Non-linear Wave Propagation and Non-Equilibrium Thermodynamics - Part 3

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Overview

Balance Law of Energy

If we enlarge the framework from mechanics to thermodynamics we have also the balance law of energy. In this case

$$\Psi = \rho e + \frac{\rho v^2}{2}, \quad \Phi_i = -t_{ij} v_j + q_i, \quad f = \mathbf{F} \cdot \mathbf{v} + r = \rho \mathbf{b} \cdot \mathbf{v} + r.$$

Then

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho e\right) + \frac{\partial}{\partial x_i}\left\{\left(\frac{1}{2}\rho v^2 + \rho e\right) v_i - t_{ij} v_j + q_i\right\} = \rho b_i v_i + r. \qquad (1)$$

Therefore in thermo-mechanics we have the following system of Balance Laws:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial \rho v_i}{\partial x_i} = 0 \\ \dfrac{\partial \rho v_j}{\partial t} + \dfrac{\partial}{\partial x_i}\left(\rho v_i v_j - t_{ij}\right) = \rho b_j \qquad (j = 1, 2, 3) \\ \dfrac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho e\right) + \dfrac{\partial}{\partial x_i}\left\{\left(\frac{1}{2}\rho v^2 + \rho e\right) v_i - t_{ij} v_j + q_i\right\} = \rho b_j v_j + r. \end{cases} \qquad (2)$$

Generic System of Balance Laws

The previous system of balance laws is a particular case of

$$\frac{\partial \mathbf{F}^0}{\partial t} + \frac{\partial \mathbf{F}^i}{\partial x^i} = \mathbf{f}, \quad \leftrightarrow \quad \frac{\partial \mathbf{F}^\alpha}{\partial x^\alpha} = \mathbf{f} \tag{3}$$

$$\mathbf{F}^0 = \begin{pmatrix} \rho \\ \rho v_1 \\ \rho v_2 \\ \rho v_3 \\ \frac{1}{2}\rho v^2 + \rho e \end{pmatrix}; \quad \mathbf{F}^i = \begin{pmatrix} \rho v_i \\ \rho v_i v_1 - t_{i1} \\ \rho v_i v_2 - t_{i2} \\ \rho v_i v_3 - t_{i3} \\ \left(\frac{1}{2}\rho v^2 + \rho e\right) v_i - t_{ij} v_j + q_i \end{pmatrix}; \quad \mathbf{f} = \begin{pmatrix} 0 \\ \rho b_1 \\ \rho b_2 \\ \rho b_3 \\ \rho b_j v_j + r \end{pmatrix}. \tag{4}$$

In general $\mathbf{F}^0$, $\mathbf{F}^i$ e $\mathbf{f}$ $\mathbb{R}^n$ vectors.

Lagrangian form of Balance Laws

It is convenient in some case to use Lagrangian variables $(\mathbf{X}, t)$ referring to the initial configuration. We know

$$dV = JdV^*, \quad \mathbf{n}d\Sigma = \mathbf{F}^C\mathbf{n}^*d\Sigma^*. \tag{5}$$

Therefore

$$\frac{d}{dt}\int_V \Psi dV = -\int_\Sigma \Phi_i n_i d\Sigma + \int_V f dV \tag{6}$$

can be rewritten

$$\frac{d}{dt}\int_{V^*} J\Psi dV^* = -\int_{\Sigma^*} \Phi_i F^C_{iA} n^*_A d\Sigma^* + \int_{V^*} fJdV^*. \tag{7}$$

Using Cauchy-Green theorem

$$\int_{V^*} \frac{\partial}{\partial t}(J\Psi)\ dV^* + \int_{V^*} \frac{\partial \Phi_i F^C_{iA}}{\partial X_A} dV^* = \int_{V^*} fJdV^*$$

and assuming regularity conditions

$$\frac{\partial \Psi^*}{\partial t} + \frac{\partial \Phi^*_A}{\partial X_A} = f^* \tag{8}$$

where

$$\Psi^* = \frac{\rho^*}{\rho}\Psi, \qquad \Phi^*_A = \Phi_i F^C_{iA}, \qquad f^* = \frac{\rho^*}{\rho} f. \tag{9}$$

For example the Balance Laws of mass, momentum and energy becomes in Lagrangian variables:

$$\begin{cases} \rho = \rho^*/J \\ \dfrac{\partial \rho^* v_j}{\partial t} - \dfrac{\partial T_{jA}}{\partial X_A} = \rho^* b_j \\ \dfrac{\partial}{\partial t}\left(\rho^* \dfrac{v^2}{2} + \rho^* e\right) + \dfrac{\partial}{\partial X_A}\left(Q_A - T_{iA} v_i\right) = \rho^* b_i v_i + r^*. \end{cases} \tag{10}$$

where $\mathbf{T} \equiv (T_{iA})$ is the Piola-Kirchhoff stress tensor

$$T_{jA} = t_{ij} F^C_{iA} \quad \Longleftrightarrow \quad \mathbf{T} = \mathbf{t}\mathbf{F}^C. \tag{11}$$

$\mathbf{Q} \equiv (Q_A)$ the Lagrangian heat flux

$$Q_A = q_i F^C_{iA} \quad \Longleftrightarrow \quad \mathbf{Q} = \mathbf{F}^{CT}\mathbf{q} \tag{12}$$

and

$$r^* = r\frac{\rho^*}{\rho}. \tag{13}$$

Constitutive Equations and Universal Principles

To close the system we need the constitutive equations that to be physical consistent need to verifies the following two principles:

1. *The Frame Indifference Principle – Objectivity* that require that the constitutive equations are independent of the Observer
2. *The Entropy Principle*, that require that admissible constitutive equations are such that every solution of the closed system is compatible with the second law of thermodynamics

Objectivity Principle

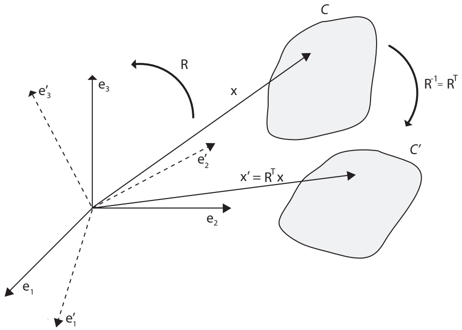


Figure: Material Indifference

Therefore the stress, the heat flux, the internal energy, are the same in $\mathcal{C}$ and in $\mathcal{C}'$ except for a similitude transformation:

$$\mathbf{t}' = \mathbf{R}^T \mathbf{t} \mathbf{R}, \quad \mathbf{q}' = \mathbf{R}^T \mathbf{q}, \quad e' = e. \tag{14}$$

The Entropy Principle

i) *There exists an additive and objective scalar, which we call entropy;*

ii) *The entropy density s the flux of the entropy Φ_i are constitutive functions to be determined*;

iii) *The entropy production Σ is non-negative for all thermodynamic processes*.

$$\frac{\partial \rho s}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho s\, v_i + \Phi_i\right) = \Sigma \tag{15}$$

This formulations is due to MÜLLER. In the classical vision (COLEMAN-NOLL) was considered the so called CLAUSIUS-DUHEM entropy principle in which

$$\mathbf{\Phi} = \frac{\mathbf{q}}{\vartheta}. \tag{16}$$

In this lectures we consider the simple case of Clausius-Duhem:

$$\frac{\partial \rho s}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho s\, v_i + \frac{q_i}{\vartheta}\right) = \Sigma \geq 0. \tag{17}$$

In the case of energy production the entropy principle is modified in

$$\frac{\partial \rho s}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho s\, v_i + \frac{q_i}{\vartheta}\right) = \frac{r}{\vartheta} + \Sigma, \qquad \Sigma \geq 0. \tag{18}$$

In Lagrangian variables the balance law of entropy becomes:

$$\frac{\partial \rho^* s}{\partial t} + \frac{\partial}{\partial X_A}\left(\frac{Q_A}{\vartheta}\right) = \frac{r^*}{\vartheta} + \Sigma^*; \qquad \Sigma^* \geq 0 \tag{19}$$

with

$$r^* = \frac{\rho^*}{\rho}\, r\,, \qquad \Sigma^* = \frac{\rho^*}{\rho}\,\Sigma\,.$$

