

# Interplay of superradiance and disorder in the Anderson model

G. L. Celardo<sup>1,2,\*</sup>, A. Biella<sup>3</sup>, L. Kaplan<sup>4</sup>, and F. Borgonovi<sup>1,2</sup>

<sup>1</sup> Dipartimento di Matematica e Fisica and Interdisciplinary Laboratories for Advanced Materials Physics, Università Cattolica, via Musei 41, 25121 Brescia, Italy

<sup>2</sup> Istituto Nazionale di Fisica Nucleare, Sezione di Pavia, via Bassi 6, 27100 Pavia, Italy

<sup>3</sup> Dipartimento di Matematica e Fisica, Università Cattolica, via Musei 41, 25121 Brescia, Italy

<sup>4</sup> Tulane University, Department of Physics, New Orleans, Louisiana 70118, USA

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Using a non-Hermitian Hamiltonian approach to open systems, we study the interplay of disorder and superradiance in a one-dimensional Anderson model. Analyzing the complex eigenvalues of the non-Hermitian Hamiltonian, a transition to a superradiant regime is shown to occur. As an effect of openness the structure of eigenstates undergoes a strong change in the superradiant regime: we show that the sensitivity to disorder of the superradiant and the subradiant subspaces is very different; superradiant states remain delocalized as disorder increases, while subradiant states are sensitive to the degree of disorder.

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## 1 Introduction

Nanoscale systems in the quantum coherent regime are at the center of many research fields in physics, ranging from quantum computing and cold atoms to transport in nanoscale and mesoscopic systems. Transport in the quantum coherent regime can be considered one of the central subjects in modern solid state physics [1,2] and in cold atom physics [3]. Transport properties depend strongly on the degree of openness of the system.

As a consequence of quantum coherence many interesting features arise. Here we focus on two important effects induced by quantum coherence: Anderson localization [4] and Dicke superradiance [5]. Anderson localization is driven by intrinsic disorder and consists in a suppression of diffusion due to an exponential localization of the eigenfunctions of the system. Dicke superradiance is driven by the fact that the system is open, namely coupled to an external environment characterized by a continuum of states. To explain the superradiance effect, consider a discrete quantum system coupled to an environment having a continuum of states. The system-environment coupling alters the unperturbed energy levels: it causes an energy shift and the appearance of a resonance width (inverse lifetime) for each level. For weak coupling strength, all resonance widths are roughly equal. However, once the coupling strength reaches a critical value, the widths start to overlap, and width segregation occurs. In this regime, almost the entire (summed up) decay width is allocated to just a few short-lived “superradiant” states, while all other states are long-lived (and effectively decoupled from the environment). We call this segregation the “superradiance transition” (ST).

In the superradiant regime, the effect of the opening is large, and cannot be treated perturbatively. Thus, a consistent way to take the effect of the opening into account for arbitrary coupling strength between the system and the outside world is highly desirable. The effective non-Hermitian Hamiltonian approach

\* Corresponding author E-mail: nicedirac@gmail.com

to open quantum systems has been shown to be a very effective tool in addressing this issue [6]. Non-Hermitian Hamiltonians have been already employed to study realistic open quantum systems, such as transport through quantum dots [7], superradiance in cold atoms [3] and nuclear physics [8]. The superradiance effect has also been studied using random matrix theory [9, 10], in microwave billiards [11] and in paradigmatic models of coherent quantum transport [12, 13]. As an example of the importance of the ST, maximum transmission in a realistic model for quantum transport was shown to be achieved exactly at the ST [12].

In this paper we analyze a one-dimensional Anderson model, where a particle hops from site to site in the presence of disorder, and is also allowed to escape the system from any site. When the wavelength of the particle is comparable with the sample size, an effective long-range hopping is created between the sites. This coupling can induce the ST, which affects in a non-trivial way the transport properties of the system. Similar models of quantum transport with coherent dissipation have been already considered in the literature [14], but a detailed analysis of the interplay of localization and superradiance has been lacking. This interplay has been recently analyzed in [12], but there the particle was allowed to escape only from the end sites, while in the situation analyzed in this work, all sites are coupled to the external environment. This situation occurs in many important physical situations, such as in cold atoms, where a single photon is injected in the atomic cloud [3], or in quantum dots [15].

Intrinsic disorder and opening to the environment have opposing effects: while disorder tends to localize the wave functions, the opening tends to delocalize them, since it induces a long range interaction. The aim of this paper is to study the interplay of disorder and opening, and the relation to superradiance. We show that while below the ST, all states are affected by the disorder and the opening in a similar way, above it, the effects are quite different for superradiant and subradiant subspaces, the latter being more affected by disorder than the former.

