Magnetoconvection of a micropolar fluid in a vertical channel

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1. Introduction

The recent industrial processes are characterized by the use of new materials which cannot be described by Newtonian fluids. Due to this reason, many non-Newtonian models have been proposed. Among these models, the micropolar fluids have been introduced by Eringen [1] in order to take into account the effects of local structure and micro-motions of the fluid particles which cannot be described by the classical models. The incompressible micropolar fluids represent liquids consisting of rigid, randomly oriented spherical particles suspended in a viscous medium, where the deformation of fluid particles is ignored. The related mathematical model is based on the introduction of a new vector field (the microrotation) which describes the total angular velocity field of the particles rotation. Hence, one new equation is added representing the balance law of local angular momentum.

There are many examples of flows of micropolar fluids that are relevant for practical applications as flows of biological fluids in thin vessels, polymeric suspensions, liquid crystals, slurries, colloidal fluids, exotic lubricants, etc. Extensive reviews of the theory and its applications can be found in [2,3].

The purpose of the present paper is to study the influence of an external uniform magnetic field on the mixed convection in the fully developed flow of a micropolar fluid filling a vertical channel under the Oberbeck–Boussinesq approximation. A systematic and rigorous derivation of this approximation is provided in [4].

Convection flow of an electrically conducting fluid in a channel under the effect of a transverse magnetic field has a relevant technical significance because of its many industrial applications such as geothermal reservoirs, cooling of nuclear reactors, electric transmission cables, thermal insulation and petroleum reservoirs, to name a few.

In our study we solve the problem of the mixed convection of a Boussinesqian electrically conducting micropolar fluid which steadily flows in a vertical channel under the action of a uniform magnetic field applied normal to the direction of the velocity. The walls are maintained at constant temperatures $T_1$ and $T_2$ ($T_1 < T_2$).

The first paper on the fully developed free convection of a micropolar fluid in a vertical channel is [5]; this work has been generalized in [6] in order to consider also the mass transfer. In [7–9] mixed convection flow with symmetric and asymmetric heating is examined. To the best of our knowledge, few results are known concerning the influence of an external magnetic field on the convective flow of a micropolar fluid in a vertical channel [10,11], while in recent years the same situation in a double channel has been studied in [12]. However, in most of the previous papers, a restrictive condition on the material parameters has been imposed following the work of Ahmadi [13]. We point out that in our research we have not required any condition so that two...
material parameters describe the micropolar nature of the fluid, instead of one as in the simplified Ahmadi’s approach.

In our paper, as it is usual in the Oberbeck–Boussinesq approximation [14], we neglect the dissipation terms in the energy equation, so that we can obtain the explicit solution of the problem which takes into account the induced magnetic field. We point out that the induced magnetic field is neglected in most of the works concerning the convective flow in a vertical channel, also in the simpler case of a Newtonian fluid.

The paper is organized in this way:

In Section 2 we formulate the problem from the physical point of view. In order to determine the analytical solution, we have to distinguish three cases which are related to the strength of the external uniform magnetic field.

Section 3 is devoted to integrate the boundary value problem which describes the motion in the three cases.

In Section 4 we make some comments about the flow and we give the solution when the heating is symmetric, in the case of natural convection, in the absence of magnetic field and in the same problems for the Newtonian fluid.

The trend of the solution is plotted in Section 5 in order to show the influence of the relevant parameters on the flow. The behavior of the micropolar flow differs highly from the Newtonian one as the coupling number increases and the micropolar parameter decreases. For suitable values of the buoyancy parameter \( \lambda \), the reverse flow occurs near the coldest (hottest) wall if \( \lambda > 0 \) (\( \lambda < 0 \)). The presence of the external magnetic field tends to prevent the occurrence of the reverse flow. If the buoyancy parameter vanishes (symmetric heating), then the phenomenon of the reverse flow does not appear.

Section 6 summarizes the results.

2. Formulation of the problem

Let us consider a Boussinesquian, electrically conducting micropolar fluid filling the region \( S \) between two infinite rigid, fixed, non-electrically conducting vertical plates \( \Pi_1, \Pi_2 \) separated by a distance \( 2d \) (Fig. 1).

We assume the regions outside the plane to be a vacuum (free space). The coordinate axes are fixed in order to have

\[
S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_3) \in \mathbb{R}^2, \ x_2 \in (-d, d)\}, \\
\Pi_i = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_3) \in \mathbb{R}^2, \ x_2 = (-1)^i d\}, \ i = 1, 2
\]

(1)

and \( x_1 \)-axis is vertical upward.

Our aim is to study the steady mixed convection in the fully developed flow of the fluid under the action of an external uniform magnetic field \( H_0 \mathbf{e}_2 \) normal to planes \( \Pi_{1,2} \) (\( H_0 > 0 \)).

