Effect of a non-uniform external magnetic field on the 3D stagnation-point flow

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A B S T R A C T

This paper is concerned with the study of the influence of a non-uniform external magnetic field on the steady three-dimensional stagnation-point flow of a Newtonian fluid when the total magnetic field is parallel to the velocity at infinity. The fluid occupies the half-space over a solid obstacle which is a rigid uncharged dielectric at rest. We solve the problem both in the fluid and in the solid. We prove that the fluid flow is possible only in the axisymmetric case and it is described by an ordinary differential boundary value problem. The numerical integration shows that the viscosity appears only in a boundary layer whose thickness depends on the Reynolds and the Alfvén numbers.

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1. Introduction

In this paper we study the influence of a non-uniform external magnetic field on the steady three-dimensional stagnation-point flow of a Newtonian fluid. Such a flow occurs when a jet of fluid impinges on a rigid body and it is an important example of motion where the three velocity components appear. Since it appears very often in nature as a component of more complicated fluid motions, this flow has been object of many investigations [1–3] starting from the paper of Homman in 1936. Through similarity transformations, the study of the flow is reduced to a non-linear ordinary differential boundary value problem where there is the parameter \( c > -1 \) which is a measure of three-dimensionality.

Magnetohydrodynamic stagnation point flow has given rise to many papers in the literature due to its applications in several physical and engineering situations. As far as the 3D flow is concerned, few results are known: Gribben in [4] studied the axisymmetric flow of a Newtonian fluid embedded in a magnetic field whose field lines are circles having the centres on the axis of symmetry; in [5] it is proved that, if an external uniform magnetic field is impressed, and the induced magnetic field is neglected, then the steady three-dimensional MHD stagnation-point flow is possible if, and only if, the external magnetic field has three particular directions. Previously, in a different physical situation [7], a uniform external magnetic field orthogonal to the obstacle has been considered and the induced magnetic field has been neglected.

In our study we investigate the three dimensional problem which is the generalization of the one considered in the plane case [8]. We examine the 3D stagnation-point flow of a Newtonian fluid filling the half-space \( x_2 > 0 \) when the total magnetic field \( \mathbf{H} \) is aligned to the velocity at infinity. We suppose \( \mathbf{H} \) depending on two sufficiently regular unknown functions \( h = h(x_2), k = k(x_2) \). The whole space is permeated by a non-uniform external magnetic field \( \mathbf{H} \), while the external electric field is absent. The region occupied by the fluid is bordered by the boundary of a solid obstacle which is a rigid uncharged dielectric at rest.

We first consider the electromagnetic field in the solid \((\mathbf{H}_s, \mathbf{E}_s)\): we underline that many Authors ignore the details of the electromagnetic field in the solid region but the relevance of the problem to any physical situation may be in doubt if we do not join the solution in the fluid to a suitable solution in the solid. Independently of the fluid model over the solid, we find \( \mathbf{E}_s = 0 \) and we determine \( \mathbf{H}_s \) by asking that its non-degenerate field lines belong to surfaces which asymptote to the plane \( x_2 = 0 \) as \( |x_1|, |x_3| \) go to infinity.

Before studying the problem for the Newtonian fluid, we examine the case of an inviscid fluid. The analysis of the inviscid case is very important because, as it is reasonable from the physical point of view, we assume that far from the wall the flow of the viscous fluid approaches the flow of an inviscid fluid for which the stagnation-point is shifted from the origin. In the inviscid fluid we find that \( \mathbf{(H, E)} = (\mathbf{H}_s, \mathbf{0}) \) and the pressure field is not modified by the presence of \( \mathbf{H} \).

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If the fluid is Newtonian, then we prove that the steady 3D MHD stagnation point flow is possible only if the flow is axisymmetric. The study of the problem leads to a nonlinear ordinary differential problem which depends on two electromagnetic parameters $R_m$ (Reynolds number) and $\beta_m$ (Alfvén number) related to the magnetic nature of the flow. The Alfvén number is also a measure of the strength of the external magnetic field.

Some numerical examples and pictures are given in order to illustrate the effects due to the magnetic field on the behaviour of the solution. The numerical results are obtained by using the MATLAB routine bvp4c, which is described in [9].