In Sect. 2 we introduce the model, in Sect. 3 we analyze the ST in our system, and in Sect. 4 we present our main numerical results, which we partly justify in Sect. 5 using perturbation theory. Finally in Sect. 6 we present our conclusions.

## 2 Model

Our starting point is the standard one-dimensional Anderson model [2, 4], for the motion of a particle in a disordered potential. The Hamiltonian of the Anderson model can be written as:

$$H_0 = \sum_{j=1}^N E_j |j\rangle\langle j| + \Omega \sum_{j=1}^{N-1} (|j\rangle\langle j+1| + |j+1\rangle\langle j|), \quad (1)$$

where  $E_j$  are random variables uniformly distributed in  $[-W/2, +W/2]$ ,  $W$  is a disorder parameter, and  $\Omega$  is the tunneling transition amplitude (in our numerical simulations we set  $\Omega = 1$ ). For  $W = 0$  the eigenstates are extended and we have for the eigenvalues:

$$E_q = -2\Omega \cos\left(\frac{\pi q}{N+1}\right), \quad (2)$$

and the eigenstates:

$$\psi_q(j) = \sqrt{\frac{2}{N+1}} \sin\left(\frac{\pi q}{N+1} j\right), \quad (3)$$

where  $q = 1, \dots, N$  is a quantum number and  $j = 1, \dots, N$  is a discrete coordinate. In this case, the eigenvalues lie in the interval  $[-2\Omega, 2\Omega]$ , so the mean level spacing can be estimated as  $D = 4\Omega/N$ . For  $W \neq 0$ , the eigenstates of the one-dimensional Anderson model are exponentially localized on the system

sites, with exponential tails given by  $|\psi(j)| \sim \exp(-|j - j_0|/\xi)$ , where for weak disorder, the localization length  $\xi$  can be written as:

$$\xi \approx 96 (1 - (E/2\Omega)^2) \left( \frac{\Omega}{W} \right)^2. \quad (4)$$

For  $E = 0$ , Eq. (4) has to be modified and we have [16]:

$$\xi \approx 105.2 \left( \frac{\Omega}{W} \right)^2.$$

The phenomenon of Anderson localization was studied in a closed disordered chain, while in our case we can vary the degree of openness of the system. In particular we consider the model in which all the sites are coupled to a common channel in the continuum, with equal coupling strength  $\gamma$ . This situation can arise when the wavelength of the decaying particle is much larger than the size of the system. This results in a coherent dissipation, which differs from the usual dissipation where every site decays independently to a different channel in the continuum. A comparison between these two different mechanisms will be the subject of a future work. The continuum coupling can be taken into account with the aid of an effective non-Hermitian Hamiltonian [12], which in general can be written as,

$$H_{\text{eff}}(E) = H_0 + \Delta(E) - iQ(E),$$

where  $H_0$  is the Hermitian Hamiltonian of the closed system decoupled from the environment and  $\Delta(E)$  and  $Q(E)$  are the induced energy shift and the dissipation, respectively. Neglecting the energy dependence and the energy shift we have

$$(H_{\text{eff}})_{ij} = (H_0)_{ij} - \frac{i}{2} \sum_c A_i^c (A_j^c)^*, \quad (5)$$

where  $A_i^c$  are the transition amplitudes from the discrete states  $i$  to the continuum channels  $c$ .

In the case under study, we have only one decay channel,  $c = 1$ , and all couplings are equal, so that  $A_i^1 = \sqrt{\gamma}$ . Thus the effective Hamiltonian can be written as:

$$H_{\text{eff}} = H_0 - i\frac{\gamma}{2}Q, \quad (6)$$

where  $H_0$  is the Anderson Hamiltonian with diagonal disorder, Eq. (1), and  $Q_{ij} = 1 \forall i, j$ .

In order to study the interplay of Anderson localization and superradiance we analyze the participation ratio ( $PR$ ) of the eigenstates of  $H_{\text{eff}}$ , defined as,

$$PR = \left\langle \frac{1}{\sum_i |\langle i|\psi\rangle|^4} \right\rangle, \quad (7)$$

where the average is over disorder. For completely extended states we have  $PR = N$  and for completely localized states we have  $PR = 1$ .

