Balance laws in Continuum Mechanics: a measure-theoretical approach

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A Elena, mia moglie
e fonte del mio entusiasmo

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Introduction

Balance laws are a basic concept in Continuum Mechanics and apply to many situations, since they do not depend on constitutive relations. They have a natural integral structure from which one can deduce local versions, provided that suitable continuity is assumed.

At the end of the 50's, a new approach to such equations was introduced by dealing with set functions associated with physical quantities, rather than with functions evaluated at single points [22]. Such set functions should satisfy some reasonable conditions as additivity, hence it was natural to put the matter in the setting of Measure Theory. In this way, the existence of densities did not have to be supposed as an axiom, but it could be deduced. The proof of Cauchy's Stress Theorem under weaker assumptions was one of the main outcomes. In particular, hypotheses about dependence of the stress density only on the normal to the surface and on the position (the so called *Cauchy postulate*) could be dropped, as well as continuity of the density with respect to the position [16, 15]. Moreover, even the case of unbounded densities could be considered [26, 28], by weakening the usual assumption of Lipschitz continuity with respect to volume measure. On the other hand, in this general setting a full converse cannot be expected, namely a vector filed whose divergence is summable induces a flux only on "almost every" surface. An efficient definition of "almost every" and a complete equivalence between vector fields and fluxes is one of the major results of [28].

However, the problem of choosing the class of *subbodies*, i.e. the suitable class of sets upon which one has to formulate a balance law, remained open. Although they may be used to describe the situation arising in the body in a very satisfactory way, subbodies are not completely physical, since the class of subsets which have to represent them is a matter of choice. One needs a family of sets which is somewhat stable with respect to union and intersection, smooth enough for the Gauss-Green Theorem to hold, and quite rich, in order to state a local version of the balance law. Some years ago, the matter was set in the framework of Geometric Measure Theory: the sets with finite perimeter are a very general class on which the Gauss-Green Theorem applies, hence they turned out to be a good model for the subbodies [34, 26, 19, 28].

In this dissertation we study set functions which are not absolutely continuous with respect to the n-dimensional Lebesgue measure, but rather to a generic Radon measure. In this way, some concentration of the flux density is allowed, such as a Dirac's delta around a point. The main result is the proof of an integral representation of such functions by means of locally summable densities with divergence measure. Moreover, the equivalence between the integral form of the balance law and the distributional form is proved even in this general setting. One of the tools in obtaining this result is a suitable concept of "almost all", introduced on the class of subbodies; it means that the representation formula does not hold for every subbody, but for a lot of them, in a sense that will be clarified. This kind of notion was first introduced in [28], but for general Radon measures some modifications are required (see [5]).

A remarkable result which follows from this development is that it is enough to have information only on a very simple class of sets, the n-intervals (defined with respect to a grid, to have the notion translation-rotation invariant), in order to extend the set function to almost all the sets with finite perimeter, in a unique way. This situation is typical of topics in measure theory, where often one can extend a result to a wider class of sets. The choice of sets with finite perimeter as the subbodies, even if it could be considered too much involved and with no physical meaning (see [24]), finds in this extension a good explanation. Moreover, this kind of result can help some numerical applications. On the other hand, the concept of almost all is more explicit on the class of n-intervals: it means that the coordinates of the end-points of the intervals must lay in a set which contains almost every real number.

In the first chapter the concept of *Cauchy interaction* between two subbodies is introduced and a decomposition in a "volume" part and a "contact" part is proved. Moreover, a representation theorem for the volume part, which is dominated by a measure of the Cartesian product of the whole subbodies, is given. In the second chapter we study the case of contact interactions and it is proved that the interaction between two subbodies can be seen as a flux through the contact surface, which we call a *Cauchy flux*. In particular, a version of Cauchy's Stress Theorem is proved under weaker assumptions. In the third chapter we study balance laws of entropy kind, where the balance is stated by an inequality and the concept of *entropy production* as a superadditive function is introduced. In this case one gets a weak formulation of the classical Clausius-Duhem inequality and the existence of temperature. In the last chapter we propose a different approach to a balance law, as a balance of the mechanical power, following the ideas in [11, 6]. If the body B is a subset of $\mathbb{R}^n$, the two formulations turn out to be equivalent. However, if B is an abstract differentiable manifold, the approach by Cauchy fluxes seems to be possible only in the scalar case (e.g. for the heat flux). On the contrary, for vector-valued contact interactions the power approach seems to be mandatory. Finally, in Appendix A we show that a form of continuity of the Cauchy interaction with respect to the subbody can be recovered, also when dealing with general Radon measures, provided that the boundary of the subbody is supposed to have a nonzero thickness. In Appendix B we give sufficient assumptions in order to consider as subbodies also the sets whose boundary meets the boundary of the body; in this way we can state a boundary value problem for Cauchy interactions.