By solving numerically the problem, we find that the thickness of the layer where the influence of the viscosity appears depends on $R_m$ and $\beta_m$. More precisely, it increases as $\beta_m$ increases, while it decreases as $R_m$ increases.

The paper is organized in this way:

In Section 2 we formulate the problem from the physical point of view and we study the situation of the solid obstacle and of the inviscid fluid. The result obtained in the solid shows that, independently of the kind of fluid, the expression of $\mathbf{H}$ is formally the same and depends on $H(0)$, $k(0)$.

Section 3 is devoted to the Newtonian fluids. The main result states that the only possible motion is axisymmetric.

In Section 4 we solve numerically the boundary value problem using suitable transformations related to the values of $R_m$ and $\beta_m$ and we discuss the results. Section 5 contains the conclusions.

2. Position of the problem. Inviscid fluids

In order to clarify the physics of the problem, we begin by considering the steady three-dimensional MHD stagnation-point flow of a homogeneous, incompressible, electrically conducting inviscid fluid which fills the half-space $\delta$ (see Fig. 1), given by

$$\delta = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_3) \in \mathbb{R}^2, x_2 > 0\}. \tag{1}$$

The coordinate axes are chosen in order that the stagnation-point coincides with the origin and the canonical base of $\mathbb{R}^3$ is denoted by $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. $\delta \mathbf{\sigma}$, i.e. the plane $x_2 = 0$, is the boundary of a solid which is a rigid uncharged dielectric at rest occupying

$$\delta^- = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_3) \in \mathbb{R}^2, x_2 < 0\}. \tag{2}$$

As it is well known [10,11], in the three-dimensional stagnation-point flow the components of the velocity field are

$$v_1 = \alpha x_1, \quad v_2 = -\alpha(1 + \lambda) x_2, \quad v_3 = \alpha x_3, \tag{3}$$

$$(x_1, x_3) \in \mathbb{R}^2, \quad x_2 \geq 0,$$

where $\alpha, \lambda$ are some constants. We suppose $\alpha > 0$, $\lambda \neq 0$ and we exclude the case $\lambda \leq -1$, because we impose the condition $v_2 < 0$, so that the fluid moves towards the wall $x_2 = 0$. The parameter $\alpha$ is a measure of the third-dimensionality of the motion because when $\alpha = 0$ we find the plane orthogonal stagnation-point flow.

If $\alpha = 1$, then the velocity is axisymmetric:

$$v_1 = \alpha x_1, \quad v_2 = -2\alpha x_2, \quad v_3 = \alpha x_3. \tag{4}$$

The equations governing such a flow in the absence of external mechanical body forces and free electric charges are

$$\rho \mathbf{V} \cdot \nabla \mathbf{V} = -\nabla p + \mu_e (\nabla \times \mathbf{H}) \times \mathbf{H},$$
$$\nabla \cdot \mathbf{V} = 0,$$
$$\nabla \times \mathbf{E} = \sigma_e (\mathbf{E} + \mu_e \nabla \times \mathbf{H}),$$
$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{H} = 0, \quad \nabla \times \mathbf{E} = 0, \quad \nabla \times \mathbf{H} = 0, \quad \text{in} \; \delta, \tag{5}$$

where $\mathbf{V}$ is the velocity field, $p$ is the pressure, $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields, respectively, $\rho$ is the mass density, $\mu_e$ is the magnetic permeability, $\sigma_e$ is the electrical conductivity ($\rho, \mu_e, \sigma_e =$ constants $> 0$).

As usual, we impose the no-penetration condition to the velocity field and we ask that the tangential components of $\mathbf{H}$ and $\mathbf{E}$ and the normal components of $\mathbf{B} = \mu_e \mathbf{H}$ and $\mathbf{D} = \varepsilon \mathbf{E}$ ($\varepsilon =$ dielectric constant) are continuous across the plane $x_2 = 0$.

We suppose that the external magnetic field

$$\mathbf{H}_e = H_\infty [x_1 \mathbf{e}_1 - (1 + \lambda) x_2 \mathbf{e}_2 + \lambda x_3 \mathbf{e}_3], \tag{6}$$

$H_\infty =$ constant, $(x_1, x_2, x_3) \in \mathbb{R}^3$,

permeates the whole physical space and that the external electric field $\mathbf{E}_e$ is absent.

