Non-linear Wave Propagation and Non-Equilibrium Thermodynamics - Part 3

Tommaso Ruggeri

Department of Mathematics and Research Center of Applied Mathematics
University of Bologna

January 21, 2017

Overview

1. The Riemann Problem
2. The Riemann Problem with Structure
3. Bibliography

The Physical laws in continuum theories are *balance laws*: Let $\mathbf{F}^0(\mathbf{x}, t)$; $\mathbf{x} \in \Omega$, $t \in R^+$, a generic density. The time derivative in the domain Ω is expressed by

$$\frac{d}{dt}\int_\Omega \mathbf{F}^0\, d\Omega = -\int_\Sigma \mathbf{G}^i n_i d\Sigma + \int_\Omega \mathbf{f}\, d\Omega, \tag{1}$$

where the first integral on the r.h.s. represents the flux of some quantities $\mathbf{G^i}$ trough the surface Σ of unit normal $\overrightarrow{n}$ and velocity $\overrightarrow{v}$, while the last integral represents the productions.

Under regularity assumptions the system can be put in the local form:

$$\frac{\partial \mathbf{F}^0}{\partial t} + \frac{\partial \mathbf{F}^i}{\partial x^i} = \mathbf{f}, \qquad \mathbf{F}^i = \mathbf{F}^0 v^i + \mathbf{G}^i \tag{2}$$

For example in the case of fluids:

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x^i} = 0 && \text{(mass balance)} \\
&\frac{\partial (\rho v_j)}{\partial t} + \frac{\partial}{\partial x^i}(\rho v_i v_j - t_{ij}) = 0 && \text{(balance of momentum)} \\
&\frac{\partial E}{\partial t} + \frac{\partial}{\partial x^i}(E v_i + q_i - t_{ij} v_j) = 0 && \text{(energy conservation).}
\end{aligned}
\tag{3}
$$

where $E = \rho \frac{v^2}{2} + \rho \varepsilon$ and $\rho, \mathbf{v} \equiv (v_i), \mathbf{t} \equiv (t_{ij}), \mathbf{q} \equiv (q_i), \varepsilon$ are the density, the velocity, the stress tensor, the heat flux and the internal energy.
Of course the system is not closed and we need the so called *constitutive equations*.

In the modern constitutive theory all the constitutive equations must obey the two principles:

- The objectivity principle: the proper constitutive equations are independent of the Observer;
- The second principle of thermodynamics that in the Rational Thermodynamics requires that any solutions of the full system satisfies the inequality of Clausius-Duhem (Coleman-Noll 1963):

$$\frac{\partial \rho S}{\partial t} + \frac{\partial}{\partial x^i}\left(\rho S v^i + \frac{q^i}{T}\right) \geq 0 \qquad \text{for all processes} \tag{4}$$

For instance in the case of classical approach of fluids with Fourier Navier-Stokes assumptions

$$\mathbf{t} = -p\mathbf{I} + \boldsymbol{\sigma}$$

$$q^i = -\chi \frac{\partial T}{\partial x^i}; \qquad \sigma_{<ij>} = \mu \frac{\partial v_{<i}}{\partial x^{j>}}; \qquad \sigma_{ll} = \nu div\mathbf{v},$$

the constitutive equations compatible with (4) require the existence of a *free energy* ψ, function of the density ρ and temperature T, such that:

$$p = \rho^2 \frac{\partial \psi}{\partial \rho}, \qquad S = -\frac{\partial \psi}{\partial T}, \qquad \varepsilon = \psi - T\frac{\partial \psi}{\partial T}, \tag{5}$$

while

$$\chi \text{ (heat conductivity)},\ \mu \text{ (shear viscosity)},\ \nu \text{ (bulk viscosity)} \ \geq \ 0.$$

The entropy principle is also supported by the kinetic theory of gases. In fact from the Boltzmann equation

$$\frac{\partial f}{\partial t} + c^i \frac{\partial f}{\partial x^i} = Q; \quad f \equiv f(\mathbf{x}, t, \mathbf{c})$$

introducing as moment:

$$\rho S = \int (-k \log f) f \; d\mathbf{c}; \qquad \phi^i = \int (-k \log f) f c^i \; d\mathbf{c};$$

we have the so called H-theorem:

$$\frac{\partial \rho S}{\partial t} + \frac{\partial}{\partial x^i} \left(\rho S v^i + \phi^i \right) \geq 0 \tag{6}$$

but the non convective entropy flux ϕ^i is in general different from q^i / T.