Elastic body

A body is elastic if the stress tensor depend only on the gradient of deformation:

$$\mathbf{t} \equiv \mathbf{t}\left(\mathbf{F}\right). \tag{20}$$

In this case we have

$$\frac{\partial^2 \rho^* u_i}{\partial t^2} - \frac{\partial T_{iA}\left(\frac{\partial u_k}{\partial X_B}\right)}{\partial X_A} = \rho^* b_i, \tag{21}$$

that can be rewritten as system of first order:

$$\begin{cases} \rho^* \dfrac{\partial v_i}{\partial t} - \dfrac{\partial T_{iA}(F_{kB})}{\partial X_A} = \rho^* b_i \\[2ex] \dfrac{\partial F_{iA}}{\partial t} - \dfrac{\partial v_i}{\partial X_A} = 0 \end{cases} \tag{22}$$

in the unknown field $\mathbf{v} \equiv \mathbf{v}\left(\mathbf{X},t\right)$ and $\mathbf{F} \equiv \mathbf{F}\left(\mathbf{X},t\right)$.

Consequences of the Objectivity Principle in elasticity

Let $\mathbf{S}$ the so called second tensor of PIOLA-KIRCHHOFF:

$$\mathbf{S} = \mathbf{F}^{-1}\mathbf{T} \in \mathcal{Sym}, \quad \mathbf{t} = \frac{1}{J}\,\mathbf{F}\,\mathbf{S}\,\mathbf{F}^{T}. \tag{23}$$

Theorem

Necessary and sufficient condition such that the objectivity principle hold is that the second tensor of PIOLA-KIRCHHOFF *depends on* $\mathbf{F}$ *only trough the deformation matrix* $\mathbf{E}$*:*

$$\mathbf{S} \equiv \mathbf{S}(\mathbf{E}) \quad \textit{or that is the same} \quad \mathbf{S} \equiv \mathbf{S}(\mathbf{C}).$$

Proof: As the body is elastic we have:

$$\mathbf{t}(\mathbf{F}') = \mathbf{R}^{T}\mathbf{t}(\mathbf{F})\mathbf{R} \quad \forall\, \mathbf{R} \in \mathcal{Rot} \qquad \text{con } \mathbf{F}' = \mathbf{R}^{T}\mathbf{F}$$

then

$$\mathbf{t}\left(\mathbf{R}^{T}\mathbf{F}\right) = \mathbf{R}^{T}\mathbf{t}(\mathbf{F})\,\mathbf{R} \qquad \forall\, \mathbf{R} \in \mathcal{Rot}. \tag{24}$$

Substituting $(23)_2$ in (24) we have

$$\frac{1}{J}\mathbf{R}^T\mathbf{F}\mathbf{S}\left(\mathbf{R}^T\mathbf{F}\right)\mathbf{F}^T\mathbf{R} = \frac{1}{J}\mathbf{R}^T\mathbf{F}\mathbf{S}\left(\mathbf{F}\right)\mathbf{F}^T\mathbf{R} \qquad \forall\mathbf{R}\in\mathcal{Rot}$$

i.e.

$$\mathbf{S}\left(\mathbf{R}^T\mathbf{F}\right) = \mathbf{S}\left(\mathbf{F}\right) \qquad \forall\mathbf{R}\in\mathcal{Rot} \tag{25}$$

Recalling the polar theorem

$$\mathbf{F} = \hat{\mathbf{R}}\mathbf{U} \tag{26}$$

and requiring that (25)is satisfied also for $\mathbf{R} = \hat{\mathbf{R}}$ we obtain

$$\mathbf{S}\left(\mathbf{U}\right) = \mathbf{S}\left(\mathbf{F}\right),$$

and therefore $\mathbf{S}$ depends on $\mathbf{F}$ only trough the dilatation $\mathbf{U}$ or equivalently $\mathbf{S}$ depends on $\mathbf{C}$ $\left(\mathbf{C} = \mathbf{U}^2\right)$ or $\mathbf{S}$ depends on $\mathbf{E}$ $\left(\mathbf{E} = \frac{1}{2}\left(\mathbf{C} - \mathbf{I}\right)\right)$:

$$\mathbf{S} \equiv \mathbf{S}\left(\mathbf{E}\right) \qquad\qquad c.v.d.$$

From $(23)_2$ we have that the dependence of the stress tensor $\mathbf{t}$ on $\mathbf{F}$ must be

$$\mathbf{t}(\mathbf{F}) = \frac{1}{J}\,\mathbf{F}\,\mathbf{S}(\mathbf{E})\,\mathbf{F}^T \tag{27}$$

while the first PIOLA-KIRCHHOFF tensor

$$\mathbf{T}(\mathbf{F}) = \mathbf{F}\mathbf{S}(\mathbf{E}).$$

A body is thermoelastic if the constitutive equations depends on the gradient of deformation, the temperature and eventually of gradient of temperature:

$$\begin{cases} \mathbf{t} \equiv \mathbf{t}\left(\mathbf{F}, \vartheta\right) \\ e \equiv e\left(\mathbf{F}, \vartheta\right) \\ \mathbf{q} = -\chi\left(\mathbf{F}, \vartheta\right) \nabla\vartheta. \end{cases} \tag{28}$$

The last equation of (28) is the Fourier law in which χ denotes the heat conductibility.

Theorem

Necessary and sufficient condition such that the objectivity principle hold is that the second tensor of PIOLA-KIRCHHOFF, *internal energy and heat conductivity depends on* **F** *only trough the deformation matrix* **E** *and on the temperature:*

$$\mathbf{S} \equiv \mathbf{S}\left(\mathbf{E}, \vartheta\right), \quad e \equiv e(\mathbf{E}, \vartheta), \quad \chi \equiv \chi(\mathbf{E}, \vartheta). \tag{29}$$

The field equations

$$\begin{cases} \rho^* \dfrac{\partial v_i}{\partial t} - \dfrac{\partial T_{iA}}{\partial X_A} = \rho^* b_i \\[2ex] \dfrac{\partial F_{iA}}{\partial t} - \dfrac{\partial v_i}{\partial X_A} = 0 \\[2ex] \rho^* \dfrac{\partial}{\partial t}\left(\dfrac{v^2}{2} + e\right) - \dfrac{\partial}{\partial X_A}\left(T_{iA} v_i - Q_A\right) = \rho^* b_i v_i + r^*. \end{cases} \tag{30}$$

can be rewritten taking into account that $e \equiv e(\mathbf{F}, \vartheta)$ in the form

$$\begin{cases} \rho^* \dfrac{\partial v_i}{\partial t} = \dfrac{\partial T_{iA}}{\partial X_A} + \rho^* b_i \\[2ex] \dfrac{\partial F_{iA}}{\partial t} = \dfrac{\partial v_i}{\partial X_A} \\[2ex] \dfrac{\partial \vartheta}{\partial t} = \dfrac{1}{\frac{\partial e}{\partial \vartheta}} \left\{ \left(\dfrac{T_{iA}}{\rho^*} - \dfrac{\partial e}{\partial F_{iA}} \right) \dfrac{\partial v_i}{\partial X_A} - \dfrac{1}{\rho^*} \dfrac{\partial Q_A}{\partial X_A} + \dfrac{r^*}{\rho^*} \right\} \end{cases} \tag{31}$$

The Fourier law become in Lagrangian variables:

$$Q_A = -\chi F^C_{iA} \frac{\partial \vartheta}{\partial x_i} = -\chi F^C_{iA} \frac{\partial \vartheta}{\partial X_B} \frac{\partial X_B}{\partial x_i} = -\chi J F^{-1}_{Ai} F^{-1}_{Bi} \frac{\partial \vartheta}{\partial X_B}$$

i.e.

