Lecture 3:
The Traveling Salesman Problem:
Inequalities and Separation

Outline:

1. The ILP formulation of the symmetric TSP
2. Survey of valid inequalities and facets
3. Exact separation based on templates
4. Heuristic separation based on templates
5. Non-template-based separation
6. The asymmetric TSP
7. Conclusions and open problems
1. ILP formulation of the symmetric TSP

Standard formulation due to Dantzig, Fulkerson & Johnson (1954):

Let $V$ be the vertex set and $E$ the edge set.
Define a 0-1 variable $x_e$ for each $e \in E$.
For any $S \subseteq V$, let:

\begin{align*}
E(S) &= \text{edges with both end-vertices in } S. \\
\delta(S) &= \text{edges with one end-vertex in } S.
\end{align*}

For any $F \subseteq E$, $x(F)$ denotes $\sum_{e \in F} x_e$.

If $c_e$ is the cost of edge $e$, we have:

Minimise $\sum_{e \in E} c_e x_e$
Subject to:

\begin{align*}
x(\delta(i)) &= 2 \quad (\forall i \in V) \quad \text{(degree equations)} \\
x(E(S)) &\leq |S| - 1 \quad (\forall S \subseteq V) \quad \text{(SECs)} \\
x &\in \{0, 1\}^{|E|} \quad \text{(binary condition)}
\end{align*}
2. Survey of valid inequalities and facets.

The degree equations define the affine hull. The trivial bounds $0 \leq x_e \leq 1$ induce facets.

The SECs induce facets. Note that there are an exponential number of them.

(An upper bound $x_e \leq 1$ is equivalent to an SEC with $|S| = 2$. )

The polyhedron defined by the degree equations, non-negativity inequalities, and SECs is called the *subtour elimination polytope* and denoted by SEP($n$).

The convex hull of integer solutions is called the *symmetric traveling salesman polytope* and denoted by STSP($n$).

For $3 \leq n \leq 5$, SEP($n$) = STSP($n$).

However, for $n \geq 6$, STSP($n$) is strictly contained in SEP($n$) and more inequalities are needed.
2.1. 2-matching inequalities

Discovered by Edmonds (1965) in the context of matching problems.

For any set $H \subset V$ and any edge set $F \subset \delta(H)$ such that $|F|=p$ is odd, we have:

$$x(E(H)) + x(F) \leq |H| + \left\lfloor p/2 \right\rfloor.$$

Can be written in various other ways, for example:

$$x(\delta(H) \setminus F) \geq x(F) - |F| + 1.$$

The set $H$ is now called the handle and the edges in $F$ are called teeth.
2.2. Comb inequalities

Due to Grötschel and Padberg (1979).

A comb consists of a vertex set $H$ (the handle) and vertex sets $T_1, \ldots, T_p$ (the teeth) such that:

- $p \geq 3$ and odd;
- all teeth are disjoint;
- $H \cap T_j$ and $T_j \setminus H$ are non-empty for all $j$.

The comb inequality is:

$$x(E(H)) + \sum_{j=1}^{p} x(E(T_j)) \leq |H| + \sum_{j=1}^{p} |T_j| - \left\lceil \frac{3p}{2} \right\rceil.$$
A *Chvátal* comb inequality also satisfies:
\[ |H \cap T_j| = 1 \text{ for all } j \] (Chvátal, 1973).

Letchford & Lodi (2002) define *simple* comb inequalities, in which, for all \( j \), either \( |H \cap T_j| = 1 \) or \( |T_j \setminus H| = 1 \).

