore the Cauchy problem
\[ \begin{cases}
\frac{\partial \eta}{\partial t}(u, t) = W(\eta(u, t)) \\
\eta(u, 0) = u
\end{cases} \]
defines a continuous map $\eta : X \times \mathbb{R} \to X$ such that $\eta(u, t) = u$ whenever $|f(u) - c| \geq 2 \hat{\varepsilon}$, whence assertion (d). From (1.11) also (b) and (c) easily follow.

By (1.10) we have
\[ \|\eta(u, t) - u\| \leq \int_0^t \|W(\eta(u, \tau))\| \, d\tau \leq \mu \int_0^t (1 + \|\eta(u, \tau)\|) \, d\tau \leq \mu \int_0^t \|\eta(u, \tau) - u\| \, d\tau + \mu(1 + \|u\|)t, \]
hence
\[ \int_0^t \|\eta(u, \tau) - u\| \, d\tau \leq \frac{1 + \|u\|}{\mu} (\exp(\mu t) - 1) - (1 + \|u\|)t. \]
If $0 \leq t \leq 1$, it follows
\[ \|\eta(u, t) - u\| \leq (1 + \|u\|) (\exp(\mu t) - 1) \leq (1 + \|u\|) (\exp \mu - 1) t \leq (1 + \|u\|) \lambda t, \]
whence assertion (a). Since $\eta(u, t_2) = \eta(u, t_1), t_2 - t_1)$, we also have
\[ 0 \leq t_1 \leq t_2 \leq 1 \implies \|\eta(u, t_2) - \eta(u, t_1)\| \leq \lambda(1 + \|\eta(u, t_1)\|)(t_2 - t_1). \]

Finally, to prove assertion (c), consider $R > 0$ such that $B_{2\hat{\varepsilon}}(K_c) \subseteq B_R(0)$ and $\varepsilon \in [0, \hat{\varepsilon}]$ such that
\[ 8\varepsilon \leq \sigma \mu, \quad 8\lambda(1 + R)\varepsilon \leq \sigma \mu \varrho. \]
Let $u \in f^{-1}(c + \varepsilon)$ and assume, for a contradiction, that $f(\eta(u, 1)) > c - \varepsilon$ and $\eta(u, 1) \notin U$. First of all, we have $c - \varepsilon < f(\eta(u, t)) \leq c + \varepsilon$ for every $t \in [0, 1]$. Moreover, it is not possible to have $\eta([u] \times [0, 1]) \cap B_{2\hat{\varepsilon}}(K_c) = \emptyset$, for otherwise from (1.11) it would follow
\[ 2\varepsilon > f(u) - f(\eta(u, 1)) \geq \frac{1}{4} \sigma \mu. \]
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Therefore there exist $0 \leq t_1 < t_2 \leq 1$ such that

$$d(\eta(u, t_1), K_c) = 2\rho, \quad d(\eta(u, t_2), K_c) = 3\rho,$$

$$\forall t \in ]t_1, t_2[: 2\rho < d(\eta(u, t), K_c) < 3\rho.$$  

We have

$$2\varepsilon > f(\eta(u, t_1)) - f(\eta(u, t_2)) \geq \frac{1}{4} \sigma \mu (t_2 - t_1),$$

hence

$$\rho \leq \|\eta(u, t_2) - \eta(u, t_1)\| \leq \lambda(1 + \|\eta(u, t_1)\|)(t_2 - t_1) < \lambda(1 + R) \frac{8\varepsilon}{\sigma \mu},$$

and a contradiction follows. ■

We end this section by providing a useful criterion for the verification of condition $(CPS)_c$.

1.12 Definition  Let $Y, Z$ be two normed spaces. A map $F : Y \to Z$ is said to be completely continuous, if

(a) $F$ is continuous;

(b) for every bounded sequence $(u_h)$ in $Y$, $(F(u_h))$ admits a (strongly) convergent subsequence in $Z$.

1.13 Theorem  Assume that

$$f'(u) = Lu - F(u)$$

where $L : X \to X^*$ is linear, continuous, with closed range and finite dimensional null space and $F : X \to X^*$ is completely continuous.

Then for every $c \in \mathbb{R}$ the following assertions are equivalent:

(a) $f$ satisfies condition $(CPS)_c$;

(b) every $(CPS)_c$-sequence for $f$ is bounded in $X$.

Proof.

(a) $\implies$ (b) If $(u_h)$ is an unbounded $(CPS)_c$-sequence for $f$, there exists a subsequence $(u_{h_k})$ with $\|u_{h_k}\| \to \infty$. Then $(u_{h_k})$ is a $(CPS)_c$-sequence which cannot admit any convergent subsequence.

(b) $\implies$ (a) Let $(u_h)$ be a $(CPS)_c$-sequence for $f$. In particular, we have $f'(u_h) \to 0$ in $X^*$. Since $(u_h)$ is bounded in $X$, up to a subsequence $(F(u_h))$ is convergent in $X^*$. Consequently, also $(Lu_h)$ is convergent in $X^*$. Let $Y$ be a closed subspace of $X$ with $X = \mathcal{N}(L) \oplus Y$ and let $P_0 : X \to \mathcal{N}(L)$, $P_1 : X \to Y$ be the projections associated with the direct decomposition. Of course, we have $LP_1 u_h = Lu_h$. Since $L : Y \to \mathcal{R}(L)$ is bijective and $\mathcal{R}(L)$ is closed, from the Open Mapping Theorem we deduce that $(P_1 u_h)$ is convergent in $Y$, hence in $X$. On the other hand, up to a subsequence also $(P_0 u_h)$ is convergent, as $\mathcal{N}(L)$ is finite dimensional. Then the assertion follows. ■
2 Mountain pass theorems

Throughout this section, we will consider again a real Banach space $X$ and a function $f : X \to \mathbb{R}$ of class $C^1$.

\((2.1)\) Definition \textbf{Let $A, B \subseteq X$. We say that $B$ links $A$, if $B \cap A = \emptyset$ and $B$ is not contractible in $X \setminus A$.}

\((2.2)\) Remark \textbf{Of course any $B \subseteq X$ is contractible in $X$.}

The next result is a general mountain pass theorem which will be specialized in some corollaries later. Our kind of approach is taken from [5, 11]. We want also to recall that the possibility to consider also the large inequality in the sup $- \inf$ estimate involving $B$ and $A$ is due to [7].

