Mathematical problems in Fluid Dynamics: 20 minutes of flow

Marco Sammartino joint work with
R,Caflisch, Courant Institute, New York
M.Cannone, Marne la Vallée
F.Gargano, M.C.Lombardo, V.Sciacca, Palermo
I.Kukavica, USC Los Angeles

Department of Mathematics
University of Palermo

60 minutes of Continuum Mechanics
in honor of Alfredo Marzocchi
Brescia, 16 November 2020

The equations of incompressible fluid dynamics

Let $\mathbf{u}(\mathbf{x}, t)$ a velocity field:

$$\begin{aligned} \frac{D\mathbf{u}}{Dt} &= \mathbf{F} && \text{Newton's law} \\ \nabla \cdot \mathbf{u} &= 0 && \text{incompressibility: volume is preserved, a geometrical constraint} \end{aligned}$$

The velocity is self-convected, therefore the (material) time derivative:

$$\frac{D\mathbf{u}}{Dt} = \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}$$

The equations of incompressible fluid dynamics

Let $\mathbf{u}(\mathbf{x}, t)$ a velocity field:

$$\begin{aligned} \frac{D\mathbf{u}}{Dt} &= \mathbf{F} \qquad \text{Newton's law} \\ \nabla \cdot \mathbf{u} &= 0 \qquad \text{incompressibility: volume is preserved, a geometrical constraint} \end{aligned}$$

The velocity is self-convected, therefore the (material) time derivative:

$$\frac{D\mathbf{u}}{Dt} = \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}$$

In the simplest case, ideal flows, the "force" is just related to the isotropic pressure:

$$\mathbf{F} = -\nabla p$$

Euler equations: appeared Académie Royale des Sciences et des Belles-Lettres de Berlin, Mémoires 1757

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} + \boldsymbol{\nabla} p &= 0 \\ \boldsymbol{\nabla} \cdot \mathbf{u} &= 0 \end{aligned}$$

After more than 250 years they are one of the most intriguing mathematical model, still presenting fundamental and challenging problems for mathematicians.

The equations of incompressible fluid dynamics: dissipation

When friction is present the force **F** has a more complicated expression Introduce the vorticity of the flow, giving the intensity of the spinning of the flow:

$$\omega = \boldsymbol{\nabla} \times \mathbf{u}$$

The equation for the vorticity, in 2D, is:

$$\partial_t \omega + \mathbf{u} \cdot \boldsymbol{\nabla} \omega = 0 \quad \rightarrow$$

The equations of incompressible fluid dynamics: dissipation

When friction is present the force $\mathbf{F}$ has a more complicated expression Introduce the vorticity of the flow, giving the intensity of the spinning of the flow:

$$\omega = \boldsymbol{\nabla} \times \mathbf{u}$$

The equation for the vorticity, in 2D, is:

$$\partial_t \omega + \mathbf{u} \cdot \boldsymbol{\nabla} \omega = 0 \quad \rightarrow \qquad \frac{D\omega}{Dt} = 0$$

However vorticity is not only convected, but also "diffuses":

The equations of incompressible fluid dynamics: dissipation

When friction is present the force **F** has a more complicated expression Introduce the vorticity of the flow, giving the intensity of the spinning of the flow:

$$\omega = \nabla \times \mathbf{u}$$

The equation for the vorticity, in 2D, is:

$$\partial_t \omega + \mathbf{u} \cdot \nabla \omega = 0 \quad \rightarrow \qquad \frac{D\omega}{Dt} = 0$$

However vorticity is not only convected, but also "diffuses":

$$\partial_t \omega + \mathbf{u} \cdot \nabla \omega = \nu \Delta \omega$$

Navier-Stokes equations: Navier 1822, Stokes 1845

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla\, \mathbf{u} + \nabla p &= \nu \Delta\, \mathbf{u}\,, \\ \nabla \cdot \mathbf{u} &= 0\,, \end{aligned}$$

The major mathematical problems concerning the NS and Euler eqs are...

- Do smooth solutions exist for all time? Or do singularities develop in finite time? (Millennium Prize problem)
- Do non smooth but physically significant initial data lead to classical weak solutions?
- Do solutions of the Navier-Stokes equation converge to solutions of the Euler equations away from boundaries?
- The long time dynamics of the solutions.

P.Constantin, Bull.Am.Math.Soc., 2007.

Regularity or blow-up?