This flow in the absence of external mechanical body forces, body couples and free electric charges under the Oberbeck–Boussinesq approximation is governed by [3,1,2]

\[
\rho_0 \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + (\mu + \mu_i) \Delta \mathbf{v} + 2\mu_i (\nabla \times \nabla \mathbf{w}) + \mu_i (\nabla \times \mathbf{H}) \times \mathbf{H} + \rho_0 (1 - \chi T (T - T_0)) \mathbf{g}, \\
\rho_0 \mathbf{v} \cdot \nabla \mathbf{w} = (c_0 + c_2) \Delta \mathbf{w} + (c_0 + c_2 - c_0) \nabla (\nabla \cdot \mathbf{w}) + 2\mu_i (\nabla \times \nabla \mathbf{w}) + 2\mu_i (\nabla \times \mathbf{v} - 2\mathbf{w}), \\
\nabla \cdot \mathbf{v} = 0, \\
\eta_e \Delta \mathbf{H} = \nabla \times (\mathbf{H} \times \mathbf{v}), \\
\nabla \cdot \mathbf{H} = 0, \\
\n\nabla T \cdot \mathbf{v} = b \triangle T, \ \text{in} \ S.
\]

Greek symbols

\( \alpha, \beta, \gamma \) dimensionless constants defined by (11)

\( \alpha_F \) thermal expansion coefficient

\( \varepsilon \) electrical permittivity \( (\varepsilon = \frac{1}{\varepsilon_{s\varepsilon}}) \)

\( \eta_e \) dimensionless temperature defined by (6)_{14}

\( \lambda \) buoyancy coefficient defined by (6)_{10}

\( \mu \) Newtonian viscosity coefficient \( (\mu > 0) \)

\( \mu_2 \) magnetic permeability

\( \mu_i \) dynamic microrotation viscosity coefficient

\( \eta \) constant defined by (6)_{9}

\( \eta_e \) microinertia coefficient

\( k \) fluid thermal conductivity

\( I \) characteristic length defined by (6)_{2}

\( l \) dimensionless constant defined by (6)_{3}

\( M_l \) Hartman number defined by (6)_{4}

\( M_w \) micropolar parameter defined by (6)_{6}

\( N_l \) coupling number defined by (6)_{1}

\( (0 < N_l^2 < 1) \)

\( \nu \) Nusselt number

\( p \) pressure

\( \rho \) arbitrary constant

\( q \) heat flux vector

\( Re \) Reynolds number defined by (6)_{8}

\( T = T(x_2) \) temperature

\( T_0 \) reference temperature

\( T_1, T_2 \) uniform temperatures \( (T_2 \geq T_1) \)

\( \mathbf{v} \) velocity field

\( \nu(y) \) dimensionless function describing the velocity defined by (6)_{12}

\( \nu_1(x_2) \) characteristic velocity defined by (6)_{6}

\( \mathbf{w} \) velocity component in the \( x_1 \)-direction

\( \mathbf{w}(y) \) dimensionless function describing the microrotation defined by (6)_{13}

\( \mathbf{w}_3(x_2) \) microrotation component in the \( x_3 \)-direction

\( \mathbf{y} \) dimensionless transverse coordinate defined by (6)_{11}

Nomenclature

\( b \) thermal diffusivity

\( C \) constant such that \( P = -C_1 + P_0 \)

\( C_0, C_d, C_a \) angular viscosity coefficients

\( 2d \) channel width

\( E \) electric field

\( g = -ge_1 \) gravity acceleration

\( H \) total magnetic field

\( h(y) \) dimensionless function describing the induced magnetic field defined by (6)_{15}

\( H_0 \mathbf{e}_2 \) external uniform magnetic field \( (H_0 > 0) \)

\( H_1(x_2) \) induced magnetic field component in the \( x_1 \)-direction

\( I \) microinertia coefficient

\( k \) fluid thermal conductivity

\( k_{1,2} \) heat transfer coefficients evaluated at \( \Pi_{1,2} \)

\( l \) characteristic length defined by (6)_{2}

\( L \) dimensionless constant defined by (6)_{3}

\( M_l \) Hartman number defined by (6)_{4}

\( M_w \) micropolar parameter defined by (6)_{6}

\( N_l \) coupling number defined by (6)_{1}

\( (0 < N_l^2 < 1) \)

\( Nu \) Nusselt number

\( p \) pressure

\( P = p + \mu_i \frac{\mu_i}{2} + \rho_0 \gamma_1 \) difference between the hydromagnetic pressure and the hydrostatic pressure

\( p_0 \) arbitrary constant

\( \mathbf{q} \) heat flux vector

\( Re \) Reynolds number defined by (6)_{8}

\( T = T(x_2) \) temperature

\( T_0 \) reference temperature

\( T_1, T_2 \) uniform temperatures \( (T_2 \geq T_1) \)

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\( \mathbf{y} \) dimensionless transverse coordinate defined by (6)_{11}
All the material parameters are positive constants and $\mu_0$ is equal to the magnetic permeability of free space. As it is usual in the Boussinesq approximation [14], in Eq. (2) the dissipative terms have been neglected.

