Sign changing solutions for a Yamabe type equation

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"Metodi topologici e variazionali nello studio di fenomeni non lineari", Brescia 2010

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Torus action on S^n and sign changing solutions for conformally invariant equations. (preprint)

The problem on S^n

Find sign changing solutions to the equation

$$(1) \qquad \mathcal{L}_{g_o} u = |u|^{\frac{4}{n-2}} u \quad \text{in} \quad (S^n, g_o)$$

where

- (S^n, g_o) is the unit sphere with the standard metric
- $\mathcal{L}_{g_o} = -\Delta_{g_o} - \frac{n-2}{4(n-1)} R_{g_o}$ is the conformal Laplacian
- $R_{g_o} = n(n-1)$ is the scalar curvature

An equivalent problem

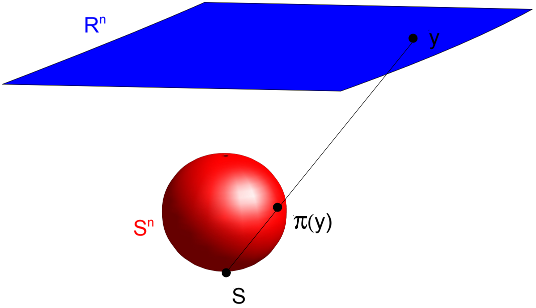
The problem on R^n

Find sign changing solutions to the equation

$$(2) \qquad -\Delta u = |u|^{\frac{4}{n-2}} u \quad \text{in} \quad \mathbb{R}^n$$

The stereographic projection π

- $S = (0, \ldots, 0, -1)$ is the south pole of S^n
- $\pi : \mathbb{R}^n \to S^n \setminus \{S\}$ is defined by $\pi(y) = \left(\frac{2y}{1+|y|^2}, \frac{1-|y|^2}{1+|y|^2}\right)$



Problems (1) and (2) are equivalent

π is a local conformal diffeomorphism, i.e.

$$\pi^* g_o = \phi^{\frac{4}{n-2}} dy, \ \phi(y) := \left(\frac{2}{1+|y|^2} \right)^{\frac{n-2}{2}}$$

$$\Downarrow$$

$\pi^* \left(\mathcal{L}_{g_o} v \right) = \phi^{-\frac{n+2}{n-2}} \Delta \left(\phi \pi^* v \right)$ for any function v defined on S^n

$$\Downarrow$$

u is a solution to (1) $\quad \Leftrightarrow \quad$ $w = \phi \pi^* u$ is a solution to (2)

Existence of positive solutions: Obata 1972, Talenti 1976, Caffarelli-Gidas-Spruck 1989

On S^n ...

all the positive solutions to (1), up to rotations, are given by u_ϵ

$$\pi^* u_\epsilon(y) = \epsilon^{\frac{n-2}{2}} \left(\frac{1 + |y|^2}{\epsilon^2 + |y|^2} \right)^{\frac{n-2}{2}}, \quad \epsilon > 0$$

On $\mathbb{R}^n$ ...

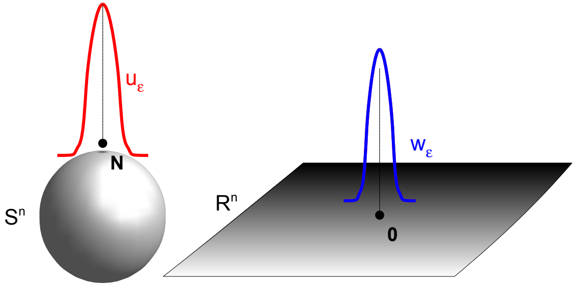
all the positive solutions to (2), up to traslations, are given by w_ϵ

$$w_\epsilon(y) = \phi(y)\pi^* u_\epsilon(y) = \epsilon^{-\frac{n-2}{2}} \left(\frac{2\epsilon^2}{\epsilon^2 + |y|^2} \right)^{\frac{n-2}{2}}, \quad \epsilon > 0$$

Remark

- $u_1(x) \equiv 1$ is a (trivial) solution to (1)
- u_ϵ blows-up at the north pole as $\epsilon \to 0$

- $w_1(y) = \phi(y) = \left(\frac{2}{1+|y|^2}\right)^{\frac{n-2}{2}}$ is a solution to (2)
- w_ϵ blows-up at the origin as $\epsilon \to 0$



Ding's result (1986)

(1) has infinitely many sign changing solutions,

which are invariant under the action of

$O(h) \times O(n+1-h)$ for any $h = 2, \ldots, n-1$

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Remark

Nothing is known about the profile of Ding's solution!