The energy inequality

The basic property of the Euler equations is conservation of energy: $\|\mathbf{u}\|_{L^2}^2 = \int |\mathbf{u}|^2 dx$

$$\int [\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p] \cdot \mathbf{u} dx = 0$$

and derive

$$\frac{d\|\mathbf{u}\|_{L^2}^2}{dt} = 0$$

The reason is integration by parts:

$$\int (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \mathbf{u} dx = \frac{1}{2} \int \nabla \cdot \left(u^2 \mathbf{u}\right) dx = 0$$

$$\frac{d\|\mathbf{u}\|_{L^2}^2}{dt} = -2\nu \|\nabla \mathbf{u}\|^2 < 0 \qquad \text{for Navier-Stokes}$$

If one goes to more regular Sobolev spaces:

$$\frac{1}{2}\frac{d\|\mathbf{u}\|_{H^s}^2}{dt} \leq c\|\nabla \mathbf{u}\|_{L^\infty}\|\mathbf{u}\|_{H^s}^2 \leq c\|\mathbf{u}\|_{H^s}^3$$

$$\dot{y} = y^{3/2} \;\longrightarrow\; y(t) = \frac{y(0)}{\left(1 - 2t\sqrt{y(0)}\right)}$$

If one goes to more regular Sobolev spaces:

$$\frac{1}{2}\frac{d\|\mathbf{u}\|_{H^s}^2}{dt} \leq c\|\nabla\mathbf{u}\|_{L^\infty}\|\mathbf{u}\|_{H^s}^2 \leq c\|\mathbf{u}\|_{H^s}^3$$

$$\dot{y} = y^{3/2} \longrightarrow y(t) = \frac{y(0)}{\left(1 - 2t\sqrt{y(0)}\right)}$$

Difference between 2D and 3D

In 2D the equation of vorticity is $\partial_t\omega + \mathbf{u}\cdot\nabla\omega = 0$:

$$\frac{D\omega}{dt} = 0 \quad \text{vorticity is conserved along Lagrangian paths}$$

while in 3D: $\partial_t\omega + \mathbf{u}\cdot\nabla\omega - \omega\cdot\nabla u = 0$

It is the vortex-stretching term that can lead to strong amplification of vorticity.

Regularity criteria for Euler equations

Theorem (Beale-Kato-Majda result)

As long as

$$\int_0^T \|\omega(\cdot, s)\|_{L^\infty} ds < \infty$$

the solution of the Euler equations $\mathbf{u}$ *stays regular.*

A criterion realted to the BKM result was found by Constantin, Fefferman and Majda: it shows that it is (mostly) the variation in the direction of the vorticity that could produce singularities.

$$\xi = \frac{\omega}{|\omega|} \quad \text{and evaluate } K \equiv \sup_{x,y} \frac{|\xi(x,t) - \xi(y,t)|}{|x-y|}$$

Theorem (Constantin, Fefferman and Majda 1996)

As long as

$$\int_0^T K(s) ds < \infty$$

the solution of the Euler equations $\mathbf{u}$ *stays regular.*

Criteria for Navier-Stokes

Theorem (Constantin and Fefferman 1993)

sin-Lipschitz condition: Define $\theta(x, y, t)$ to be the angle between $\xi(x, t)$ and $\xi(y, t)$, and suppose that in regions of large vorticity

$$\frac{|\sin\theta(x, y, t)|}{|y|} \leq C$$

Then the solution of the NS equations remains regular.

Theorem (Beirao da Veiga and Berselli 2002)

sin-Holder condition: Define $\theta(x, y, t)$ to be the angle between $\xi(x, t)$ and $\xi(y, t)$, and suppose that in regions of large vorticity

$$\frac{|\sin\theta(x, y, t)|}{|y|^{1/2}} \leq C$$

Then the solution of the NS equations remains regular.

Terence Tao's recent result

If one consider the "cheap" Navier-Stokes equations (Montgomery 2001, Gallagher and Paicu 2009)

$$\partial_t u = \Delta u + \sqrt{-\Delta}(u^2)$$

which shares a lot of features with NS (not conservation of energy, though!!), one can prove the existence of solutions blowing up in finite time. Tao writes NS equations eliminating the pressure using the Leray projector (i.e. the Hodge decomposition):

$$\partial_t \mathbf{u} + B(\mathbf{u}, \mathbf{u}) = \Delta \mathbf{u} \qquad \text{where } B(\mathbf{u}, \mathbf{u}) = \mathbb{P}\left(\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u}\right)$$

$$\langle B(\mathbf{u}, \mathbf{u}), \mathbf{u} \rangle = 0 \qquad \text{is equivalent to conservation of energy}$$

Is it possible to construct a $\tilde{B}(\mathbf{u}, \mathbf{u})$ which satisfies conservation of energy, of the same strength (from the harmonic analysis point of view) as B, so that

$$\partial_t \mathbf{u} + \tilde{B}(\mathbf{u}, \mathbf{u}) = \Delta \mathbf{u}$$

exhibits finite time blow up?