We note that in [1] and in [2] Eqs. (2) are slightly different because they are deduced as a special case of a much more general model of microfluids. For the details we refer to [3, p. 23]. In particular the relation between the parameters of Eringen ($\mu_v, k_v, \gamma_v$) used in [1,2] and the parameters of Lukaszewicz ($\mu_v, k_v, c_0, c_3, c_4$) used in [3] is the following:

$$\mu = \mu_v + \frac{k_v}{2}, \quad k_v = \frac{k_v}{2}, \quad c_0 + c_d = \gamma.$$  

As it is underlined by Eringen ([2, p. 23], II), the Newtonian viscosity coefficient is equivalent to $\mu_v + \frac{k_v}{2}$, i.e. $\mu$.

We search $v$, $w$, $H$, $T$ in the following form:

$$v = v_1(x_3) e_1, \quad w = w_3(x_2) e_1, \quad H = H_0 e_2 + H_1(x_2) e_1, \quad T = T(x_2),$$

where $(e_1, e_2, e_3)$ is the canonical base of $\mathbb{R}^3$. Thank to (3), $v$, $w$, $H$ are divergence free.

The previous unknown functions satisfy the boundary conditions

$$v_1(\pm d) = 0, \quad w_3(\pm d) = 0, \quad H_1(\pm d) = 0, \quad T(-d) = T_1, \quad T(d) = T_2.$$  

We choose the reference temperature $T_0 = \frac{T_1 + T_2}{2}$. If $T_1 = T_2$, then the heating is called symmetric, if $T_2 > T_1$, then the heating is called asymmetric.

At the moment we suppose $T_2 > T_1$.

By virtue of (3) and (2), we deduce $P = P(x_3) = -C x_3 + p_0 (C, p_0$ some constants) so that the governing equations give:}

$$ (\mu + \mu_v) v_1'' + 2 \mu v_3' + \mu H_0 H_1' + \rho_0 x_3(T - T_0) g = -C, \quad (c_0 + c_d) w_3'' - 2 \mu v_3' - 4 \mu H_0 v_3' = 0, \quad \eta_0 H_1' + H_0 v_3' = 0, \quad T'' = 0, \quad \text{in } [-d, d].$$

By putting

$$N^2 = \frac{H_0^2}{\mu + \mu_v}, \quad l^2 = \frac{c_0 + c_d}{4\mu}, \quad L = \frac{d}{T}, \quad M_0^2 = N_0^2 l^2, \quad M^2 = \frac{\sigma_0}{\mu} l^2 H_0^2 l^2,$$

$$V_0 = \frac{c_0}{\mu}, \quad V_0 = \frac{\mu}{\rho_0}, \quad \Re = \frac{V_0 d}{V_0}, \quad \Gr = \frac{\sigma_0 g(T_2 - T_1) d^3}{\rho_0 V_0^5}, \quad \frac{\lambda}{\Re} \frac{\Gr}{\Re} = \frac{y}{x_2},$$

$$\gamma = \frac{v_1'(dy)}{V_0}, \quad w(y) = \frac{d w_3(dy)}{V_0}, \quad \vartheta(y) = \frac{T(dy) - T_0}{T_2 - T_1}, \quad h(y) = \frac{H_1(dy)}{V_0 \sqrt{\sigma_0 d \mu}}.$$  

Eqs. (5) can be written in dimensionless form:

$$v'' + 2N^2 w' + M(1 - N^2) h' + \lambda \vartheta + 1 - N^2 = 0,$$

$$w'' - \frac{M_0^2}{2(1 - N^2)} v' - \frac{M_0^2}{1 - N^2} w' = 0,$$

$$h'' + M_0^2 v' = 0,$$

$$\vartheta' = 0, \quad \text{in } [-1, 1].$$

We recall that the coupling number $N^2$ belongs to $(0,1)$. Boundary conditions (4) in dimensionless form become

$$v(\pm 1) = 0, \quad w(\pm 1) = 0, \quad h(\pm 1) = 0, \quad \vartheta(\pm 1) = \pm \frac{1}{2}.$$  

Eq. (7), together with (8), imply

$$\vartheta(y) = \frac{y}{2}, \quad \text{in } [-1, 1].$$

By differentiating (7) and (8) of $w$ and (9), $w$ has to satisfy the following linear ordinary differential equation:

$$w'' - \alpha w' + \beta w = \gamma,$$

where

$$\alpha = M_0^2 (1 - N^2) M^2, \quad \beta = M_0^2 M^2, \quad \gamma = -\frac{j M_0^2}{2(1 - N^2)}.$$  

The general solution of Eq. (10) depends on the sign of the discriminant $\Delta = \alpha^2 - 4 \beta$ of the algebraic equation

$$\xi^2 - \alpha \xi + \beta = 0.$$  

In the next section, we solve problem (7), (8) in the three cases: $\Delta > 0, \Delta = 0, \Delta < 0$. These cases are all possible from the physical point of view because they represent the following situations:

- if $\Delta > 0$, then $0 < H_0 < \sqrt{\frac{\mu}{\rho_0} \frac{M_0^2}{M^2}}$ or $H_0 > \sqrt{\frac{\mu}{\rho_0} \frac{M_0^2}{M^2}}$ weak or strong external uniform magnetic field;
- if $\Delta = 0$, then $H_0 = \sqrt{\frac{\mu}{\rho_0} \frac{M_0^2}{M^2}}$, or $H_0 = \sqrt{\frac{\mu}{\rho_0} \frac{M_0^2}{M^2}}$ critical external uniform magnetic field;
- if $\Delta < 0$, then $\sqrt{\frac{\mu}{\rho_0} \frac{M_0^2}{M^2}} < H_0 < \sqrt{\frac{\mu}{\rho_0} \frac{M_0^2}{M^2}}$ bounded external uniform magnetic field.