The technique involves classical tools of Measure Theory, such as the Radon-Nikodym Theorem, the Lebesgue points and the Besicovitch Theorem, together with some notions about sets with finite perimeter, in particular the generalized Gauss-Green Theorem and the approximation by means of multi-intervals.

A part of the above results has been published in [5, 20, 21].

Chapter 1

Review of measure theory and Cauchy interactions

In [19], Gurtin, Williams and Ziemer introduced the concept of *Cauchy interaction* in order to represent an interaction between two disjoint subbodies, possibly having a part of their boundary in common. This is, roughly speaking, a set function I of two variables, the subbodies, which is additive on each variable and which is Lipschitz continuous with respect to the area measure of the common part of the boundaries and the volume measure. In that paper it is proved that:

(*a*) I can be decomposed as the sum of a "body interaction" I_b and a "contact interaction" I_c, satisfying the bounds

$$|I_b(A, C)| \leqslant K_C \mathscr{L}^n(A), \qquad |I_c(A, C)| \leqslant K \mathscr{H}^{n-1}(\partial_* A \cap \partial_* C);$$

(*b*) I_b can be represented as

$$I_b(A, C) = \int_{A \times C} b(x, y)\, dx\, dy, \quad \text{for a suitable } b \in L^1(B \times B);$$

(*c*) in the balanced case, i.e. when $|I(A, \mathbb{R}^n \setminus A)| \leqslant K \mathscr{L}^n(A)$, the contact part I_c is a Cauchy flux which can be represented as

$$I_c(A, C) = \int_{\partial_* A \cap \partial_* C} \boldsymbol{q} \cdot \boldsymbol{n}_{\partial_* A \cap \partial_* C}\, d\mathscr{H}^{n-1},$$

where $\boldsymbol{q} : B \to \mathbb{R}^n$ is a bounded vector field with bounded divergence.

In this chapter we extend this definition of Cauchy interaction in order to allow the corresponding densities to be also distributions of order zero; thus, also interactions which are singular can be considered. We deal with the notion of "almost all subbodies", already introduced by Šilhavý [28] and extended by Degiovanni, Marzocchi and Musesti [5] for the formulation of the Cauchy Stress Theorem. In particular, for almost all subbodies we first show that:

(a') I can be decomposed in a unique way as the sum of a "body interaction" I_b and a "contact interaction" I_c, satisfying the bounds

$$|I_b(A,C)| \leqslant \eta(A \times C), \qquad |I_c(A,C)| \leqslant \int_{\partial_* A \cap \partial_* C} h(x)\, d\mathscr{H}^{n-1}(x)$$

where η is a Radon measure and h a positive function in L^1_{loc};

(b') I_b is represented by

$$I_b(A,C) = \int_{A\times C} b(x,y)\, d\mu(x,y)$$

where μ is a Radon measure and $b : B \times B \to \{-1,1\}$ is a Borel function.

We remark that in this chapter we do not assume any kind of balance. The corresponding of the above-mentioned (c), i.e.

(c') in the balanced case, i.e. when $|I(A,\mathbb{R}^n \setminus A)| \leqslant \lambda(A)$ for a Radon measure λ, the contact part I_c is a Cauchy flux which is represented by

$$I_c(A,C) = \int_{\partial_* A \cap \partial_* C} \boldsymbol{q} \cdot \boldsymbol{n}_{\partial_* A \cap \partial_* C}\, d\mathscr{H}^{n-1}$$

where $\boldsymbol{q} : B \to \mathbb{R}^n$ is a locally integrable vector field with divergence measure

will be proved in Chapter 2, as well as a correspondence between contact interactions and Cauchy fluxes (Theorem 2.1.8).