YES, passing through a dyadic model, T.Tao 2015

Dyadic models are local version of the NS, where the mechanism of energy cascade is oversimplified. The $j-th$ Fourier mode interacts only with neighboring modes.
Define

$$u_j^2(t) = \|\mathbf{u}^j\|_{L^2}^2 \qquad \text{energy in the j-th shell}$$

$$\Pi_j = \lambda^j u_j^2 u_j \qquad \text{energy flux from } j-th \text{ to } (j+1)-th \text{ shell}$$

and write the energy balance equation as Kats-Pavlovic model

$$\frac{1}{2}\frac{d}{dt}u_j^2 = -\Pi_j + \Pi_{j-1} - \nu\lambda^{2j\alpha}$$

$\lambda = 2$, while α rules the strength of the dissipative effects.
Tao recalls a result of Barbato Morandin and Romito 2011, where it is shown that "dispersive" effects do not allow blow up for this toy model: energy does not flow fast enough from the mode j to mode $j+1$ and remains trapped (to be eventually dissipated) without having the chance to build a singularity.

Tao is able to modify the Kats-Pavlovic model, substituting u_j^2 a vector (a circuit). He is therefore able to build a dyadic model which blows up in fine time.

He shows that to his dyadic model corresponds an averaged Euler operator $\tilde{B}$ which conserves energy and elicity

$$\partial_t \mathbf{u} + \tilde{B}(\mathbf{u}, \mathbf{u}) = \nu \nabla \mathbf{u}$$

has solutions which develop a singularity in finite time.

$\tilde{B}$ is "weaker" than the Euler operator B.

Tao's example rigorously demonstrates that any attempt to positively resolve the NavierStokes global regularity problem in three dimensions has to use finer (still unknown) structure of the equation beyond the energy identity and incompressibility.

Singular data: Vortex sheets

Vortex sheets are data where vorticity is concentrated on a curve

Existence of solutions, Delort '91 *J. Am. Math. Soc.*, Majda '93, *Comm. Pure Appl. Math.*

If ω_0 is a Radon measure with positive singular part, such that $v_0 \in L^2_{loc}$. Then Euler equations (in the weak form) admit as solution a bounded measure ω_t, and $\mathbf{u} \in L^\infty_{loc}(\mathbb{R}, L^2_{loc})$.

Duchon and Robert '88, Di Perna and Majda '87, Chemin '95, Evans and Muller 94'

But:

- the question of uniqueness is still unanswered
- it gives no information on the structure of the solution.
- no relation with the Birkhoff-Rott equation

Euler equations admit vortex layer solutions

Theorem (Existence and uniqueness for the Euler-Vortex Layer equations, *R.Caflisch, MC.Lombardo, and M.S., CPAM 2020)*

Let $\tilde{\omega}_0 \in B^1_{\rho_0,\sigma_0,\mu_0}$, $\varphi_0 \in B^2_{\rho_0}$, *and suppose there are two constants* S_0, $R_0 < 1/2$ *such that:*

$$
\begin{aligned}
\|\tilde{\omega}_0\|_{1,\rho_0,\sigma_0,\mu_0} &\leq S_0 \\
\|\varphi_0\|_{2,\rho_0} &\leq R_0 \,.
\end{aligned}
$$

Then there exist $\beta > 0$, S, $R < 1/2$ *such that the Euler-Vortex Layer equations, with initial data* $(\tilde{\omega}_0, \varphi_0)$, *admit a unique solution* $(\tilde{\omega}, \varphi)$ *with* $\tilde{\omega} \in B^1_{\rho_0,\sigma_0,\mu_0,\beta,T}$, $\varphi \in B^2_{\rho_0,\beta,T}$, *where* $T < \rho_0/\beta$, *and with:*

$$
\begin{aligned}
\|\tilde{\omega}\|_{1,\rho_0,\sigma_0,\mu_0,\beta,T} &\leq S \\
\|\varphi\|_{2,\rho_0,\beta,T} &\leq R \,.
\end{aligned}
$$