3. Solution of the flow

3.1. $\Delta > 0$: weak or strong external uniform magnetic field

In this case, Eq. (12) admits the following real routes

$$\xi_1 = \sqrt{\frac{\alpha + \Delta}{2}} - \frac{1}{2} \sqrt{(M_0 + M)^2 - N_0^2 M_0^2 - \sqrt{(M_0 + M)^2 - N_0^2 M_0^2}},$$

$$\xi_2 = \sqrt{\frac{\alpha + \Delta}{2}} - \frac{1}{2} \sqrt{(M_0 + M)^2 - N_0^2 M_0^2 + \sqrt{(M_0 + M)^2 - N_0^2 M_0^2}},$$

$$\xi_3 = -\xi_1, \quad \xi_4 = -\xi_2.$$  

\(\text{Fig. 1. Physical configuration and coordinate system.}\)
so that the general solution of (10) is given by
\[ w(y) = C_1 \cosh(\zeta_1 y) + C_2 \sinh(\zeta_1 y) + C_3 \cosh(\zeta_2 y) + C_4 \sinh(\zeta_2 y) + \frac{y}{\beta} \] (14)

Thanks to this last equation, we obtain
\[ v(y) = 2\left[ A_1 |C_1 | \sinh(\zeta_1 y) + C_2 \cosh(\zeta_1 y) + A_2 |C_3 | \cosh(\zeta_2 y) + C_4 \cosh(\zeta_2 y) - \frac{y}{\beta} y \right] + C_5, \]

where
\[ D_1^+ = \cosh \xi_1 + A_1^+ \sinh \xi_1, \quad D_2^+ = \cosh \xi_2 + A_2^+ \sinh \xi_2. \] (18)

3.2. \( \Lambda = 0 \): critical external uniform magnetic field

In this situation, Eq. (12) admits the following real routes
\[ \xi_1 = \xi_2 = \sqrt{\frac{2}{A}} = \sqrt{M_p M} = \xi, \quad \xi_3 = \xi_4 = -\xi. \] (19)

After determining the general solution of (7) and imposing the boundary conditions (8)1, 2, 3 and imposing the boundary conditions (8)1, 2, 3, we arrive at:

\[ v(y) = \frac{M_p^2 A_1^+}{2\xi_1} \sinh(\zeta_1 y) - y \sinh(\zeta_1 y) \cosh(\zeta_1 y) - 1 + A_2^+ \sinh(\zeta_1 y) \cosh(\zeta_1 y) + A_1^+ D_1^+ y \cosh(\zeta_1 y) + A_2^+ D_1^+ y \sinh(\zeta_1 y) - 2 \frac{\gamma y}{\beta}, \]

where
\[ D_1^+ = \cosh \xi_1 + A_1^+ \sinh \xi_1, \quad D_2^+ = \cosh \xi_2 + A_2^+ \sinh \xi_2. \] (18)

3.2. \( \Lambda = 0 \): critical external uniform magnetic field

In this situation, Eq. (12) admits the following real routes
\[ \xi_1 = \xi_2 = \sqrt{\frac{2}{A}} = \sqrt{M_p M} = \xi, \quad \xi_3 = \xi_4 = -\xi. \] (19)

After determining the general solution of (7) and imposing the boundary conditions (8)1, 2, 3, we arrive at:

\[ w(y) = \frac{M_p^2 A_2^+}{2\xi_2} \cosh(\zeta_2 y) - y \cosh(\zeta_2 y) \sinh(\zeta_2 y) - 1 + A_1^+ \cosh(\zeta_2 y) \sinh(\zeta_2 y) + A_1^+ D_2^+ y \cosh(\zeta_2 y) + A_2^+ D_2^+ y \sinh(\zeta_2 y) - 2 \frac{\gamma y}{\beta}, \]

where
\[ D_1^+ = \cosh \xi_1 + A_1^+ \sinh \xi_1, \quad D_2^+ = \cosh \xi_2 + A_2^+ \sinh \xi_2. \] (18)

3.3. \( \Lambda < 0 \): bounded external uniform magnetic field

In this situation, Eq. (12) admits the following complex routes
\[ \xi_1 = \frac{1}{2} \sqrt{(M_p + M)^2 - N^2 M^2 - i \sqrt{N^2 M^2 - (M_p - M)^2}}, \]
\[ \xi_2 = \frac{1}{2} \sqrt{(M_p + M)^2 - N^2 M^2 + i \sqrt{N^2 M^2 - (M_p - M)^2}}, \] (22)
\[ \xi_3 = -\xi_1, \quad \xi_4 = -\xi_2. \]

and \( C_i, \ i = 1, \ldots, 6 \) are arbitrary constants.