$$\mathbf{Q} = -\chi J \mathbf{F}^{-1} \left(\mathbf{F}^{-1}\right)^T \mathrm{Grad}\ \vartheta \quad \mathbf{Q} = -\chi(\mathbf{E}, \vartheta)\ J\mathbf{C}^{-1} \mathrm{Grad}\ \vartheta$$

where

$$\mathrm{Grad}\ \vartheta \equiv \left(\frac{\partial \vartheta}{\partial X_1}, \frac{\partial \vartheta}{\partial X_2}, \frac{\partial \vartheta}{\partial X_3} \right).$$

Entropy principle in thermoelasticity

We now require the compatibility with the entropy principle, i.e. every solution of (31) must be solution of (19) assuming that the entropy principle is also a constitutive equation:

$$s \equiv s\left(\mathbf{F}, \vartheta\right). \tag{32}$$

we have the following

Theorem (Entropy Principle)

Necessary an sufficient condition such that the entropy principle is satisfied is that there exists a scalar function the free energy, $\psi \equiv \psi(\mathbf{E}, \vartheta)$, such that

$$s = -\frac{\partial \psi}{\partial \vartheta}, \qquad \mathbf{S} = \rho^* \frac{\partial \psi}{\partial \mathbf{E}}, \qquad e = \psi - \vartheta \frac{\partial \psi}{\partial \vartheta}. \tag{33}$$

Moreover the heat conductivity must be non negative:

$$\chi\left(\mathbf{E}, \vartheta\right) \geq 0.$$

Proof : From (19) and (32) we have

$$\rho^* \left(\frac{\partial s}{\partial F_{iA}} \frac{\partial F_{iA}}{\partial t} + \frac{\partial s}{\partial \vartheta} \frac{\partial \vartheta}{\partial t} \right) + \frac{1}{\vartheta} \frac{\partial Q_A}{\partial X_A} - \frac{1}{\vartheta^2} Q_A \frac{\partial \vartheta}{\partial X_A} - \frac{r^*}{\vartheta} = \Sigma^* \geq 0.$$

then

$$\rho^* \left\{ \frac{\partial s}{\partial F_{iA}} + \frac{\frac{\partial s}{\partial \vartheta}}{\frac{\partial e}{\partial \vartheta}} \left(\frac{T_{iA}}{\rho^*} - \frac{\partial e}{\partial F_{iA}} \right) \right\} \frac{\partial v_i}{\partial X_A} + \left\{ \frac{1}{\vartheta} - \frac{\frac{\partial s}{\partial \vartheta}}{\frac{\partial e}{\partial \vartheta}} \right\} \left(\frac{\partial Q_A}{\partial X_A} - r^* \right) + \tag{34}$$

$$+ \frac{1}{\vartheta^2} \chi J \mathbf{C}^{-1} \mathrm{Grad}\ \vartheta \cdot \mathrm{Grad}\ \vartheta = \Sigma^* \geq 0.$$

then

$$\begin{cases} \vartheta \dfrac{\partial s}{\partial \vartheta} = \dfrac{\partial e}{\partial \vartheta} \\ \vartheta \dfrac{\partial s}{\partial F_{iA}} = \dfrac{\partial e}{\partial F_{iA}} - \dfrac{T_{iA}}{\rho^*} \\ \chi \geq 0. \end{cases} \tag{35}$$

The first two of (35) are equivalent to the so called GIBBS equation (local equilibrium)

$$\vartheta ds = de - \frac{1}{\rho^*}\mathbf{T} \cdot d\mathbf{F} \tag{36}$$

or equivalently

$$\vartheta ds = de - \frac{1}{\rho^*}\mathbf{S}\cdot d\mathbf{E}. \tag{37}$$

Let

$$\psi = e - \vartheta s, \tag{38}$$

the free energy we obtain

$$d\psi = -s\, d\vartheta + \frac{1}{\rho^*}\mathbf{S}\cdot d\mathbf{E}. \tag{39}$$

and we proved the conditions (33). Taking into account that

$$\Sigma^* = \frac{\chi J}{\vartheta^2}\mathbf{C}^{-1}\mathrm{Grad}\,\vartheta \cdot \mathrm{Grad}\,\vartheta \geq 0. \tag{40}$$

then $\chi \geq 0$ and theorem is proved, In particular the first PIOLA-KIRCHHOFF tensor must be in the form

$$\mathbf{T} = \rho^* \mathbf{F}\frac{\partial \psi}{\partial \mathbf{E}}. \tag{41}$$

Ideal Fluids and Euler system

Definition

A fluid is ideal if the specific stress is normal and have pressure character:

$$\mathbf{t}_n = -p_n \mathbf{n} \qquad p_n \geq 0 \qquad \forall\ \mathbf{n}. \tag{42}$$

For the Cauchy theorem we have

$$p_n \mathbf{n} = p_1 n_1 \mathbf{e}_1 + p_2 n_2 \mathbf{e}_2 + p_3 n_3 \mathbf{e}_3$$

and therefore we obtains soon the PASCAL result

$$p_n = p_1 = p_2 = p_3 = p \qquad \forall\ \mathbf{n}.$$

Then

$$\mathbf{t}_n = -p\ \mathbf{n} \qquad p \geq 0 \qquad \forall\ \mathbf{n} \tag{43}$$

that implies that the stress tensor is isotropic

$$\mathbf{t} = -p\mathbf{I}, \qquad \mathbf{t} \equiv \begin{vmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{vmatrix}, \qquad t_{ij} = -p\delta_{ij}. \tag{44}$$

In an ideal fluid is supposed also negligible the heat conductivity and then:

$$\mathbf{q} = 0. \tag{45}$$

Therefore the balance laws assume the form

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial \rho v_i}{\partial x_i} = 0 \\[2ex] \dfrac{\partial \rho v_j}{\partial t} + \dfrac{\partial}{\partial x_i}\left(\rho v_i v_j + p\delta_{ij}\right) = \rho b_j \qquad (j = 1, 2, 3) \\[2ex] \dfrac{\partial}{\partial t}\left(\dfrac{\rho v^2}{2} + \rho e\right) + \dfrac{\partial}{\partial x_i}\left\{\left(\dfrac{\rho v^2}{2} + \rho e + p\right) v_i\right\} = \rho b_j v_j + r. \end{cases} \tag{46}$$

For prescribed thermal and caloric equation of state

$$p \equiv p(\rho, \vartheta), \qquad e \equiv e(\rho, \vartheta). \tag{47}$$

the system is closed and is called EULER system.

Fourier-Navier-Stokes Dissipative fluids

Definition

A real fluid have character of pressure only in equilibrium.

Then in nonequilibrium

$$\mathbf{t} = -p\mathbf{I} + \boldsymbol{\sigma}, \qquad t_{ij} = -p\delta_{ij} + \sigma_{ij}, \tag{48}$$

where $\boldsymbol{\sigma} \equiv \|\sigma_{ij}\| \in \mathcal{Sym}$ is the viscous stress tensor and is assumed that depends in the symmetric part of velocity gradient $\mathbf{D}$:

$$\boldsymbol{\sigma} \equiv \boldsymbol{\sigma}(\mathbf{D}), \quad \boldsymbol{\sigma}(0) = 0, \quad \mathbf{D} = \frac{1}{2}\left(\nabla\mathbf{v} + (\nabla\mathbf{v})^T\right). \tag{49}$$

For a real fluid the heat flux is not zero and depends on gradient of the temperature

$$\mathbf{q} \equiv \mathbf{q}(\nabla\vartheta), \quad \mathbf{q}(0) = 0. \tag{50}$$

The most simple case is to suppose linear constitutive equations. For the heat flux we have seen the FOURIER law:

$$\mathbf{q} = -\chi\,\nabla\vartheta, \quad \chi \equiv \chi(\rho, \vartheta). \tag{51}$$

While for the viscous stress tensor there is the assumptions of NAVIER STOKES) :

$$\boldsymbol{\sigma} = \lambda \, \mathrm{div}\mathbf{v} \; \mathbf{I} + 2\mu \; \mathbf{D}, \tag{52}$$

i.e.