Convergence: Justification of the BR equations

One can prove the following result:

Theorem (Convergence, *Caflisch, Lombardo, and S., CPAM 2020*)

Suppose

- *All the Assumptions of Theorem giving well-posedness for Euler Vortex Layer equations [Caflisch, Lombardo,S.] are satisfied;*
- *All the Assumptions of Theorem giving well-posedness for BR equations [Sulem,Sulem,Bardos,Frish] are satisfied;*
- *Initially the configuration are, i.e.* $\gamma_0^\varepsilon \equiv \gamma_0^{BR}$ *and* $\varphi_0^\varepsilon = \varphi_0^{BR}$

Then there exist $\beta_C > \max[\beta, \beta_{BR}]$ *such that,*

$$\|\gamma^\varepsilon - \gamma^{BR}\|_{1,\rho_0,\beta_C,T_C} + \|\varphi^\varepsilon - \varphi^{BR}\|_{2,\rho_0,\beta,T_C} \leq c\varepsilon^\kappa$$

being $T_C < \min[\rho_0, \rho_{BR}]/\beta_C$ *and* $1 > \kappa > 0$.

Reality is more singular than mathematical models

link to Video1

link to Video2

link to Video3

Last video *F.Gargano, M.S., V.Sciacca, submitted to J.Fluid.Mech.* , and a careful study of the complex singularities of the Navier-Stokes equations, show concentration *à la Di Perna-Majda.*

Il layer converge al VS per $t \to t_s^-$*, ma non per* $t \to t_s^+$

Zero viscosity: do Navier-Stokes converge to Euler?

link to Video4

Zero viscosity: D'Alembert Paradox: 1752

This paradox puzzled theoretical Fluid Dynamics for more than 150 years

It seems to me that the theory (potential flow), developed in all possible rigor, gives, at least in several cases, a strictly vanishing resistance, a singular paradox which I leave to future Geometers to elucidate.

Consider the Euler equation in a steady situation:

$$\nabla \cdot (\mathbf{u}\mathbf{u}) + \nabla p = 0$$

and integrate in space, on a volume enclosed in a surface S

$$\int_S p\mathbf{n}dS = -\int_S \mathbf{u}\mathbf{u} \cdot \mathbf{n}dS$$

Consider now the situation of a body immersed in a very long channel...

Zero viscosity: D'Alembert Paradox: 1752

This paradox puzzled theoretical Fluid Dynamics for more than 150 years

It seems to me that the theory (potential flow), developed in all possible rigor, gives, at least in several cases, a strictly vanishing resistance, a singular paradox which I leave to future Geometers to elucidate.

Consider the Euler equation in a steady situation:

$$\nabla \cdot (\mathbf{uu}) + \nabla p = 0$$

and integrate in space, on a volume enclosed in a surface S

$$\int_S p\mathbf{n}dS = -\int_S \mathbf{uu} \cdot \mathbf{n}dS$$

Consider now the situation of a body immersed in a very long channel...

$$\int_{S_1} p_1 dS - \int_{S_2} p_2 dS - D = \int_{S_2} u_2^2 dS - \int_{S_1} u_1^2 dS$$

However $u_1 = u_2$ for mass conservation.

Then $p_1 = p_2$ for the Bernoulli law, stating that in steady condition $p + u^2/2$ is constant along a streamline. It follows:

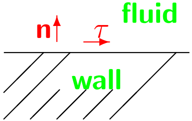
D'Alembert paradox: no drag on a moving body!!!!!

$$D \equiv 0$$

Ludwig Prandtl's Transition layer theory, 1905

During the week of 8 August 1904, a small group of mathematicians and scientists gathered in picturesque Heidelberg, Germany, (....)Heidelberg was a natural venue for the Third International Mathematics Congress.
One of the presenters at the congress was Ludwig Prandtl, a 29-year-old professor at the Technische Hochschule. His presentation, and the subsequent paper that was published in the congress proceedings one year later, introduced the concept of the boundary layer in a fluid flow over a surface.
The modern world of aerodynamics and fluid dynamics is still dominated by Prandtl's idea. By every right, his boundary-layer concept was worthy of the Nobel Prize. He never received it, however; some say the Nobel Committee was reluctant to award the prize for accomplishments in classical physics.

J.D.Anderson Jr



When $\nu \to 0$ one gets a singular limit.
The fluid shows two different regimes.

- Far away from the boundary:
  viscous forces $\ll$ *inertial forces* $\Longrightarrow$ Euler eqs. might be OK
- Close to the boundary:
  viscous forces are NOT neglegible $\Longrightarrow$ Boundary layer eqs.