Remark 1. As it is easy to verify, the field lines of $\mathbf{H}_e$ have the following parametric equations

$$x_1 = A_1 e^{H_\infty \lambda},$$
$$x_2 = A_2 e^{-(1+c)H_\infty \lambda},$$
$$x_3 = A_3 e^{H_\infty \lambda}, \quad \lambda \in \mathbb{R}, \tag{7}$$

where $A_1, A_2, A_3$ are arbitrary constants. These field lines degenerate at least one of the three constants vanishes. Otherwise they belong to the Titeica surfaces

$$x_1 x_2 x_3 = A_1 A_2 A_3,$$

which asymptote to the plane $x_2 = 0$ as $|x_1|, |x_3| \rightarrow +\infty$ (see Fig. 2).
We seek the total magnetic fields in the fluid and in the solid as
\[ \mathbf{H} = H_\infty (x_1 h'(x_2) \mathbf{e}_1 - [h'(x_2) + ck(x_2)] \mathbf{e}_2 + c x_2 \mathbf{e}_3), \]
\[ x_2 \geq 0, \quad \text{and} \]
\[ \mathbf{H}_s = H_\infty (x_1 h'(x_2) \mathbf{e}_1 - [h'(x_2) + ck(x_2)] \mathbf{e}_2 + c x_2 \mathbf{e}_3), \]
\[ x_2 \leq 0. \]  
respectively, where \( h, k, h_s, k_s \) are sufficiently regular unknown functions to be determined.

We ask that \( \mathbf{H} \) tends to \( \mathbf{H}_s \) as \( x_2 \to +\infty \) so that \( \mathbf{H} \) and \( \mathbf{v} \) are parallel at infinity and
\[ \lim_{x_2 \to +\infty} h'(x_2) = 1, \quad \lim_{x_2 \to +\infty} k'(x_2) = 1, \]
\[ \lim_{x_2 \to +\infty} [h(x_2) - x_2] = 0, \quad \lim_{x_2 \to +\infty} [k(x_2) - x_2] = 0. \]  

We now analyse the situation in the solid. First of all, we assume that \( \mathbf{H}_s \) satisfies the following conditions
(i) it is not uniform;
(ii) its non-degenerate field lines belong to surfaces which asymptote to the plane \( x_2 = 0 \) as \( |x_1|, |x_3| \to +\infty \).

The properties of the solid imply
\[ \mathbf{E}_s = 0, \quad \nabla \times \mathbf{H}_s = 0, \quad \text{in } \delta^-, \]
from which we get
\[ h_s(x_2) = C_1 x_2 + C_2, \quad k_s(x_2) = C_3 x_2 + C_4, \quad x_2 \leq 0, \]
where \( C_1, C_2, C_3, C_4 \in \mathbb{R} \).

By virtue of the continuity of the tangential components of the magnetic field across the plane \( x_2 = 0 \), we find
\[ C_1 = h'(0), \quad C_3 = k'(0), \]
so that
\[ \mathbf{H}_s = H_\infty [h'(0) x_1 \mathbf{e}_1 - [(h'(0) + ck'(0)) x_2 + C_2 + c C_4] \mathbf{e}_2 + c (0) x_2 \mathbf{e}_3]. \]  

\( \mathbf{H}_s \) is uniform if \( h'(0) = k'(0) = 0 \); hence to satisfy (i) we proceed assuming \( h'(0) \neq 0 \) or \( k'(0) \neq 0 \). In this case the magnetic field lines in the solid are
\[ x_1 = B_1 e^{h_\infty h'(0) x_2}, \quad x_2 = B_2 e^{h_\infty [h'(0) + ck'(0)] x_2 + c C_4}, \]
where \( B_1, B_2, B_3 \) are some real constants.

The non-degenerate field lines belong to the surface
\[ x_1 x_2 x_3 = B_1 B_3, \quad x_2 \leq 0, \quad C_2 + C_4 = 0, \]
so that
\[ \mathbf{H}_s = H_\infty [h'(0) x_1 \mathbf{e}_1 - [(h'(0) + ck'(0)) x_2 \mathbf{e}_2 + c k'(0) x_2 \mathbf{e}_3], \]
\[ x_2 \leq 0. \]  