It is worth pointing out that our definition of Cauchy interaction, as well as that of [19], is modeled on the situation in which the set function represents the sum of the heat generated in the subbody and the heat transferred through its boundary. This leads to the peculiar choice of the subbodies in Definition 1.2.1: it is requested that either the subsets lie in the interior of the body, or their complements have this property.

1.1 Preliminary lemmas from Geometric Measure Theory

Let $M \subseteq \mathbb{R}^n$. We denote by $\operatorname{cl} M$ and $\operatorname{int} M$ the closure and the interior of M in $\mathbb{R}^n$, respectively. When M is a Borel set, we also denote by $\mathfrak{B}(M)$ the σ-algebra of Borel subsets of M.

We denote by $\mathscr{L}^n$ the Lebesgue outer measure on $\mathbb{R}^n$ and by $\mathscr{H}^k$ the k-dimensional Hausdorff outer measure. Denoting by $B_r(x)$ the open ball with center x and radius r, we introduce

$$M_* = \left\{ x \in \mathbb{R}^n : \lim_{r \to 0^+} \frac{\mathscr{L}^n(B_r(x) \setminus M)}{\mathscr{L}^n(B_r(x))} = 0 \right\}$$

and

$$\partial_* M = \mathbb{R}^n \setminus [M_* \cup (\mathbb{R}^n \setminus M)_*]\,,$$

(the so called *measure-theoretic interior and measure-theoretic boundary of* M, respectively). It is well-known that M_* and $\partial_* M$ are Borel subsets of $\mathbb{R}^n$. We say that a set M is *normalized* (cf. [28]), if $M_* = M$.

Now let $M \subseteq \mathbb{R}^n$, $x \in \partial_* M$ and $u \in \mathbb{R}^n$ with $|u| = 1$. We say that u is *a unit exterior normal vector* to M at x if

$$\lim_{r \to 0^+} \frac{\mathscr{L}^n\left(\{\xi \in B_r(x) \cap M : (\xi - x) \cdot u > 0\}\right)}{\mathscr{L}^n(B_r(x))} = 0,$$

$$\lim_{r \to 0^+} \frac{\mathscr{L}^n\left(\{\xi \in B_r(x) \setminus M : (\xi - x) \cdot u < 0\}\right)}{\mathscr{L}^n(B_r(x))} = 0.$$

If u and v are two unit exterior normal vectors to M at x, it turns out that $u = v$, so we can define a map $\boldsymbol{n}^M : \partial_* M \to \mathbb{R}^n$, setting $\boldsymbol{n}^M(x)$ equal to the unit exterior normal vector to M at x, where it exists, and $\boldsymbol{n}^M(x) = 0$ otherwise. Then $\boldsymbol{n}^M$ is a Borel and bounded map, that is called *the unit exterior normal* to M.
We say that M *has finite perimeter* if $\mathscr{H}^{n-1}(\partial_* M) < +\infty$ (this implies the $\mathscr{L}^n$-measurability of M). Such sets are also called *Caccioppoli sets*. In that case, $|\boldsymbol{n}^M(x)| = 1$ for $\mathscr{H}^{n-1}$-a.e. $x \in \partial_* M$ and the Gauss-Green Theorem

$$\int_M \boldsymbol{v} \cdot \operatorname{grad} f \, d\mathscr{L}^n = \int_{\partial_* M} f \, \boldsymbol{v} \cdot \boldsymbol{n}^M \, d\mathscr{H}^{n-1} - \int_M f \operatorname{div} \boldsymbol{v} \, d\mathscr{L}^n$$

holds whenever $f : \mathbb{R}^n \to \mathbb{R}$ and $\boldsymbol{v} : \mathbb{R}^n \to \mathbb{R}^n$ are Lipschitz continuous with compact support (see e.g. [8, Theorem 4.5.6] or [35, Theorem 5.8.2]).
Let Ω be an open subset of $\mathbb{R}^n$. We denote by $\mathfrak{M}(\Omega)$ the collection of Borel measures $\mu : \mathfrak{B}(\Omega) \to [0, +\infty]$ finite on compact subsets of Ω and by $\mathcal{L}^1_{loc,+}(\Omega)$ the set of Borel functions $h : \Omega \to [0, +\infty]$ with $\int_K h \, d\mathscr{L}^n < +\infty$ for every compact subset $K \subseteq \Omega$. For $\mu, \eta \in \mathfrak{M}(\Omega)$ we will say that μ is *absolutely continuous* with respect to η, and write $\mu \ll \eta$, if

