Non-linear Wave Propagation and Non-Equilibrium Thermodynamics - Part 1

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Overview

1. Deformation of a continuous body

2. Forces and Stress

3. Balance Laws in Continuum Mechanics

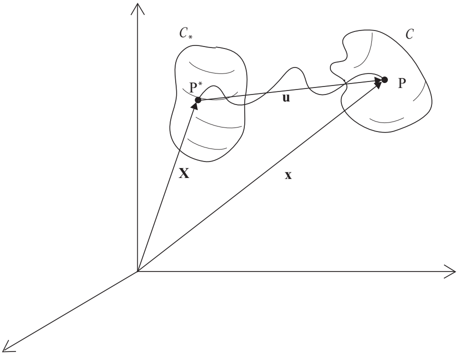


Figure: Configuration of a continuous body

We identify a continuous body as a region of Euclidean space. At time $t = 0$, is C^* the configuration of the body and let C the configuration at time t:

$$\mathbf{X} \equiv (X_1, X_2, X_3) \quad ; \quad \mathbf{x} \equiv (x_1, x_2, x_3)\,. \tag{1}$$

The vectorial function

$$\mathbf{x} \equiv \mathbf{x}(\mathbf{X},t) \quad \Longleftrightarrow \quad x_i \equiv x_i(X_1, X_2, X_3, t), \qquad i = 1, 2, 3. \tag{2}$$

permits to reconstruct the configuration at time t from the one at $t = 0$. The linear operator

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \equiv \|F_{iA}\|; \qquad F_{iA} = \frac{\partial x_i}{\partial X_A} \tag{3}$$

is called *deformation gradient*(Jacobian matrix).
We need

$$J = \det \mathbf{F} \neq 0 \quad \forall\ \mathbf{X} \text{ e } \forall\ t \geq 0.$$

As for $t = 0$

$$\mathbf{x}(\mathbf{X},0) = \mathbf{X} \ \Rightarrow \ \mathbf{F}(\mathbf{X},0) = \mathbf{I}$$

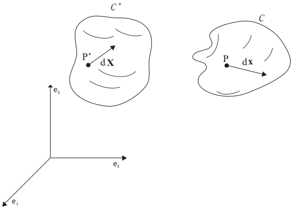
we have

$$J(0) = 1,$$

and therefore

$$J(\mathbf{X}, t) > 0.$$

Deformation Tensors



$$dx_i = \frac{\partial x_i}{\partial X_A}\, dX_A\,, \qquad \text{i.e.} \quad d\mathbf{x} = \mathbf{F} d\mathbf{X}. \tag{4}$$

$$|d\mathbf{x}|^2 = d\mathbf{x}\cdot\, d\mathbf{x} = \mathbf{F}\; d\mathbf{X}\cdot\;\, \mathbf{F}\; d\mathbf{X} = \mathbf{F}^T\mathbf{F}\; d\mathbf{X}\cdot d\mathbf{X} \tag{5}$$

Let

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} \in \mathcal{S}ym^+ \qquad C_{AB} = \frac{\partial x_i}{\partial X_A}\frac{\partial x_i}{\partial X_B}, \tag{6}$$

we have

$$|d\mathbf{x}|^2 = \mathbf{C}\; d\mathbf{X}\cdot d\mathbf{X} = |d\mathbf{X}|^2 + 2\mathbf{E}\; d\mathbf{X}\cdot d\mathbf{X}. \tag{7}$$

with

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) \in \mathcal{S}ym, \qquad E_{AB} = \frac{1}{2}\left(\frac{\partial x_i}{\partial X_A}\frac{\partial x_i}{\partial X_B} - \delta_{AB}\right). \tag{8}$$

In the case of rigid transformation $\mathbf{E} = 0$ and $\mathbf{C} = \mathbf{I}$.
Therefore $\mathbf{E}$ is called *deformation tensor* of GREEN -SAINT VENANT, while $\mathbf{C}$ is called the *right deformation tensor* of CAUCHY-GREEN.
We recall the so called *polar theorem*:

$$\mathbf{F} = \mathbf{R}\mathbf{U}, \tag{9}$$

with

$$\mathbf{R} \in \mathcal{Rot} \quad \text{and} \quad \mathbf{U} \in \ \mathcal{Sym}^+, \quad \text{such that} \quad \mathbf{U}^2 = \mathbf{F}^T\mathbf{F} = \mathbf{C}. \tag{10}$$