As far as the electric field \( \mathbf{E} = E_1 \mathbf{e}_1 + E_2 \mathbf{e}_2 + E_3 \mathbf{e}_3 \) is concerned, the boundary conditions require that
\[ E_1 = 0, \quad E_3 = 0 \quad \text{at } x_2 = 0. \]  
From (5) follows
\[ \mathbf{E} = -\nabla \psi. \quad (\text{with } \psi \in C^2(\delta) \text{ electrostatic potential}), \]
and (5) furnishes
\[ \frac{\partial \psi}{\partial x_1} = -\frac{H_\infty}{\sigma_e} c x_3 [k'(0) + a (1 + c) k'(0)](x_2) x_2 - (h(x_2) + ck(x_2))]). \]  

We now consider the inviscid fluid filling the half-space \( \delta \) and suppose \( \mathbf{H} \) given by (8). First of all, thanks to the continuity of the normal component of \( \mathbf{B} \) across the boundary \( x_2 = 0 \), we deduce
\[ h(0) + ck(0) = 0, \quad \forall c \in (-1, +\infty), \quad c \neq 0. \]  

Remark 2. The previous result holds even if \( \delta \) is occupied by a viscous fluid for which \( \mathbf{H} \) has the form (8).

Finally, we remark that \( \nabla \times \mathbf{H} = 0 \) so that the pressure field is not influenced by the magnetic field:
\[ p = -\frac{1}{2} \rho a^2 [x_1^2 + (1 + c)^2 x_2^2 + c^2 x_3] + p_0, \]
\[ (x_1, x_3) \in \mathbb{R}^2, \quad x_2 \geq 0, \]  
where \( p_0 \) is the pressure at the stagnation-point.

Remark 3. The previous results state that if \( \delta \) is occupied by an inviscid fluid, then the induced magnetic field in the solid and in the fluid vanishes.

Remark 4. In order to study the MHD three-dimensional stagnation-point flow for a viscous fluid, we suppose that the inviscid fluid impinges on the flat plane \( x_2 = C \) (constant called displacement thickness), so that
\[ \mathbf{v} = a [x_1 \mathbf{e}_1 - (1 + c)(x_2 - C) \mathbf{e}_2 + c x_3 \mathbf{e}_3], \]
\[ \mathbf{H}_s = H_\infty [x_1 \mathbf{e}_1 - (1 + c)(x_2 - C) \mathbf{e}_2 + c x_3 \mathbf{e}_3], \]
\[ (x_1, x_3) \in \mathbb{R}^2, \quad x_2 \geq C, \]
\[ \mathbf{H} \to H_\infty [x_1 \mathbf{e}_1 - (1 + c)(x_2 - C) \mathbf{e}_2 + c x_3 \mathbf{e}_3] \quad \text{as } x_2 \to +\infty. \]
In this way, the stagnation point is \((0, C, 0)\) and
\[
p = -\frac{1}{2} \rho a^2 [x_1^2 + (1 + c)^2(x_2 - C)^2 + C^2] + p_0,
\]
\[
\mathbf{H} = \mathbf{H}_e = H_{\infty} [x_1 \mathbf{e}_1 - (1 + c)(x_2 - C) \mathbf{e}_2 + C \mathbf{e}_3],
\]
\((x_1, x_2) \in \mathbb{R}^2, \ x_2 \geq C.\)  \(\text{(24)}\)

### 3. Newtonian fluids

Consider now the previous problem for a Newtonian fluid. The equations governing the flow in the absence of external mechanical body forces and free electric charges are \((5)_{2-6}\) together with
\[
\mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p + \nu \Delta \mathbf{v} + \frac{\mu_f}{\rho} (\nabla \times \mathbf{H}) \times \mathbf{H}, \quad \text{in } \Omega,
\]
where \(\nu\) is the kinematic viscosity.

As far as the boundary conditions are concerned, we modify only the condition for \(\mathbf{v}\), assuming the no-slip boundary condition
\[
\mathbf{v}|_{x_2=0} = \mathbf{0}.
\]
\(\text{(26)}\)

The three-dimensional stagnation-point flow of such a fluid is determined by a velocity field of the form
\[
v_1 = ax f'(x_2), \quad v_2 = -af(x_2) + cg(x_2),
\]
\[
v_3 = cax \left[\psi(x_1, x_2) \mathbf{e}_3, \quad (x_1, x_2) \in \mathbb{R}^2, \ x_2 \geq 0,
\]
with \(f, g\) sufficiently regular functions satisfying
\[
f(0) = 0, \quad f'(0) = 0, \quad g(0) = 0, \quad g'(0) = 0.\]
\(\text{(28)}\)

As for the inviscid fluid, we suppose that the external magnetic field
\[
\mathbf{H}_e = H_{\infty} [x_1 \mathbf{e}_1 - (1 + c)x_2 \mathbf{e}_2 + C \mathbf{e}_3]
\]
permeates the whole physical space and that the external electric field is absent.