\((2.3)\) Theorem \textbf{Let $A$ be a nonempty closed subset of $X$, $B$ a nonempty subset of $X$ and let $\mathcal{C}_B$ be the family of all contractions of $B$ in $X$. Assume that $B$ links $A$, that}

$$\sup_B f \leq \inf_A f,$$

$$c := \inf_{\mathcal{H} \in \mathcal{C}_B} \sup_{B \times [0,1]} f \circ \mathcal{H} < +\infty$$

\textbf{and that $f$ satisfies (CPS)$_c$.}

Then $c \geq \inf_A f$ and $c$ is a critical value of $f$. Moreover, if $c = \inf_A f$, there exists a critical point $u$ of $f$ with $f(u) = c$ and $u \in A$.

\textbf{Proof.} Since $B$ links $A$, we have $\mathcal{H}(B \times [0,1]) \cap A \neq \emptyset$ for every $\mathcal{H} \in \mathcal{C}_B$. It follows $c \geq \inf_A f$.

Now, consider first the case $c = \inf_A f$ and assume, for a contradiction, that $K_c \cap A = \emptyset$. Let $U$ be a neighbourhood of $K_c$ with $U \cap A = \emptyset$ and let $\varepsilon > 0$ and $\eta : X \times [0,1] \to X$ be as in the Deformation Theorem. Let also $\mathcal{H} \in \mathcal{C}_B$ be such that $f(\mathcal{H}(u,t)) \leq c + \varepsilon$ for every $(u,t) \in B \times [0,1]$. If we define $\mathcal{K} : B \times [0,1] \to X$ by

$$\mathcal{K}(u,t) = \begin{cases} 
\eta(u,2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\
\eta(\mathcal{H}(u,2t-1),1) & \text{if } \frac{1}{2} \leq t \leq 1,
\end{cases}$$

it is readily seen that $\mathcal{K} \in \mathcal{C}_B$. For every $u \in B$ we have either $\eta(u,2t) = u$ or $f(\eta(u,2t)) < f(u) \leq \inf_A f$. In both cases it follows $\eta(u,2t) \notin A$. On the other hand

$$\eta(\mathcal{H}(u,2t-1),1) \notin f_{c-\varepsilon} \cup U$$

and $(f_{c-\varepsilon} \cup U) \cap A = \emptyset$. Therefore $\mathcal{K}$ is a contraction of $B$ in $X \setminus A$ and this contradicts the assumption that $B$ links $A$.

Finally, consider the case $c > \inf_A f$ and assume, for a contradiction, that $K_c = \emptyset$. Let $U = \emptyset$ and let $\varepsilon > 0$ and $\eta : X \times [0,1] \to X$ be as in the Deformation Theorem. Let also $\mathcal{H} \in \mathcal{C}_B$ be such that $f(\mathcal{H}(u,t)) \leq c + \varepsilon$ for every $(u,t) \in B \times [0,1]$. If we define $\mathcal{K} : B \times [0,1] \to X$ by

$$\mathcal{K}(u,t) = \begin{cases} 
\eta(u,2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\
\eta(\mathcal{H}(u,2t-1),1) & \text{if } \frac{1}{2} \leq t \leq 1,
\end{cases}$$

we have again $\mathcal{K} \in \mathcal{C}_B$. On the other hand, for every $u \in B$ we have

$$f(\eta(u,2t)) \leq f(u) \leq \sup_B f,$$
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\[ f(\eta(\mathcal{H}(u, 2t - 1), 1)) \leq c - \epsilon. \]

Since \( \sup_B f < c \), this contradicts the definition of \( c \). \[ \square \]

\[ \text{(2.4) Corollary} \]

Let \( A \) be a nonempty closed subset of \( X \), \( B \) a nonempty subset of \( X \) and let \( \mathcal{C}_B \) be the family of all contractions of \( B \) in \( X \). Assume that \( B \) links \( A \), that \( \sup_B f \leq \inf_A f \) and that \( c := \inf_{\mathcal{H} \in \mathcal{C}_B} \sup_B \times [0, 1] f \circ \mathcal{H} < +\infty \).

Then \( c \geq \inf_A f \) and there exists a \((CPS)_c\)–sequence \((u_h)\) for \( f \).

Proof. As before, one easily verifies that \( c \geq \inf_A f \). Now assume, for a contradiction, that there are no \((CPS)_c\)–sequences for \( f \). Then there exists \( \sigma > 0 \) such that

\[ c - \sigma \leq f(u) \leq c + \sigma \implies (1 + \|u\|)\|f'(u)\| \geq \sigma. \]

Therefore condition \((CPS)_c\) holds and from Theorem (2.3) we deduce that \( c \) is a critical value of \( f \). This contradicts (2.5). \[ \square \]

The first particular case we consider is the classical mountain pass theorem of Ambrosetti-Rabinowitz (see [1, 10]).

\[ \text{(2.6) Corollary} \ (\text{Mountain Pass Theorem}) \]

Assume there exist \( u_1 \in X \) and \( r > 0 \) such that \( \|u_1\| > r \) and

\[ \max\{f(0), f(u_1)\} \leq \inf \{f(u) : \|u\| = r\}. \]

Set

\[ \Gamma = \{\gamma \in C([0, 1]; X) : \gamma(0) = 0, \gamma(1) = u_1\}, \]

\[ c = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} f(\gamma(t)) \]

and suppose that \( f \) satisfies \((CPS)_c\).

Then \( c \geq \inf \{f(u) : \|u\| = r\} \) and \( c \) is a critical value of \( f \). Moreover, if \( c = \inf \{f(u) : \|u\| = r\} \), there exists a critical point \( u \) of \( f \) with \( f(u) = c \) and \( \|u\| = r \).

Proof. Set \( A = \{u \in X : \|u\| = r\} \) and \( B = \{0, u_1\} \). It is evident that \( B \) links \( A \) and that \( c < +\infty \). If \( \gamma \in \Gamma \), then

\[ \mathcal{H}(u, t) = \begin{cases} \gamma(t) & \text{if } u = 0 \\ u_1 & \text{if } u = u_1 \end{cases} \]

is clearly a contraction of \( B \) in \( X \). Therefore

\[ c \geq \inf_{\mathcal{H} \in \mathcal{C}_B} \sup_{B \times [0, 1]} f \circ \mathcal{H}. \]
Conversely, if \( \mathcal{H} \) is a contraction of \( B \) in \( X \), then
\[
\gamma(t) = \begin{cases} 
\mathcal{H}(0, 2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\
\mathcal{H}(u_1, 2 - 2t) & \text{if } \frac{1}{2} \leq t \leq 1 
\end{cases}
\]
belong to \( \Gamma \), whence
\[
c \leq \inf_{\mathcal{H} \in C_B} \sup_{B \times [0,1]} f \circ \mathcal{H}.
\]
From Theorem (2.3) the assertion follows. \( \blacksquare \)

(2.7) Lemma  Let \( Y \) be a finite dimensional normed space, \( U \) a bounded open subset of \( Y \) and \( y_0 \in U \).