So from most general to least: comb, simple comb, Chvátal comb, 2-matching.
2.3. Other ‘handle-tooth’ inequalities

Many other known inequalities can be expressed in the form:

\[ \sum_{j=1}^{p} x(E(H_j)) + \sum_{j=1}^{q} x(E(T_j)) \leq \text{RHS}, \]

or, equivalently, in the form

\[ \sum_{j=1}^{p} x(\delta(H_j)) + \sum_{j=1}^{q} x(\delta(T_j)) \geq \text{RHS}. \]

These include:

- Clique-tree inequalities
  (Grötschel & Pulleyblank, 1986)

- Path inequalities
  (Cornuéjols, Fonlupt & Naddef, 1985)

- Star inequalities (Fleischmann, 1988)

- Hyperstar inequalities
  (Fleischmann, unpublished)

- Bipartition inequalities
  (Boyd & Cunningham, 1991)

- Binested inequalities (Naddef, 1992)
2.4. Miscellaneous inequalities

Not all inequalities are of “handle-tooth” type:

Hypohamiltonian (Grötschel, 1980),
Chain (Padberg & Hong, 1980),
Crown (Naddef & Rinaldi, 1992),
Ladder (Boyd et al., 1995)

Indeed, Christof, Jünger & Reinelt (1991, 1995, 1996) have studied STSP($n$) for $n \leq 10$ and the majority of the facets are not of handle-tooth type.

Various other facets have been discovered, e.g., by Naddef & Rinaldi.
3. Exact separation based on templates

Following Applegate, Bixby, Chvátal & Cook, we say that a specified class of facet-inducing inequalities (SECs, combs, etc.) is a template.

To use inequalities from a template in a cutting plane algorithm, we need to solve the following separation problem (Grötschel, Lovász & Schrijver 1988):

Given a vector \( x^* \in \mathbb{R}^{|E|} \) as input, either find an inequality in the template which is violated by \( x^* \), or prove that none exists.

Exact polynomial time separation algorithms are known for:

i) SECs (Crowder & Padberg, 1980)
ii) 2-matching ineq.s (Padberg & Rao, 1982)
iii) clique tree inequalities with a fixed number of handles and teeth (Carr, 1997)
iv) certain inequalities defined by lifting (Carr, 1996, 1997).
3.1 Subtour elimination constraints

If we write the SECs in the form $x(\delta(S)) \geq 2$, we see that the separation problem reduces to a minimum cut problem.

The support graph $G^*$ is the graph with vertex set $V$ and edge set $E^* = \{ e \in E : x^*_e > 0 \}$.

Each $e \in E^*$ is given the weight $x^*_e$.

Then, a violated SEC exists if and only if there is a cut in $G^*$ whose $x^*$-weight is less than 2.

We can therefore use any min-cut algorithm (e.g., Gomory – Hu, Padberg – Rinaldi, Hao – Orlin…).

Current fastest is $O(nm + n^2 \log n)$ (Nagamochi, Ono & Ibaraki, 1994).
3.2 2-matching inequalities

Padberg & Rao (1982) noted that the inequality can be re-written as:

\[ x(\delta(H) \setminus F) + \sum_{e \in F} (1 - x_e) \geq 1, \]

where \( F \subset \delta(H) \) is the set of teeth.

This enabled them to reduce the separation problem to that of finding a minimum weight odd cut in the so-called ‘split graph’.

Their algorithm uses \( O(m) \) max-flows in a graph with \( m + n \) vertices and \( 2m \) edges.

Grötschel & Holland (1987) reduced this to \( O(m) \) max-flows in graphs with \( n + 2 \) vertices and \( m + 2 \) edges.

Letchford, Reinelt & Theis (2003) showed how to reduce this to \( n - 1 \) max-flows in \( G^* \) itself. This leads to \( O(n^2 m \log (n^2/m)) \) time.
3.3 Carr’s separation algorithms

Carr’s original separation algorithm was based on enumeration of possible configurations (“backbones”), plus the solution of a sequence of max-flows.

E.g., for a comb with $p$ teeth, there are $O(n^{2p})$ possible configurations, thus $O(n^{2p})$ max-flow problems need to be solved.

The same idea works for any handle-tooth template, but the order of the polynomial grows rapidly with the number of handles and teeth.