Our main result

We prove that

(1) has infinitely many sign changing solutions,

which are the superposition of the constant solution u_1

with a large number of copies of negative solutions of (1)

which blow-up at points which in turn are regularly arranged

along some special submanifolds of S^n

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Remark

Our solutions are not invariant under the action of $O(2) \times O(n-1)$

$$\Downarrow$$

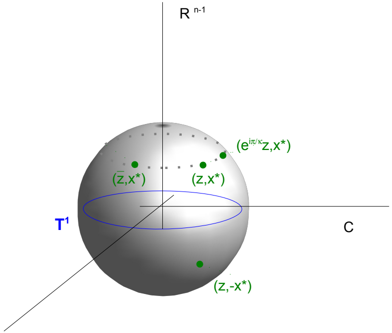
Our solutions are different from Ding's solutions!

Theorem 1

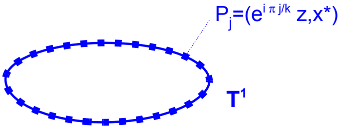
- $S^n \subset \mathbb{C} \times \mathbb{R}^{n-1}$, $x \in S^n \Leftrightarrow x = (z, x^*) \in \mathbb{C} \times \mathbb{R}^{n-1}$
- $\mathbb{T}^1 := S^1 \times \{0\}$ is a great circle of S^n

There exists $k_0 > 0$ such that for any $k \geq k_0$ there exists u_k solution to (1) such that

- $u_k(z, x^*) = u_k(\bar{z}, x^*) = u_k(z, -x^*)$
- $u_k(z, x^*) = u_k\left(e^{\frac{i\pi}{k}} z, x^*\right)$

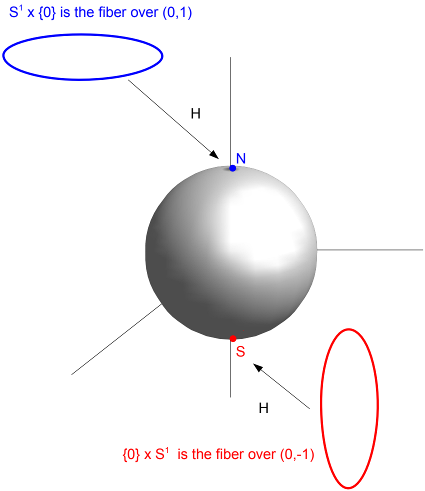


- $u_k \to 1$ uniformly on compact sets of $S^n \setminus \mathbb{T}^1$ as $k \to \infty$
- u_k blow-up negatively at the $2k$ points $P_j := \left(e^{\frac{\pi i}{k} j}, 0^*\right) \in \mathbb{T}^1$, $j = 1, \ldots, 2k$ as $k \to \infty$



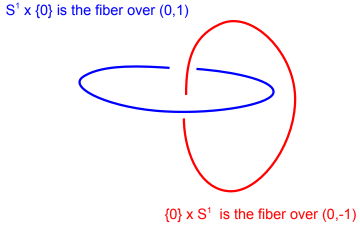
Hopf fibration

- $S^3 \subset \mathbb{C} \times \mathbb{C}$ and $S^2 \subset \mathbb{C} \times \mathbb{R}$
- $H : S^3 \to S^2$, $H(z_1, z_2) := \left(2z_1\bar{z}_2, |z_1|^2 - |z_2|^2\right)$ is the Hopf map
- Each fiber over a point of S^2 is a great circle in S^3
- Fibers over different points are different great circles in S^3



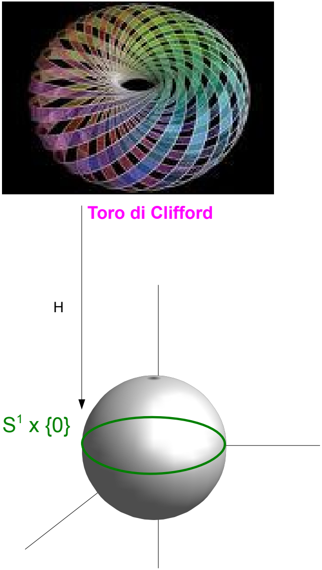
Hopf link

- Stereographic projection of S^3 to $\mathbb{R}^3$ maps two different great circles in S^3 into two linking circles in $\mathbb{R}^3$, i.e. a Hopf link in $\mathbb{R}^3$



The Clifford torus

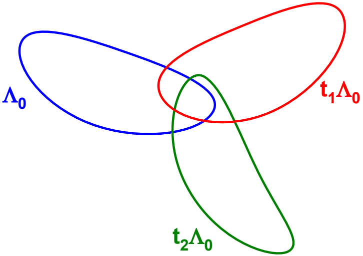
- The Clifford torus is the fiber over $S^1 \times \{0\}$



Concentration on q great circles which are linked ($n \geq 4$)