The solution of our problem is determined by asking that the functions given by (14), (15) satisfy the boundary conditions (8)1, 2, 3. More precisely:
Proceeding as in the previous cases, the solution of the problem (7), (8) is given by:

\[
v(y) = \frac{M^2}{2k} \frac{1}{\sin^2 \sigma + \sinh^2 \delta} \left[ (A_1 \sin \sigma \cosh \delta - A_1 \cos \sigma \sinh \delta) \cos(\sigma y) \cosh(\delta y) - (A_1 \cos \sigma \sinh \delta + A_2 \sin \sigma \cosh \delta) \sin(\sigma y) \sinh(\delta y) + A_2 \sinh \delta \cosh \delta - A_1 \cos \sigma \sin \delta \right] + 2 \frac{\gamma}{\beta} \frac{(A_1 D_1 - A_2 D_2)}{A_1 \sin \sigma \cosh \delta + A_2 \sin \sigma \cosh \delta}
\]

\[
w(y) = \frac{M^2}{4\sigma} \frac{1}{\sin^2 \sigma + \sinh^2 \delta} \left[ (B_1 \sin \sigma \cosh \delta - B_1 \cos \sigma \sinh \delta) \cos(\sigma y) \cosh(\delta y) - (B_1 \cos \sigma \sinh \delta + B_2 \sin \sigma \cosh \delta) \sin(\sigma y) \sinh(\delta y) + B_2 \sinh \delta \cosh \delta - B_1 \cos \sigma \sin \delta \right]
\]

\[
h(y) = \frac{M^3}{2M^2\sigma} \frac{1}{\sin^2 \sigma + \sinh^2 \delta} \left[ (B_1 \sin \sigma \cosh \delta - B_1 \cos \sigma \sinh \delta) \cos(\sigma y) \cosh(\delta y) - (B_1 \cos \sigma \sinh \delta + B_2 \sin \sigma \cosh \delta) \sin(\sigma y) \sinh(\delta y) + B_2 \sinh \delta \cosh \delta - B_1 \cos \sigma \sin \delta \right]
\]

where

\[
\delta = \frac{1}{2} \sqrt{(M_0 M_2)^2 - N^2 M^2}, \quad \sigma = \frac{1}{2} \sqrt{N^2 M^2 - (M_0 - M)^2},
\]

\[
A_1 = \left( 1 - N^2 / M_0 M_2 \right) \delta, \quad A_2 = \left( 1 - N^2 / M_0 M_2 \right) \sigma,
\]

\[
B_1 = M^2 - M^2(1 - N^2), \quad B_2 = 2 M^2 \sigma \delta,
\]

\[
D_1 = A_1 \cos \sigma \sin \delta - A_2 \sin \sigma \cosh \delta + \cos \sigma \cosh \delta, \quad D_2 = A_1 \sin \sigma \cosh \delta + A_2 \cos \sigma \sin \delta + \sin \sigma \sin \delta.
\]

4. Remarks on the flow

In this section, we first make some considerations on the flow which hold in all the three previous cases when the heating is asymmetric.

- It is easy to prove that all the denominators in (17), (20), (23) do not vanish.
- It is interesting to compute the electric field \( E \) associated to the magnetic field. From the Maxwell equation

\[
E = \frac{1}{\sigma_e} \nabla \times H + \mu_e H \times \nu
\]

and \( (3), (3) \) we get that \( E \) is parallel to \( \nu \):

\[
E = -\frac{\mu_e H_0 v_0}{M} |h(y)| + M \nu(y) |e_2|.
\]

Taking into account (7) and the boundary conditions, after some straightforward and long calculations, we obtain that \( E \) is constant and when \( \Delta > 0, \Delta = 0, \Delta < 0 \) it assumes the following form

\[
E_{E_0 e_2} = \frac{\mu_e H_0 v_0}{M} \left[ -\frac{M^2 M_0^2 A_1^2 \sin \delta \cosh \delta - A_1 \sinh \delta \cosh \delta}{\sinh \delta \cosh \delta} \right] e_1,
\]

\[
E_{E_0 e_3} = \frac{\mu_e H_0 v_0}{2M^2 \sigma} \left[ \frac{M^2 M_0^2 A_1 \sin \delta \cosh \delta - A_1 \sinh \delta \cosh \delta}{2 \sinh \delta \cosh \delta} \right] e_1,
\]

\[
E_{E_0 e_3} = \frac{\mu_e H_0 v_0}{M^2} \left[ -\frac{M^2 M_0^2 A_1 \sinh \delta \cosh \delta - A_1 \sin \sigma \cosh \sigma}{\sin^2 \sigma + \sinh^2 \delta} \right] e_3.
\]

We notice that the computation of the electric field is omitted in most of the papers concerning MHD flows.

