ISOTROPIC LINEAR CONSTITUTIVE RELATIONS FOR NONSIMPLE FLUIDS

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Abstract. We investigate the general constitutive relation of an isotropic linear fluid when the stress tensor can depend on higher-order spatial gradients of the velocity. We apply the results to the case of second-grade and third-grade fluids, be compressible or not. However, the expression of the general isotropic tensor can be a matter of interest also for other classes of nonsimple materials.

1. Introduction

In a recent paper, [2], Fried and Gurtin developed a theory for a class of second-grade incompressible fluids. In particular, the main properties of the material are expressed by means of two tensor fields: the usual second-order Stress Tensor T, which depends on the symmetric gradient D of velocity, and a further third-order tensor G, which depends on the second gradient $\operatorname{grad}\operatorname{grad}\boldsymbol{v}$ of the velocity. Facing the problem of the constitutive assumptions, they choose the standard isotropic linear dependence, both for T and for G.

With that choice for T, by standard arguments one gets the well-known incompressible Cauchy–Poisson relation

$$\mathsf{T}_{ij} = -p\delta_{ij} + 2\mu\mathsf{D}_{ij} \,,$$

where δ_{ij} is the usual Kronecker symbol and μ a constant (positive by a dissipation principle). On the contrary, the form of G is not so clear. By setting

$$\mathsf{G}^0_{ijk} = \mathsf{G}_{ijk} + \delta_{ij}\pi_k \,,$$

where $\boldsymbol{\pi}$ is the so-called *hyperpressure*, we will deal with the problem of finding the most general isotropic linear relation between G^0 and $\operatorname{grad}\operatorname{grad}\boldsymbol{v}$.

Fried and Gurtin assumed that such a relation is

$$\mathsf{G}^0_{ijk} = \eta_1 v_{i,jk} + \eta_2(v_{k,ij} + v_{j,ik} - v_{i,rr}\delta_{jk}) \tag{1}$$

(see [2, $(72)_2$]). Indeed, in Footnote 13 they declare: "We conjecture that [the above formula] is the most general linear, isotropic relation possible between G^0 and $\operatorname{grad}\operatorname{grad}\boldsymbol{v}$".

The aim of this note is to prove that indeed the conjecture is "almost" true. Precisely, the following theorem will be proved.

2000 *Mathematics Subject Classification.* Primary 74A20; Secondary 74A30, 76A02.

I wish to thank Alfredo Marzocchi for providing many useful discussions. The research was partially supported by the Italian Research Project "Mathematical Models for Materials Science".

Theorem 1.1. *Under the hypotheses of incompressibility and regularity of $\boldsymbol{v}$, every isotropic linear relation between G^0 and* $\operatorname{grad}\operatorname{grad}\boldsymbol{v}$ *has the form*

$$\begin{aligned}\mathsf{G}^0_{ijk} = \eta_1 v_{i,jk} &+ \eta_2(v_{k,ij} + v_{j,ik} - v_{i,rr}\delta_{jk}) \\ &+ \eta_3(v_{k,rr}\delta_{ij} + v_{j,rr}\delta_{ik} - 4v_{i,rr}\delta_{jk}) \,, \end{aligned} \qquad i,j,k = 1,\dots,N \tag{2}$$

where N is the dimension of the space.

Hence, the only fault in the conjecture (1) is the assumption that $\eta_3 = 0$. The proof of Theorem 1.1 relies on [3], where the link between isotropic tensors and Weyl's orthogonal invariant polynomial functions is made explicit.

2. Isotropic linear functions of even order

A constitutive relations, under the assumption of linearity, is usually prescribed by means of a tensor of even order, relating, for instance, the stress and the gradient of velocity, or the hyperstress and the second gradient of velocity. Hence from now on, we will consider only linear functions of even order.

Denote with I the second order identity tensor, *i.e.* $\mathsf{I}_{hk} = \delta_{hk}$. Let us introduce the *orthogonal group*

$$\mathrm{Orth} = \{\mathsf{R} \text{ is a second order tensor and } \mathsf{R}\mathsf{R}^T = \mathsf{I}\}.$$

Definition 2.1. A linear function is *isotropic* if its components are the same in any orthogonal reference frame. In particular, $\mathsf{F}_{i_1\dots i_n}$ is an isotropic linear function of order n if and only if

$$\mathsf{F}_{j_1\dots j_n} = \mathsf{R}_{j_1 i_1}\dots\mathsf{R}_{j_n i_n}\mathsf{F}_{i_1\dots i_n} \tag{3}$$

for every $\mathsf{R} \in \mathrm{Orth}$.