Then \( \partial U \) is not contractible in \( Y \setminus \{y_0\} \).

Proof. Assume, for a contradiction, that \( \mathcal{H} : \partial U \times [0, 1] \to Y \setminus \{y_0\} \) is a contraction of \( \partial U \) in \( Y \setminus \{y_0\} \) to some point \( y_1 \). If \( F : \overline{U} \to Y \) is the map with constant value \( y_1 \), by well known properties of Brouwer’s degree (see e.g. [6, 12]), we have
\[
1 = \deg (\text{Id}, U, y_0) = \deg (F, U, y_0) = 0,
\]
which is clearly absurd. \( \blacksquare \)

Now we come to the saddle theorem of Rabinowitz (see [10]).

(2.8) Corollary  (Saddle Theorem) Assume that

(a) \( X = X_- \oplus X_+ \), where \( \dim X_- < \infty \) and \( X_+ \) is closed in \( X \);

(b) there exists \( R > 0 \) such that
\[
\max \{ f(u) : u \in X_-, \|u\| = R \} \leq \inf \{ f(u) : u \in X_+ \};
\]

(c) \( f \) satisfies \((CPS)_c\), where
\[
c = \inf_{\varphi \in \Phi} \max_{u \in D} f(\varphi(u)),
\]
\[
D = \{ u \in X_- : \|u\| \leq R \},
\]
\[
\Phi = \{ \varphi \in C(D; X) : \varphi(u) = u \text{ whenever } \|u\| = R \}.
\]

Then \( c \geq \inf_{X_+} f \) and \( c \) is a critical value of \( f \). Moreover, if \( c = \inf_{X_+} f \), there exists a critical point \( u \) of \( f \) with \( f(u) = c \) and \( u \in X_+ \).

Proof. Set \( A = X_+ \) and
\[
B = \{ u \in X_- : \|u\| = R \}.
\]
Since \( D \) is compact, it is evident that \( c < +\infty \). Moreover, if \( \mathcal{H} \) is a contraction of \( B \) in \( X \setminus X_+ \) and \( P_- : X \to X_- \) is the projection associated with the direct decomposition, then
\[
\mathcal{K}(u, t) = P_- \mathcal{H}(u, t)
\]
is a contraction of \( B \) in \( X_- \setminus \{0\} \). Since this contradicts Lemma (2.7), it follows that \( B \) links \( A \).
If $\varphi \in \Phi$, then
\[ H(u, t) = \varphi((1 - t)u) \]
is a contraction of $B$ in $X$. Therefore
\[ c \geq \inf_{H \in C} \sup_{B \times [0, 1]} f \circ H. \]
Conversely, if $H$ is a contraction of $B$ in $X$ to some point $u_1$, we can define a continuous map
\[ \psi : (B \times [0, 1]) \cup (D \times \{1\}) \to X \]
by
\[ \psi(u, t) = \begin{cases} H(u, t) & \text{if } (u, t) \in B \times [0, 1], \\ u_1 & \text{if } (u, t) \in D \times \{1\}. \end{cases} \]
There exists a homeomorphism
\[ F : D \to (B \times [0, 1]) \cup (D \times \{1\}) \]
with $F(B) = B \times \{0\}$. Then we have that $\psi \circ F \in \Phi$, whence
\[ c \leq \inf_{H \in C} \sup_{B \times [0, 1]} f \circ H. \]
From Theorem (2.3) the assertion follows. $\blacksquare$

Finally, we derive the linking theorem of Benci-Rabinowitz (see [10] and [2] for the corresponding version in the strongly indefinite case).

(2.9) Corollary (Linking Theorem) Assume that

(a) $X = X_- \oplus X_+$, where $\dim X_- < \infty$ and $X_+$ is closed in $X$;

(b) there exist $0 < r < R$ and $v \in X_+$ with $\|v\| = 1$ such that
\[ \max \{ f(u) : u \in B \} \leq \inf \{ f(u) : u \in S \}, \]
where $B$ is the boundary of
\[ D := \{ u + tv : u \in X_-, t \geq 0, \|u + tv\| \leq R \} \]
in $X_- \oplus \mathbb{R} v$ and
\[ S = \{ u \in X_+ : \|u\| = r \}; \]

(c) $f$ satisfies (CPS)$_c$, where
\[ c = \inf_{\varphi \in \Phi} \max_{u \in D} f(\varphi(u)), \]
\[ \Phi = \{ \varphi \in C(D; X) : \varphi(u) = u \text{ whenever } u \in B \}. \]

Then $c \geq \inf_S f$ and $c$ is a critical value of $f$. Moreover, if $c = \inf_S f$, there exists a critical point $u$ of $f$ with $f(u) = c$ and $u \in S$. 
Proof. Since $D$ is compact, it is evident that $c < +\infty$. If $H$ is a contraction of $B$ in $X \setminus S$, consider the projections $P_{\pm} : X \to X_{\pm}$ associated with the direct decomposition. Then

$$K(u, t) = P_{-} H(u, t) + \|P_{+} H(u, t)\|v$$

is a contraction of $B$ in $(X_{-} \oplus \mathbb{R}v) \setminus \{rv\}$. Since this contradicts Lemma (2.7), it follows that $B$ links $A$.

Now, the same argument used in the proof of the Saddle Theorem shows that $c = \inf_{H \in C} \sup_{B \times [0,1]} f \circ H$. From Theorem (2.3) the assertion follows.

3 Nemytskij operator

Throughout this section, $E$ will denote a measurable subset of $\mathbb{R}^n$ and $\| \cdot \|_p$ the usual norm of $L^p$ $(1 \leq p \leq \infty)$.

(3.1) Definition We say that $g : E \times \mathbb{R}^N \to \mathbb{R}^k$ is a Carathéodory function, if

(a) for every $s \in \mathbb{R}^N$ the function $\{x \mapsto g(x, s)\}$ is measurable on $E$;

(b) for a.e. $x \in E$ the function $\{s \mapsto g(x, s)\}$ is continuous on $\mathbb{R}^N$.