### 3 Superradiance transition

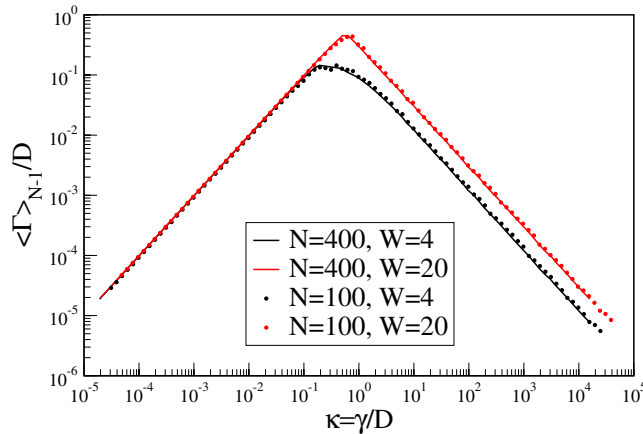
ST can be analyzed by studying the complex eigenvalues  $\mathcal{E}_r = E_r - i\Gamma_r/2$  of  $H_{\text{eff}}$  defined in Eq. (6). As the coupling between the states and the continuum increases, one observes a rearrangement of the widths  $\Gamma_r$ . ST is expected to occur for  $\langle \Gamma \rangle / D \simeq 1$  [6, 12]. The average width  $\langle \Gamma \rangle$  is  $\gamma$ , so we can define

$$\kappa = \gamma / D \quad (8)$$

as the parameter controlling the coupling strength to the continuum. In the deep localized regime where disorder is strong ( $W \gg \Omega$ ) we can write  $D \approx W/N$ , so that the effective coupling strength can be written as:

$$\kappa = \frac{\gamma N}{W}. \quad (9)$$

In Fig. 1 we show that the ST occurs at  $\kappa \sim 1$  for different values of  $W/\Omega$  and  $N$ .



**Fig. 1** (online colour at: [www.fp-journal.org](http://www.fp-journal.org)) The average width of the  $N - 1$  subradiant states, normalized by the mean level spacing  $D$ , versus the effective coupling strength  $\kappa$  for different values of  $N$  and  $W$ , and  $\Omega = 1$ . Here we average over 100 disordered configurations.

For  $\kappa \gg 1$ , we can treat the matrix  $Q$  as the leading term in Eq. (6), and  $H_0$  as a perturbation. The superradiant state  $|SR\rangle$  is given to zeroth order by the only eigenstate of  $Q$  with nonzero eigenvalue:  $|d\rangle = \frac{1}{\sqrt{N}}(1, \dots, 1)^T$ , and the energy of  $|SR\rangle$  is evaluated at first order as

$$\langle d|H_{\text{eff}}|d\rangle = \epsilon - i\frac{\gamma}{2}N, \quad (10)$$

where

$$\epsilon = \frac{1}{N} \sum_{i=1}^N E_i + 2\Omega \frac{N-1}{N}$$

and  $E_i$  are the random diagonal elements of  $H_0$ . Averaging over disorder and taking into account that  $E_i$  are distributed uniformly in  $[-W/2, W/2]$  we obtain,

$$\langle \epsilon \rangle = 2\Omega \frac{N-1}{N} \quad (11)$$

and

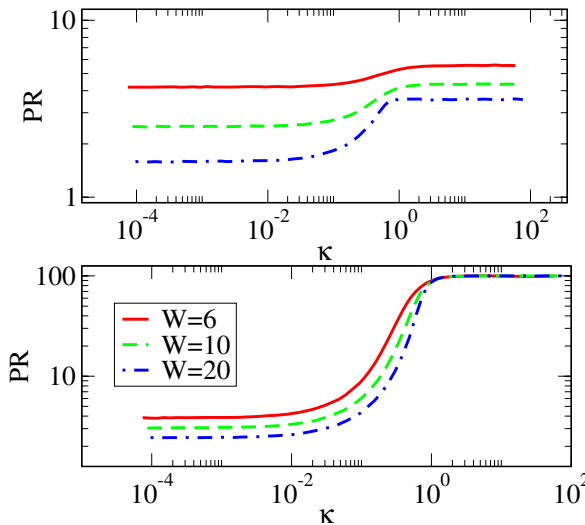
$$\text{Var}(\epsilon) = \langle \epsilon^2 \rangle - \langle \epsilon \rangle^2 = \frac{W^2}{12N}. \quad (12)$$

These results agree with our numerical simulations for different values of  $N$  and allow one to know the position in the energy band of the superradiant state in the limit  $\kappa \gg 1$ . From Eq. (11) we deduce that the mean energy  $\langle \epsilon \rangle$  of the superradiant state is independent of  $W$ .