Remark 2.2. If the space dimension N is odd, then the previous definition does not change replacing Orth by the *positive orthogonal subgroup*

$$\mathrm{Orth}^+ = \{\mathsf{R} \in \mathrm{Orth} : \det\mathsf{R} = +1\}.$$

On the contrary, if N is even the two notions are different, and an alternative definition of invariance can be introduced, namely the so-called *odd isotropy*, where (3) is replaced by

$$\mathsf{F}_{j_1\dots j_n} = (\det\mathsf{R})\mathsf{R}_{j_1 i_1}\dots\mathsf{R}_{j_n i_n}\mathsf{F}_{i_1\dots i_n} \tag{3'}$$

for every $\mathsf{R} \in \mathrm{Orth}$.

Suiker & Chang [3] observed that an isotropic linear function is related to a so-called, in Weyl's terminology, *even orthogonal invariant* and an odd isotropic linear function is related to an *odd orthogonal invariant* (see [4, pp. 52-53]). In particular we are interested in the following result, which is proved in [4]:

Theorem 2.3 (Weyl)**.** *An isotropic linear function is represented by the most general combination of Kronecker symbols δ_{ij}, while an odd isotropic linear function is represented by combinations of Kronecker symbols and permutation symbols $\epsilon_{i_1\dots i_n}$.*

In Appendix A we will prove that the number of free coefficients of an isotropic linear function of even order $2n$ is

$$\prod_{i=1}^{n}(2i-1)\,, \tag{4}$$

which is denoted sometimes by the *double factorial* $(2n-1)!!$. For instance, a second-order isotropic linear function has only one free component

$$\mathsf{F}_{ij} = C_1 \delta_{ij}$$

and a fourth-order isotropic linear function has three free components

$$\mathsf{F}_{ijhk} = C_1 \delta_{ij}\delta_{hk} + C_2 \delta_{ih}\delta_{jk} + C_3 \delta_{ik}\delta_{jh} \,.$$

Let us show the effectiveness of Weyl's Theorem by a well-known example.

Example 2.4 (Compressible Newtonian fluids)**.** We investigate the case of the classical compressible Newtonian fluids, where the tensor

$$\mathsf{T}^0 = \mathsf{T} + p\mathsf{I}$$

is assumed to be an isotropic linear function of D, the symmetric part of $\operatorname{grad} \boldsymbol{v}$. In this case the isotropy condition is usually expressed as

$$\forall \mathsf{R} \in \mathrm{Orth}: \quad \mathsf{R}^T \mathsf{T}^0(\mathsf{D})\mathsf{R} = \mathsf{T}^0(\mathsf{R}^T\mathsf{D}\mathsf{R}) \,,$$

which can be rewritten as

$$\forall \mathsf{R} \in \mathrm{Orth}: \quad \mathsf{T}^0(\mathsf{D}) = \mathsf{R}\mathsf{T}^0(\mathsf{R}^T\mathsf{D}\mathsf{R})\mathsf{R}^T \,. \tag{5}$$

Moreover, by linearity we can introduce a fourth-order tensor F such that

$$\mathsf{T}^0_{ij} = \mathsf{F}_{ijlm}\mathsf{D}_{lm} \,.$$

Then, equation (5) implies

$$\forall \mathsf{R} \in \mathrm{Orth}: \quad \mathsf{F}_{ijlm}\mathsf{D}_{lm} = \mathsf{F}_{i'j'l'm'}\mathsf{R}_{ii'}\mathsf{R}_{jj'}\mathsf{R}_{ll'}\mathsf{R}_{mm'}\mathsf{D}_{lm} \,,$$

hence F is isotropic in the sense of Definition 2.1.

Taking into account the symmetries of i, j and of l, m, Weyl's Theorem yields

$$\mathsf{F}_{ijlm} = \lambda \delta_{ij}\delta_{lm} + \mu(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}) \,,$$

where λ and μ are free coefficients. Hence

$$\mathsf{T}^0_{ij} = \mathsf{F}_{ijlm}\mathsf{D}_{lm} = \frac{1}{2}\mathsf{F}_{ijlm}(v_{l,m} + v_{m,l}) = \lambda\delta_{ij}v_{r,r} + \mu(v_{i,j} + v_{j,i})$$

and we deduce the usual compressible Cauchy–Poisson relation

$$\mathsf{T} = -p\mathsf{I} + \lambda(\mathrm{tr}\,\mathsf{D})\mathsf{I} + 2\mu\mathsf{D} \,. \tag{6}$$

Now we prove the main theorem.