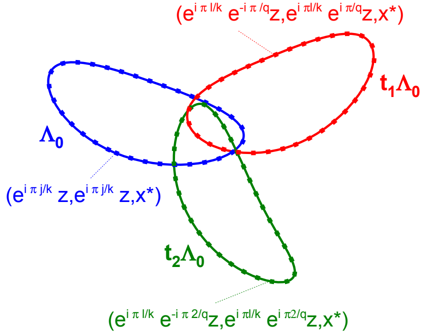
Theorem 2

- $S^n \subset \mathbb{C} \times \mathbb{C} \times \mathbb{R}^{n-3}$, $x \in S^n \Leftrightarrow x = (z_1, z_2, x^*) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}^{n-3}$
- $\Lambda_0 := \left\{ \frac{1}{\sqrt{2}}(z, z, 0^*) : \ z \in S^1 \right\}$ is a great circle of S^n
- $q \geq 1$ and $t_q : S^n \to S^n$ be $t_q(z_1, z_2, x^*) = \left(e^{-\frac{i\pi}{q}} z_1, e^{\frac{i\pi}{q}} z_2, x^* \right)$
- $\Lambda_0, t_q\Lambda_0, \ldots, t_q^{q-1}\Lambda_0$ are q different great circles
- Any two such great circles are linked
- $\Lambda := \Lambda_0 \cup t_q\Lambda_0 \cup \cdots \cup t_q^{q-1}\Lambda_0$ is the union of q great circles



There exists $k_0 > 0$ such that for any $k \geq k_0$ there exists u_k solution to (1) such that

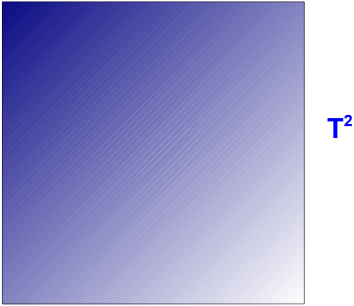
- $u_k(z_1, z_2, x^*) = u_k(\bar{z}_1, \bar{z}_2, x^*) = u_k(z_1, z_2, -x^*) = u_k(z_2, z_1, x^*)$
- $u_k(z_1, z_2, x^*) = u_k\left(e^{\frac{i\pi}{k}} z_1, e^{\frac{i\pi}{k}} z_2, x^*\right) = u_k\left(e^{-\frac{i\pi}{q}} z_1, e^{\frac{i\pi}{q}} z_2, x^*\right)$
- $u_k \to 1$ uniformly on compact sets of $S^n \setminus \Lambda$ as $k \to \infty$
- u_k blow-up negatively at the $2k \times q$ points in Λ as $k \to \infty$



Concentration on the Clifford torus ($n \geq 5$)

Theorem 3

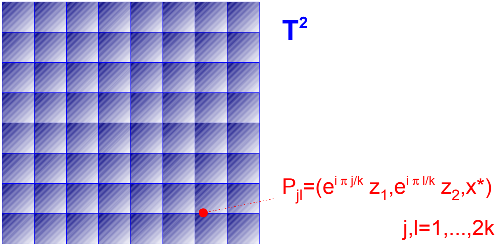
- $S^n \subset \mathbb{C} \times \mathbb{C} \times \mathbb{R}^{n-3}$, $x \in S^n \Leftrightarrow x = (z_1, z_2, x^*) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}^{n-3}$
- $\mathbb{T}^2 := \frac{1}{\sqrt{2}} \left(S^1 \times S^1 \right) \times \{0^*\}$ is the Clifford torus of S^n



Concentration on the Clifford torus ($n \geq 5$)

There exists $k_0 > 0$ such that for any $k \geq k_0$ there exists u_k solution to (1) such that

- $u_k(z_1, z_2, x^*) = u_k(\bar{z}_1, \bar{z}_2, x^*) = u_k(z_1, z_2, -x^*) = u_k(z_2, z_1, x^*)$
- $u_k(z_1, z_2, x^*) = u_k\left(e^{\frac{i\pi}{k}} z_1, e^{\frac{i\pi}{k}} z_2, x^*\right) = u_k\left(e^{-\frac{i\pi}{k}} z_1, e^{\frac{i\pi}{k}} z_2, x^*\right)$
- $u_k \to 1$ uniformly on compact sets of $S^n \setminus \mathbb{T}^2$ as $k \to \infty$
- u_k blow-up negatively at the $(2k)^2$ points of $\mathbb{T}^2$ as $k \to \infty$



Ansatz

We look for a solution of (2) as

$$u_k(y) = w_1(y) - \sum_{j=1}^{2k} w_\delta(y - \xi_j) + v(y), \; y \in \mathbb{R}^n = \mathbb{C} \times \mathbb{R}^{n-1}$$