## 4 Numerical results

In order to study the interplay of superradiance and disorder we have analyzed the  $PR$  of the eigenstates of the non-Hermitian Hamiltonian, Eq. (6). As explained in the previous section, as the coupling with the

continuum is increased we have the formation of one superradiant state (the one with the largest width) and  $N - 1$  subradiant ones. In Fig. 2 (upper panel) we analyze the  $PR$  as a function of  $\kappa$  for the states that become subradiant for  $\kappa > 1$ , and in Fig. 2 (lower panel) we analyze the case of the state with the largest width, which becomes superradiant for  $\kappa > 1$ . As the opening, determined by the parameter  $\kappa$ , increases, the  $PR$  of both superradiant and subradiant states increases, showing that the opening has a delocalizing effect. But the consequences of the opening are very different for superradiant and subradiant states. For the latter, the  $PR$  reaches a plateau value above the ST ( $\kappa \approx 1$ ), which is slightly higher than the  $PR$  for  $\kappa \ll 1$ . Moreover on increasing the disorder, the  $PR$  of the subradiant states decreases, both below and above the ST, see Fig. 2 upper panel. The situation is different for the superradiant states. Above the ST these states become completely delocalized ( $PR = N$ ) and their delocalization is not affected by an increase in  $W$ , see Fig. 2 lower panel.



**Fig. 2** (online colour at: [www.fp-journal.org](http://www.fp-journal.org)) The participation ratio  $PR$  is shown as a function of  $\kappa$  for different disorder strengths. In the upper panel we consider states with  $-1.5 \leq E/\Omega \leq -0.5$ , which become subradiant for large  $\kappa$ , while in the lower panel we consider the state with the largest width, which corresponds to the superradiant state for large  $\kappa$ . Here  $N = 100$ ,  $\Omega = 1$ , and the  $PR$  is averaged over 4000 disorder realizations.

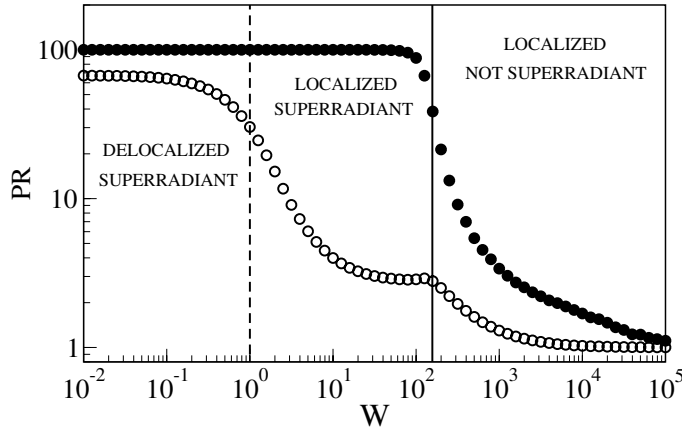
We now look more closely at how the subradiant and superradiant states are affected differently by increasing the disorder strength  $W$ . In Fig. 3, we consider the case of  $N = 100$  and  $\gamma = \Omega = 1$ . For small disorder we have  $D \approx 4\Omega/N$ , so that

$$\kappa = \gamma/D = \gamma N/4\Omega \approx 25 \gg 1.$$

This implies that we are in the superradiant regime. Moreover for sufficiently small disorder we have that the localization length is larger than the system size,  $\xi \approx 100 \Omega^2/W^2 > N$ , so that both superradiant and subradiant states are delocalized. For larger disorder, here  $W > 1$ , we enter the localized regime, for which  $\xi < N$ . In this regime the  $PR$  of the subradiant states starts to decrease, while the  $PR$  of the superradiant state remains unchanged ( $PR = N$ ), signaling a superradiant state that remains completely delocalized. As we increase disorder further,  $\kappa$  decreases according to Eq. (9). The ST occurs at  $W \approx \gamma N$ , here  $W \approx 100$ , and above this value the superradiance effect disappears. Summarizing, we have a critical value of disorder ( $W \approx 100$  indicated as a full vertical line in Fig. 3) separating the superradiant regime ( $\kappa > 1$ ), from the non-superradiant one ( $\kappa < 1$ ). Only for  $W > 100$ , i.e., below the ST, do the superradiant states begin to localize, and, for very large disorder, corresponding to very small  $\kappa$ , they behave the same as the subradiant states.

## 5 Discussion

In this section we will justify (using perturbation theory) and briefly discuss the interesting results presented previously: for small  $\kappa$  (below the ST) all the states are affected in a similar way by the opening



**Fig. 3** The participation ratio is shown as a function of the disorder strength  $W$ . Open circles stand for the subradiant states, while full circles indicate the superradiant state. Each point is obtained by averaging over 100 disorder realizations for the superradiant state, while for the subradiant states, an additional average over all the subradiant states is performed. The right and left vertical lines indicate the ST and the delocalization transition, respectively. Here  $N = 100$  and  $\gamma = \Omega = 1$ .

and disorder, while for large  $\kappa$  (above the ST), the superradiant states remain completely delocalized, independent of the degree of disorder, while the subradiant states are still sensitive to disorder, and their  $PR$  decreases with increasing disorder.