Later, he showed how to separate inequalities from other templates, not necessarily handle-tooth based, by solving an LP for each backbone.

Impractical but theoretically elegant.
4. Heuristic separation based on templates

A *heuristic* separation algorithm (for a given template) outputs either one or more inequalities in the template violated by $x^*$, or a failure message.

**SECs:** connected components (folklore)  
shrinking (Crowder & Padberg, 1980)  
spanning trees (Fischetti et al.)  
segments (ABCC).

2-matching / Chvátal comb: usually based on *blocks* in the *fractional graph* (Grötschel & Holland, 1991; Padberg & Rinaldi, 1990; Naddef & Clochard, 1994; Naddef & Thienel, 2002…)

General comb: tend to be based on shrinking, followed by heuristics for Chvátal comb separation.

Other inequalities: Clochard & Naddef, 1993; Naddef & Thienel, 2002; ABCC…
The small instance approach (Heidelberg):

List all facets of STSP($n$) up to $n = 10$.

Put them into equivalence classes

For each class:

Shrink $G^*$ to a small graph.
Heuristically solve a QAP.

Not competitive at present (explained later...)
But can be easily parallelised.
5. Non-template-based separation

Instead of concentrating on a specific class of inequalities, it seems to be better to look at the way in which the inequalities are derived.

E.g., the comb inequalities can be derived as so-called $\{0, \frac{1}{2}\}$-cuts (Caprara, Fischetti & Letchford, 2000).

CFL (2000): $O(n^2m)$ exact algorithm for detecting maximally violated $\{0, \frac{1}{2}\}$-cuts.

Letchford (2000): $O(n^3)$ exact separation algorithm for a class of $\{0, \frac{1}{2}\}$-cuts containing all combs, when the support graph is planar.

Letchford & Lodi (2002): $O(n^3m^3 \log n)$ exact separation algorithm for a class of $\{0, \frac{1}{2}\}$-cuts containing all simple combs.

Fleischer, Letchford & Lodi (2003) reduce running time to $O(n^2m^2 \log (n^2/m))$. 
Caprara & Letchford (2003) show that in the case of the STSP, the \( \{0, \frac{1}{2}\} \)-cuts can be derived by a special disjunctive technique based on handles.

This led to an exact separation algorithm when the handle is \textit{fixed}.

Finally, Applegate, Bixby, Chvátal & Cook introduced the idea of \textit{local cuts}: as in the Heidelberg approach, the support graph is shrunk to a smaller one.

However, instead of resorting to a long list of templates, they use column generation.

The templates are dealt with “implicitly”.
6. The Asymmetric TSP

Every inequality for the STSP has a counterpart for the ATSP.

So any separation algorithm for the STSP can also be used for the ATSP.

There are also many asymmetric inequalities known:

- $D_k$, C3 (Grötschel & Padberg)
- Odd CAT (Balas)
- Source-Destination (Balas & Fischetti)
- A-path (Chopra & Rinaldi)
- Lifted cycle (Balas & Fischetti)

Fischetti & Toth found an exact separation algorithm for $D_k$ inequalities…

… which was later proven polynomial.

They also gave a good separation heuristic for the Odd CAT inequalities, again based on the idea of $\{0, \frac{1}{2}\}$-cuts.
7. Conclusions and Open Problems

Not just of theoretical interest. On average TSPLIB instance:

- SECs bring within 98% of optimal
- Combs within 99.5%
- Local cuts/DPIs within 99.8%.

Only with these last inequalities is it possible to solve very large STSP instances.

**Main lesson:** seems better to look at *methods for deriving inequalities*, rather than templates.

**Key Open Problems:**

- complexity of comb separation (or a superclass such as DPIs).
- separation for ATSP, especially Odd CAT, SD and lifted cycle inequalities.
- worst-case ratios (4/3 for STSP using SECs? 6/5 using combs?)