The necessity to extend the entropy principle with a general entropy flux was proposed by INGO MÜLLER (1967).
At present the general form (6)

$$\frac{\partial \rho S}{\partial t} + \frac{\partial}{\partial x^i}\left(\rho S v^i + \phi^i\right) \geq 0$$

is universally accepted in the continuum community and all the constitutive equations in new models are tested by the entropy principle.

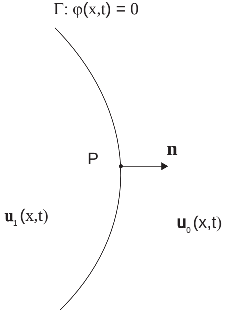
For systems (2) we can define the so called *weak solution*:

$$\int_V \left(\mathbf{u}\partial_t\Phi + \mathbf{F}^i\partial_i\Phi\right) dV \;=\; 0 \tag{7}$$

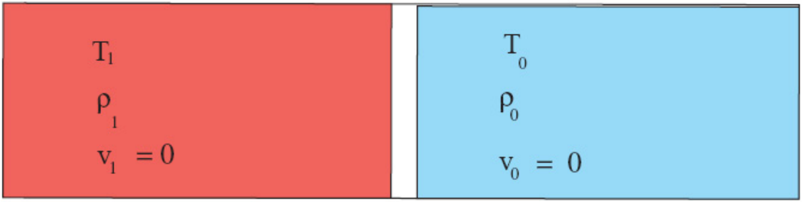
for any test function Φ.

In a particular a shock wave is a weak solution of (2) iff across the shock front the Rankine-Hugoniot equations are fulfilled:

$$-s\,[\mathbf{u}] + \left[\mathbf{F}^i\right] n_i = 0.$$



The Riemann problem was originated by the following well know problem in fluidynamics:



The Riemann Problem

Let us consider quasi-linear system of conservation laws compatible with an entropy principle with a convex entropy density:

$$\partial_t \mathbf{u} + \partial_x \mathbf{F}(\mathbf{u}) = 0$$

$$\partial_t h(\mathbf{u}) + \partial_x k(\mathbf{u}) \leq 0. \tag{8}$$

with initial data

$$\mathbf{u}(x,0) = \begin{cases} \mathbf{u}_0 & \text{for } x > 0 \\ \mathbf{u}_1 & \text{for } x < 0. \end{cases}$$

The Riemann Problem and the non uniqueness of weak solutions

Before we recall how the Riemann Problem was solved we remember a peculiarity of the weak solutions: *the non uniqueness*!
To explain this let's consider the Burger equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

equivalent to

$$u_t + u u_x = 0$$

with initial data

$$u\,(x,0) = \begin{cases} u_0 \text{ for } x > \; 0 \\ u_1 \text{ for } x < \; 0 \end{cases}$$

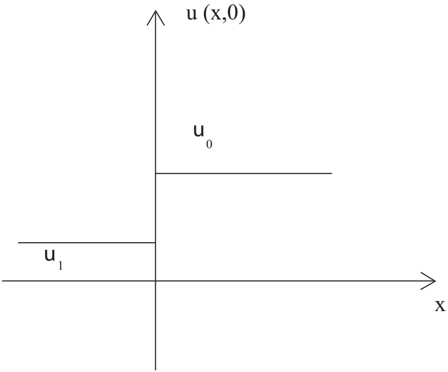


Figure: Initial data

If we are looking for a shock solution, along the curve $dx/dt = s$ the R-H condition must be satisfied. In this case this turns into:

$$-s\,[u] + \left[u^2/2\right] = 0 \tag{9}$$

and we can get the velocity of the shock, s

$$s = \frac{\left[u^2/2\right]}{[u]} = \frac{1}{2}\left(u_0 + u_1\right). \tag{10}$$

So we have the weak solution

$$u\left(x,t\right) = \begin{cases} u_0 \;\; \text{per } x > \; st \\ \\ u_1 \quad \text{per } x < st \end{cases} \quad ; \quad \text{with } s = \frac{1}{2}(u_0 + u_1). \tag{11}$$