### 5.1 Perturbative approach for $\kappa \ll 1$

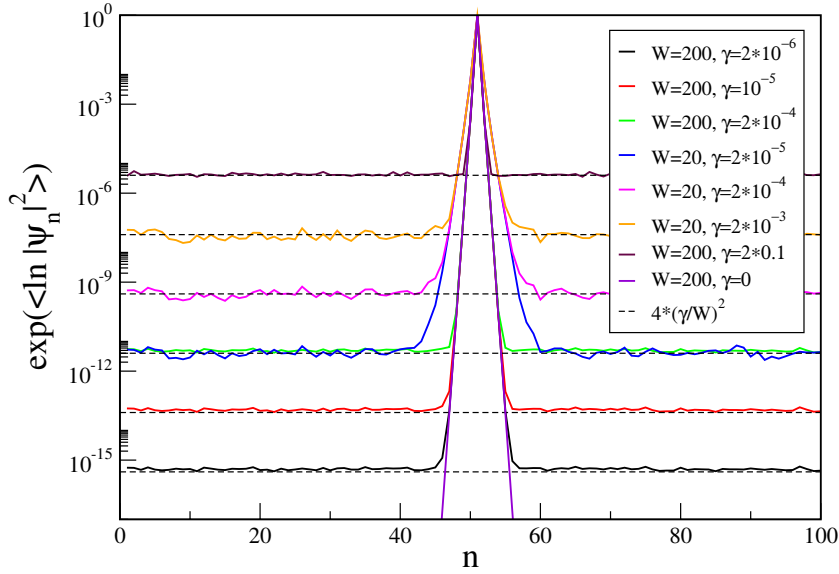
In the limit  $\kappa \ll 1$ , the eigenstates of  $H_{\text{eff}}$  at first order in perturbation theory can be written as:

$$|n\rangle = \frac{1}{\sqrt{C_n}} \left[ |n^0\rangle - i \frac{\gamma}{2} \sum_{k^0 \neq n^0} \frac{\langle k^0 | Q | n^0 \rangle}{E_{n^0} - E_{k^0}} |k^0\rangle \right], \quad (13)$$

where  $|n^0\rangle$  are the eigenstates of the closed system, i.e., of the Anderson model. Of course, the perturbation expansion makes sense only when each coefficient in the sum in Eq. (13) is much less than one. This cannot be true in general since the eigenvalues  $E_{n^0}$  are random numbers uniformly distributed in the interval  $[-W/2, W/2]$ . Thus perturbation theory cannot be applied *tout court*, but only for those states whose energies are not too close.

This simple observation has deep consequences for the structure of the eigenstates. Indeed we observe numerically that on the one hand many single-peaked eigenstates become double- or multiple-peaked as  $\gamma$  increases, while on the other hand, they all develop a constant plateau proportional to  $(\gamma/W)^2$ , see Fig. 4.

This last fact can easily be explained using first-order perturbation theory as given by Eq. (13): in the deep localized regime  $W \gg \Omega$ , the matrix elements  $\langle k^0 | Q | n^0 \rangle$  are of order unity and the average distance between two random energies is  $W/3$ , so the typical coefficients  $\langle k_0 | n \rangle$  in Eq. (13) are  $\sim \gamma/W$ . Furthermore, the mean level spacing is  $D \approx W/N$ , and thus the few largest coefficients in Eq. (13) are typically  $\sim \gamma N/W \sim \kappa$  (using Eq. (9)). Thus for weak opening ( $\kappa \ll 1$ ), the *typical* eigenstate consists of a single Anderson model eigenstate with a  $O(\kappa^2)$  admixture of other states, and therefore the *typical*  $PR$  for small  $\kappa$  differs only by  $O(\kappa^2)$  from the  $PR$  of the Anderson model.