$$\forall E \in \mathfrak{B}(\Omega) : \eta(E) = 0 \implies \mu(E) = 0.$$

If X is a finite-dimensional normed space, we denote by $\mathcal{L}^1_{loc}(\Omega; X)$ the set of Borel maps $\boldsymbol{v} : \Omega \to X$ with $\int_K \|\boldsymbol{v}\| \, d\mathscr{L}^n < +\infty$ for any compact subset K of Ω. We also denote by $L^1_{loc}(\Omega, \mu)$ the quotient set of Borel functions $f : \Omega \to \mathbb{R}$ such that $\int_K |f| \, d\mu < +\infty$ for every compact subset $K \subseteq \Omega$, where we identify the functions that agree μ-almost everywhere in Ω.
In the remainder of the section we establish some general properties of measure-theoretic boundary which will be used in the sequel.

Proposition 1.1.1. *Let M, N be two $\mathscr{L}^n$-measurable subsets of $\mathbb{R}^n$. Then we have*

$$\begin{aligned}\left[((\partial_* M) \setminus N_*) \cup ((\partial_* N) \setminus M_*)\right] \setminus (\partial_* M \cap \partial_* N) \subseteq \partial_*(M \cup N) \subseteq \\ \subseteq ((\partial_* M) \setminus N_*) \cup ((\partial_* N) \setminus M_*),\end{aligned}$$

$$\begin{aligned}(N_* \cap \partial_* M) \cup (M_* \cap \partial_* N) \subseteq \partial_*(M \cap N) \subseteq \\ \subseteq (N_* \cap \partial_* M) \cup (M_* \cap \partial_* N) \cup (\partial_* M \cap \partial_* N),\end{aligned}$$

$$\begin{aligned}\left[((\partial_* M) \setminus N_*) \cup (M_* \cap \partial_* N)\right] \setminus (\partial_* M \cap \partial_* N) \subseteq \partial_*(M \setminus N) \subseteq \\ \subseteq ((\partial_* M) \setminus N_*) \cup (M_* \cap \partial_* N).\end{aligned}$$

Proof. It is well-known that $\mathscr{L}^n((M\setminus M_*)\cup(M_*\setminus M))=0$ if and only if M is $\mathscr{L}^n$-measurable. In particular, this implies that $\partial_* M=\partial_*(M_*)$ for every $\mathscr{L}^n$-measurable subset $M\subseteq\mathbb{R}^n$. Thus we can suppose that M and N are normalized. Now the claimed properties follow easily by [19, Lemma 3.2] and [28, Proposition 2.1]. $\square$

The following refines Proposition 1.1.1, stating useful decompositions of the measure-theoretic boundary of $M\cup N$, $M\cap N$ and $M\setminus N$ up to sets of zero surface measure.

Proposition 1.1.2. *Let M, N be two $\mathscr{L}^n$-measurable subsets of $\mathbb{R}^n$ of finite perimeter and let $A=(\partial_*M\setminus(N_*\cup\partial_*N))$, $B=(\partial_*N\setminus(M_*\cup\partial_*M))$, $C=(M_*\cap\partial_*N)$, $D=(N_*\cap\partial_*M)$,*

$$E=\{x\in\partial_*M\cap\partial_*N:\boldsymbol{n}^M(x)\neq 0,\boldsymbol{n}^N(x)\neq 0,\boldsymbol{n}^M(x)\neq-\boldsymbol{n}^N(x)\},$$

$$F=\{x\in\partial_*M\cap\partial_*N:\boldsymbol{n}^M(x)\neq 0,\boldsymbol{n}^N(x)\neq 0,\boldsymbol{n}^M(x)\neq\boldsymbol{n}^N(x)\}.$$

Then there exist three sets $R_k\subseteq\partial_*M\cap\partial_*N$, for $k=1,2,3$, such that $\mathscr{H}^{n-1}(R_k)=0$ and

$$\begin{aligned}\partial_*(M\cup N)&=A\cup B\cup E\cup R_1,\\ \partial_*(M\cap N)&=C\cup D\cup E\cup R_2,\\ \partial_*(M\setminus N)&=A\cup C\cup F\cup R_3,\end{aligned}$$

where the unions are disjoint.