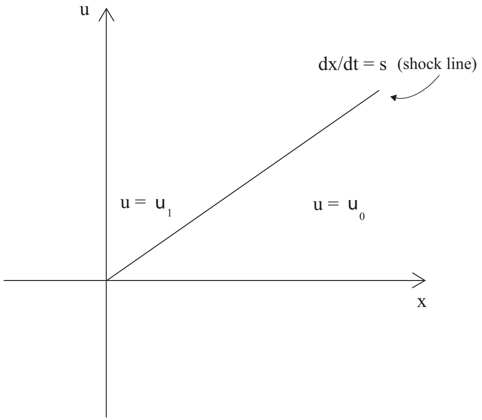


Figure: Shock wave

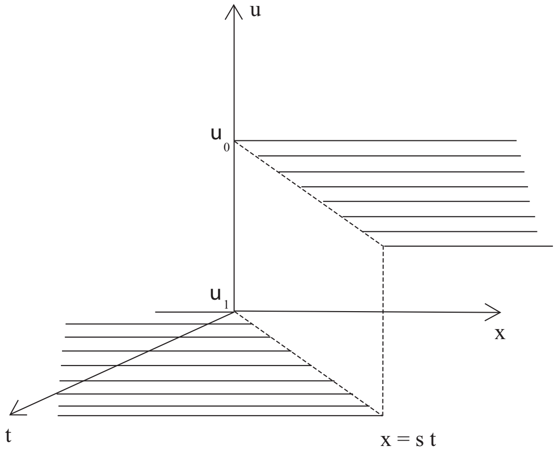
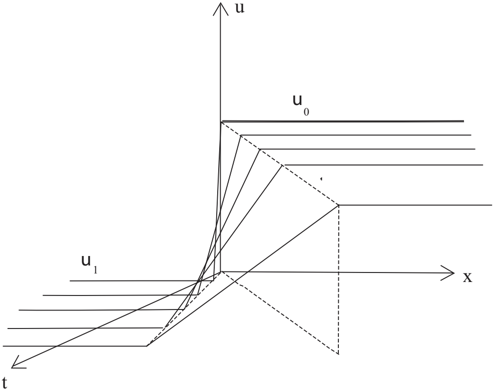


Figure: Shock wave in space-time

We can see that we have also the following solution $u(x,t)$

$$u(x,t) = \begin{cases} u_1 & \text{per } \frac{x}{t} < u_1 \\ \dfrac{x}{t} & \text{per } u_1 \leq \dfrac{x}{t} \leq u_0 \\ u_0 & \text{per } \frac{x}{t} > u_0 \end{cases} \tag{12}$$



The solution (12)

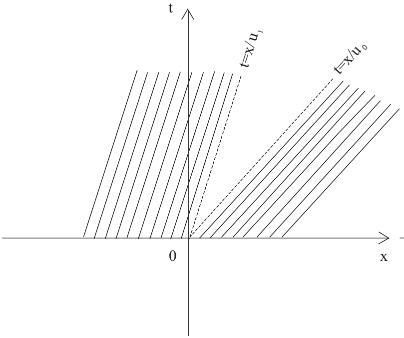
- is compatible with the initial data;
- is continuous for any $t > 0$;
- is differentiable;
- is a weak solution

So we have found two different solutions of this Riemann problem. Both of them are acceptable from a mathematical point of view, but from a physical point of view only one of them should be acceptable.

Which is the physically acceptable solution?

In the present case the characteristic velocity $\lambda = u$ and the characteristic curves are straight lines represented by the equations $x = u_0 t + x_0$ and $x = u_1 t + x_0$.

- If $\lambda(u_1) < \lambda(u_0)$ the two families of characteristics do not intersect for any $t > 0$;
- If $\lambda(u_1) > \lambda(u_0)$ the two families of characteristics do intersect for $t > 0$.