If $u : E \to \mathbb{R}^N$ is a function, we denote by $g(x, u)$ the function

$$E \quad \mapsto \quad \mathbb{R}^k$$

$$x \quad \mapsto \quad g(x, u(x))$$

(3.2) Theorem Let $g : E \times \mathbb{R}^N \to \mathbb{R}^k$ be a Carathéodory function.

Then for every measurable function $u : E \to \mathbb{R}^N$ we have that $g(x, u) : E \to \mathbb{R}^k$ is measurable. Moreover, if $u, v$ agree a.e. in $E$, then also $g(x, u)$ and $g(x, v)$ agree a.e. in $E$.

Proof. Let $u : E \to \mathbb{R}^N$ be a simple function, namely a measurable function with a finite number of values. If $u(E) = \{s_1, \ldots, s_m\}$, set $E_h = u^{-1}(s_h)$. Then $\{E_1, \ldots, E_m\}$ is a measurable partition of $E$ and we have

$$\forall x \in E : g(x, u(x)) = \sum_{h=1}^m \chi_{E_h}(x) g(x, s_h).$$

Therefore $g(x, u)$ is measurable.

Let now $u : E \to \mathbb{R}^N$ be a measurable function. It is well known that there exists a sequence $(u_h)$ of simple functions pointwise convergent to $u$. Then we have

$$\lim_h g(x, u_h) = g(x, u) \quad \text{a.e. in } E,$$

whence the measurability of $g(x, u)$.

It is evident that, if $u, v$ agree a.e. in $E$, then also $g(x, u)$ and $g(x, v)$ agree a.e. in $E$.■
(3.3) Theorem Let \( g : E \times \mathbb{R}^N \rightarrow \mathbb{R}^k \) be a Carathéodory function and let \( p, q \in [1, \infty] \). Assume there exist \( a \in L^q(E) \) and \( b \in \mathbb{R} \) such that
\[
|g(x, s)| \leq a(x) + b|s|^\frac{q}{p}
\]
for a.e. \( x \in E \) and every \( s \in \mathbb{R}^N \).

Then for every \( u \in L^p(E; \mathbb{R}^N) \) we have \( g(x, u) \in L^q(E; \mathbb{R}^k) \) and the map
\[
L^p(E; \mathbb{R}^N) \rightarrow L^q(E; \mathbb{R}^k)
\]
\( u \mapsto g(x, u) \)
is continuous.

Proof. For any \( u \in L^p(E; \mathbb{R}^N) \) we have
\[
|g(x, u)|^q \leq \left( a(x) + b|u|^\frac{q}{p} \right)^q \leq 2^{q-1} \left( a(x)^q + b^q|u|^p \right).
\]
Combining this fact with Theorem (3.2), we deduce that \( g(x, u) \in L^q(E; \mathbb{R}^k) \).

Now, let \( (u_h) \) be a sequence convergent to some \( u \in L^p(E; \mathbb{R}^N) \). Up to a subsequence, \( (u_h) \) is convergent a.e. to \( u \) and there exists \( w \in L^p(E) \) such that
\[
|u_h| \leq w \quad \text{a.e. in } E
\]
(see e.g. [3, Theorem IV.9]). Therefore we have
\[
\lim_h g(x, u_h) = g(x, u) \quad \text{a.e. in } E,
\]
\[
|g(x, u_h) - g(x, u)|^q \leq 2^{q-1} \left( |g(x, u_h)|^q + |g(x, u)|^q \right) \leq 4^{q-1} \left( 2a(x)^q + b^q|u_h|^p + b^q|u|^p \right) \leq 4^{q-1} \left( 2a(x)^q + b^q w^p + b^q|u|^p \right) \quad \text{a.e. in } E.
\]
From Lebesgue’s Theorem we deduce that \( (g(x, u_h)) \) is convergent to \( g(x, u) \) in \( L^q(E; \mathbb{R}^k) \).

(3.4) Theorem Let \( g : E \times \mathbb{R}^N \rightarrow \mathbb{R}^k \) be a Carathéodory function and let \( q \in [1, \infty] \). Assume that for every \( M > 0 \) there exists \( a_M \in L^q(E) \) such that
\[
|g(x, s)| \leq a_M(x)
\]
for a.e. \( x \in E \) and every \( s \in \mathbb{R}^N \) with \( |s| \leq M \).

Then for every \( u \in L^\infty(E; \mathbb{R}^N) \) we have \( g(x, u) \in L^q(E; \mathbb{R}^k) \) and the map
\[
L^\infty(E; \mathbb{R}^N) \rightarrow L^q(E; \mathbb{R}^k)
\]
\( u \mapsto g(x, u) \)
is continuous.

Proof. If \( u \in L^\infty(E; \mathbb{R}^N) \) and \( M = \|u\|_\infty \), we have \( |g(x, u)| \leq a_M(x) \) a.e. Taking into account Theorem (3.2), we easily deduce that \( g(x, u) \in L^q(E; \mathbb{R}^k) \).
If \((u_h)\) is a sequence convergent to some \(u\) in \(L^\infty(E; \mathbb{R}^N)\), there exists \(M > 0\) such that \(\|u_h\|_\infty \leq M\) for every \(h\). It follows that \((g(x, u_h))\) is convergent to \(g(x, u)\) a.e. and

\[
|g(x, u_h) - g(x, u)|^q \leq 2^q a_M(x)^q.
\]

From Lebesgue’s Theorem we deduce that \((g(x, u_h))\) is convergent to \(g(x, u)\) in \(L^q(E; \mathbb{R}^k)\).

**3.5 Definition** The map

\[
L^p(E; \mathbb{R}^N) \longrightarrow L^q(E; \mathbb{R}^k) \quad u \mapsto g(x, u)
\]

is called Nemytskij operator or superposition operator associated with \(g\).

**3.6 Definition** We say that \(G : E \times \mathbb{R}^N \to \mathbb{R}^k\) is a \(C^1\)-Carathéodory function, if

(a) for every \(s \in \mathbb{R}^N\) the function \(x \mapsto G(x, s)\) is measurable on \(E\);

(b) for a.e. \(x \in E\) the function \(s \mapsto G(x, s)\) is of class \(C^1\) on \(\mathbb{R}^N\).

**3.7 Proposition** Let \(g : E \times \mathbb{R} \to \mathbb{R}^k\) be a Carathéodory function and set \(G(x, s) = \int_0^s g(x, t) dt\).