Proof. As in Proposition 1.1.1, we can suppose that M and N are normalized. We start from the last decomposition. From Proposition 1.1.1 we have that $\partial_*M\setminus(N\cup\partial_*N)\subseteq\partial_*(M\setminus N)$ and $M\cap\partial_*N\subseteq\partial_*(M\setminus N)$. Let $x\in F$ and consider the cone

$$C_e=\{\xi\in\mathbb{R}^n:(\xi-x)\cdot\boldsymbol{n}^N(x)<0<(\xi-x)\cdot\boldsymbol{n}^M(x)\};$$

for every $r>0$ we have that

$$(C_e\cap B_r(x))\subseteq\{\xi\in B_r(x)\cap M:(\xi-x)\cdot\boldsymbol{n}^M(x)>0\}\cup[B_r(x)\setminus(M\setminus N)].$$

By the definition of unit exterior normal vector, this implies that

$$\limsup_{r\to 0^+}\frac{\mathscr{L}^n(B_r(x)\setminus(M\setminus N))}{\mathscr{L}^n(B_r(x))}\geqslant\lim_{r\to 0^+}\frac{\mathscr{L}^n(C_e\cap B_r(x))}{\mathscr{L}^n(B_r(x))}>0,$$

hence $x\notin(M\setminus N)_*$.
In the same way, setting

$$C_i=\{\xi\in\mathbb{R}^n:(\xi-x)\cdot\boldsymbol{n}^M(x)<0<(\xi-x)\cdot\boldsymbol{n}^N(x)\},$$

for every $r>0$ one can prove that

$$\begin{aligned}&\big[(C_i\cap B_r(x))\setminus\{\xi\in B_r(x)\setminus M:(\xi-x)\cdot\boldsymbol{n}^M(x)<0\}\big]\setminus\\ &\qquad\setminus\{\xi\in B_r(x)\cap N:(\xi-x)\cdot\boldsymbol{n}^N(x)>0\}\subseteq B_r(x)\cap(M\setminus N),\end{aligned}$$

hence $x\notin(\mathbb{R}^n\setminus(M\setminus N))_*$. Thus $F\subseteq\partial_*(M\setminus N)$.
Now let $x\in\partial_*M\cap\partial_*N$ be such that $\boldsymbol{n}^M(x)=\boldsymbol{n}^N(x)\neq 0$; for every $r>0$ we have that the set

$$\{\xi\in B_r(x)\cap M:(\xi-x)\cdot\boldsymbol{n}^M(x)>0\}\cup\{\xi\in B_r(x)\setminus N:(\xi-x)\cdot\boldsymbol{n}^N(x)\leqslant 0\}$$

contains $B_r(x) \cap (M \setminus N)$, hence

$$\lim_{r \to 0^+} \frac{\mathscr{L}^n(B_r(x) \cap (M \setminus N))}{\mathscr{L}^n(B_r(x))} = 0.$$

This means that $x \in (\mathbb{R}^n \setminus (M \setminus N))_*$, thus $x \notin \partial_*(M \setminus N)$. Setting

$$R_3 = \partial_*(M \setminus N) \setminus [(\partial_* M \setminus (N \cup \partial_* N)) \cup (M \cap \partial_* N) \cup F],$$

it follows that

$$R_3 \subseteq \{\xi \in \partial_* M \cap \partial_* N : \boldsymbol{n}^M(x) = 0 \text{ or } \boldsymbol{n}^N(x) = 0\}$$

and, by the properties of the unit exterior normal, we have $\mathscr{H}^{n-1}(R_3) = 0$. This prove the third decomposition.
The other two formulas turn out if we write $M \cup N$ as $\mathbb{R}^n \setminus ((\mathbb{R}^n \setminus M) \cap (\mathbb{R}^n \setminus N))$ and $M \cap N$ as $M \setminus (\mathbb{R}^n \setminus N)$. □