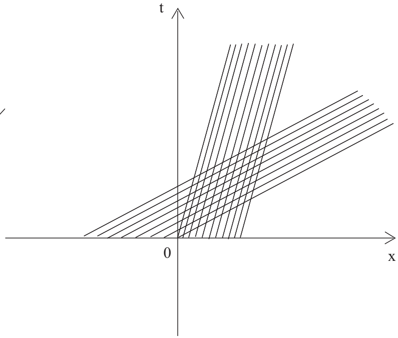


Figure: Non-intersecting and intersecting characteristic lines

The **Lax condition** select the admissible shock:

$$\lambda(u_0) < s < \lambda(u_1). \tag{13}$$

The entropy growth:

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \quad -s[u] + \left[\frac{u^2}{2}\right] = 0, \quad u_1 = 2s - u_0;$$

$$\left(\frac{u^2}{2}\right)_t + \left(\frac{u^3}{3}\right)_x = 0, \qquad \eta = -s\left[\frac{u^2}{2}\right] + \left[\frac{u^3}{3}\right];$$

$$\eta = \frac{2}{3}(s - \lambda_0)^3 > 0 \quad \rightarrow \quad s > \lambda_0$$

and as

$$s = (\lambda_1 + \lambda_0)/2 \quad \rightarrow \quad s < \lambda_1$$

i.e. the equivalence between the Lax condition and the entropy growth condition:

$$\lambda(u_0) < s < \lambda(u_1) \quad \Leftrightarrow \quad \eta > 0. \tag{14}$$

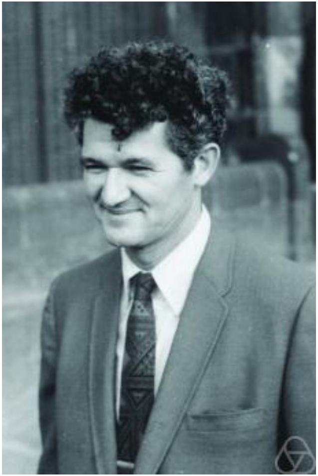
Both are also justified by the artificial viscosity method:

$$u_t + uu_x = \nu u_{xx} \tag{15}$$

taking the limit of $\nu \to 0$.

Peter Lax

The general Lax solution:



Let

$$\mathbf{A} = \nabla \mathbf{F}; \quad (\mathbf{A} - \lambda \mathbf{I})\mathbf{r} = 0.$$

The Riemann problem for initial sufficiently small jump is solved as a "superposition" of shocks, characteristic shocks, rarefaction waves and constant states. The physical shocks are those for which:

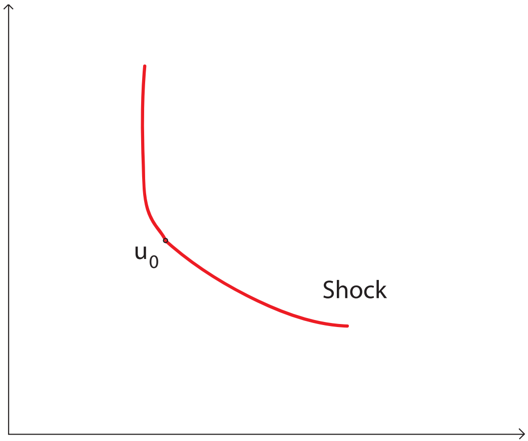
Shock:

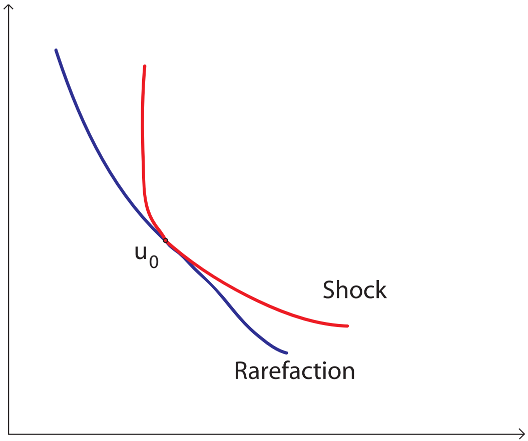
$$\text{if} \quad \nabla\lambda \cdot \mathbf{r} \neq 0, \qquad \lambda(\mathbf{u}_0) < s < \lambda(\mathbf{u}_1) \iff \eta > 0$$

Characteristic Shock:

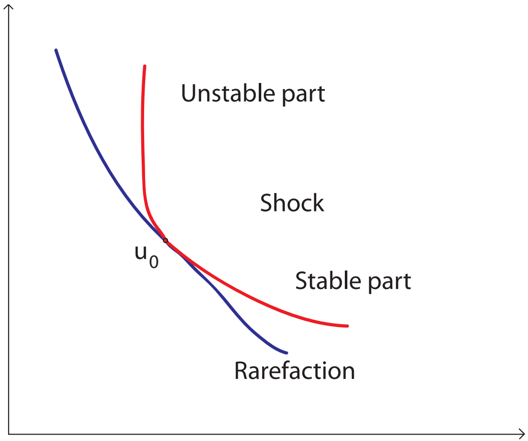
$$\text{if} \quad \nabla\lambda \cdot \mathbf{r} \equiv 0, \qquad \lambda(\mathbf{u}_0) = s = \lambda(\mathbf{u}_1) \iff \eta = 0$$