Then \(G : E \times \mathbb{R} \to \mathbb{R}^k\) is a \(C^1\)-Carathéodory function with \(G(x, 0) = 0\).

**Proof.** It is evident that \(\{s \mapsto G(x, s)\}\) is of class \(C^1\) for a.e. \(x \in E\). Moreover, for every \(s \in \mathbb{R}\) we have

\[
G(x, s) = \lim_{k \to \infty} \left( \sum_{h=1}^{k} \frac{s}{k} g \left( x, \frac{s}{k} h \right) \right) \quad \text{a.e. in } E.
\]

Therefore \(\{s \mapsto G(x, s)\}\) is measurable for every \(s \in \mathbb{R}\).

**3.8 Theorem** Let \(G : E \times \mathbb{R}^N \to \mathbb{R}^k\) be a \(C^1\)-Carathéodory function, let \(1 \leq q < p < \infty\), let \(r > 1\) be such that

\[
\frac{1}{r} + \frac{1}{p} = \frac{1}{q}
\]

and set \(g(x, s) = D_s G(x, s)\). Assume that \(G(x, 0) \in L^q(E)\) and that there exist \(a \in L^r(E)\) and \(b \in \mathbb{R}\) such that

\[
|g(x, s)| \leq a(x) + b|s|^{\frac{r-1}{r}}
\]

for a.e. \(x \in E\) and every \(s \in \mathbb{R}^N\).

Then \(g : E \times \mathbb{R}^N \to \mathbb{R}^{nk}\) is a Carathéodory function, we have \(G(x, u) \in L^q(E; \mathbb{R}^k)\) for every \(u \in L^p(E; \mathbb{R}^N)\) and the Nemytskij operator

\[
G : L^p(E; \mathbb{R}^N) \longrightarrow L^q(E; \mathbb{R}^k) \quad u \mapsto G(x, u)
\]

associated with \(G\) is of class \(C^1\). Moreover we have

\[
\forall u, v \in L^p(E; \mathbb{R}^N) : G'(u)v = g(x, u)v.
\]
3. NEMYTSKIJ OPERATOR

Proof. It is evident that \( \{ s \mapsto g(x, s) \} \) is continuous for a.e. \( x \in E \). Moreover, for every \( s, \sigma \in \mathbb{R}^N \) we have
\[
g(x, s)\sigma = \lim_k k \left( G \left( x, s + \frac{1}{k} \sigma \right) - G(x, s) \right) \quad \text{a.e. in } E.
\]
Therefore \( \{ x \mapsto g(x, s) \} \) is measurable for every \( s \in \mathbb{R} \).

Since \( \frac{p}{p-q} = \frac{q}{q} \), from (3.9) and Young’s inequality we deduce that
\[
\| G(x, s) \| \leq |G(x, 0)| + a(x) |s| + \frac{bq}{p} |s|^\frac{q}{p} \leq |G(x, 0)| + \frac{p-q}{p} a(x) \frac{r}{p} + \frac{q}{p} |s|^\frac{q}{p} + \frac{bq}{p} |s|^\frac{q}{p} =
\]
\[
= |G(x, 0)| + \frac{p-q}{p} a(x) \frac{r}{p} + (1 + b) \frac{q}{p} |s|^\frac{q}{p}.
\]

Since \( a^\frac{r}{p} \in L^q(E) \), from Theorem (3.3) it follows that the Nemytskij operator
\[
G : \quad L^p(E; \mathbb{R}^N) \quad \longrightarrow \quad L^q(E; \mathbb{R}^k)
\]
is well defined and continuous. Let \( v \in L^p(E; \mathbb{R}^N) \) and define
\[
L^p(E; \mathbb{R}^N) \quad \longrightarrow \quad L^q(E; \mathbb{R}^k)
\]
is well defined and continuous.

Now let \( u \in L^p(E; \mathbb{R}^N) \). By Hölder’s inequality it is readily seen that the map
\[
L^p(E; \mathbb{R}^N) \quad \longrightarrow \quad L^q(E; \mathbb{R}^k)
\]
is well defined, linear and continuous. Let \( (v_h) \) be a sequence convergent to \( 0 \) in \( L^p(E; \mathbb{R}^N) \). Up to a subsequence, \( (v_h) \) is convergent to \( 0 \) also a.e. and there exists \( w \in L^p(E) \) such that \( |v_h| \leq w \) a.e. Set \( z_h = \frac{v_h}{\| v_h \|_p} \) and define
\[
\alpha_h = \begin{cases} 
\frac{G(x, u + v_h) - G(x, u) - g(x, u)v_h}{|v_h|} & \text{where } v_h(x) \neq 0, \\
0 & \text{where } v_h(x) = 0.
\end{cases}
\]

Then \( (\alpha_h) \) is convergent to \( 0 \) a.e. and, by Lagrange’s Inequality, we have
\[
|\alpha_h|^r \leq |g(x, u + \vartheta_h v_h) - g(x, u)|^r \leq (a + |u| + \vartheta_h |v_h|^\frac{r}{q} + a + b|u|^\frac{r}{q})^r \leq (a + b|u| + |w|)|^\frac{r}{q} + a + b|u|^\frac{r}{q}
\]
where \( 0 < \vartheta_h < 1 \). Therefore \( (\alpha_h) \) is convergent to \( 0 \) also in \( L^r(E; \mathbb{R}^k) \). From Hölder’s inequality it follows
\[
\int_E \left| \frac{G(x, u + v_h) - G(x, u) - g(x, u)v_h}{\| v_h \|_p} \right|^q dx = \int_E |\alpha_h|^q |z_h|^q dx \leq \|\alpha_h\|_r^q \|z_h\|_q^q = \|\alpha_h\|_r^q,
\]
hence
\[
\lim_h \int_E \left| \frac{G(x, u + v_h) - G(x, u) - g(x, u)v_h}{\| v_h \|_p} \right|^q dx = 0.
\]
Therefore \( G \) is Fréchet differentiable at \( u \) and \( G'(u)v = g(x, u)v \).
Finally, for every $u_1, u_2, v \in L^p(E; \mathbb{R}^N)$ we have
\[
\|G'(u_1)v - G'(u_2)v\|_q = \|g(x, u_1)v - g(x, u_2)v\|_q \leq \|g(x, u_1) - g(x, u_2)\|_r \|v\|_p ,
\]
hence
\[
\|G'(u_1) - G'(u_2)\|_{\mathcal{L}(L^p; L^q)} \leq \|g(x, u_1) - g(x, u_2)\|_r .
\]
Therefore $G$ is of class $C^1$. □

(3.10) Theorem Let $G : E \times \mathbb{R}^N \to \mathbb{R}^k$ be a $C^1$–Carathéodory function, let $1 \leq q < \infty$ and set $g(x, s) = D_s G(x, s)$. Assume that $G(x, 0) \in L^q(E; \mathbb{R}^k)$ and that for every $M > 0$ there exists $a_M \in L^q(E)$ such that
\[
|g(x, s)| \leq a_M(x)
\]
for a.e. $x \in E$ and every $s \in \mathbb{R}^N$ with $|s| \leq M$.