Therefore

$$\mathbf{U} = \sqrt{\mathbf{C}}.$$

Deformation Coefficients

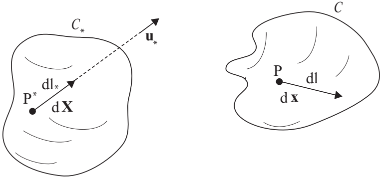


Figure: linear dilatation

$$\delta_{u^*} = \frac{dl - dl^*}{dl^*} = \sqrt{\mathbf{C}\ \mathbf{u}_* \cdot \mathbf{u}_*} - 1. \tag{11}$$

$$dl = dl^* \sqrt{\mathbf{C}\ \mathbf{u}_* \cdot \mathbf{u}_*} \tag{12}$$

In particular

$$\begin{aligned} \delta_1 &= \sqrt{C_{11}} - 1 = \sqrt{1 + 2E_{11}} - 1 \\ \delta_2 &= \sqrt{C_{22}} - 1 = \sqrt{1 + 2E_{22}} - 1 \\ \delta_3 &= \sqrt{C_{33}} - 1 = \sqrt{1 + 2E_{33}} - 1. \end{aligned} \tag{13}$$

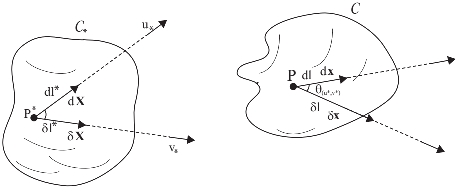


Figure: Angles deformation

$$\cos\theta_{(\mathbf{u}^*,\mathbf{v}^*)} = \frac{\mathbf{C}\mathbf{u}_*\cdot\mathbf{v}_*}{\sqrt{\mathbf{C}\mathbf{u}_*\cdot\mathbf{u}_*}\sqrt{\mathbf{C}\mathbf{v}_*\cdot\mathbf{v}_*}}. \tag{14}$$

$$\cos\theta_{12} = \frac{C_{12}}{\sqrt{C_{11}}\sqrt{C_{22}}} = \frac{2E_{12}}{\sqrt{1+2E_{11}}\sqrt{1+2E_{22}}}$$
$$\cos\theta_{13} = \frac{C_{13}}{\sqrt{C_{11}}\sqrt{C_{33}}} = \frac{2E_{13}}{\sqrt{1+2E_{11}}\sqrt{1+2E_{33}}} \tag{15}$$
$$\cos\theta_{23} = \frac{C_{23}}{\sqrt{C_{22}}\sqrt{C_{33}}} = \frac{2E_{23}}{\sqrt{1+2E_{22}}\sqrt{1+2E_{33}}}.$$

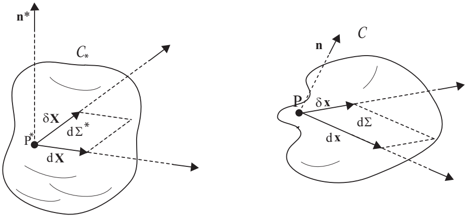


Figure: Superficial Deformation

$$\delta_\Sigma = \frac{d\Sigma - d\Sigma^*}{d\Sigma^*} = J\sqrt{\mathbf{C}^{-1}\mathbf{n}^* \cdot \mathbf{n}^*} - 1.$$

$$\mathbf{n} = \frac{\mathbf{F}^C \mathbf{n}^*}{J\sqrt{\mathbf{C}^{-1}\mathbf{n}^* \cdot \mathbf{n}^*}}, \quad J = \sqrt{(\det C)}. \tag{16}$$

In a rigid motion $\mathbf{F} = \mathbf{R}$ e $\mathbf{C} = \mathbf{I}$ and therefore $\mathbf{n} = \mathbf{R}\mathbf{n}^*$ and $\delta_\Sigma = 0$.