- Outside the planes where there is a vacuum, by virtue of the usual transmission conditions for the electromagnetic field across \( \Pi_{1,2} \), we have

\[
E = E_0 e_1, \quad H = H_0 e_2.
\]

- From the practical point of view, it is interesting to compute the Nusselt number at \( \Pi_{1,2} \):

\[
Nu_{1,2} = \frac{k_{1,2} d}{k} \left[ \frac{d}{T_2 - T_1} \frac{d}{dx} \right] = \frac{(\Delta + 1)}{2}.
\]

This allows us to compute the heat transfer coefficients \( k_{1,2} \) evaluated at the walls.

The Nusselt number is related to the heat flux vector in the channel, which is given by

\[
q = -\frac{(T_2 - T_1) k}{2d} e_2.
\]

This expression is physically quite reasonable because the heat transfer occurs from the hot wall to the cold one.

- The skin friction (\( \tau_{1,2} \)) and the skin couple friction (\( \tau_{p,1,2} \)) at both plates are given by

\[
\tau_{1,2} = (\mu + \mu_e) \frac{V_0}{d} \nu(\pm 1) e_1, \quad \tau_{p,1,2} = (\mu_e + \mu) \frac{V_0}{d} \nu(\pm 1) e_2.
\]

The expression of \( \tau_{1,2} \) is related to the occurrence of the reverse flow, as we will see in the next section. We now consider other interesting physical situations.

- It is very easy to obtain the corresponding results to Sections \( 3.1, 3.2, 3.3 \) when \( T_1 = T_2 \) (symmetric heating). From (5) we get \( T = T_1 = T_2 \) so that \( \lambda = 0 \) and the expression of \( \nu, w, h \) is deduced by putting \( \gamma = 0 \) in (17), (20), (23).

- In the previous Sections we have considered the mixed convection case assuming the constant \( C = 0 \). In the case of natural convection (\( C = 0 \)), when the heating is asymmetric, the solution is obtained by (17), (20), (23) writing only the terms having coefficient \( \beta \) This fact implies that the electric field vanishes. Of
course in the dimensionless variables, the reference velocity $V_0$ cannot be expressed in terms of $C$.

If $C = 0$ and the heating is symmetric, then the fluid is at rest and the induced magnetic field vanishes ($\nu = w = h = 0$).

- In the absence of external magnetic field and when the heating is asymmetric, the motion is given by

$$v(y) = \frac{1}{2} (1 - y^2) \left[ \frac{\lambda}{6(1 - N^2)} y + 1 \right]$$

$$+ \frac{N^2 \cosh(M_p y) - \cosh(M_p)}{M_p \sinh M_p} + \frac{\lambda N^2 [M_p^2 + 3(1 - N^2)] [\sinh(M_p y) - y \sinh M_p]}{6M_p^2 (1 - N^2) (M_p \cosh M_p - N^2 \sinh M_p)}$$

$$w(y) = -\frac{\lambda (1 - y^2) + y \sinh M_p - \sinh(M_p y)}{8(1 - N^2)}$$

$$+ \frac{\lambda [M_p^2 + 3(1 - N^2)] [\cosh M_p - \cosh(M_p y)]}{12M_p (1 - N^2) (M_p \cosh M_p - N^2 \sinh M_p)}$$

(26)

If $T_1 = T_2$, then $v, w$ can be deduced by putting $\lambda = 0$ in the previous expressions. This result is in agreement with the one obtained by Lukaszewicz for the Poiseuille flow of a homogeneous, incompressible micropolar fluid [3].

- If $S$ is occupied by a Boussinesqian, electrically conducting Newtonian fluid, then its MHD mixed convective flow is governed by (7) with $w = 0$. $N = 0$, (7)$_{3,4}$ and (8)$_{1,3,4}$ and it is given by

$$v(y) = \frac{1}{2} (1 - y^2) \left[ \frac{\lambda}{6(1 - N^2)} y + 1 \right]$$

$$+ \frac{N^2 \cosh(M_p y) - \cosh(M_p)}{M_p \sinh M_p} + \frac{\lambda N^2 [M_p^2 + 3(1 - N^2)] [\sinh(M_p y) - y \sinh M_p]}{6M_p^2 (1 - N^2) (M_p \cosh M_p - N^2 \sinh M_p)}$$

$$w(y) = -\frac{\lambda (1 - y^2) + y \sinh M_p - \sinh(M_p y)}{8(1 - N^2)}$$

$$+ \frac{\lambda [M_p^2 + 3(1 - N^2)] [\cosh M_p - \cosh(M_p y)]}{12M_p (1 - N^2) (M_p \cosh M_p - N^2 \sinh M_p)}$$

(26)