$$\sigma_{ij} = \lambda \frac{\partial v_k}{\partial x_k} \, \delta_{ij} + \mu \, \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).$$

It is more convenient to use orthogonal tensors and for this reason is convenient to decompose the matrix **D** in the deviatoric part (traceless) and the isotropic part:

$$\mathbf{D} = \mathbf{D}^D + \frac{1}{3} \, \mathrm{div}\mathbf{v} \, \mathbf{I}$$

and the (52) becomes

$$\boldsymbol{\sigma} = \nu \, \mathrm{div}\mathbf{v} \; \mathbf{I} + 2\mu \; \mathbf{D}^D \tag{53}$$

con

$$\nu = \frac{1}{3}(3\lambda + 2\mu).$$

The scalars ν e and μ are the so called *bulk viscosity* and *shear viscosity* respectively and they depends on the density and temperature:

$$\nu \equiv \nu(\rho, \vartheta), \qquad \mu \equiv \mu(\rho, \vartheta). \tag{54}$$

The stress tensor becomes

$$\mathbf{t} = -(p+\pi)\,\mathbf{I} + 2\mu\,\mathbf{D}^D \tag{55}$$

where we have put

$$\pi = -\nu\,\mathrm{div}\mathbf{v}.$$

For this reason π is called *dynamic pressure*, while

$$\mathbf{t}^D = \boldsymbol{\sigma}^D = 2\mu\mathbf{D}^D$$

is the *deviatoric viscous stress tensor*. A fluid is called *Stokesian* if $\nu = 0$ (example are the Monatomic gases). The balance laws for Fourier-Navier-Stokes fluids becomes

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial \rho v_i}{\partial x_i} = 0 \\[2ex] \dfrac{\partial \rho v_j}{\partial t} + \dfrac{\partial}{\partial x_i}\left(\rho v_i v_j + p\delta_{ij} - \sigma_{ij}\right) = \rho b_j \qquad (j = 1,2,3) \\[2ex] \dfrac{\partial}{\partial t}\left(\dfrac{\rho v^2}{2} + \rho e\right) + \dfrac{\partial}{\partial x_i}\left\{\left(\dfrac{\rho v^2}{2} + \rho e + p\right) v_i - \sigma_{ij} v_j + q_i\right\} = \rho b_j v_j + r \end{cases} \tag{56}$$

To close the system we need the constitutive equations:

$$\begin{aligned} &p \equiv p(\rho, \vartheta) \quad \text{(pressure)}, \quad e \equiv e(\rho, \vartheta) \;\; \text{(internal energy)}, \\ &\chi \equiv \chi(\rho, \vartheta) \quad \text{(heat conductivity)}, \\ &\nu \equiv \nu(\rho, \vartheta) \quad \text{e} \quad \mu \equiv \mu(\rho, \vartheta) \quad \text{(bulk and shear viscosity)}. \end{aligned} \tag{57}$$

It is easy to prove that the previous system can be rewritten fro classical solutions in the form:

$$\begin{cases} \dfrac{d\rho}{dt} + \rho \operatorname{div} \mathbf{v} = 0 \\[2ex] \rho \dfrac{dv_j}{dt} + \dfrac{\partial}{\partial x_i} \left(p\delta_{ij} - \sigma_{ij} \right) = \rho b_j \\[2ex] \rho \dfrac{de}{dt} + p \operatorname{div} \mathbf{v} - \boldsymbol{\sigma} \cdot \mathbf{D} + \operatorname{div} \mathbf{q} = r \end{cases} \tag{58}$$

Taking into account that $e \equiv e(\rho, \vartheta)$ the last equation can be write

$$\frac{d\vartheta}{dt} = \frac{1}{\rho \frac{\partial e}{\partial \vartheta}} \left\{ r + \left(\rho^2 \frac{\partial e}{\partial \rho} - p \right) \operatorname{div} \mathbf{v} + \boldsymbol{\sigma} \cdot \mathbf{D} - \operatorname{div} \mathbf{q} \right\}. \tag{59}$$

Entropy principle for a fluid

The entropy principle require that every solutions of the fluid system satisfy also

$$\rho \frac{ds}{dt} + \frac{\partial}{\partial x_i}\left(\frac{q_i}{\vartheta}\right) - \frac{r}{\vartheta} = \Sigma \geq 0. \tag{60}$$

As the entropy density is a constitutive function $s \equiv s(\rho, \vartheta)$ we have

$$\left\{\frac{1}{\vartheta} - \frac{\frac{\partial s}{\partial \vartheta}}{\frac{\partial e}{\partial \vartheta}}\right\}(\operatorname{div}\mathbf{q} - r) + \operatorname{div}\mathbf{v}\left\{-\rho^2\frac{\partial s}{\partial \rho} + \left(\rho^2\frac{\partial e}{\partial \rho} - p\right)\frac{\frac{\partial s}{\partial \vartheta}}{\frac{\partial e}{\partial \vartheta}}\right\}$$
$$+\frac{\frac{\partial s}{\partial \vartheta}}{\frac{\partial e}{\partial \vartheta}}\boldsymbol{\sigma}\cdot\mathbf{D} - \frac{1}{\vartheta^2}\mathbf{q}\cdot\nabla\vartheta = \Sigma \geq 0. \tag{61}$$

Then

$$\begin{cases} \vartheta \dfrac{\partial S}{\partial \vartheta} = \dfrac{\partial e}{\partial \vartheta} \\[2ex] \vartheta \dfrac{\partial S}{\partial \rho} = \dfrac{\partial e}{\partial \rho} - \dfrac{p}{\rho^2}. \end{cases} \tag{62}$$

i.e. the Gibbs equation hold

$$\vartheta ds = de - \frac{p}{\rho^2} d\rho. \tag{63}$$

The residual inequality becomes for Fourier-Navier-Stokes

$$\Sigma = \frac{1}{\vartheta}\left(\lambda \ \mathrm{div}\,\mathbf{v}\ \mathbf{I} + 2\mu\mathbf{D}^D\right)\cdot\mathbf{D} + \frac{\chi}{\vartheta^2}|\nabla\vartheta|^2 =$$

$$= \frac{1}{\vartheta}\left(\lambda\left(\mathrm{div}\,\mathbf{v}\right)^2 + 2\mu\|\mathbf{D}^D\|^2\right) + \frac{\chi}{\vartheta^2}|\nabla\vartheta|^2 \geq 0,$$

that implies

$$\lambda(\rho,\vartheta) \geq 0, \qquad \mu(\rho,\vartheta) \geq 0, \qquad \chi(\rho,\vartheta) \geq 0. \tag{64}$$

Introducing the free energy

$$\psi = e - \vartheta s \tag{65}$$

we obtain

$$d\psi = \frac{p}{\rho^2}d\rho - s d\vartheta. \tag{66}$$

and then

$$p = \rho^2\frac{\partial\psi}{\partial\rho}, \qquad s = -\frac{\partial\psi}{\partial\vartheta}, \qquad e = \psi - \vartheta\frac{\partial\psi}{\partial\vartheta}. \tag{67}$$

Heat equation

In the case of a rigid heat conductor we have only the energy balance

$$\rho^* \frac{\partial e}{\partial t} + \operatorname{div} \mathbf{q} = r \tag{68}$$

with the constitutive FOURIER equation:

$$\mathbf{q} = -\chi \nabla \vartheta \tag{69}$$

and

$$e \equiv e(\vartheta) \quad \text{e} \quad \chi \equiv \chi(\vartheta). \tag{70}$$

Substituting the (69) and (70) in (68) we have

$$\rho^* \, c_V(\vartheta) \, \frac{\partial \vartheta}{\partial t} - \operatorname{div} (\chi \nabla \vartheta) = r \tag{71}$$

with

$$c_V(\vartheta) = e'(\vartheta) = \frac{de}{d\vartheta} \quad \text{(specific heat).}$$