We seek the total magnetic field in the fluid in the form \((8)\), where, thanks to the continuity of the normal component of \(\mathbf{B}\) across the boundary, \(h, k\) satisfy
\[
h(0) + ck(0) = 0, \quad \forall c \in (-1, +\infty), \ c \neq 0.\]
\(\text{(29)}\)

Further we impose

**Condition P.** The MHD three-dimensional stagnation-point flow of a viscous fluid approaches at infinity the flow of an inviscid fluid whose velocity, pressure and magnetic field are given by \((23)\) and \((24)\).

We then append to \((5)_{2-6}\) and \((25)\) the following conditions
\[
\lim_{x_2 \to +\infty} f'(x_2) = 1, \quad \lim_{x_2 \to +\infty} g'(x_2) = 1, \quad \text{together with } (9)_{1, 2}.\]
\(\text{(30)}\)

The asymptotic behaviour of the functions \(f, g, h\) and \(k\) at infinity depends on the constant \(c\) in \((23)\) as
\[
\lim_{x_2 \to +\infty} f(x_2) = -A, \quad \lim_{x_2 \to +\infty} g(x_2) - x_2 = -B, \quad \text{(31)}
\]
\[
\lim_{x_2 \to +\infty} h(x_2) = -A, \quad \lim_{x_2 \to +\infty} k(x_2) = -B, \quad \text{(32)}
\]
\[
\lim_{x_2 \to +\infty} f(x_2) + cg(x_2) - (1 + c)x_2 = -(1 + c)C,
\]
\[
\lim_{x_2 \to +\infty} h(x_2) + ck(x_2) - (1 + c)x_2 = -(1 + c)C,
\]
so that
\[
C = \frac{A + cB}{1 + c}
\]
and
\[
\mathbf{v} \times \mathbf{H} = \mathbf{0} \quad \text{at infinity}. \quad \text{(34)}
\]

The constants \(A, B, C\) are not assigned a priori, but their values can be found by solving numerically the problem.

Our purpose is now to prove that, under the no-restrictive hypothesis on \(h\):
\(h\) vanishes at most at isolated points,
the only possibility is the axisymmetric flow, i.e. \(c = 1, f = g, \ h = k\).