**Fig. 2.** The effect of $N$ on the velocity, on the microrotation and on the induced magnetic field. If $N = 0.2, 0.5, 0.7, 0.9$, then $\Delta > 0$, $\Delta = 0$, $\Delta < 0$, respectively.
\( T_2 > T_1 : \)
\[ \theta(y) = \frac{y}{2}. \]

\[ v(y) = \frac{\cosh M - \cosh(My)}{M \sinh M} - \frac{\sinh(My) - y \sinh M}{2M^2 \sinh M}, \]
\[ h(y) = \frac{\sinh(My) y}{M \sinh M} + \frac{\cosh(My) - \cosh M}{2M^2 \sinh M} + \frac{\kappa}{4M} (1 - y^2), \]
\[ E = E_0 \mathbf{e}_1 = \frac{\mu_0 H_0 V_0}{M^2} \left[ 1 - \frac{M \cosh M}{\sinh M} \right] \mathbf{e}_1, \]
\[ T_2 = T_1 : \]
\[ T = T_1 = T_2 ; \]

\[ T_2 > T_1 : \]
\[ v(y) = \frac{y}{2} \quad v(y) = \frac{1}{2} (1 - y^2) \left( \frac{y}{6} + 1 \right) \]
\[ h(y) = \frac{\sinh(My) y}{M \sinh M} \quad \frac{\kappa}{4M} (1 - y^2). \]

In the case of symmetric heating the previous solution coincides with Hartmann flow (see for example [15]).

In the case of natural convection, either when the heating is symmetrical or not, the solution can be easily obtained by the previous relations writing only the terms having coefficient \( \kappa \).

Finally, as in [16,17], in the absence of the external magnetic field we obtain

\[ T_2 > T_1 : \]
\[ \theta(y) = \frac{y}{2} \quad v(y) = \frac{1}{2} (1 - y^2) \left( \frac{y}{6} + 1 \right) \]
\[ T_2 = T_1 : \]
\[ T = T_1 = T_2 ; \]

Due to the geometry of the problem, all the previous relations hold for all \( y \in [-1, 1] \).

5. Results and discussions

The problem of the mixed magnetoconvection in the fully developed flow of a micropolar fluid filling a vertical channel has been analytically solved. As it is proved in Section 3, in the general case of asymmetric heating, the solution is given by (17) or (20) or (23) according to the strength of the external magnetic field. In any

Fig. 3. The effect of \( M_p \) on the velocity, on the microrotation and on the induced magnetic field. If \( M_p = 1, 2, 5, 10 \), then \( \Delta = 0, \Delta < 0, \Delta > 0, \Delta > 0 \), respectively.
case, the solution depends on the values of some relevant physical dimensionless parameters:

- the coupling parameter $N^0$ which is related to the Newtonian and microrotation viscosity coefficients. $0 < N < 1$ and when

$$N \to 0^+ \quad \text{Eq. (2)}$$

reduces to the corresponding equation for a Newtonian fluid;

- the micropolar parameter $M_p$ which is related to $N$, to the geometry of the problem through $L$ and to the particles size by means of $l$. Actually, the more the particles sizes are small, the more $M_p$ is big;

- the Hartmann number $M^2$ which characterizes the strength of the external magnetic field and the electromagnetic properties of the fluid;

- the buoyancy coefficient $\lambda$ which appears in the analytical solution through $\gamma$ and which is related to the buoyancy forces due to the gravity. When the heating is symmetric, it vanishes. Its sign depends on the one of the characteristic velocity $V_0$.

The micropolar properties of the fluid are described by two parameters ($N$ and $M_p$) unlike most of the papers in the literature.

Table 1
Micropolar: Critical value of $M$

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Table 2
Newtonian: Critical value of $M$

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because we have not employed any condition on the material constants.

The aim of this Section is to present a selected set of graphical results illustrating the effects on the flow of the various parameters involved in the problem.

We first provide Fig. 2 in order to show the influence of the parameter $N$ on the velocity, on the microrotation and on the induced magnetic field.

The velocity $v$ decreases as the coupling number increases. It can be noticed that the velocity in the case of micropolar fluid is

![Graphical results](image-url)

**Fig. 5.** The effect of $\lambda$ on the velocity, on the microrotation and on the induced magnetic field ($\lambda = 0$).
less than that of the Newtonian fluid. Since $N$ is an increasing function in $\mu$, the difference of the velocity between the micropolar and the Newtonian case grows with $N$, as it can be expected.

As far as the microrotation field is concerned, it takes a minimum and a maximum, which are more pronounced as $N$ increases.

The absolute value of $h$ (describing the induced magnetic field) decreases as $N$ increases.

Fig. 3 reveals the effect of the micropolar parameter $M_p$ on the flow.

When $N$ and the geometry of the problem are fixed, different values of $M_p$ represent different values of $c_a$ and $c_d$, i.e. different sizes of the particles. The smaller the particles sizes ($M_p$ increases), the greater the non-Newtonian effects on the velocity. The minimum and the maximum of the microrotation decrease as $M_p$ increases.

We now provide Fig. 4 in order to show the influence of the strength of the external magnetic field and the electromagnetic properties of the fluid on $v$, $w$, $h$.

Fig. 4 illustrates that the velocity decreases as $M$ increases. The main effect of the transverse external magnetic field is to generate electric currents which retard the fluid in the central regions and accelerate the fluid near the boundaries thus flattening the velocity profile in the absence of the magnetic field. This behavior is the same as in the Hartmann flow [15].