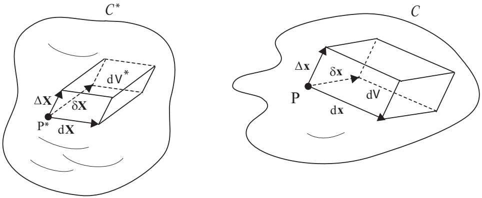


Figure: Volume Dilatation

$$\delta_V = \frac{dV - dV^*}{dV^*} = J - 1. \qquad (17)$$

Therefore a body is *Incompressible* if and only if

$$J = 1. \qquad (18)$$

Forces

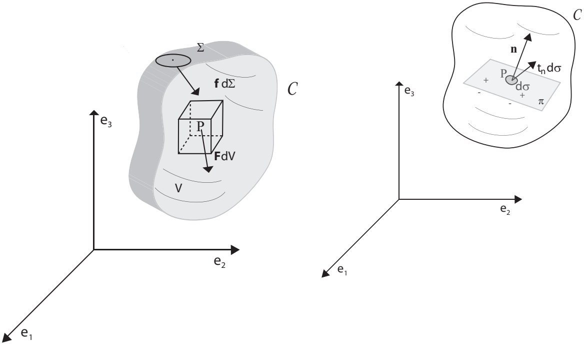


Figure: Different kind of Forces.

- **Body forces or mass forces**: $\mathbf{F}dV = \mathbf{b}\,dm$, $(\mathbf{F} = \rho\mathbf{b})$.
- **Superficial forces**: $\mathbf{f}\ d\Sigma$.
- **Contact forces**: $\mathbf{t}_n\ d\sigma$, $\mathbf{t}_n$ is the *specific stress* in the $\mathbf{n}$-direction $(\mathbf{t}_{-n} = -\mathbf{t}_n)$.

Theorem (Cauchy)

For any unit vector $\mathbf{n} \equiv (n_1, n_2, n_3)$

$$\mathbf{t}_n = \mathbf{t}_1 n_1 + \mathbf{t}_2 n_2 + \mathbf{t}_3 n_3 \tag{19}$$

where $\mathbf{t}_i$ $(i = 1, 2, 3)$ *are the specific stress in the axis directions* $\mathbf{n} \equiv \mathbf{e}_i$ *.*

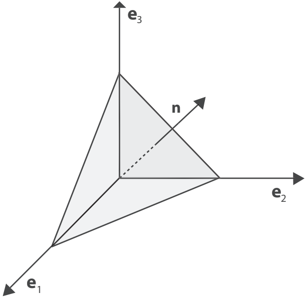


Figure: Cauchy Tetrahedron

Proof:

$$\mathbf{R}^{(e)}=0.$$

$$\mathbf{R}^{(e)} = \int_{\Delta V} \mathbf{F} dV + \int_{\Delta\sigma} \mathbf{t}_n d\sigma = 0. \tag{20}$$

$$\mathbf{F}(\bar{P})\Delta V + \mathbf{t}_n(\tilde{P})\Delta\sigma - \mathbf{t}_1(\tilde{P}_1)\Delta\sigma_1 - \mathbf{t}_2(\tilde{P}_2)\Delta\sigma_2 - \mathbf{t}_3(\tilde{P}_3)\Delta\sigma_3 = 0 \tag{21}$$

$$\Delta V = \frac{\Delta\sigma h}{3}; \qquad \Delta\sigma_i = n_i \Delta\sigma$$

with h the height of the tetrahedron, and then the (21) becomes:

$$\mathbf{F}(\bar{P})\frac{h}{3} + \mathbf{t}_n(\tilde{P}) - \mathbf{t}_1(\tilde{P}_1)n_1 - \mathbf{t}_2(\tilde{P}_2)n_2 - \mathbf{t}_3(\tilde{P}_3)n_3 = 0 \tag{22}$$

when $h \to 0$ the theorem is proved.

Let

$$\mathbf{t}_1 \equiv (t_{11}, t_{12}, t_{13}), \quad \mathbf{t}_2 \equiv (t_{21}, t_{22}, t_{23}), \quad \mathbf{t}_3 \equiv (t_{31}, t_{32}, t_{33}).$$

We define the CAUCHY - *stress tensor*:

$$\mathbf{t} \equiv \begin{pmatrix} t_{11} & t_{21} & t_{31} \\ t_{12} & t_{22} & t_{32} \\ t_{13} & t_{23} & t_{33} \end{pmatrix},$$

and we have

$$\mathbf{t}_n = \mathbf{t}\ \mathbf{n}. \tag{23}$$

Mass Balance

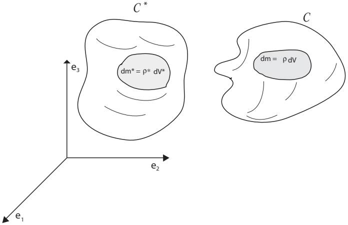


Figure: Mass Conservation

$$m = \int_C \rho \, dV \quad \rightarrow \quad \int_{C^*} \rho^* dV^* = \int_C \rho dV, \quad \rightarrow \quad \rho = \frac{\rho^*}{J}$$

Lagrangian point of view. But

$$\frac{dJ}{dt} = J \operatorname{div} \mathbf{v} \quad \leftrightarrow \quad \frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{v} = 0,$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} = 0 \tag{24}$$

Eulerian point of view.