Proof of Theorem 1.1. We have to study the most general linear relation between G^0 and $\operatorname{grad}\operatorname{grad} \boldsymbol{v}$, which is a sixth-order linear function F_{ijklmn} such that

$$\mathsf{G}^0_{ijk} = \mathsf{F}_{ijklmn}v_{l,mn} \,. \tag{7}$$

In view of formula (4), there are 15 free components, and by Weyl's Theorem the general representation for an isotropic sixth-order linear function (see also [3, eq. 35]) is

$$\begin{aligned}
\mathsf{F}_{ijklmn} = {} & C_1\delta_{ij}\delta_{kl}\delta_{mn} + C_2\delta_{ij}\delta_{km}\delta_{ln} + C_3\delta_{ij}\delta_{kn}\delta_{lm} \\
& + C_4\delta_{ik}\delta_{jl}\delta_{mn} + C_5\delta_{ik}\delta_{jm}\delta_{ln} + C_6\delta_{ik}\delta_{jn}\delta_{lm} \\
& + C_7\delta_{il}\delta_{jk}\delta_{mn} + C_8\delta_{il}\delta_{jm}\delta_{kn} + C_9\delta_{il}\delta_{jn}\delta_{km} \\
& + C_{10}\delta_{im}\delta_{jk}\delta_{ln} + C_{11}\delta_{im}\delta_{jl}\delta_{kn} + C_{12}\delta_{im}\delta_{jn}\delta_{kl} \\
& + C_{13}\delta_{in}\delta_{jk}\delta_{lm} + C_{14}\delta_{in}\delta_{jl}\delta_{km} + C_{15}\delta_{in}\delta_{jm}\delta_{kl}
\end{aligned} \tag{8}$$

where $C_1, \dots, C_{15}$ are independent coefficients.

In the present case there are some further constraints: the velocity field is assumed to be at least of class C^2, yielding the symmetry of the last two indices of $v_{l,mn}$. Moreover, the fluid is incompressible, hence $v_{l,l} = 0$. Since G^0 is in the dual space of $\operatorname{grad}\operatorname{grad}\boldsymbol{v}$, without loss of generality we can impose the same constraints also on G^0. Summarizing,

$$\mathsf{G}^0_{ijk} = \mathsf{G}^0_{ikj}\,, \quad v_{l,mn} = v_{l,nm}\,, \tag{9}$$

$$\mathsf{G}^0_{iik} = \mathsf{G}^0_{iki} = 0\,, \quad v_{l,ln} = v_{l,nl} = 0\,, \tag{10}$$

where the summation convention over repeated indices is tacitly assumed. By the symmetries (9) and formula (23) in Appendix B it follows that six free coefficients remain. Namely, in formula (8) we can choose

$$\begin{aligned} &A := C_1 = C_4\,, && B := C_2 = C_3 = C_5 = C_6\,, && C := C_7\,,\\ &D := C_8 = C_9\,, && E := C_{10} = C_{13}\,, && F := C_{11} = C_{12} = C_{14} = C_{15}\,, \end{aligned}$$

hence G^0 has the form

$$\begin{aligned} \mathsf{G}^0_{ijk} = \mathsf{F}_{ijklmn} v_{l,mn} &= A(v_{k,rr}\delta_{ij} + v_{j,rr}\delta_{ik}) \\ &+ B(v_{r,kr}\delta_{ij} + v_{r,jr}\delta_{ik}) + C v_{i,rr}\delta_{jk} \\ &+ D v_{i,jk} + E v_{r,ir}\delta_{jk} + F(v_{j,ik} + v_{k,ij})\,. \end{aligned} \tag{11}$$

Moreover, by (10) it follows that

$$0 = \mathsf{G}^0_{iik} = 4A v_{k,rr} + C v_{k,rr} + F v_{k,rr}\,,$$

hence $C = -4A - F$. Substituting in (11) we get

$$\begin{aligned} \mathsf{G}^0_{ijk} &= A(v_{k,rr}\delta_{ij} + v_{j,rr}\delta_{ik} - 4v_{i,rr}\delta_{jk}) \\ &\quad + D v_{i,jk} + F(v_{j,ik} + v_{k,ij} - v_{i,rr}\delta_{jk})\,. \end{aligned} \tag{12}$$