The absolute value of the microrotation decreases as $M$ increases.

From picture 4$_2$, it appears that the absolute value of $h$ is an increasing function in $M$ until $M$ reaches a critical values $M'$. If $M > M'$, then the absolute value of $h$ becomes a decreasing function in $M$.

This behavior of $h$ has never been observed previously in the study of the MHD flow of a fluid in a vertical channel.

The critical value $M'$ depends on the other parameters, as it is shown in Table 1.

It is interesting to compare these values with the corresponding value of $M'$ in the Newtonian case (Table 2).

In the micropolar fluid, $M'$ is always greater than in the Newtonian fluid and its value increases as $N$ increases. This behavior can be expected when $N \rightarrow 1^{-}$ the fluid differs highly from the Newtonian one.

Finally, the influence of the buoyancy parameter $\kappa$ on the flow is provided in Fig. 5.

As Figs. 5$_1, 2$ reveal, the reverse flow occurs. This well known phenomenon has been first discovered for the Newtonian fluid in [16]. The reverse flow appears when the dimensional velocity and the gradient of $P$ have the same direction. If $\lambda = 0$ (symmetric heating, i.e. $T_1 = T_2$), then the pictures show that $v$ is always positive so that the dimensional velocity ($v = v_r e_i$) and the gradient of $P$ ($\nabla P = -C e_i$) have opposite direction, provided $C \neq 0$. Different choices of the values of the other parameters do not modify the profile of the velocity. Hence, in the case $\lambda = 0$ the reverse flow does not occur. Therefore, the occurrence of the reverse flow is a feature of the mixed convection in the case of asymmetric heating. Actually, this phenomenon appears for suitable values of $\lambda$. It is possible to compute a critical value $\lambda'$ of $\lambda$ such that

- $C > 0$ ($\iff \lambda < \lambda'$), then the reverse flow does not appear; if $\lambda > \lambda'$, then the reverse flow occurs near the coldest wall;
- $C < 0$ ($\iff \lambda > \lambda'$), then the reverse flow does not appear; if $\lambda < \lambda'$, then the reverse flow occurs near the hottest wall.

The value of $\lambda'$ depends on the choice of the other parameters and it is computed by putting $\tau_{1,2}$ equal to zero. In Table 3 we furnish the values of $\lambda'$ when $C > 0$. From this Table we can easily obtain the corresponding critical values of $\lambda'$ when $C < 0$ because the profiles of $v$ for negative values of $\lambda$ can be found by symmetry from the corresponding graphics of $v$ when $\lambda > 0$.

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From this Table it appears that
\( \frac{C_{15}}{C_3} \) increases as \( M \) increases;
\( \frac{C_{15}}{C_3} \) decreases as \( N \) increases;
\( \frac{C_{15}}{C_3} \) is not influenced in a relevant way by \( M_p \).

Hence, the influence of \( M \) on \( \frac{C_{15}}{C_3} \) shows that the presence of the external magnetic field tends to prevent the occurrence of the reverse flow. This behavior has been observed also in other physical situations [18].

For fixed values of \( M \), when \( N \to 1 \) the value of \( \frac{C_{15}}{C_3} \) differs highly from the corresponding value in the Newtonian case (see Tables 3 and 4).

We point out that the value of \( \frac{C_{15}}{C_3} \) is never computed in the papers concerning micropolar fluid.

From pictures 53, 4, 5, 6 we see that the absolute values of \( w \) and \( h \) increase as \( |\lambda| \) increases.

In order to complete the description of the flow, in Table 5 we furnish the values of \( \frac{E_0}{M_p} \) as \( M_p \), \( N \), and \( M \) change. We have that \( E \) has always opposite direction of \( e_1 \).

For the sake of completeness, we provide Fig. 6 which displays the behavior of the flow in the case of natural convection.

6. Conclusions

The analytical solution is obtained for the MHD mixed convection in the fully developed flow of an electrically conducting micropolar fluid filling a vertical channel with symmetric and asymmetric wall temperatures. In our analysis, we determine also the induced magnetic field, which is usually neglected in the literature.

The following facts have been reported:

1. The behavior of the micropolar flow differs highly from the Newtonian one as the coupling number \( N \) increases to 1 and the micropolar parameter \( M_p \) decreases.
2. The absolute value of the function \( h \) describing the induced magnetic field is an increasing function in the Hartmann number \( M \) until \( M \) reaches a critical value \( M^* \). If \( M > M^* \), then the absolute value of \( h \) becomes a decreasing function in \( M \).
3. For suitable values of the buoyancy parameter \( \lambda \), the reverse flow occurs near the coldest (hottest) wall if \( \lambda > 0 \) (\( \lambda < 0 \)). If the buoyancy parameter vanishes (symmetric heating), then the phenomenon of the reverse flow does not appear.
4. The presence of the external magnetic field tends to prevent the occurrence of the reverse flow.
Conflict of interest

None declared.

Acknowledgment

The authors are grateful to the referees for the useful and valuable suggestions.

References