Then (71) becomes:

$$\frac{\partial \vartheta}{\partial t} - \mu \, \Delta \vartheta - \nu \, (\operatorname{grad} \vartheta)^2 = r \tag{72}$$

where

$$\mu(\vartheta) = \frac{\chi(\vartheta)}{\rho^* c_V(\vartheta)}\,; \quad \nu(\vartheta) = \frac{\chi'(\vartheta)}{\rho^* c_V(\vartheta)}\,; \quad \chi'(\vartheta) = \frac{d\chi(\vartheta)}{d\vartheta}.$$

in the simple case in which χ and c_V are constants and we have no supply $r = 0$, the equation assume the usual form of heat equation:

$$\frac{\partial\vartheta}{\partial t} - \mu\,\Delta\vartheta = 0. \tag{73}$$

that is the typical parabolic equation If we prescribe the initial data

$$\vartheta(\mathbf{x}, 0) = \vartheta_0(\mathbf{x}), \tag{74}$$

the solution is

$$\vartheta(\mathbf{x}, t) = \frac{1}{(4\pi\mu t)^{3/2}} \int_{-\infty}^{+\infty} \vartheta_0(\mathbf{y}) \exp\left(-\frac{(\mathbf{y}-\mathbf{x})^2}{4\mu t}\right) d\mathbf{y}, \tag{75}$$

and for an initial data having support compact we have the so called heat paradox of infinite propagation.

Cattaneo Equation

Carlo Cattaneo propose to modify the Fourier law:

$$\mathbf{q} = -\chi \nabla \vartheta + \chi \tau \nabla \dot{\vartheta} \qquad (\dot{\vartheta} = \partial \vartheta / \partial t), \tag{76}$$

where τ is a relaxation time. The (76) can be rewritten as :

$$\mathbf{q} = -\chi \left(1 - \tau \frac{\partial}{\partial t}\right) \nabla \vartheta. \tag{77}$$

If τ is small enough the inverse operator is

$$\left(1 - \tau \frac{\partial}{\partial t}\right)^{-1} \simeq 1 + \tau \frac{\partial}{\partial t} \tag{78}$$

then from (78) and (77) we obtain the *Cattaneo equation*

$$\tau \frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -\chi \nabla \vartheta. \tag{79}$$

Combining this equation with the energy equation we obtain the hyperbolic *telegraphist equation*

$$\tau \frac{\partial^2 \vartheta}{\partial t^2} + \frac{\partial \vartheta}{\partial t} - \mu \Delta \vartheta = 0. \tag{80}$$

Hyperbolic Systems

The hyperbolic systems of continuum mechanic are balance laws

$$\frac{\partial \mathbf{F}^\alpha(\mathbf{u})}{\partial x^\alpha} = \mathbf{f}(\mathbf{u}) \tag{81}$$

with $\mathbf{F}^\alpha$ e $\mathbf{f}$ local function of $\mathbf{u}$. The (81) can be rewritten

$$\mathbf{A}^\alpha(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x^\alpha} = \mathbf{f}(\mathbf{u}), \quad \mathbf{A}^\alpha = \frac{\partial \mathbf{F}^\alpha}{\partial \mathbf{u}}. \tag{82}$$

Definition (Hyperbolic System)

A system (82) is hyperbolic in the time direction if
a) $\det \mathbf{A}^0 \neq 0$*;*
b) The following eigenvalue problem

$$(\mathbf{A}_n - \lambda \mathbf{A}^0)\mathbf{d} = \mathbf{0}, \qquad (\mathbf{A}_n = \mathbf{A}^i n_i) \tag{83}$$

$\forall \mathbf{n} \in \mathbb{R}^3 : \|\mathbf{n}\| = 1$, *admits real eigenvalues* λ *and the eigenvectors* $\mathbf{d}$ *are linearly independent*

Choosing as field

$$\mathbf{u} \equiv (\rho, v_1, v_2, v_3, s)^T$$

the Euler system becomes in the form (82) with:

$$\mathbf{A}^0 \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{A}^1 \equiv \begin{pmatrix} v_1 & \rho & 0 & 0 & 0 \\ p_\rho/\rho & v_1 & 0 & 0 & p_S/\rho \\ 0 & 0 & v_1 & 0 & 0 \\ 0 & 0 & 0 & v_1 & 0 \\ 0 & 0 & 0 & 0 & v_1 \end{pmatrix},$$

$$\mathbf{A}^2 \equiv \begin{pmatrix} v_2 & 0 & \rho & 0 & 0 \\ 0 & v_2 & 0 & 0 & 0 \\ p_\rho/\rho & 0 & v_2 & 0 & p_S/\rho \\ 0 & 0 & 0 & v_2 & 0 \\ 0 & 0 & 0 & 0 & v_2 \end{pmatrix}, \quad \mathbf{A}^3 \equiv \begin{pmatrix} v_3 & 0 & 0 & \rho & 0 \\ 0 & v_3 & 0 & 0 & 0 \\ 0 & 0 & v_3 & 0 & 0 \\ p_\rho/\rho & 0 & 0 & v_3 & p_S/\rho \\ 0 & 0 & 0 & 0 & v_3 \end{pmatrix},$$

then ($v_n = \mathbf{v} \cdot \mathbf{n}$)

$$\mathbf{A}_n = \mathbf{A}^i\, n_i \equiv \begin{pmatrix} v_n & \rho\, n_1 & \rho\, n_2 & \rho\, n_3 & 0 \\ \frac{p_\rho}{\rho}\, n_1 & v_n & 0 & 0 & \frac{p_S}{\rho}\, n_1 \\ \frac{p_\rho}{\rho}\, n_2 & 0 & v_n & 0 & \frac{p_S}{\rho}\, n_2 \\ \frac{p_\rho}{\rho}\, n_3 & 0 & 0 & v_n & \frac{p_S}{\rho}\, n_3 \\ 0 & 0 & 0 & 0 & v_n \end{pmatrix}$$

that have eigenvalues

$$\lambda^{(1)} = v_n - c; \quad \lambda^{(2)} = \lambda^{(3)} = \lambda^{(4)} = v_n; \quad \lambda^{(5)} = v_n + c, \tag{84}$$

and eigenvectors

$$\mathbf{d}^{(1)} \equiv (\rho, -cn_1, -cn_2, -cn_3, 0)^T, \quad \mathbf{d}^{(5)} \equiv (\rho, cn_1, cn_2, cn_3, 0)^T,$$
$$\mathbf{d}^{(2)} \equiv (-p_s, 0, 0, 0, c^2)^T, \quad \mathbf{d}^{(3)} \equiv (0, -n_3, 0, n_1, 0)^T, \quad \mathbf{d}^{(4)} \equiv (0, -n_2, n_1, 0, 0)^T, \tag{85}$$

Example of Cattaneo system

Instead to construct the matrix is more convenient to apply the following rule from the system

$$\frac{\partial}{\partial t} \to -\lambda\delta, \qquad \frac{\partial}{\partial x_i} \to n_i\delta, \qquad \mathbf{f} \to 0 \tag{86}$$

obtaining immediately

$$(\mathbf{A}_n - \lambda \mathbf{A}^0)\delta\mathbf{u} = 0 \tag{87}$$

from which we deduce that $\delta\mathbf{u}$ coincide qith the right eigenvector $\mathbf{d}$.
For example from Cattaneo system

$$\begin{cases} \rho^* c_V(\vartheta)\dfrac{\partial\vartheta}{\partial t} + \dfrac{\partial q_i}{\partial x_i} = r \\ \\ \tau(\vartheta)\dfrac{\partial q_i}{\partial t} + \chi(\vartheta)\dfrac{\partial\vartheta}{\partial x_i} = -q_i\,, \end{cases}$$

we have

$$\begin{cases} -\rho^* c_V(\vartheta)\lambda\, \delta\vartheta + \delta q_n = 0 \\ \\ -\lambda\, \tau(\vartheta)\, \delta q_i + \chi(\vartheta)\, n_i\, \delta\vartheta = 0 \end{cases} \tag{88}$$

con $\delta q_n = \delta q_i\, n_i$. Da (88) and we have

$$\lambda^{(1)} = -\sqrt{\frac{\chi}{\rho^*\, c_V\, \tau}}; \qquad \lambda^{(2)} = \lambda^{(3)} = 0; \qquad \lambda^{(4)} = \sqrt{\frac{\chi}{\rho^*\, c_V\, \tau}};$$

$$\mathbf{d}^{(1)} \equiv \left(1, \frac{\chi}{\lambda_1 \tau}\mathbf{n}\right); \quad \mathbf{d}^{(2)} \equiv (0, \mathbf{w}_1); \quad \mathbf{d}^{(3)} \equiv (0, \mathbf{w}_2); \quad \mathbf{d}^{(4)} \equiv \left(1, \frac{\chi}{\lambda_4 \tau}\mathbf{n}\right);$$

con $\mathbf{w}_1$ e $\mathbf{w}_2$: $\mathbf{w}_1 \cdot \mathbf{n} = \mathbf{w}_2 \cdot \mathbf{n} = \mathbf{w}_1 \cdot \mathbf{w}_2 = 0$ and the system is hyperbolic provided $\tau > 0$:

Wave equation and method of the characteristics

Let consider the wave equation in one space dimension

$$U_{tt} - c^2 U_{xx} = 0. \tag{89}$$

Let

$$U_t = v, \quad U_x = w. \tag{90}$$

that the equation can be rewritten as system of first order

$$\begin{cases} v_t - c^2 w_x = 0 \\ w_t - v_x = 0 \end{cases}, \tag{91}$$

that belong on the form

$$\mathbf{u}_t + \mathbf{A}\mathbf{u}_x = 0 \tag{92}$$

with

$$\mathbf{u} = (v, w)^T, \quad \mathbf{A} = \begin{pmatrix} 0 & -c^2 \\ -1 & 0 \end{pmatrix}. \tag{93}$$

The eigenvalues and the right eigenvectors are

$$\lambda^{(1)} = -c; \qquad \lambda^{(2)} = c; \tag{94}$$

$$\mathbf{d}^{(1)} \equiv \begin{pmatrix} c \\ 1 \end{pmatrix}; \qquad \mathbf{d}^{(2)} \equiv \begin{pmatrix} -c \\ 1 \end{pmatrix}. \tag{95}$$

Let $\mathbf{l}$ the left eigenvector of $\mathbf{A}$, i.e.:

$$\mathbf{l}\,\mathbf{A} = \lambda \mathbf{l} \tag{96}$$

then

$$\mathbf{l}^{(1)} \equiv (1, c) \; ; \qquad \mathbf{l}^{(2)} \equiv (1, -c). \tag{97}$$

We assign the initial data

$$U(x,0) = \varphi(x), \qquad U_t(x,0) = \psi(x). \tag{98}$$

i.e. for the first order system

$$v(x,0) = \psi(x), \qquad w(x,0) = \varphi'(x); \qquad \text{con} \quad \varphi'(x) = \frac{d\varphi}{dx} \tag{99}$$

Multiplying the (92) for the left eigenvector we have

$$\mathbf{l}\,\{\mathbf{u}_t + \mathbf{A}\mathbf{u}_x\} = 0 \quad \Leftrightarrow \quad \mathbf{l}\,\{\mathbf{u}_t + \lambda \mathbf{u}_x\} = 0 \tag{100}$$

and we define the characteristic line in the space-time as:

$$\frac{dx}{dt} = \lambda. \tag{101}$$

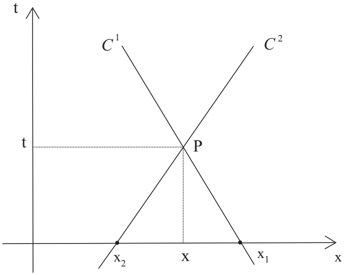


Figure: Characteristic lines

In the present case we have two characteristic lines

$$\mathcal{C}^1 \;\Rightarrow\; \frac{dx}{dt} = \lambda^{(1)} \;\Rightarrow\; x = -ct + x_1 \tag{102}$$

$$\mathcal{C}^2 \;\Rightarrow\; \frac{dx}{dt} = \lambda^{(2)} \;\Rightarrow\; x = ct + x_2 \tag{103}$$

We consider the directional derivative

$$\frac{d}{dt} = \partial_t + \lambda\, \partial_x \tag{104}$$

obtaining

$$\mathbf{l} \cdot \frac{d\mathbf{u}}{dt} = 0. \tag{105}$$

Then

$$\frac{d}{dt}(\mathbf{l}^{(1)} \cdot \mathbf{u}) = 0 \quad (\text{su } \mathcal{C}^1), \qquad \frac{d}{dt}(\mathbf{l}^{(2)} \cdot \mathbf{u}) = 0 \quad (\text{su } \mathcal{C}^2). \tag{106}$$

Therefore

$$\mathbf{l}^{(1)} \cdot \mathbf{u}(x,t) = \mathbf{l}^{(1)} \cdot \mathbf{u}(x_1, 0) \tag{107}$$

$$\mathbf{l}^{(2)} \cdot \mathbf{u}(x,t) = \mathbf{l}^{(2)} \cdot \mathbf{u}(x_2, 0) \tag{108}$$

Let $\mathbf{u}_0(x)$ the initial data of $\mathbf{u}$:

$$\mathbf{u}(x,0) = \mathbf{u}_0(x).$$

Therefore

$$\begin{cases} \mathbf{l}^{(1)} \cdot \mathbf{u}(x,t) = \mathbf{l}^{(1)} \cdot \mathbf{u}_0(x + ct) \\ \\ \mathbf{l}^{(2)} \cdot \mathbf{u}(x,t) = \mathbf{l}^{(2)} \cdot \mathbf{u}_0(x - ct). \end{cases} \tag{109}$$

In the present case we have the algebraic system:

$$\begin{cases} v(x,t) + c\,w(x,t) = \psi(x+ct) + c\varphi'(x+ct) \\ v(x,t) - c\,w(x,t) = \psi(x-ct) - c\varphi'(x-ct). \end{cases}$$

Then the solution

$$v(x,t) = \frac{1}{2}\left\{\psi(x+ct) + \psi(x-ct) + c\left[\varphi'(x+ct) - \varphi'(x-ct)\right]\right\} \tag{110}$$

$$w(x,t) = \frac{1}{2c}\left\{\psi(x+ct) - \psi(x-ct) + c\left[\varphi'(x+ct) + \varphi'(x-ct)\right]\right\}. \tag{111}$$

or

$$U(x,t) = \frac{1}{2c}\left\{\Gamma(x+ct) - \Gamma(x-ct) + c\left[\varphi(x+ct) + \varphi(x-ct)\right]\right\} \tag{112}$$

with

$$\Gamma(\xi) = \int_0^{\xi} \psi(\tau)d\tau.$$

Linear systems

In the case of a generic first order system of N equations if we represent the initial data in the basis of right eigenvectors

$$\mathbf{u}(x,t) = \sum_{j=1}^{N} \Pi^j(x,t)\mathbf{d}^{(j)} \tag{113}$$

proceeding in the same way with the method of characteristic it is possible to prove that the solution is a combination of N waves:

$$\mathbf{u}(x,t) = \sum_{j=1}^{N} \Pi_0^j(x-\lambda^j t)\mathbf{d}^{(j)}. \tag{114}$$

A non linear example: the Burgers equation

Let consider the non linear BURGERS equation

$$u_t + uu_x = 0. \tag{115}$$

The characteristc is

$$\frac{dx}{dt} = u\,(x,t) \tag{116}$$

But

$$\frac{du}{dt} = u_t + \lambda u_x = u_t + uu_x = 0. \tag{117}$$

i.e.

$$u(x,t) = u_0(x_0). \tag{118}$$

Therefore also in this case the charcateristc is a line and

$$x = x_0 + u_0(x_0)\,t \tag{119}$$

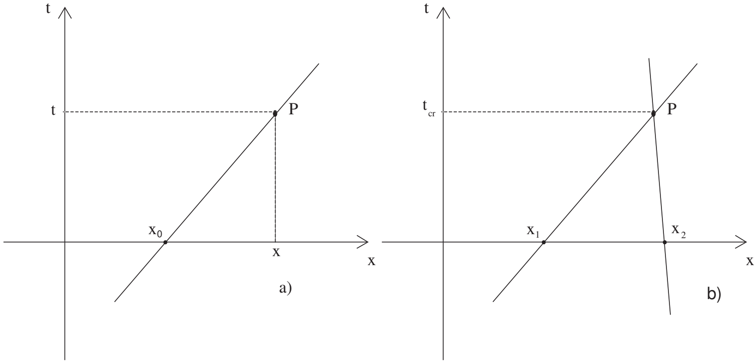


Figure: Caratteristica e tempo critico dell'equazione di Burgers

but the slope of the line depends on x_0 (see figure).