Switching to a notation similar to [2], setting $\eta_1 = D$, $\eta_2 = F$ and $\eta_3 = A$ we obtain the final form

$$\mathsf{G}^0_{ijk} = \eta_1 v_{i,jk} + \eta_2(v_{j,ik} + v_{k,ij} - v_{i,rr}\delta_{jk}) + \eta_3(v_{k,rr}\delta_{ij} + v_{j,rr}\delta_{ik} - 4v_{i,rr}\delta_{jk})$$

which proves Theorem 1.1. □

3. The flow equation for second-grade incompressible fluids

Following [2], we will briefly show how the equation for incompressible second-grade fluids can be deduced. The essential tool is the Principle of Virtual Power,[(1)] *i.e.* the assumption that

$$W_{\text{int}}(R, \boldsymbol{v}) = W_{\text{ext}}(R, \boldsymbol{v})\,, \tag{13}$$

where R is a generic subbody, $\boldsymbol{v}$ any possible *virtual velocity*, W_{int} and W_{ext} the internal and external power, resp.

For a second-grade material, the internal power is assumed to have the form

$$W_{\text{int}}(R, \boldsymbol{v}) = \int_R (\mathsf{T} \cdot \operatorname{grad}\boldsymbol{v} + \mathsf{G} \cdot \operatorname{grad}\operatorname{grad}\boldsymbol{v})\,dx$$

[(1)] See also [1] for a general approach to the subject.

with T a second-order tensor (the *Cauchy stress tensor*) and G a third-order tensor (the *hyperstress*). In particular, using the divergence theorem one has

$$
\begin{aligned}
W_{\mathrm{int}}(R, \boldsymbol{v}) = &\int_R (-\operatorname{div}\mathsf{T} + \operatorname{div}\operatorname{div}\mathsf{G}) \cdot \boldsymbol{v}\, dx \\
&+ \int_{\partial R} [\mathsf{G}\boldsymbol{n} \cdot \operatorname{grad}\boldsymbol{v} + (\mathsf{T}\boldsymbol{n} - (\operatorname{div}\mathsf{G})\boldsymbol{n}) \cdot \boldsymbol{v}]\, dS\,.
\end{aligned} \tag{14}
$$

On the contrary, introducing the external body force $\boldsymbol{b} = \boldsymbol{f} - \rho\boldsymbol{a}$ (accounting also for inertial forces), the *surface stress* $\boldsymbol{t}_{\partial R}$ and the *surface hyperstress* $\boldsymbol{m}_{\partial R}$, the external power is assumed to be of the form

$$
W_{\mathrm{ext}}(R, \boldsymbol{v}) = \int_{\partial R} \left(\boldsymbol{t}_{\partial R} \cdot \boldsymbol{v} + \boldsymbol{m}_{\partial R} \cdot \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} \right) dS + \int_R \boldsymbol{b} \cdot \boldsymbol{v}\, dx\,.
$$

By imposing the balance (13), keeping into account the arbitrariness of R and $\boldsymbol{v}$, under natural regularity assumptions one gets the local balance

$$
-\operatorname{div}\mathsf{T} + \operatorname{div}\operatorname{div}\mathsf{G} = \boldsymbol{b} \tag{15}
$$

equipped with some surface conditions.

Considering now the Cauchy–Poisson relation, the term $\operatorname{div}\mathsf{T}$ gives rise to the classical block $\mu\Delta\boldsymbol{v} - \operatorname{grad} p$. Moreover, keeping into account Theorem 1.1 and the incompressibility, one has

$$
\begin{aligned}
(\operatorname{div}\mathsf{G})_{ij} = \mathsf{G}_{ijk,k} = &-\delta_{ij}\pi_{k,k} + \eta_1 v_{i,jkk} \\
&+ \eta_2(v_{j,ikk} - v_{i,jrr}) + \eta_3(v_{j,irr} - 4v_{i,jrr})
\end{aligned}
$$

and

$$
(\operatorname{div}\operatorname{div}\mathsf{G})_i = \mathsf{G}_{ijk,kj} = -\pi_{k,ki} + (\eta_1 - \eta_2 - 4\eta_3)v_{i,jjkk}\,,
$$

hence

$$
\operatorname{div}\operatorname{div}\mathsf{G} = -\operatorname{grad}\operatorname{div}\boldsymbol{\pi} + (\eta_1 - \eta_2 - 4\eta_3)\Delta\Delta\boldsymbol{v}\,.
$$