First of all, as for the inviscid fluid, from \((5)\) follows that
\[
\mathbf{E} = -\nabla \psi.
\]
\(\text{Moreover } (5)\) provides
\[
\frac{\partial \psi}{\partial x_1} = -\frac{H}{\sigma_e} cax [k'(x_2) + \sigma_e \mu_e a(f(x_2) + cg(x_2))k'(x_2)] - (h(x_2) + ck(x_2))g'(x_2),
\]
\[
\frac{\partial \psi}{\partial x_2} = H_{\infty} \mu_e acx_1 x_2 [h'(x_2)g'(x_2) - k'(x_2)f'(x_2)],
\]
\[
\frac{\partial \psi}{\partial x_3} = -\frac{H}{\sigma_e} x_1 [h''(x_2) - \sigma_e \mu_e a(f(x_2) + cg(x_2))h'(x_2) - (h(x_2) + ck(x_2))f'(x_2)] + (35)
\]
Since \(\mathbf{E}\) is divergence free, from \((35)\), we get
\[
H_{\infty} \mu_e acx_1 x_2 [h'(x_2)g'(x_2) - k'(x_2)f'(x_2)] = 0,
\]
\(\forall(x_1, x_2) \in \mathbb{R}^2, \ x_2 \geq 0,\)
which for the conditions at infinity \((9)\) and \((30)\), gives
\[
h'(x_2)g'(x_2) = k'(x_2)f'(x_2), \quad \forall x_2 \geq 0. \quad \text{(36)}
\]
The equality \((36)\) implies the following relationships of proportionality
\[
k'(x_2) = l(x_2)h'(x_2), \quad g'(x_2) = l(x_2)f'(x_2), \quad \text{where } l = l(x_2) \text{ is a sufficiently regular unknown function satisfying the condition}
\]
\[
l(x_2) = 1. \quad \text{(38)}
\]
From \((36)\) we have \(\psi = \psi(x_1, x_2)\).
Then from \((35)\) and from the behaviour at infinity we deduce
\[
l'(x_2)h'(x_2) = h'(x_2), \quad \forall x_2 \geq 0, \quad \text{(41)}
\]
From relation \((41)\), hypothesis \((h)\) and \((38)\), we find \(l(x_2) = 1, \ \forall x_2 \geq 0,\)
so that the relationships \((37)\) are reduced to
\[
k'(x_2) = h'(x_2), \quad g'(x_2) = f'(x_2). \quad \text{(42)}
\]
The relation \((42)\) together with boundary conditions \((28)\) involves
\[
g(x_2) = f(x_2) \text{ and } \ A = B = C, \quad \forall x_2 \geq 0, \quad \text{(43)}
so that
\[
k(x_2) = h(x_2), \quad \forall x_2 \geq 0, \quad \text{(44)}
\]
and from \((29)\)
\[
h(0) = 0, \quad k(0) = 0. \quad \text{(45)}
\]
On substituting \((42)\) into \((39)\), we find that \(h\) has to satisfy
\[
h''(x_2) + \sigma_e \mu_e a(1 + c)[f(x_2)h'(x_2) - h(x_2)f'(x_2)] = 0. \quad \text{(46)}
\]
We now proceed in order to determine $p$, $f$. We substitute (43), (44) and (27) into (25) to obtain
\[
ax_1 \left[ \nu f''' + a(1+c)ff'' - af'^2 - \frac{\mu e}{\rho a} H^2 \nu (1+c)hh'' \right] = \frac{\partial p}{\partial x_1},
\]
\[
\nu a(1+c)f'' - a^2(1+c)^2ff' + \frac{\mu e}{\rho a} H^2 (x_1^2 + c^2 x_2^2) hh'' = \frac{\partial p}{\partial x_2},
\]
\[
ax_3 \left[ \nu f''' + a(1+c)ff'' - acf'^2 - \frac{\mu e}{\rho a} H^2 c(1+c)hh'' \right] = \frac{\partial p}{\partial x_3}.
\]
(47)

Then, by integrating \((47)_2\) and by supposing that, far from the wall, the pressure $p$ has the same behaviour as for an inviscid fluid, whose pressure is given by (24), we get
\[
p = -\rho \frac{\partial^2}{\partial x_1^2} \left[ x_1^2 + (1+c)^2 f''(x_2) + c^2 x_2^2 \right] - \rho a(1+c) f'(x_2)
- \frac{\mu e}{2} H^2 (x_2^2 + c^2 x_2^2) [h'^2(x_2) - 1] + p_0.
\]
(48)

In consideration of (48), we have
\[
\frac{\nu}{a} f'''' + (1+c)ff'' - f'^2 + 1 - \beta_m[(1+c)hh'' - h'^2 + 1] = 0,
\]
\[
\frac{\nu}{a} f'''' + (1+c)ff'' - cf'^2 + c
- \beta_m[(1+c)hh'' - ch'^2 + c] = 0
\]
with $\beta_m = \frac{\mu_e h^2}{\rho c}$ (Alfvén number).

Eqs. (49) are compatible if, and only if,
\[
(c - 1)[f'^2 - 1 - \beta_m(h'^2 - 1)] = 0.
\]
(50)

This equation implies either $c = 1$ or $f'^2 - 1 - \beta_m(h'^2 - 1) = 0$. We will prove that $c = 1$.

Actually, if we suppose
\[
f'^2 - 1 - \beta_m(h'^2 - 1) = 0, \quad \forall x_2 \geq 0,
\]
(51)

we get
\[
h^2(0) = \frac{\beta_m - 1}{\beta_m},
\]
(52)

which gives an absurdum if $\beta_m < 1$.

We now turn to the case $\beta_m \geq 1$.