Momentum Equation

$$\mathbf{R}^{(e)} = \int_{\Delta V} \mathbf{F} dV + \int_{\Delta \sigma} \mathbf{t}_n d\sigma = 0. \tag{25}$$

$$\int_{\Delta V} F_i dV + \int_{\Delta \sigma} t_{ij} n_j d\sigma = 0. \tag{26}$$

For Gauss-Green theorem:

$$\int_{\Delta V} \left(F_i + \frac{\partial t_{ij}}{\partial x_j} \right) dV = 0, \tag{27}$$

then under regularity assumptions

$$F_i + \frac{\partial t_{ij}}{\partial x_j} = 0, \qquad \forall P \in V. \tag{28}$$

Angular Momentum

The angular momentum in equilibrium:

$$\mathbf{M}_O^{(e)} = \int_{\Delta V} \mathbf{x} \wedge \mathbf{F}\; dV + \int_{\Delta \sigma} \mathbf{x} \wedge \mathbf{t}_n \, d\sigma = 0,$$

$$\int_{\Delta V} \varepsilon_{ilm} x_l F_m \; dV + \int_{\Delta \sigma} \varepsilon_{ilm} x_l t_{mk} n_k \; d\sigma = 0.$$

Using the Gauss-Green Theorem

$$\int_{\Delta V} \varepsilon_{ilm} x_l F_m \; dV + \int_{\Delta V} \frac{\partial}{\partial x_k} (x_l t_{mk})\, \varepsilon_{ilm} \; dV = 0,$$

$$\varepsilon_{ilm} x_l \left(F_m \; + \frac{\partial t_{mk}}{\partial x_k} \right) + \varepsilon_{ilm} t_{ml} = 0 \qquad \forall P \in V.$$

then

$$\varepsilon_{ilm} t_{ml} = 0, \quad i = 1, 2, 3$$

then we obtain the symmetry of the stress tensor:

$$t_{23} = t_{32} \;\; ; \;\; t_{31} = t_{13} \; ; \;\; t_{12} = t_{21}$$

In the dynamical case using the D'Alembert Principle the momentum equation becomes:

$$\Leftrightarrow \quad \rho\frac{dv_j}{dt} - \frac{\partial t_{ij}}{\partial x_i} = F_j, \quad (j = 1, 2, 3), \tag{29}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_k\frac{\partial}{\partial x_k}$$

denotes the material derivative. Combining with mass conservation we obtain the momentum balance equation in the form:

$$\frac{\partial \rho v_j}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho v_i v_j - t_{ij}\right) = \rho b_j. \tag{30}$$

General Balance Laws

The previous balance equations

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial \rho v_i}{\partial x_i} = 0 & \text{(Mass Conservation)} \\ \dfrac{\partial \rho v_j}{\partial t} + \dfrac{\partial}{\partial x_i}\left(\rho v_i v_j - t_{ij}\right) = \rho b_j & \text{(Balance of Momentum)} \end{cases} \tag{31}$$

are particular case of a general balance law:

$$\frac{d}{dt}\int_V \Psi dV = -\int_\Sigma \Phi_i n_i d\Sigma + \int_V f dV \tag{32}$$

Using the so called Transport theorem

$$\frac{d}{dt}\int_V \Psi dV = \int_V \left(\frac{d\Psi}{dt} + \Psi div \mathbf{v}\right) dV, \tag{33}$$

we have

$$\int_V \left(\frac{d\Psi}{dt} + \Psi div \mathbf{v} \right) \; dV + \int_V \frac{\partial \Phi_i}{\partial x_i} \; dV = \int_V f dV. \tag{34}$$

i.e.

$$\frac{\partial \Psi}{\partial t} + \frac{\partial}{\partial x_i} \left(\Psi v_i + \Phi_i \right) = f. \tag{35}$$

For example the conservation of mass is obtained when

$$\Psi = \rho, \quad \Phi_i = 0, \quad f = 0$$

while if

$$\Psi = \rho v_j, \quad \Phi_i = -t_{ij} \; , \quad f = \rho b_j \qquad\qquad (j = 1, 2, 3)$$

we obtain the momentum equation.