A dissipation principle imposes the further conditions

$$
\mathsf{T} \cdot \operatorname{grad}\boldsymbol{v}\,, \quad \mathsf{G} \cdot \operatorname{grad}\operatorname{grad}\boldsymbol{v} \geq 0 \tag{16}
$$

for any test velocity $\boldsymbol{v}$. By $(16)_1$ it follows that $\mu \geq 0$. Moreover, inequality $(16)_2$ can be rewritten as

$$
\eta_1 v_{i,jk}v_{i,jk} + \eta_2(v_{k,ij}v_{i,jk} + v_{j,ik}v_{i,jk} - v_{i,rr}v_{i,jj}) - 4\eta_3 v_{i,rr}v_{i,jj} \geq 0\,.
$$

In particular, if the dimension of the space is greater that 2, one can choose $\boldsymbol{v} = \frac{c}{2}x_1^2\boldsymbol{e}_2$, where c is a coefficient with dimensions of $(\text{length})^{-1}(\text{time})^{-1}$, yielding $v_{2,11} = c$ and $v_{i,jk} = 0$ otherwise. The dissipation principle then implies $\eta_1 - \eta_2 - 4\eta_3 \geq 0$.

Setting then $\xi = \eta_1 - \eta_2 - 4\eta_3$ and recalling that $\boldsymbol{b} = \boldsymbol{f} - \rho\boldsymbol{a}$, we come to the flow equation for second-grade incompressible fluids

$$
\rho\boldsymbol{a} = \boldsymbol{f} - \operatorname{grad} P + \mu\Delta\boldsymbol{v} - \xi\Delta\Delta\boldsymbol{v}\,, \tag{17}
$$

where $P = p - \operatorname{div}\boldsymbol{\pi}$, $\mu \geq 0$, $\xi \geq 0$.

We do not deal in this paper with the important problem of boundary conditions for a nonsimple fluid. We refer the reader to [2] for a deep discussion on the subject.

4. Second-grade compressible fluids

The generality of Weyls's Theorem, and in particular of formula (8), allows us to remove quite easily the incompressibility assumption, giving a more general description of second-grade fluids. In such a case, the only constraints on the form of F are the symmetry of j,k and the symmetry of m,n, which come from the symmetry in the last two indices of $\operatorname{grad}\operatorname{grad}\boldsymbol{v}$. Then we can start directly from equation (11), obtaining:

$$\begin{aligned}\mathsf{G}_{ijk} &= -\delta_{ij}\pi_k + A(v_{k,rr}\delta_{ij} + v_{j,rr}\delta_{ik}) \\ &\quad + B(v_{r,kr}\delta_{ij} + v_{r,jr}\delta_{ik}) + Cv_{i,rr}\delta_{jk} \\ &\quad + Dv_{i,jk} + Ev_{r,ir}\delta_{jk} + F(v_{j,ik} + v_{k,ij})\,.\end{aligned}$$

Then one has

$$\begin{aligned}\mathsf{G}_{ijk,k} &= -\delta_{ij}\pi_{k,k} + A(v_{k,krr}\delta_{ij} + v_{j,irr}) \\ &\quad + B(v_{r,kkr}\delta_{ij} + v_{r,ijr}) + Cv_{i,jrr} \\ &\quad + Dv_{i,jkk} + Ev_{r,ijr} + F(v_{j,ikk} + v_{k,ijk})\,.\end{aligned}$$

and

$$\begin{aligned}\mathsf{G}_{ijk,kj} &= -\pi_{k,ki} + 2Av_{k,ikrr} \\ &\quad + 2Bv_{r,ikkr} + Cv_{i,jjrr} \\ &\quad + Dv_{i,jjkk} + Ev_{r,ijjr} + 2Fv_{j,ijkk}\,.\end{aligned}$$

Setting $\xi_1 = 2A + 2B + E + F$ and $\xi_2 = C + D$, it follows that

$$\operatorname{div}\operatorname{div}\mathsf{G} = -\operatorname{grad}\operatorname{div}\boldsymbol{\pi} + \xi_1\operatorname{grad}\operatorname{div}\Delta\boldsymbol{v} + \xi_2\Delta\Delta\boldsymbol{v}\,. \tag{18}$$