The determination of the critical time is simple In fact the (118) and (119) are solution in paramteric form: for a fixed time t the depenmdence of u from x is tyrouth the paarmeter x_0. Then invertibility is lost when $dx/dx_0 = 0$ and then:

$$t_c(x_0) = -\frac{1}{u'(x_0)}.$$

the critical time is

$$t_{cr} = \inf_{x_0}\{t_c(x_0) > 0\}. \tag{120}$$

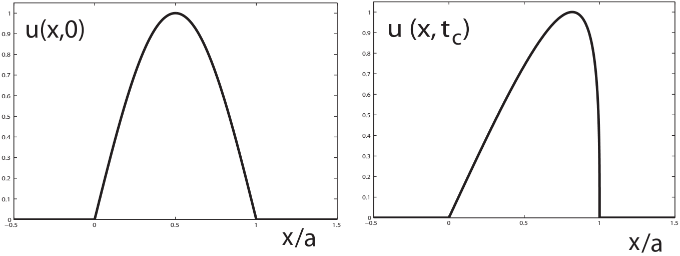


Figure: Critical time for Burtgers equation

Weak solutions

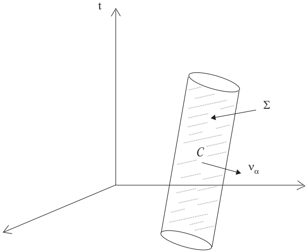


Figure: weak solutions

Let a system of balance laws

$$\partial_\alpha \mathbf{F}^\alpha(\mathbf{u}) = \mathbf{f}(\mathbf{u}). \tag{121}$$

Let $\mathcal{C}$ a domain in the space-time and multiply (121) for a test function ϕ, with support in $\mathcal{C}$ and we have

$$\int_{\mathcal{C}} \phi \left(\partial_\alpha \mathbf{F}^\alpha - \mathbf{f} \right) d\mathcal{C} = 0 \tag{122}$$

or

$$\int_{\mathcal{C}} \partial_\alpha \left(\phi \mathbf{F}^\alpha\right) d\mathcal{C} - \int_{\mathcal{C}} \left(\mathbf{F}^\alpha \partial_\alpha \phi + \mathbf{f}\phi\right) d\mathcal{C} \; = 0. \qquad (123)$$

Using the Gauss-Green theorem

$$\int_{\Sigma} \nu_\alpha \left(\phi \mathbf{F}^\alpha\right) d\Sigma - \int_{\mathcal{C}} (\mathbf{F}^\alpha \partial_\alpha \phi + \mathbf{f}\phi) d\mathcal{C} = 0. \qquad (124)$$

then (assuming zero initial data)

$$\int_{\mathcal{C}} \mathbf{F}^\alpha \partial_\alpha \phi d\mathcal{C} + \int_{\mathcal{C}} \mathbf{f}\phi d\mathcal{C} = 0. \qquad (125)$$

A solution of (125) *for any test function ϕ is called a weak solution* of (121).

Shock waves

If exists a regular surface Γ *with unit normal* $\mathbf{n}$ *moving with normal velocity s separating the space in two sub-spaces in which there are classical smooth solutions* $\mathbf{u}_0$ *and* $\mathbf{u}_1$ *such that their limit values in the surface are different we call this kind of solution a shock wave.* Let denoting the jump with a square bracket:

$$[\mathbf{u}] = \mathbf{u}_1 \big|_{\varphi^-} - \mathbf{u}_0 \big|_{\varphi^+} \qquad (\text{su } \Gamma).$$

We want to prove that a shock wave is a weak solution if and only if across the surface there exists some compatibility conditions called RANKINE-HUGONIOT conditions. Let condider a surface in the space time σ of normal φ_α (see Figura 7). We have from (124)

$$\int_{\Sigma^+ \cup \sigma^+} \varphi_\alpha^+ \phi \mathbf{F}_+^\alpha d\Sigma^* - \int_{\mathcal{C}^+} \left(\mathbf{F}_+^\alpha \partial_\alpha \phi + \mathbf{f}_+ \phi\right) d\mathcal{C}^+ = 0$$

$$\int_{\Sigma^- \cup \sigma -} \varphi_\alpha^- \phi \mathbf{F}_-^\alpha d\Sigma^* - \int_{\mathcal{C}^-} \left(\mathbf{F}_-^\alpha \partial_\alpha \phi + \mathbf{f}_- \phi\right) d\mathcal{C}^- = 0.$$

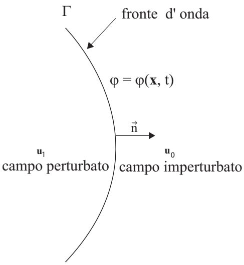


Figure: Onda d'urto

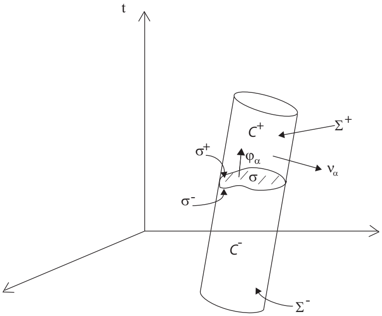


Figure: Soluzioni deboli tipo onde d'urto.

Then

$$\int_{\sigma^+} \varphi_\alpha^+ \phi \mathbf{F}_+^\alpha d\sigma^+ - \int_{\mathcal{C}^+} \left(\mathbf{F}_+^\alpha \partial_\alpha \phi + \mathbf{f}_+ \phi \right) d\mathcal{C}^+ = 0 \tag{126}$$

$$\int_{\sigma-} \varphi_\alpha^- \phi \mathbf{F}_-^\alpha d\sigma^- - \int_{\mathcal{C}^-} \left(\mathbf{F}_-^\alpha \partial_\alpha \phi + \mathbf{f}_- \phi \right) d\mathcal{C}^- = 0. \tag{127}$$

Rankine-Hugoniot equations

Summing (126) e (127) we have

$$\int_{\sigma^+} \varphi_\alpha^+ \phi \mathbf{F}_+^\alpha d\sigma^+ + \int_{\sigma-} \varphi_\alpha^- \phi \mathbf{F}_-^\alpha d\sigma^- - \int_{\mathcal{C}} (\mathbf{F}^\alpha \partial_\alpha \phi + \mathbf{f}\phi) \, d\mathcal{C} \; = 0. \tag{128}$$

The last integral vanish and therefore we mush have in the surface:

$$\int_\sigma \varphi_\alpha \left(\mathbf{F}_+^\alpha - \mathbf{F}_-^\alpha\right) \phi \, d\sigma = 0. \tag{129}$$

and then

$$\left(\mathbf{F}_+^\alpha - \mathbf{F}_-^\alpha\right) \varphi_\alpha = 0 \tag{130}$$

i.e.

$$[\mathbf{F}^\alpha] \, \varphi_\alpha = 0. \tag{131}$$

This means that the normal components in the space time of $\mathbf{F}^\alpha$ must be continuous across the surface. Dividing space and time

$$\varphi_0 = -s, \qquad \varphi_i = n_i \tag{132}$$

and assuming $\mathbf{u} \equiv \mathbf{F}^0$ we can rewrite the R-H conditions in the usual form:

$$-s \, [\mathbf{u}] + \left[\mathbf{F}^i\right] n_i = 0 \tag{133}$$

or explicitly

$$-s\mathbf{u}_1 + \mathbf{F}^i\left(\mathbf{u}_1\right) n_i = -s\mathbf{u}_0 + \mathbf{F}^i\left(\mathbf{u}_0\right) n_i, \tag{134}$$