Then $g : E \times \mathbb{R}^N \to \mathbb{R}^{Nk}$ is a Carathéodory function, we have $G(x, u) \in L^q(E; \mathbb{R}^k)$ for every $u \in L^\infty(E; \mathbb{R}^N)$ and the Nemytskij operator
\[
\mathcal{G} : L^\infty(E; \mathbb{R}^N) \to L^q(E; \mathbb{R}^k)
\]
associated with $G$ is of class $C^1$. Moreover we have
\[
\forall u, v \in L^\infty(E; \mathbb{R}^N) : \mathcal{G}'(u)v = g(x, u)v .
\]

Proof. As before, we have that $g$ is a Carathéodory function. Moreover,
\[
|G(x, s)| \leq |G(x, 0)| + a_M(x)|s| \leq |G(x, 0)| + Ma_M(x)
\]
for a.e. $x \in E$ and every $s \in \mathbb{R}^N$ with $|s| \leq M$. From Theorem (3.4) it follows that the Nemytskij operators
\[
\mathcal{G} : L^\infty(E; \mathbb{R}^N) \to L^q(E; \mathbb{R}^k)
\]
and
\[
L^\infty(E; \mathbb{R}^N) \to L^q(E; \mathbb{R}^{Nk})
\]
are well defined and continuous. Then it is possible to argue, with minor variants, as in the proof of Theorem (3.8). □

Let now $\Omega$ be an open subset of $\mathbb{R}^n$.

(3.11) Theorem The following facts hold:

(a) if $1 \leq p < n$, then we have $W_0^{1,p}(\Omega; \mathbb{R}^N) \subseteq L^{\frac{np}{n-p}}(\Omega; \mathbb{R}^N)$ and there exists $c(n, p) > 0$ such that
\[
\forall u \in W_0^{1,p}(\Omega; \mathbb{R}^N) : \|u\|_{L^{\frac{np}{n-p}}} \leq c(n, p) \|\nabla u\|_p ;
\]
(b) if $n < p < \infty$, then we have $W^{1,p}_0(\Omega; \mathbb{R}^N) \subseteq L^\infty(\Omega; \mathbb{R}^N)$ and there exists $c(n,p) > 0$ such that

$$\forall u \in W^{1,p}_0(\Omega; \mathbb{R}^N) : \|u\|_\infty \leq c(n,p) \left(\|\nabla u\|_p + \|u\|_p\right)^\frac{1}{p}.$$ 

(c) if $a, b \in \mathbb{R}$ and $1 \leq p \leq \infty$, then we have $W^{1,p}(a,b; \mathbb{R}^N) \subseteq L^\infty(a,b; \mathbb{R}^N)$ and there exists $c(a,b) > 0$ such that

$$\forall u \in W^{1,p}(a,b; \mathbb{R}^N) : \|u\|_\infty \leq c(a,b) \left(\|u\|_p + \|u\|_p\right)^\frac{1}{p}.$$ 

**Proof.** See for instance [3, Theorems IX.9, IX.12 and VIII.7].

(3.12) **Theorem** Let $1 \leq p \leq \infty$. Then every bounded sequence $(u_k)$ in $W^{1,p}(\Omega; \mathbb{R}^N)$ admits a subsequence convergent a.e. to some $u \in L^p(\Omega; \mathbb{R}^N)$.

**Proof.** If $\Omega$ is an open ball, the Rellich-Kondrachov Theorem (see e.g. [3, Theorem IX.16]) implies that there exists a subsequence $(u_{k})$ strongly convergent in $L^p(\Omega; \mathbb{R}^N)$ to some $u$. Then a further subsequence is convergent to $u$ a.e.

Since any open subset of $\mathbb{R}^n$ is a countable union of open balls, also in the general case we may find a subsequence convergent a.e. to some $u$. From Fatou's Lemma it is easy to deduce that $u \in L^p(\Omega; \mathbb{R}^N)$.

(3.13) **Theorem** Let $1 \leq p < n$, let $X$ be a subspace of $W^{1,p}(\Omega; \mathbb{R}^N)$ continuously imbedded in $L^{\frac{np}{n-p}}(\Omega; \mathbb{R}^N)$, let $G : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a $C^1$–Carathéodory function and set $g(x,s) = \nabla_s G(x,s)$. Assume that $G(x,0) \in L^1(\Omega)$ and that there exist $a \in L^{\frac{np}{n-p}}(\Omega; \mathbb{R}^N)$ and $b \in \mathbb{R}$ such that

$$|g(x,s)| \leq a(x) + b|s|^{\frac{np}{n-p} - 1}$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}^N$.

Then for every $u \in X$ we have $G(x,u) \in L^1(\Omega)$ and the functional

$$f : \begin{array}{c} X \to \mathbb{R} \\ u \mapsto \int_\Omega G(x,u) \, dx \end{array}$$

is of class $C^1$. Moreover we have

$$\forall u, v \in X : f'(u)v = \int_\Omega g(x,u) \cdot v \, dx.$$ 

**Proof.** Since

$$\frac{n(p-1)}{np} + \frac{n-p}{np} = 1,$$

it follows from Theorem (3.8) that the Nemytckij operator

$$G : L^{\frac{np}{n-p}}(\Omega; \mathbb{R}^N) \to L^1(\Omega)$$

is of class $C^1$ with $G'(u)v = g(x,u) \cdot v$. 

On the other hand $X$ is continuously included in $L^{\frac{n p}{n p - 1}}(\Omega; \mathbb{R}^N)$ and $\{ w \mapsto \int_\Omega w \, dx \}$ is a continuous and linear functional on $L^1(\Omega)$. Then the assertion easily follows. ■

(3.14) **Definition** Let $1 \leq p < n$. We say that a Carathéodory function $g : \Omega \times \mathbb{R}^N \to \mathbb{R}^k$ has subcritical growth with respect to $W^{1,p}(\Omega; \mathbb{R}^N)$, if for every $\varepsilon > 0$ there exists $a_\varepsilon \in L^{\frac{n p}{n p - 1}}(\Omega)$ such that

$$|g(x, s)| \leq a_\varepsilon(x) + \varepsilon|s|^{\frac{np}{np - 1}}$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}^N$.