Since in the compressible case the Cauchy–Poisson relation (6) for the first-gradient term gives rise to the classical block

$$-\operatorname{grad}p + (\lambda+\mu)\operatorname{grad}\operatorname{div}\boldsymbol{v} + \mu\Delta\boldsymbol{v}\,,$$

by (18) and the local balance (15) we obtain the *flow equation for second-grade compressible fluids*

$$\rho\boldsymbol{a} = \boldsymbol{f} - \operatorname{grad}P + (\lambda+\mu)\operatorname{grad}\operatorname{div}\boldsymbol{v} + \mu\Delta\boldsymbol{v} - \xi_1\operatorname{grad}\operatorname{div}\Delta\boldsymbol{v} - \xi_2\Delta\Delta\boldsymbol{v}\,,$$

where $P = p - \operatorname{div}\boldsymbol{\pi}$.

5. The flow equation for third-grade fluids

We are also able to extend the above theory to a more general class of nonsimple materials, the so-called third-grade fluids, where one takes into account a dependence on the third-order gradient of the velocity. Without changing the notation, we assume the internal power to be of the form

$$W_{\text{int}}(R,\boldsymbol{v}) = \int_R (\mathsf{T}\cdot\operatorname{grad}\boldsymbol{v} + \mathsf{G}\cdot\operatorname{grad}\operatorname{grad}\boldsymbol{v} + \mathsf{H}\cdot\operatorname{grad}\operatorname{grad}\operatorname{grad}\boldsymbol{v})\,dx\,,$$

where H is a fourth-order tensor which depends linearly and isotropically on the third-order gradient of the velocity. Using the divergence theorem as in Section 3, the Principle of Virtual Power yields the balance equation

$$-\operatorname{div}\mathsf{T} + \operatorname{div}\operatorname{div}\mathsf{G} - \operatorname{div}\operatorname{div}\operatorname{div}\mathsf{H} = \boldsymbol{b} \tag{19}$$

apart from the surface conditions. Setting

$$\mathsf{H}^0_{ijhk} = \mathsf{H}_{ijhk} + \delta_{ij}\Pi_{hk}\,,$$

with $\mathsf{\Pi}$ a tensor-valued pressure, we now study the tensor H^0. We introduce an eighth-order isotropic linear function F such that

$$\mathsf{H}^0 = \mathsf{F}\,\mathrm{grad}\,\mathrm{grad}\,\mathrm{grad}\,\boldsymbol{v}\,, \quad \mathsf{H}^0_{ijhk} = \mathsf{F}_{ijhklmnp} v_{l,mnp}$$

The isotropy assumption drops to $7!! = 105$ the number of free components of F. Moreover, in analogy with (9), the symmetry of $\mathrm{grad}\,\mathrm{grad}\,\mathrm{grad}\,\boldsymbol{v}$ yield

$$\mathsf{H}^0_{ijhk} = \mathsf{H}^0_{ijkh} = \mathsf{H}^0_{ihjk}\,, \quad v_{l,mnp} = v_{l,mpn} = v_{l,nmp}\,, \tag{20}$$

In this case there remain eight free coefficients:(2)

$$\begin{aligned}\mathsf{H}^0_{ijhk} = {} & A\big(\delta_{ij}(v_{h,krr} + v_{k,hrr}) + \delta_{ih}(v_{j,krr} + v_{k,jrr}) + \delta_{ik}(v_{h,jrr} + v_{j,hrr})\big)\\ & + B(\delta_{ij}\delta_{hk} + \delta_{ih}\delta_{jk} + \delta_{ik}\delta_{jh}) v_{l,lrr}\\ & + C(\delta_{ij} v_{r,rhk} + \delta_{ih} v_{r,rjk} + \delta_{ik} v_{r,rjh}) + D(\delta_{jh} v_{i,krr} + \delta_{jk} v_{i,hrr} + \delta_{hk} v_{i,jrr})\\ & + E v_{i,jhk} + F(\delta_{hk} v_{j,irr} + \delta_{jh} v_{k,irr} + \delta_{jk} v_{h,irr})\\ & + G(v_{j,ihk} + v_{h,ijk} + v_{k,ijh}) + H(\delta_{hk} v_{r,irj} + \delta_{jk} v_{r,irh} + \delta_{jh} v_{r,irk})\,.\end{aligned}$$