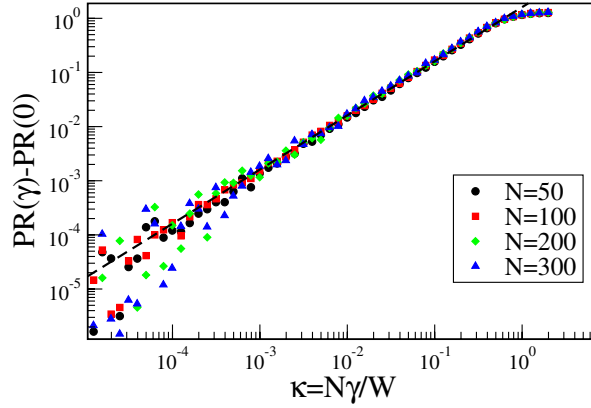


**Fig. 4** (online colour at: [www.fp-journal.org](http://www.fp-journal.org)) The averaged probability distribution of all eigenstates of the non-Hermitian Hamiltonian that are strongly peaked in the middle of the chain is shown for different coupling strength  $\gamma$  and disorder strength  $W$ , as indicated in the caption. Specifically, we average over all eigenstates having a probability  $> 0.9$  at the site  $n = N/2 + 1$  in order to avoid double-peaked states, and also average over disorder. Moreover, to reduce fluctuations, we average the logarithm of the probability distribution. In all cases we fix  $N = 100$  and  $\Omega = 1$ . Dashed horizontal lines are proportional to  $(\gamma/W)^2$  in agreement with the perturbative approach.

As already remarked previously, the perturbative approach cannot always work, because for arbitrarily small  $\kappa$  there is a small but finite probability that two energy states are too close together. This clustering behavior has important consequences for the localization properties. Specifically, since the nearest-neighbor level spacing distribution of uniform random numbers  $E_{n_0}$  is Poissonian:  $P(s) = (1/D) e^{-s/D}$ , where  $s$  is the energy difference between nearest-neighbor levels and  $D = W/N$  is the mean level spacing, we can evaluate the probability to have two levels closer than  $\gamma/2$  as  $1 - e^{-\gamma/2D} \approx \kappa/2$  for small  $\kappa$ . This means that there are  $\kappa N$  states out of  $N$ , for which perturbation theory cannot be applied. When this happens, the Anderson states mix strongly and the  $PR$  increases by an  $O(1)$  factor. Thus, even though this behavior is rare, it makes an  $O(\kappa)$  contribution to the *average*  $PR$  of the weakly open system, which exceeds the  $O(\kappa^2)$  contribution from the typical states. Indeed the average  $PR$  can be evaluated as follow:

$$PR = \frac{N\kappa PR_2 + (1 - \kappa)NPR_1}{N} = PR_1 + \kappa(PR_2 - PR_1)$$

where  $PR_1$  and  $PR_2$  refer to the  $PR$  of the states for which perturbation theory can and cannot be applied. Since  $PR_1 \simeq PR(\gamma = 0) + O(\kappa^2)$ , and  $PR_2 \simeq O(1)$ , we have that  $PR(\gamma) - PR(\gamma = 0) \simeq \kappa$ . The numerical results in Fig. 5 confirm that the effect of the opening on the  $PR$  grows as  $\kappa$ , instead of the  $\kappa^2$  growth predicted by first-order perturbation theory. Here we present the average (over disorder) of  $PR(\gamma) - PR(\gamma = 0)$ , as a function of  $\kappa = N\gamma/W$  for fixed disorder strength and different values of the system size. In any case this is quite a delicate point and we postpone its full analysis to a future work.



**Fig. 5** (online colour at: [www.fp-journal.org](http://www.fp-journal.org)) The average increase in the participation ratio, compared with the closed system, is calculated as a function of  $\kappa$ , for fixed disorder  $W = 20$  and different system sizes  $N$  as indicated in the legend. In each case the average is performed over 50000 different eigenstates. The line is  $PR(\gamma) - PR(0) = 2\kappa$ .

## 5.2 Perturbative approach for $\kappa \gg 1$

In the limit  $\kappa \gg 1$  we consider two cases. First, we consider the situation where the nearest neighbor tunneling coupling is  $\Omega = 0$ , in which case we can follow the approach explained in [17]. This approach will be very useful also for the case  $\Omega \neq 0$ , which we treat below.

### 5.2.1 $\Omega = 0$ and $\kappa \gg 1$

If  $\Omega = 0$  the Anderson Hamiltonian is diagonal in the site basis  $|j\rangle$  with eigenvalues  $E_j$  distributed uniformly in the interval  $[-W/2, W/2]$ . The eigenstates of the non-Hermitian part  $-i\frac{\gamma}{2}Q$  of the effective Hamiltonian are  $|d\rangle = \frac{1}{\sqrt{N}}(1, \dots, 1)^T$  (the superradiant state) with eigenvalue  $-i\frac{\gamma}{2}N$ , and  $N - 1$  degenerate eigenstates  $|\mu\rangle$  with eigenvalue 0 (the subradiant states). We will choose  $|\mu\rangle$  in a convenient manner later. Following [17] we can rewrite  $H_{\text{eff}}$  in the basis of these eigenstates using the transformation matrix  $V$ , which has as its columns the eigenstates of  $Q$ :

$$\tilde{H}_{\text{eff}} = V^T H_0 V - i\frac{\gamma}{2}V^T Q V = \begin{pmatrix} -i\frac{\gamma}{2}N & \vec{h}^T \\ \vec{h} & \tilde{H} \end{pmatrix}. \quad (14)$$