(3.15) **Remark** If $\Omega$ has finite measure and

$$|g(x, s)| \leq a(x) + b|s|^q$$

with $a \in L^{\frac{n p}{n p - 1}}(\Omega)$, $b \in \mathbb{R}$ and $0 < q < \frac{np}{n p - 1} - 1$, then $g$ has subcritical growth with respect to $W^{1,p}(\Omega; \mathbb{R}^N)$.

**Proof.** Let $rq = \frac{np}{n p} - 1$. From Young’s inequality we deduce that

$$|g(x, s)| \leq a(x) + \frac{1}{p'} \left(\frac{b}{2}\right)^{p'} + \frac{1}{p} \delta^{r'}|s|^{\frac{np}{np - 1}}$$

for every $\delta > 0$. Since the constant $\frac{1}{p'} \left(\frac{b}{2}\right)^{p'}$ belongs to $L^{\frac{np}{np - 1}}(\Omega)$ and $\delta^{r'}$ can be made arbitrarily small, the assertion follows. ■

(3.16) **Theorem** Let $1 \leq p < n$, let $X$ be a subspace of $W^{1,p}(\Omega; \mathbb{R}^N)$ continuously imbedded in $L^{\frac{np}{np - 1}}(\Omega; \mathbb{R}^N)$, let $G : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a $C^1$–Carathéodory function and set $g(x, s) = \nabla_s G(x, s)$. Assume that $G(x, 0) \in L^1(\Omega)$ and that $g$ has subcritical growth with respect to $W^{1,p}(\Omega; \mathbb{R}^N)$.

Then the functional

$$f : \begin{array}{c} X \rightarrow \mathbb{R} \\ u \mapsto \int_\Omega G(x, u) \, dx \end{array}$$

is of class $C^1$ with

$$\forall u, v \in X : f'(u) v = \int_\Omega g(x, u) \cdot v \, dx$$

and the derivative $f' : X \to X^*$ is completely continuous.

**Proof.** Since $g$ has subcritical growth, it follows from Theorem (3.13) that $f$ is well defined and of class $C^1$ with

$$\forall u, v \in X : f'(u) v = \int_\Omega g(x, u) \cdot v \, dx.$$ 

For every $u_1, u_2, v \in X$, Hölder’s inequality yields,

$$|f'(u_1)v - f'(u_2)v| = \left| \int_\Omega (g(x, u_1) - g(x, u_2)) \cdot v \, dx \right| \leq$$

$$\leq \|g(x, u_1) - g(x, u_2)\|_{\frac{np}{n p - 1}p} \|v\|_{\frac{np}{n p}} \leq$$

$$\leq c \|g(x, u_1) - g(x, u_2)\|_{\frac{np}{n p - 1}p} \left(\|\nabla v\|_p^p + \|v\|_p^p\right)^{\frac{1}{p}}.$$
Therefore it is sufficient to prove assertion (b) of Definition (1.12) for the map

\[ X \to L^{\frac{np}{n(p-1)+p}}(\Omega; \mathbb{R}^N) \]

Without loss of generality, we can suppose \( N = 1 \). Let us treat first of all the case in which

\[ |g(x,s)| \leq a(x) \]

with \( a \in L^{\frac{np}{n(p-1)+p}}(\Omega) \). Let \((u_h)\) be a bounded sequence in \( X \). By Theorem (3.12), up to a subsequence \((u_h)\)
is convergent a.e. to some \( u \in L^p(\Omega) \). Since

\[ |g(x,u_h) - g(x,u)| \leq 2a(x), \]

from Lebesgue’s Theorem we deduce that \((g(x,u_h))\) is strongly convergent to \( g(x,u) \) in \( L^{\frac{np}{n(p-1)+p}}(\Omega) \).

In the general case, set for any \( \varepsilon > 0 \)

\[ g_{\varepsilon}(x,s) = \min \{ \max \{ g(x,s), -a_{\varepsilon}(x) \}, a_{\varepsilon}(x) \} . \]

Since \( |g_{\varepsilon}(x,s)| \leq a_{\varepsilon}(x) \), the map

\[ X \to L^{\frac{np}{n(p-1)+p}}(\Omega) \]

\[ u \mapsto g_{\varepsilon}(x,u) \]

satisfies condition (b) of Definition (1.12) by the previous step. On the other hand, we have

\[ |g_{\varepsilon}(x,s) - g(x,s)| \leq \varepsilon |s|^{\frac{np}{n(p-1)+p} - 1}, \]

hence, for every \( u \in X \),

\[ \|g_{\varepsilon}(x,u) - g(x,u)\|^{\frac{np}{n(p-1)+p}} \leq \varepsilon \|u\|^{\frac{np}{n(p-1)+p} - 1} \leq \varepsilon c^{\frac{np}{n(p-1)+p} - 1} (\|\nabla u\|_p^p + \|u\|_p^p)^{\frac{np}{n(p-1)+p} - \frac{1}{p}} . \]

We deduce that

\[ \lim_{\varepsilon \to 0} \|g_{\varepsilon}(x,u) - g(x,u)\|^{\frac{np}{n(p-1)+p}} = 0 \]

uniformly on bounded subsets of \( X \) and the assertion follows from well known properties of completely continuous operators (see e.g. [9, Proposition III.5.4]).

\[ (3.17) \textbf{Theorem} \quad \text{Let } n < p \leq \infty, \text{ let } X \text{ be a subspace of } W^{1,p}(\Omega; \mathbb{R}^N) \text{ continuously imbedded in } L^\infty(\Omega; \mathbb{R}^N), \text{ let } G : \Omega \times \mathbb{R}^N \to \mathbb{R} \text{ be a } C^1 \text{–Carathéodory function and set } g(x,s) = \nabla_s G(x,s). \text{ Assume that } G(x,0) \in L^1(\Omega) \text{ and that for every } M > 0 \text{ there exists } a_M \in L^1(\Omega) \text{ such that } \]

\[ |g(x,s)| \leq a_M(x) \]

for a.e. \( x \in \Omega \) and every \( s \in \mathbb{R}^N \) with \( |s| \leq M \).