Proposition 1.1.3. *Let M_1, M_2, M_3 be three mutually disjoint subsets of $\mathbb{R}^n$ of finite perimeter. Then*

$$\mathscr{H}^{n-1}(\partial_* M_1 \cap \partial_* M_2 \cap \partial_* M_3) = 0.$$

Proof. See e.g. [19, Proposition 3.4]. □

1.2 Main definitions

Throughout the remainder of this chapter, B will denote a bounded normalized subset of $\mathbb{R}^n$ of finite perimeter, which we call *a (continuous) body.*

Definition 1.2.1. *Let $\mathcal{M}$ be the collection of all normalized subsets of B of finite perimeter. We set*

$$\mathcal{N} = \{C \subseteq \mathbb{R}^n : C \text{ is normalized, } C \in \mathcal{M} \text{ or } (\mathbb{R}^n \setminus C)_* \in \mathcal{M}\},$$

$$\mathfrak{D} = \{(A, C) \in \mathcal{M} \times \mathcal{N} : A \cap C = \varnothing\}.$$

Moreover, we define

$$\mathcal{M}^\circ = \{A \in \mathcal{M} : \mathrm{cl}\, A \subseteq \mathrm{int}\, B\},$$

$$\mathcal{N}^\circ = \mathcal{M}^\circ \cup \{A \cup (\mathbb{R}^n \setminus B)_* : A \in \mathcal{M}^\circ\},$$

$$\mathfrak{D}^\circ = \{(A, C) \in \mathcal{M}^\circ \times \mathcal{N}^\circ : A \cap C = \varnothing\}.$$

We refer to Section 2.9 for a different definition of the class of subbodies, which works on the whole B and drops the assumption $\mathrm{cl}\, A \subseteq \mathrm{int}\, B$.

Remark 1.2.2. Sometimes the class $\mathcal{M}$ is called *a system of parts*: it is the class of subbodies involved in the formulation of a balance law. Following [28], in [5] an axiomatic way of defining $\mathcal{M}$ was chosen; indeed every class $\mathcal{F}$ of normalized subsets of B with finite perimeter was referred to be a system of parts if the following conditions were satisfied:

(a) $\varnothing, B \in \mathcal{F}$;

(b) if $M, N \in \mathcal{F}$, then $(M \cup N)_*$, $M \cap N$, $(M \setminus N)_* \in \mathcal{F}$;

(c) if $M \in \mathcal{F}$ and $H \subseteq \mathbb{R}^n$ is an open affine half-space, then $M \cap H \in \mathcal{F}$.

That kind of approach could appear more general, since in the previous definition we choose a particular system of parts (the largest one), but a balanced interaction defined on a system of parts can be uniquely extended to almost all of $\mathcal{M}$, in a sense specified below. From this point of view, all the choices of classes of subbodies are equivalent.

Definition 1.2.3. *Let $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$, $\nu \in \mathfrak{M}(\operatorname{int} B)$. We set, following an idea of [28],*

$$\mathcal{M}^\circ_{h\nu} = \left\{ A \in \mathcal{M}^\circ : \int_{\partial_* A} h \, d\mathscr{H}^{n-1} < +\infty, \, \nu(\partial_* A) = 0 \right\},$$
$$\mathcal{N}^\circ_{h\nu} = \{ C \in \mathcal{N}^\circ : (C \cap B) \in \mathcal{M}^\circ_{h\nu} \},$$
$$\mathfrak{D}^\circ_{h\nu} = \mathfrak{D}^\circ \cap (\mathcal{M}^\circ_{h\nu} \times \mathcal{N}^\circ_{h\nu}).$$