On substituting (51) into (49), and taking into account (46), we find that $(f, h)$ satisfies
\[
\frac{\nu}{a} f'''' + (1+c)ff'' - \beta_m(1+c)hh'' = 0,
\]
\[
h'' + \alpha_0 \mu e a(1+c)(fh' - hf') = 0,
\]
(53)

together with the boundary conditions
\[
f(0) = 0, \quad f'(0) = 0, \quad h(0) = 0,
\]
\[
\lim_{x_2 \to +\infty} f(x_2) = 1, \quad \lim_{x_2 \to +\infty} h'(x_2) = 1.
\]
(54)

Combining (53), its derivative and the twice differentiating of (51), we obtain $f''(0) = 0$.

If we consider the Cauchy problem obtained by adding to (53) the initial conditions
\[
f(0) = 0, \quad f'(0) = 0, \quad f''(0) = 0,
\]
\[
h(0) = 0, \quad h'(0) = \pm \sqrt{\frac{\beta_m - 1}{\beta_m}},
\]
(55)

then its unique solution is given by
\[
f(x_2) = 0, \quad h(x_2) = \pm \sqrt{\frac{\beta_m - 1}{\beta_m}} x_2, \quad \forall x_2 \geq 0,
\]
(56)

which is clearly absurdum for boundary conditions (54)_4-5. Hence $(f, h)$ satisfies problem (53), (54) with $c = 1$.

Therefore we have proved that the motion is described by
\[
v = a(x_1 f''(x_2) e_1 - 2f(x_2) e_2 + x_1 f'(x_2) e_3),
\]
\[
\vec{H} = H \times [x_1 h'(x_2) e_1 - 2h(x_2) e_2 + x_1 h'(x_2) e_3],
\]
\[
(x_1, x_2) \in \mathbb{R}^2, \quad x_2 \geq 0,
\]
where $(f, h)$ satisfies
\[
\frac{\nu}{a} f'''' + 2ff'' - f'^2 + 1 - \beta_m(2hh'' - h'^2 + 1) = 0,
\]
\[
h'' + 2\alpha_0 \mu e a(fh' - hf') = 0,
\]
\[
f(0) = 0, \quad f'(0) = 0, \quad h(0) = 0,
\]
\[
\lim_{x_2 \to +\infty} f(x_2) = 1, \quad \lim_{x_2 \to +\infty} h'(x_2) = 1.
\]
(57)

4. Numerical results and discussion

In order to solve numerically the boundary value problem (57) we write it in dimensionless form by putting
\[
\eta = \sqrt{\frac{a}{\nu}} x_2, \quad \psi(\eta) = \sqrt{\frac{a}{\nu}} f \left( \sqrt{\frac{\nu}{a}} \eta \right),
\]
\[
\Psi(\eta) = \sqrt{\frac{a}{\nu}} h \left( \sqrt{\frac{\nu}{a}} \eta \right),
\]
(58)

so that
\[
\frac{\psi'''}{a} + 2\psi'''' - \psi'^2 + 1 - \beta_m(2\psi'' - \psi'^2 + 1) = 0,
\]
\[
\frac{\psi'''}{a} + 2R_m(\psi'\psi'' - \psi''') = 0,
\]
\[
\psi(0) = 0, \quad \psi'(0) = 0, \quad \Psi(0) = 0,
\]
\[
\lim_{\eta \to +\infty} \psi'(\eta) = 1, \quad \lim_{\eta \to +\infty} \Psi'(\eta) = 1,
\]
(59)

where
\[
R_m = \alpha_0 \mu e a v
\]
is the magnetic Reynolds number (or Prandtl magnetic number).

System (59) is strongly influenced by the electromagnetic parameters $R_m$ and $\beta_m$. We solve the problem only for $\beta_m < 1$ in order to preserve the parallelism of $H$ and $v$ at infinity [8].

If $R_m \geq 1$ and $\beta_m$ is not close to 1, then problem (59) can be easily solved numerically by using the MATLAB routine bvp4c. Such a routine is a finite difference code that implements the three-stage Lobatto IIA formula. This is a collocation formula and here the collocation polynomial provides a $C^1$-continuous solution that is fourth-order accurate uniformly in $[0, 5]$. Mesh selection and error control are based on the residual of the continuous solution. We set the relative and the absolute tolerance equal to $10^{-7}$. The method was used and described in [9].