Then for every \( u \in X \) we have \( G(x,u) \in L^1(\Omega) \) and the functional

\[ f : X \to \mathbb{R} \quad u \mapsto \int_\Omega G(x,u) \, dx \]
is of class $C^1$. Moreover we have

$$\forall u, v \in X : f'(u)v = \int_{\Omega} g(x,u) \cdot v \, dx$$

and the derivative $f' : X \to X^*$ is completely continuous.

**Proof.** From Theorem (3.10) we deduce that the Nemytskij operator

$$G : L^\infty(\Omega; \mathbb{R}^N) \to L^1(\Omega)$$

is of class $C^1$ with $G'(u)v = g(x,u) \cdot v$.

On the other hand, $X$ is continuously included in $L^\infty(\Omega; \mathbb{R}^N)$ and $\{ w \mapsto \int_{\Omega} w \, dx \}$ is a continuous and linear functional on $L^1(\Omega)$. Then it is easy to show that $f$ is of class $C^1$ with $f'(u)v = \int_{\Omega} g(x,u) \cdot v \, dx$.

For every $u_1, u_2, v \in X$, Hölder’s inequality yields,

$$|f'(u_1)v - f'(u_2)v| = \left| \int_{\Omega} (g(x,u_1) - g(x,u_2)) \cdot v \, dx \right| \leq \|g(x,u_1) - g(x,u_2)\|_1 \|v\|_\infty \leq c \|g(x,u_1) - g(x,u_2)\|_1 (\|\nabla v\|_p + \|v\|_p^p)^{\frac{1}{p}}.$$

Therefore it is sufficient to show that the map

$$X \to L^1(\Omega; \mathbb{R}^N)$$

$$u \mapsto g(x,u)$$

satisfies condition (b) of Definition (1.12).

Let $(u_h)$ be a bounded sequence in $X$ and let $M > 0$ be such that $\|u_h\|_\infty \leq M$. By Theorem (3.12), up to a subsequence $(u_{h_k})$ is convergent a.e. to some $u \in L^p(\Omega; \mathbb{R}^N)$. Since

$$|g(x,u_h) - g(x,u)| \leq 2aM(x),$$

from Lebesgue’s Theorem we deduce that $(g(x,u_h))$ is strongly convergent to $g(x,u)$ in $L^1(\Omega; \mathbb{R}^N)$.

(3.18) **Corollary** Let $\Omega$ be bounded, let $G : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a $C^1 - \text{Carathéodory function}$ with $G(x,0) \in L^1(\Omega)$ and assume that $g := \nabla_s G(x,s)$ has subcritical growth with respect to $W^{1,2}(\Omega; \mathbb{R}^N)$.

Then the functional $f : W_0^{1,2}(\Omega) \to \mathbb{R}$ defined by

$$f(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \int_{\Omega} G(x,u) \, dx$$

is of class $C^1$ with

$$\forall u, v \in W_0^{1,2}(\Omega) : f'(u)v = \int_{\Omega} \nabla u \cdot \nabla v \, dx - \int_{\Omega} g(x,u) \cdot v \, dx$$

and the derivative $f' : W_0^{1,2}(\Omega; \mathbb{R}^N) \to W^{-1,2}(\Omega; \mathbb{R}^N)$ has the form required in Theorem (1.13).

**Proof.** Define $f_1 : W_0^{1,2}(\Omega; \mathbb{R}^N) \to \mathbb{R}$ by

$$f_1(u) = - \int_{\Omega} \tilde{G}(x,u) \, dx,$$
where \( \tilde{G}(x,s) = G(x,s) + \frac{1}{2} |s|^2 \). Taking into account Remark (3.15), it is easy to see that also \( \tilde{g} := \nabla_s \tilde{G} \) has subcritical growth with respect to \( W^{1,2}(\Omega; \mathbb{R}^N) \). From Theorem (3.16) it follows that \( f_1 \) is well defined, of class \( C^1 \) with \( f_1' \) completely continuous.

Since

\[
f(u) = \frac{1}{2} \int_\Omega (|\nabla u|^2 + |u|^2) \, dx - f_1(u),
\]

we have \( f'(u) = Lu - f_1'(u) \), where \( L : W^{1,2}_0(\Omega; \mathbb{R}^N) \to W^{-1,2}(\Omega; \mathbb{R}^N) \) is an isomorphism, and the assertion follows.

(3.19) Corollary Let

\[
X = \{ u \in W^{1,2}([-\pi,\pi]; \mathbb{R}^N) : u(-\pi) = u(\pi) \}
\]

let \( G : [-\pi,\pi] \times \mathbb{R}^N \to \mathbb{R} \) be a \( C^1 \)-Carathéodory function and set \( g(x,s) = \nabla_s G(x,s) \). Assume that \( G(x,0) \in L^1([-\pi,\pi]) \) and that for every \( M > 0 \) there exists \( a_M \in L^1([-\pi,\pi]) \) such that

\[
|g(x,s)| \leq a_M(x)
\]

for a.e. \( x \in [-\pi,\pi] \) and every \( s \in \mathbb{R}^N \) with \( |s| \leq M \).

Then \( X \) is a closed subspace of \( W^{1,2}([-\pi,\pi]; \mathbb{R}^N) \), the functional \( f : X \to \mathbb{R} \) defined by

\[
f(u) = \frac{1}{2} \int_{-\pi}^{\pi} |u'|^2 \, dx - \int_{-\pi}^{\pi} G(x,u) \, dx
\]

is of class \( C^1 \) with

\[
\forall u, v \in X : f'(u)v = \int_{-\pi}^{\pi} u' \cdot v' \, dx - \int_{-\pi}^{\pi} g(x,u) \cdot v \, dx
\]

and the derivative \( f' : X \to X^* \) has the form required in Theorem (1.13).

Proof. It is well known that \( W^{1,2}([-\pi,\pi]; \mathbb{R}^N) \) is continuously imbedded in \( C([-\pi,\pi]; \mathbb{R}^N) \) (see e.g. [3]). Therefore \( X \) is well defined and is in fact a closed linear subspace of \( W^{1,2}([-\pi,\pi]; \mathbb{R}^N) \).

If we define again \( f_1 : X \to \mathbb{R} \) by

\[
f_1(u) = -\int_{-\pi}^{\pi} \tilde{G}(x,u) \, dx,
\]

where \( \tilde{G}(x,s) = G(x,s) + \frac{1}{2} |s|^2 \), we deduce now from Theorem (3.17) that \( (f_1)' : X \to X^* \) is completely continuous. Then the assertion easily follows.
Bibliography