Here  $\vec{h}$  is a vector of dimension  $N - 1$  with components

$$h_\mu = \frac{1}{\sqrt{N}} \sum_{j=1}^N E_j \langle j | \mu \rangle, \quad (15)$$

while the matrix elements of the  $(N - 1) \times (N - 1)$  submatrix  $\tilde{H}$  are

$$\tilde{H}_{\mu\nu} = \sum_{j=1}^N E_j \langle \mu | j \rangle \langle j | \nu \rangle. \quad (16)$$

Now, we can diagonalize  $\tilde{H}$ ,

$$\tilde{H}_{\mu\nu} = \sum_{j=1}^N E_j \langle \mu | j \rangle \langle j | \nu \rangle = \langle \mu | H_0 | \nu \rangle = \tilde{\epsilon}_\mu \langle \mu | \nu \rangle. \quad (17)$$



Following [17] we obtain

$$|\mu\rangle = h_\mu \frac{1}{\tilde{\epsilon}_\mu - H_0} |d\rangle = \frac{h_\mu}{\sqrt{N}} \sum_{j=1}^N \frac{1}{\tilde{\epsilon}_\mu - E_j} |j\rangle, \quad (18)$$

where the normalization coefficients  $h_\mu$  are given by

$$h_\mu = \left( \langle d | \frac{1}{(\tilde{\epsilon}_\mu - H_0)^2} |d\rangle \right)^{-1/2}. \quad (19)$$

In the limit  $\kappa \gg 1$ , the eigenstates  $|\mu\rangle$  of the non-Hermitian part of  $H_{\text{eff}}$  are also eigenstates of  $H_{\text{eff}}$ . Since  $\langle d|\mu\rangle = 0$  we have,

$$\sum_{j=1}^N \frac{1}{\tilde{\epsilon}_\mu - E_j} = 0. \quad (20)$$

Therefore each eigenvalue of  $\tilde{H}$  lies between two neighboring levels  $E_n$ , so that the values  $\tilde{\epsilon}_\mu$  are also confined in the interval  $[-W/2, W/2]$ .

Let us now estimate the magnitude of the mixing matrix elements  $h_\mu$ . To do this we compute

$$\vec{h} \cdot \vec{h} = \frac{1}{N} \sum_{\mu=1}^{N-1} \sum_{i=1}^N \sum_{j=1}^N E_i E_j \langle \mu|i\rangle \langle j|\mu\rangle, \quad (21)$$

and using the completeness relation  $\sum_{\mu=1}^{N-1} \langle j|\mu\rangle \langle \mu|i\rangle = \langle j|i\rangle - 1/N$  we have

$$\vec{h} \cdot \vec{h} = \langle E^2 \rangle - \langle E \rangle^2 = \Delta E^2. \quad (22)$$

This leads to

$$|h_\mu| \sim \frac{\Delta E}{\sqrt{N-1}} = \frac{W}{\sqrt{12(N-1)}}. \quad (23)$$

Each eigenstate  $|\mu\rangle$  in Eq. (18) is a superposition of all the site states  $|j\rangle$  with amplitudes  $\frac{h_\mu}{\sqrt{N}(\tilde{\epsilon}_\mu - E_j)} \sim \frac{W}{N(\tilde{\epsilon}_\mu - E_j)}$  that depend only on the energies  $E_j$  and not on the site positions  $j$ . Nevertheless, each state  $|\mu\rangle$  is quite localized, since the amplitudes are of order unity for the  $O(1)$  number of sites whose energy is within a few mean level spacings of  $\tilde{\epsilon}$  (i.e., when  $|\tilde{\epsilon}_\mu - E_j| \sim D = W/N$ ), and small otherwise. This small value of the  $PR$  for the subradiant states should be compared with  $PR = N$  of the superradiant states.