Prandtl 1905 derived as an asymptotic approximation of NS valid near the boundary at high Reynolds numbers.
He introduced a rescaled variable $Y = y/\varepsilon$ to magnify the thin layer close to the boundary where he could see a lot of interesting phenomena occurring.

$$\varepsilon = \sqrt{\nu}$$

The scaling is suggested by a balance between viscous and inertial forces.
Notice that:

$$\partial_y = \frac{1}{\varepsilon}\partial_Y$$

Conjecture:

$$\mathbf{u}^{NS} = m(y/\sqrt{\nu})\mathbf{u}^E + (1 - m(y/\sqrt{\nu}))\mathbf{u}^P + O(\sqrt{\nu})$$

Proven for analytic (in x and y) data, S. and Caflisch *Comm.Math.Phys '98*

Conjecture:

$$\mathbf{u}^{NS} = m(y/\sqrt{\nu})\mathbf{u}^{E} + (1 - m(y/\sqrt{\nu}))\mathbf{u}^{P} + O(\sqrt{\nu})$$

Proven for analytic (in x and y) data, S. and Caflisch *Comm.Math.Phys '98*

One of the main difficulties is that it is hard to get well posedness results on Prandtl equations. Evidences of ill-posedness Gerard-Varet and Dormy *J.Am.Math.Soc.'10* Gerard-Varet and Nguyen *Asym.Analysis'12*

Recently, I.Kukavica and V.Vicol *ARMA 2020*, have improved the result, showing that analyticity is necessary only close to the boundary.

Large time behavior of the Navier-Stokes equations

Kolmogorov theory, using heuristic arguments derives that NS generates eddies down to the cut-off length

$$\lambda_K = LR^{-3/4}$$

Below that scale viscosity dissipates all the energy.
Therefore in a box $[0, L]^3$ one must resolve

$$N = (L/\lambda_K)^3 \sim R^{9/4}$$

This is the origin of the claim that 3D fully developed turbulence has $R^{9/4}$ degrees of freedom.

Large time behavior of the Navier-Stokes equations

The idea, very popular in the '90's, is to give a rigorous justification of the Kolmogorov theory using the concept of attractor. For 2D NS the program has been quite successful.

$$d_{attr} \leq (L/\lambda_{K_r})^2 \left[1 + \log (L/\lambda_{K_r})^2\right]$$

Large time behavior of the Navier-Stokes equations

The idea, very popular in the '90's, is to give a rigorous justification of the Kolmogorov theory using the concept of attractor. For 2D NS the program has been quite successful.

$$d_{attr} \leq (L/\lambda_{K_r})^2 \left[1 + \log (L/\lambda_{K_r})^2\right]$$

- V.Sciacca, M.E.Schonbek, M.Sammartino, Long time behavior for a dissipative shallow water model, Ann. I.H.Poincaré, 34(2017)731?757.

Large time behavior of the Navier-Stokes equations

The idea, very popular in the '90's, is to give a rigorous justification of the Kolmogorov theory using the concept of attractor. For 2D NS the program has been quite successful.

$$d_{attr} \leq (L/\lambda_{K_r})^2 \left[1 + \log (L/\lambda_{K_r})^2\right]$$

- V.Sciacca, M.E.Schonbek, M.Sammartino, Long time behavior for a dissipative shallow water model, Ann. I.H.Poincaré, 34(2017)731?757.
- Ghidaglia, J.M., Marzocchi, A. Exact Decay Estimates for Solutions to Semilinear Parabolic Equations (1991) Applicable Analysis, 42 (1-4), pp. 69-81.
- Bianchi, G., Marzocchi, A. Inertial manifold for the motion of strongly damped nonlinear elastic beams. NoDEA, Nonlinear differ. equ. appl. 5, 181-192 (1998).
- Giorgi, C., Pata, V. Marzocchi, A. Asymptotic behavior of a semilinear problem in heat conduction with memory. NoDEA, Nonlinear differ. equ. appl. 5, 333?354 (1998).
- Giorgi, C., Pata, V., Marzocchi, A. Uniform attractors for a non-autonomous semilinear heat equation with memory (2000) Quarterly of Applied Mathematics, 58 (4), pp. 661-683.
- Marzocchi, A., Necca, S.Z. Attractors for dynamical systems in topological spaces (2002) Discrete and Continuous Dynamical Systems, 8 (3), pp. 585-597.