Now we are able to compute the divergences of H:

$$\begin{aligned}\mathsf{H}_{ijhk,khj} = {} & -\delta_{ij}\mathsf{\Pi}_{hk,khj}\\ & + A\big((v_{h,krrkhi} + v_{k,hrrkhi}) + (v_{j,krrkij} + v_{k,jrrkij}) + (v_{h,jrrihj} + v_{j,hrrihj})\big)\\ & + B(v_{l,lrrhhi} + v_{l,lrrjij} + v_{l,lrrijj})\\ & + C(v_{r,rhkkhi} + v_{r,rjkkij} + v_{r,rjhihj})\\ & + D(v_{i,krrkjj} + v_{i,hrrjhj} + v_{i,jrrhhj})\\ & + E v_{i,jhkkhj} + F(v_{j,irrhhj} + v_{k,irrkjj} + v_{h,irrjhj})\\ & + G(v_{j,ihkkhj} + v_{h,ijkkhj} + v_{k,ijhkhj})\\ & + H(v_{r,irjhhj} + v_{r,irhjhj} + v_{r,irkkjj})\,.\end{aligned}$$

Collecting the terms we obtain

$$\mathsf{H}_{ijhk,khj} = -\delta_{ij}\mathsf{\Pi}_{hk,khj} + \gamma_1 v_{j,jhhkki} + \gamma_2 v_{i,jjhhkk}\,,$$

where $\gamma_1 = 3(2A + B + C + F + G + H)$ and $\gamma_2 = 3D + E$. In intrinsic notation,

$$\mathrm{div}\,\mathrm{div}\,\mathrm{div}\,\mathsf{H} = -\,\mathrm{grad}\,\mathrm{div}\,\mathrm{div}\,\mathsf{\Pi} + \gamma_1\,\mathrm{grad}\,\mathrm{div}\,\Delta\Delta\boldsymbol{v} + \gamma_2\Delta\Delta\Delta\boldsymbol{v}\,.$$

Then the balance (19) yields the *flow equation for third-grade (compressible) fluids*

$$\begin{aligned}\rho\boldsymbol{a} = {} & \boldsymbol{f} - \mathrm{grad}\,P + (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{v} + \mu\Delta\boldsymbol{v}\\ & - \xi_1\,\mathrm{grad}\,\mathrm{div}\,\Delta\boldsymbol{v} - \xi_2\Delta\Delta\boldsymbol{v} + \gamma_1\,\mathrm{grad}\,\mathrm{div}\,\Delta\Delta\boldsymbol{v} + \gamma_2\Delta\Delta\Delta\boldsymbol{v}\,,\end{aligned} \tag{21}$$

where $P = p - \mathrm{div}\,\boldsymbol{\pi} + \mathrm{div}\,\mathrm{div}\,\mathsf{\Pi}$.

Finally, if the fluid is assumed to be incompressible, the tensor H^0 becomes

$$\begin{aligned}\mathsf{H}^0_{ijhk} = {} & A(\delta_{ij}(v_{h,krr} + v_{k,hrr}) + \delta_{ih}(v_{j,krr} + v_{k,jrr}) + \delta_{ik}(v_{h,jrr} + v_{j,hrr}))\\ & + D(\delta_{jh} v_{i,krr} + \delta_{jk} v_{i,hrr} + \delta_{hk} v_{i,jrr})\\ & + E v_{i,jhk} + F(\delta_{hk} v_{j,irr} + \delta_{jh} v_{k,irr} + \delta_{jk} v_{h,irr})\\ & + G(v_{j,ihk} + v_{h,ijk} + v_{k,ijh})\end{aligned}$$

(2)See Appendix B.

and it follows that

$$\operatorname{div}\operatorname{div}\operatorname{div}\mathsf{H} = -\operatorname{grad}\operatorname{div}\operatorname{div}\mathsf{\Pi} + \gamma\Delta\Delta\Delta\boldsymbol{v}\,,$$

where $\gamma := 3D + E$. As in the case of a second-grade fluid, a dissipation principle imposes that

$$\mathsf{H}\cdot\operatorname{grad}\operatorname{grad}\operatorname{grad}\boldsymbol{v} \geq 0$$