The values obtained above for the subradiant and the superradiant states correspond to zeroth-order perturbation theory. On the other hand first-order perturbation theory gives:

$$\begin{aligned} |SR\rangle &= \frac{1}{\sqrt{C}} \left[ |d\rangle + \frac{W}{\sqrt{12(N-1)}} \sum_{\mu=1}^{N-1} \frac{r_\mu}{-i\frac{\gamma}{2}N - \tilde{\epsilon}_\mu} |\mu\rangle \right] \\ &= \frac{1}{\sqrt{C}} \left[ |d\rangle - \frac{1}{\kappa\sqrt{3(N-1)}} \sum_{\mu=1}^{N-1} \frac{r_\mu}{i + 2\tilde{\epsilon}_\mu/\gamma N} |\mu\rangle \right] \\ |SUB_\mu\rangle &= \frac{1}{\sqrt{C'_\mu}} \left[ |\mu\rangle + \frac{W}{\sqrt{12(N-1)}} \frac{r_\mu}{\tilde{\epsilon}_\mu + i\frac{\gamma}{2}N} |d\rangle \right] \\ &= \frac{1}{\sqrt{C'_\mu}} \left[ |\mu\rangle + \frac{1}{\kappa\sqrt{3(N-1)}} \frac{r_\mu}{i + 2\tilde{\epsilon}_\mu/\gamma N} |d\rangle \right], \end{aligned} \quad (24)$$

where  $r_\mu$  are random coefficients with  $\langle r_\mu^2 \rangle = 1$ . We see that the exact superradiant state  $|SR\rangle$  is a combination of the unperturbed superradiant state  $|d\rangle$  and a small admixture of the unperturbed subradiant states  $|\mu\rangle$ , and the mixing probability decreases as  $1/\kappa^2$  for large  $\kappa$ . Similarly, the admixture of the unperturbed superradiant state  $|d\rangle$  in each exact subradiant states  $|SUB_\mu\rangle$  decreases as  $1/(\kappa^2 N)$ . This shows that  $PR \approx N$  for the superradiant state and  $PR \sim 1$  for the subradiant states when  $\kappa \gg 1$ .

### 5.2.2 $\Omega \neq 0$ and $\kappa \gg 1$

As a first step we write the Anderson Hamiltonian  $H_0$  in terms of its eigenstates  $|n\rangle$ . Obviously the form of  $|n\rangle$  will depend on the degree of disorder  $W$ . In the following we limit our considerations to the large disorder regime, so that in the basis of the eigenstates of  $H_0$ , the matrix elements of  $Q$  remain of order one,  $Q_{nm} \sim 1$ , and we can use the results of Sect. 5.2.1, with the site states and energies  $|j\rangle$  and  $E_j$  replaced by the Anderson eigenstates and eigenenergies  $|n\rangle$  and  $E_n$ .

In Fig. 3 we see that for  $\kappa > 1$  (corresponding to  $W < 100$ ), the superradiant state remains unaffected by the increase of disorder, while the subradiant states become more localized as the disorder strength is increased. The results of the previous section can be used to understand this strongly asymmetric behavior of the  $PR$  between the subradiant states and the superradiant state. Indeed at zeroth order in perturbation theory we can see that the superradiant state  $|SR\rangle \approx |d\rangle$  is completely delocalized,  $PR = N$ , while subradiant states  $|SUB_\mu\rangle \approx |\mu\rangle$  become more and more localized as we increase disorder. Specifically, the site states  $|j\rangle$  in Eq. (18) are replaced with Anderson eigenstates  $|n\rangle$ , with localization length  $\xi \propto 1/W^2$ . This difference persists in first-order perturbation theory, since the mixing probability between the super- and sub-radiant states decreases as  $1/\kappa^2$  for large  $\kappa$ , see Eq. (24).

Our perturbative approach justifies the results presented in Fig. 3, where we can see that the subradiant states become increasingly localized as we increase disorder. At the same time Fig. 3 shows that the superradiant state remains completely delocalized even as  $W$  is increased, until we reach the value  $W \approx 142.8$  ( $\kappa = 0.7$ ) where we find numerically that the ST takes place. The perturbative approach shows that superradiant states are much less sensitive to disorder because their complex energies are at a distance greater than  $\gamma N/2 = W\kappa/2$  from the subradiant states.

## 6 Conclusions

We have studied a 1-D Anderson model with all sites coupled to a common decay channel (coherent dissipation). Our main motivation was to understand the interplay of opening and disorder. Increasing the disorder tends to localize the states. Increasing the opening, on the other hand, reduces the degree of localization, and in particular induces a superradiance transition, with the formation of a subradiant subspace and a superradiant state completely delocalized over all the sites. Our results show that, while for small opening all the states tend to be similarly affected by the disorder, for large opening the superradiant state remains delocalized even as the disorder increases, while the subradiant states are much more affected by disorder, becoming ever more localized as the disorder increases. We have explained these effects qualitatively, mainly guided by perturbation theory. Indeed we have shown that the superradiant state is not affected by disorder, since its energy is very distant, in the complex plane, from the energies of the subradiant states.

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