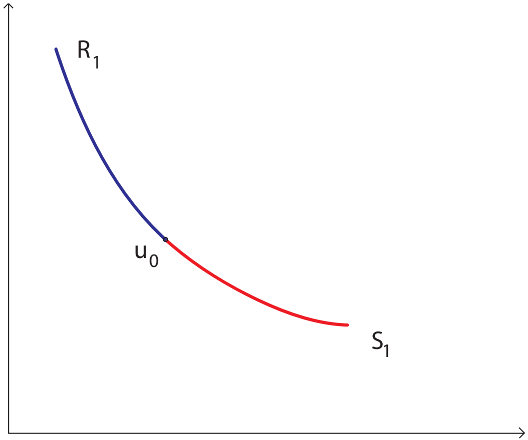
while if $\lambda(\mathbf{u}_1) < \lambda(\mathbf{u}_0)$ we have a rarefaction wave.


u0
Shock

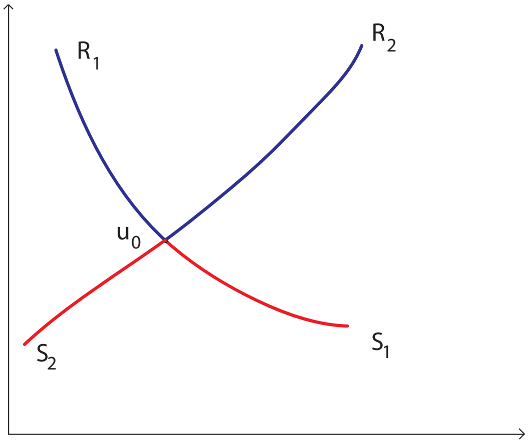


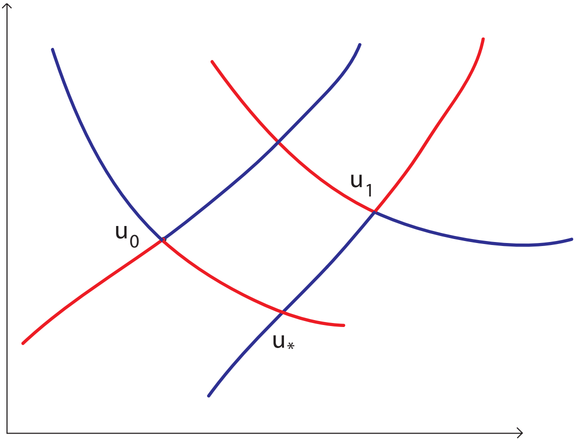
u0
Shock
Rarefaction


Unstable part
Shock
u_0
Stable part
Rarefaction



R1
u0
S1


R1
R2
u0
S2
S1


u_0
u_1
u_*

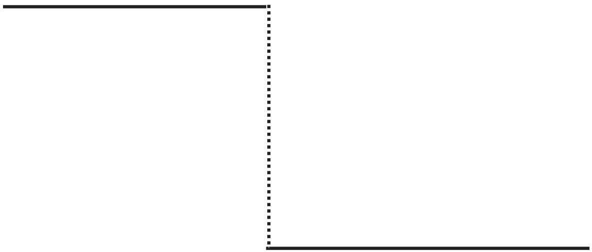


Figure: The Euler solution for the density - $t = 0$

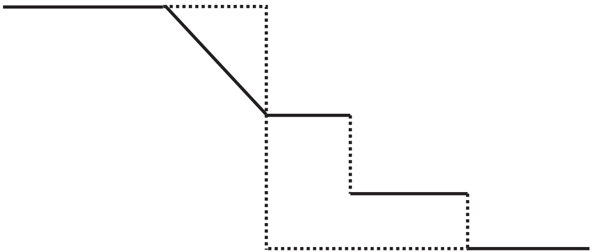
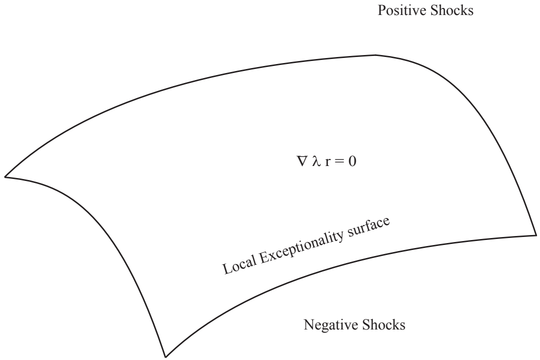