for any virtual velocity $\boldsymbol{v}$. Choosing $\boldsymbol{v} = \frac{c}{6}x_1^3\boldsymbol{e}_2$, where c is a coefficient with dimensions of (length)$^{-2}$(time)$^{-1}$, one gets $\gamma \geq 0$. Then we get the *flow equation for third-grade incompressible fluids*:

$$\rho\boldsymbol{a} = \boldsymbol{f} - \operatorname{grad}P + \mu\Delta\boldsymbol{v} - \xi\Delta\Delta\boldsymbol{v} + \gamma\Delta\Delta\Delta\boldsymbol{v}\,, \tag{22}$$

where $P = p - \operatorname{div}\boldsymbol{\pi} + \operatorname{div}\operatorname{div}\mathsf{\Pi}$, $\mu \geq 0$, $\xi \geq 0$, $\gamma \geq 0$.

A discussion about boundary conditions for third-order fluids, which goes beyond the goal of the present paper, would be desirable.

Appendix A. Free coefficients of an isotropic tensor

By Weyl's Theorem, the number D_n of free coefficients of an isotropic $(2n)$-tensor is a general combination of (products of) Kronecker symbols $\delta_{i_1i_2}\ldots\delta_{i_{2n-1}i_{2n}}$. Hence the number of free components of an isotropic linear function of order $2n$ can be computed as the number of partitions of a set with $2n$ elements into sets with two elements. Such a number can be computed in many ways: for instance, we can take the number of permutations of the indices, $(2n)!$, and divide it by $n!$, since we do not have to take into consideration the order of the sets with 2 elements. Moreover, we divide again by 2^n, since we do no want to account for the order of the elements in a set. Hence we have

$$D_n = \frac{(2n)!}{n!2^n} = \frac{(2n)!}{2\cdot 4\cdot\cdots\cdot(2n-2)\cdot(2n)} = (2n-1)!!\,.$$

Appendix B. Free coefficients of a symmetric isotropic tensor

We prove that an isotropic $(2n)$-tensor $\mathsf{F}_{ij_1\ldots j_{n-1}lm_1\ldots m_{n-1}}$ which is symmetric in $j_1\ldots j_{n-1}$ and $m_1\ldots m_{n-1}$ has a number E_n of free coefficients, where

$$E_n = \begin{cases} \frac{5n-4}{2} & \text{if } n \text{ is even,} \\ \frac{5n-3}{2} & \text{if } n \text{ is odd.} \end{cases} \tag{23}$$

Such a number can be easily written as

$$E_n = \left[\frac{5n-3}{2}\right],$$

where $[a]$ denotes the largest integer less than or equal to a.

Let us assume that $n \geq 2$ and consider a symmetric isotropic tensor of order $2n$, $\mathsf{F}_{ij_1\ldots j_{n-1}lm_1\ldots m_{n-1}}$. The tensor F is a combinations of Kronecker symbols. Focusing the attention on the indices i, l, we split the products of Kronecker symbols into five disjoint families, according as they contain a term of the type $\delta_{ij}\delta_{lj}$, $\delta_{ij}\delta_{lm}$, $\delta_{im}\delta_{lj}$, $\delta_{im}\delta_{lm}$, δ_{il}. Then we count the elements of each family, by computing how many terms of the type δ_{mn} they contain, since the other possible terms are of type δ_{mm} and δ_{nn}.

Suppose that n is even. In the first case, $\delta_{ij}\delta_{lj}$, there remain $n-3$ of j's and $n-1$ of m's. Since $n-3$ and $n-1$ are odd numbers, the term δ_{jm} can appear $1, 3, \ldots, n-3$ times, so we have $\frac{n}{2}-1$ different possibilities.

In the second case, $\delta_{ij}\delta_{lm}$, there remain $n-2$ of j's and $n-2$ of m's, which are even numbers, then the term δ_{jm} can appear $0, 2, \ldots, n-2$ times and we have $\frac{n}{2}$ different possibilities. In the same way, there are $\frac{n}{2}$ possibilities for the case $\delta_{im}\delta_{lj}$ and $\frac{n}{2}-1$ possibilities for the case $\delta_{im}\delta_{lm}$. The last case, δ_{il}, leaves $n-1$ of j's and $n-1$ of m's, which are odd numbers, giving again $\frac{n}{2}$ possibilities.

All in all, we have

$$2\left(\frac{n}{2}-1+\frac{n}{2}\right)+\frac{n}{2}=\frac{5n-4}{2}.$$

The case n odd can be managed in the same way.

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