- $w_1(y) = \left(\frac{2}{1+|y|^2}\right)^{\frac{n-2}{2}}$
- $w_\delta(y - \xi_j) = \delta^{-\frac{n-2}{2}} \left(\frac{2\delta^2}{\delta^2+|y-\xi_j|^2}\right)^{\frac{n-2}{2}}$
- $\delta := \frac{d}{k^2}$, $d > 0$ concentration rate
- $\xi_j := \left(\rho e^{\frac{\pi i}{k} j}, 0^*\right)$, $j = 1, \dots, 2k$ concentration points
- $v = v(d, \rho, k)$ is a remainder term

$k \to \infty$ is the parameter (Wei & Yan 2007)

Reduction to a finite dimensional problem

$$u_k = w_1 - \sum_{j=1}^{2k} w_\delta(\cdot - \xi_j) + v \text{ is a solution to (2),}$$

i.e. a critical point of the energy

$$J(u) := \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 dy - \frac{n-2}{2n}\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}} dy$$

$$\Updownarrow$$

$(d, \rho) \in (0, +\infty) \times (0, +\infty)$ is a critical point of the reduced energy

$$J_k(d, \rho) := J\left(w_1 - \sum_{j=1}^{2k} w_\delta(\cdot - \xi_j) + v \right)$$

Expansion of the reduced energy

$$J_k(d,\rho) - (2k+1)J(w_1) \sim$$

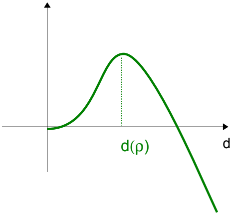
$$\sim \sum_{i=1}^{2k} \overbrace{\delta^{\frac{n-2}{2}} w_1(\xi_i)}^{\text{interaction of the peak } \xi_i \text{ with the bubble } w_1} - \sum_{\substack{i,j=1\\ i\neq j}}^{2k} \overbrace{\delta^{n-2}\left(\frac{2}{|\xi_i-\xi_j|}\right)^{n-2}}^{\text{interaction between the two peaks } \xi_i \text{ and } \xi_j} \sim$$

$$\sim 2k\left[\underbrace{\delta^{\frac{n-2}{2}}}_{\sim \frac{d^{\frac{n-2}{2}}}{k^{n-2}}}\left(\frac{2}{1+\rho^2}\right)^{\frac{n-2}{2}} - \underbrace{\delta^{n-2}}_{\sim \frac{d^{n-2}}{k^{2(n-2)}}}\frac{1}{\rho^{n-2}}\underbrace{\sum_{\substack{j=2\\ i\neq j}}^{2k}\frac{1}{(1-\cos\pi(j-1)/k)^{\frac{n-2}{2}}}}_{\sim\sigma k^{n-2},\ \sigma>0}\right] \sim$$

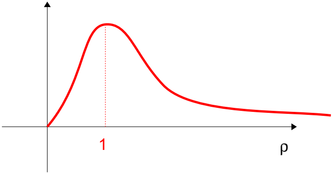
$$\sim \frac{2}{k^{n-3}}\underbrace{\left[d^{\frac{n-2}{2}}\left(\frac{2}{1+\rho^2}\right)^{\frac{n-2}{2}} - \sigma\frac{d^{n-2}}{\rho^{n-2}}\right]}_{\Phi(d,\rho)}$$

The function $\Phi(d,\rho)$ has a C^1-stable critical point

- $\Phi(d,\rho) := d^{\frac{n-2}{2}} \left(\frac{2}{1+\rho^2}\right)^{\frac{n-2}{2}} - \sigma \frac{d^{n-2}}{\rho^{n-2}}$
- for any $\rho > 0$ there exists $d(\rho)$ maximum point of $d \to \Phi(d,\rho)$



- the function $\rho \to \Phi\left(d(\rho),\rho\right) = c\left(\frac{\rho}{1+\rho^2}\right)^{n-2}$ has a maximum point at $\rho_0 = 1$



Proof completed

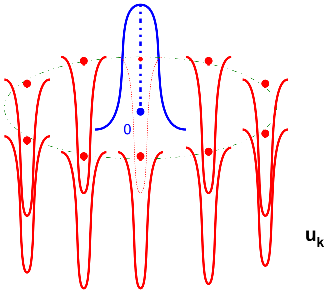
Φ has a C^1-stable critical point $(d_0, 1)$

$\Downarrow$

if $k \sim +\infty$, J_k has a critical point (d_k, ρ_k) s.t. $(d_k, \rho_k) \to (d_0, 1)$

$\Downarrow$

(2) has a sign changing solution u_k as in the picture!



Buon Compleanno Antonio!