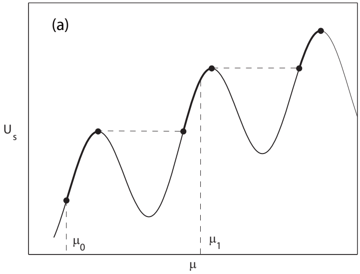
Figure: The Euler solution for the density - $t > 0$

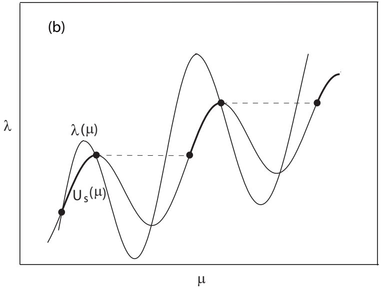
The problem fails in the special case of local exceptionality

$$\nabla\lambda \cdot \mathbf{r} = 0 \quad \text{for some} \quad \mathbf{u}.$$



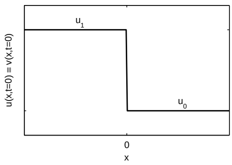
In this case the stability of the shock must be satisfy the Liu conditions that implies the generalized Lax condition $\lambda(\mathbf{u}_0) \leq s \leq \lambda(\mathbf{u}_1)$ but the entropy growth is not sufficient; it is necessary to add additional conditions for example a new superposition principle (LIU, RUGGERI, 2003).





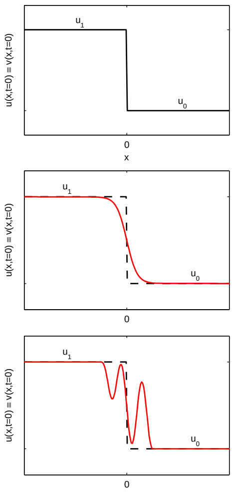
The Riemann problem is a fundamental tool for the existence theorem for solutions of the initial data (Glimm, Dafermos, Bressan, Bianchini,...) as well for numerically approach (Godunov, Russo,.....).
See the book of C. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*. Springer Verlag, Berlin.

- Riemann (R) problem



- Riemann (R) problem

- Riemann with Structure (RS) problem



The importance of the Riemann Problem is evident also for the following asymptotic results due to Tai-Ping Liu:

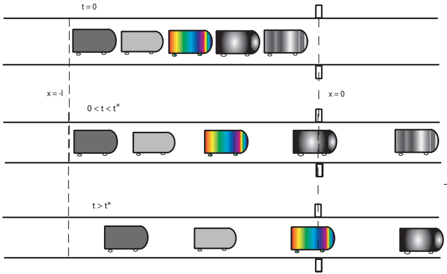
Theorem

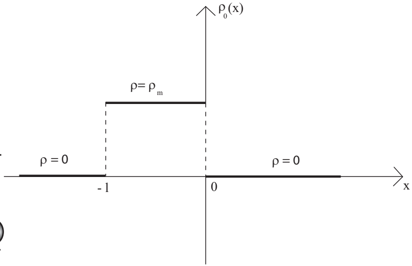
If the initial data are "perturbations" of the Riemann data, i.e. at $x = \pm\infty$ there are two constant states then for t large the solutions tends to the one of the Riemann problem for genuine non-linear characteristic, while waves of linearly degenerate characteristic velocities tend to traveling waves.

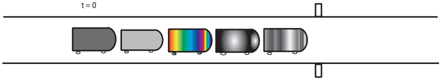
In particular if the initial data are perturbations of a constant state then for t large the solutions converge to the constant state.