For small values of $R_m$ or when $\beta_m$ is close to 1, the viscosity effects appear in a layer lining the boundary whose thickness is much larger than for other values of these parameters. Therefore the problem should be solved in a range much larger than the previous, but the numerical method does not converge. This difficulty can be overcome if we use the transformation
\[
\xi = T \eta, \quad \phi(\xi) = T \circ \psi \left( \frac{\xi}{T} \right), \quad \Psi(\xi) = T \circ \Psi \left( \frac{\xi}{T} \right),
\]
(60)
In transformation (60), the constant \( T \) takes different values depending on \( R_m \) and \( \beta_m \). More precisely:

- if \( R_m < 1 \) and \( \beta_m \) is not close to 1, then \( T = \sqrt{R_m} \) and \( \eta \in [0, 3] \) [12];
- if \( R_m \geq 1 \) and \( \beta_m \) is close to 1, then \( T = \sqrt{1 - \beta_m} \) and \( \eta \in [0, 3] \);
- if \( R_m < 1 \) and \( \beta_m \) is close to 1, then \( T = \sqrt{R_m(1 - \beta_m)} \) and \( \eta \in [0, 2] \).

The corresponding interval in which \( \eta \) varies becomes much greater as \( T \) is small. This fact explains why in the third transformation the interval of the integral in which \( \xi \) varies is smaller than in the other cases.

In Fig. 3, we furnish the profiles \( \psi, \psi', \psi'' \) for \( R_m = 1 \) and \( \beta_m = 0.5 \), while Fig. 3; shows the behaviour of \( \phi, \phi' \) for the same values of \( R_m \) and \( \beta_m \).

When \( R_m \neq 1 \) and \( \beta_m \neq 0.5 \), the profiles of \( \psi, \psi', \psi'' \), \( \phi, \phi' \) are analogous to those shown in Fig. 3.

The numerical solution \((\psi, \phi)\) of the problem has the expected behaviour, i.e.

\[
\lim_{\eta \to \infty} [\psi(\eta) - \eta] = -\alpha \quad \text{and} \quad \lim_{\eta \to \infty} [\phi(\eta)] = -\alpha,
\]

with \( \alpha = \sqrt{2 A} \).

Hence we denote by \( \delta \) the value of \( \eta \) such that \( \eta > \delta \), then \( \phi \equiv \eta - \alpha \) (\( \phi'(\delta) = 0.99 \)), so that the influence of the viscosity appears only in a layer lining the boundary whose thickness is \( \delta \).

We provide Table 1 to show the values of \( \alpha, \phi'(0), \phi'(0), \delta \) when \( R_m \) and \( \beta_m \) change.

From the Table we see that \( \alpha \) increases, while \( \phi'(0) \) and \( \phi'(0) \) decrease as \( \beta_m \) increases. Further \( \alpha, \phi'(0) \) and \( \phi'(0) \) decrease as \( R_m \) increases.

Table 1 elucidates that the thickness \( \delta \) of the boundary layer depends on \( R_m \) and \( \beta_m \) more precisely:

- \( \delta \) increases when \( \beta_m \) increases (Fig. 41);
- \( \delta \) decreases when \( R_m \) increases (Fig. 42).

This behaviour is not surprising because \( \beta_m \) is a measure of the strength of the applied magnetic field and as it is underlined in [8] when the magnetic field is strong the disturbances are no longer contained within a boundary layer along the wall. This means that

### Table 1

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<th>( R_m )</th>
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<th>( \phi'(0) )</th>
<th>( \phi'(0) )</th>
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### 5. Conclusions

In this paper we study the MHD three-dimensional stagnation-point flow of a Newtonian fluid when the total magnetic field is parallel to the velocity at infinity. The region where the fluid motion occurs is bordered by the boundary of a solid obstacle which is a rigid uncharged dielectric at rest. By means of similarity transformations, we reduce the MHD PDEs to a nonlinear system of ODEs which depends on two parameters \( R_m \) (Reynolds number) and \( \beta_m \) (Alfvén number) characterizing the magnetic effects. This system has been numerically integrated.

The results obtained show that

- The total magnetic field and the velocity are parallel at infinity only if the flow is axisymmetric.
Fig. 4. Plots showing the behaviour of the boundary layer for different $\beta_m$ and $R_m$, respectively.

Fig. 5. Plots showing the behaviour of the boundary layer.

- The thickness of the boundary layer depends on $R_m$ and $\beta_m$: it increases when $\beta_m$ increases or $R_m$ decreases.

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References