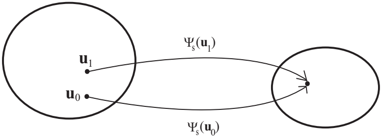
The R-H conditions formally can be write from the differential system with the operator rule

$$\partial_t \to -s\left[\cdot\right], \qquad \partial_i \to n_i\left[\cdot\right], \qquad \mathbf{f} \to 0. \tag{135}$$

Let

$$\mathbf{\Psi}_s\left(\mathbf{u}\right) = -s\mathbf{u} + \mathbf{F}^i\left(\mathbf{u}\right) n_i. \tag{136}$$

then the R-H implies



$$\mathbf{\Psi}_s\left(\mathbf{u}_1\right) = \mathbf{\Psi}_s\left(\mathbf{u}_0\right). \tag{137}$$

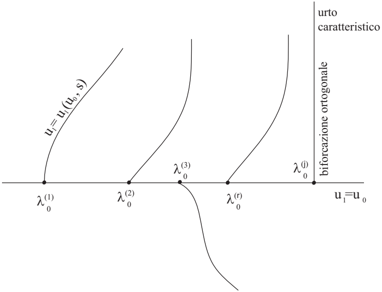
This require the non invertibility of the function $\mathbf{\Psi}_s\left(\mathbf{u}\right)$.

We have

$$\frac{\partial \mathbf{\Psi}_s}{\partial \mathbf{u}} = -s\mathbf{I} + \mathbf{A}^i n_i \tag{138}$$

then bifurcation point are the one when s meet an unperturbed eigenvalue λ_0 (k-shocks)

$$\det\left(\mathbf{A}_n - \lambda \mathbf{I}\right)_{\mathbf{u}_0} = 0. \tag{139}$$



$$\mathbf{u}_1 \equiv \mathbf{u}_1\left(\mathbf{u}_0, s, \mathbf{n}\right). \tag{140}$$

Shock waves in Euler fluid

The R-H for Euler system becomes:

$$-s\,[\rho] + [\rho v_n] = 0 \tag{141}$$

$$-s\,[\rho\,\mathbf{v}] + [\rho\, v_n\, \mathbf{v} + p\,\mathbf{n}] = 0 \tag{142}$$

$$-s\left[\rho\frac{v^2}{2} + \rho e\right] + \left[\left(\rho\frac{v^2}{2} + \rho e + p\right) v_n\right] = 0 \tag{143}$$

where $v_n = \mathbf{v}\cdot\mathbf{n}$. Let introduce the MACH number and the specific volume

$$M_0 = \frac{s - v_{0n}}{c_0}\,, \qquad V = \frac{1}{\rho}\,, \tag{144}$$

then the solution of the R-H are

$$p = p_0 + \frac{2\gamma}{\gamma+1}\, p_0\,(M_0^2 - 1). \tag{145}$$

$$V = V_0 - \frac{2}{\gamma+1}\, V_0\, \frac{M_0^2 - 1}{M_0^2} \tag{146}$$

$$\mathbf{v} = \mathbf{v}_0 + \frac{2c_0}{\gamma+1}\, \frac{M_0^2 - 1}{M_0}\, \mathbf{n}. \tag{147}$$

From (145) and (146) we have

$$\frac{[p]}{[V]} = -\frac{c_0^2\, M_0^2}{V_0^2} \leq 0. \tag{148}$$

Then we have two possibilities

i) $[p] > 0$ e $[V] < 0$: corresponding to $M_0^2 > 1$,

ii) $[p] < 0$ e $[V] > 0$: corresponding to $M_0^2 < 1$.

Mathematically both are acceptable but which of the two is physical consistent? For this reason we calculate the R-H relative to the entropy law

$$\eta = s[\rho S] - [\rho S v_n] = [\rho(s - v_n)S] \tag{149}$$

If the weak solution of teh system is also weak solution of the entropy law η must be zero, while η is not null. In fact we have

$$\eta = \rho_0 c_0 c_V \; M_0 \; \log\left\{\left(\frac{2 + M_0^2(\gamma - 1)}{M_0^2(\gamma + 1)}\right)^{\gamma} \frac{2M_0^2\gamma + 1 - \gamma}{1 + \gamma}\right\} \tag{150}$$

Entropy growth across the shock

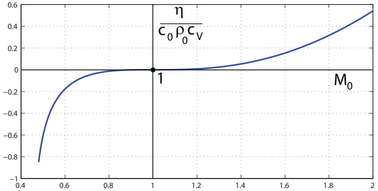


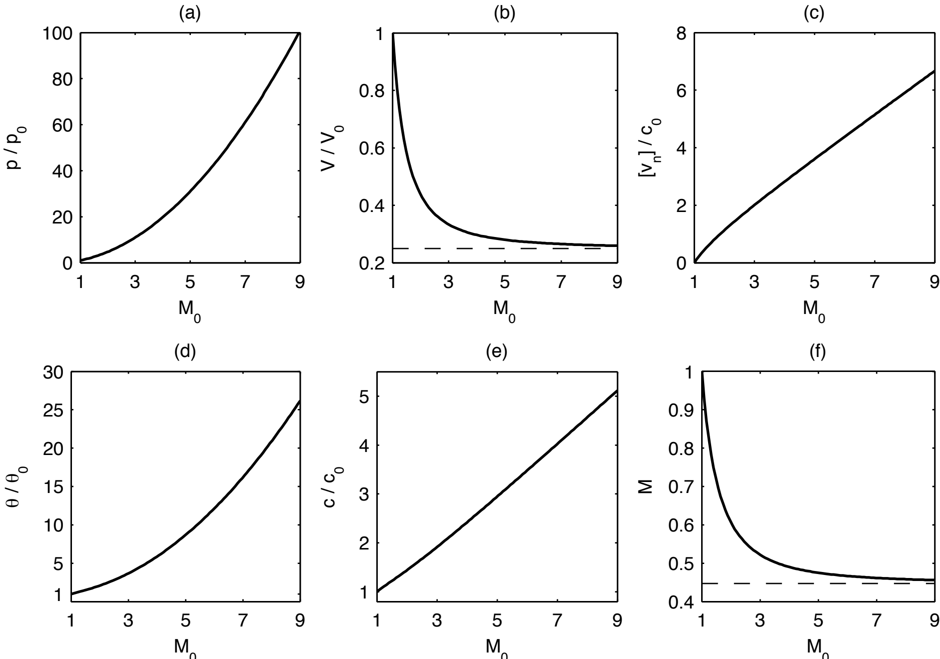
Figure: Entropy growth across the shock

As η have the meaning of the production of entopy across the shock we need to require

$$\eta \geq 0$$

and then

$$M_0^2 > 1.$$


(a)
(b)
(c)
(d)
(e)
(f)
p / p0
V / V0
[vn] / c0
θ / θ0
c / c0
M
M0
100
80
60
40
20
0
1
0.8
0.6
0.4
0.2
8
6
4
2
30
25
20
15
10
5
0.9
0.7
0.5
3
7
9

We note that

$$\lim_{M_0 \to \pm\infty} \frac{V}{V_0} = \frac{\gamma - 1}{\gamma + 1}.$$

In reality we have another solution of the R-H equations: the characteristic shock

$$v_n = v_{0n} = s, \quad p = p_0, \tag{151}$$

with

$$[\mathbf{v}_T] \text{ arbitrario}, \quad [\rho] \text{ arbitrario}, \quad \mathbf{v}_T = \mathbf{v} - v_n \mathbf{n} \tag{152}$$

where $\mathbf{v}_T$ is the tangential component of the fluid velocity. In this case $\eta = 0$.