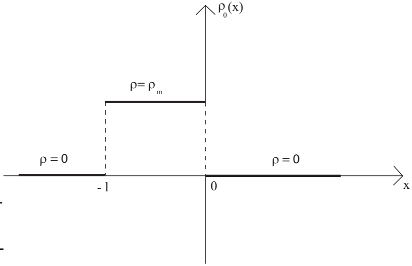
Example of the traffic problem:

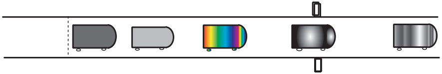
$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial x} = 0 \qquad v(\rho) = v_m \left(1 - \frac{\rho}{\rho_m}\right).$$

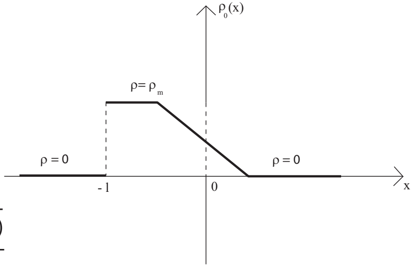




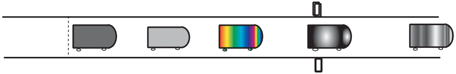

t = 0

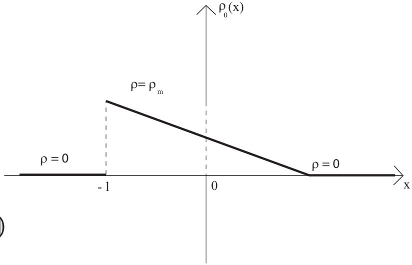

$\rho_0(x)$
$\rho = \rho_m$
$\rho = 0$
$\rho = 0$
-1
0
x

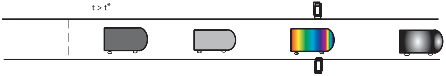


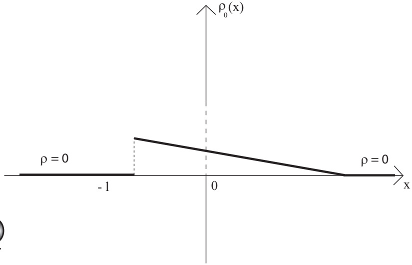


$\rho_0(x)$
$\rho = \rho_m$
$\rho = 0$
$\rho = 0$
-1
0
x




$\rho_0(x)$
$\rho = \rho_m$
$\rho = 0$
$\rho = 0$
- 1
0
x


t > t*


ρ0(x)
ρ = 0
ρ = 0
- 1
0
x

- Lax, P. D. 1973 *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*. CBMS-NSF, Regional Conference Series in Applied Mathematics 11, SIAM.
- C. Dafermos *Hyperbolic Conservation Laws in Continuum Physics* Springer-Verlag, Berlin (2001).
- R. J. LeVeque, *Numerical Methods for Conservation Laws* Lectures in Mathematics, ETH-Zurich Birkhauser-Verlag, Basel, 1990
- G. Boillat, T. Ruggeri, *On the Shock Structure Problem for Hyperbolic System of Balance Laws and Convex Entropy*, Continuum Mech. Thermodyn. **10**, 285–292 (1998)

- Liu, T.-P 1976 *The entropy condition and the admissibility of shocks.* J. Math. Anal. Appl. 53, 7888.
- Liu, T.-P. 1981 *Admissible solutions of hyperbolic conservation laws*. Mem. Am. Math. Soc. 240, 12.
- Liu, T.-P., Ruggeri, T. 2003 *Entropy production and admissibility of shocks*. Acta Math. Appl. Sinica, English Series. 19, 112.
- I. Müller, T. Ruggeri, *Rational Extended Thermodynamics*, Springer Tracts in Natual Philosophy **37**, 2nd Ed., Springer Verlag (1998)

F. Brini, T. Ruggeri, *On the Riemann Problem in Extended Thermodynamics*, Proc. of 10th Int. Conference on Hyperbolic Problems (Osaka, 2004); Yokohama Pub. Inc. (2006)

A. Mentrelli, T. Ruggeri, *Asymptotic Behavior of Riemann and Riemann with Structure Problems for a* 2×2 *Hyperbolic Dissipative System*, Suppl. Rend. Circ. Mat. Palermo, Serie II, **78**, 201–226 (2006)

F. Brini, T. Ruggeri, *On the Riemann Problem with Structure in Extended Thermodynamics*, Suppl. Rend. Circ. Mat. Palermo, Serie II, **78**, 31–44 (2006)