Remark 1.2.4. In Definition 1.2.3 we may assume, without loss of generality, that the map $h : \operatorname{int} B \to [0, +\infty]$ is a Borel function with $\int_{\operatorname{int} B} h \, d\mathscr{L}^n < +\infty$ and $\nu : \mathfrak{B}(\operatorname{int} B) \to [0, +\infty]$ is a positive Borel measure with $\nu(\operatorname{int} B) < +\infty$. In fact, given an increasing sequence (K_m) of compact subsets of $\operatorname{int} B$ with $\operatorname{int} B = \bigcup_{m=1}^{\infty} \operatorname{int} K_m$, we can set

$$\widehat{h}(x) = \begin{cases} \dfrac{h(x)}{1 + \int_{K_1} h \, d\mathscr{L}^n} & \text{if } x \in K_1, \\[2ex] \dfrac{h(x)}{2^{m-1}(1 + \int_{K_m} h \, d\mathscr{L}^n)} & \text{if } x \in K_m \setminus K_{m-1}, \, m \geqslant 2, \end{cases}$$

$$\widehat{\nu}(M) = \frac{\nu(M \cap K_1)}{1 + \nu(K_1)} + \sum_{m=2}^{\infty} \frac{\nu(M \cap (K_m \setminus K_{m-1}))}{2^{m-1}(1 + \nu(K_m))} \quad (M \in \mathfrak{B}(\operatorname{int} B)).$$

Then $\widehat{h}$, $\widehat{\nu}$ have the required properties and $\mathcal{M}^\circ_{\widehat{h}\widehat{\nu}} = \mathcal{M}^\circ_{h\nu}$.

Remark 1.2.5. For every $\eta \in \mathfrak{M}(\operatorname{int} B \times \operatorname{int} B)$ we can define a measure $\nu \in \mathfrak{M}(\operatorname{int} B)$ such that $\eta \ll \nu \times \nu$. In fact, we can take an increasing sequence (K_m) of compact subsets of $\operatorname{int} B$ with $\operatorname{int} B = \bigcup_{m=1}^{\infty} \operatorname{int} K_m$ and set

$$\forall E \in \mathfrak{B}(\operatorname{int} B) : \nu(E) = \sum_{m=1}^{\infty} \frac{\eta((E \cap K_m) \times K_m) + \eta(K_m \times (E \cap K_m))}{2^{m-1}(1 + \eta(K_m \times K_m))}.$$

In this way, given $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ we have $\eta((\partial_* A) \times \operatorname{int} B) = \eta((\operatorname{int} B) \times \partial_* A) = 0$ for every $A \in \mathcal{M}^\circ_{h\nu}$.

Remark 1.2.6. If $(A, C) \in \mathfrak{D}^\circ$, then $A \cap \partial_* C = C \cap \partial_* A = \varnothing$. In fact, from Proposition 1.1.1 we have

$$(A \cap \partial_* C) \cup (C \cap \partial_* A) \subseteq \partial_*(A \cap C) = \varnothing.$$

Definition 1.2.7. *We say that* $\mathcal{P} \subseteq \mathcal{M}^\circ$ contains almost all of $\mathcal{M}^\circ$, *if* $\mathcal{M}^\circ_{h\nu} \subseteq \mathcal{P}$ *for some* $h \in \mathcal{L}^1_{loc,+}(\mathrm{int}\, B)$ *and* $\nu \in \mathfrak{M}(\mathrm{int}\, B)$.
A property π holds almost everywhere in $\mathcal{M}^\circ$, *if the set*

$$\{A \in \mathcal{M}^\circ : \pi(A) \text{ is defined and } \pi(A) \text{ holds}\}$$

contains almost all of $\mathcal{M}^\circ$.
We say that $\mathcal{P} \subseteq \mathfrak{D}^\circ$ contains almost all of $\mathfrak{D}^\circ$, *if* $\mathfrak{D}^\circ_{h\nu} \subseteq \mathcal{P}$ *for some* $h \in \mathcal{L}^1_{loc,+}(\mathrm{int}\, B)$ *and* $\nu \in \mathfrak{M}(\mathrm{int}\, B)$.
A property π holds almost everywhere in $\mathfrak{D}^\circ$, *if the set*

$$\{(A, C) \in \mathfrak{D}^\circ : \pi(A, C) \text{ is defined and } \pi(A, C) \text{ holds}\}$$

contains almost all of $\mathfrak{D}^\circ$.

Remark 1.2.8. Definition 1.2.7 should be compared with [28, Definition 4.1], where a similar notion appeared for the first time. There the case $\eta = 0$ was considered. The main point is that we want to consider vector fields $\boldsymbol{q}$ whose distributional divergence is a measure and to prove a Gauss-Green formula like

$$\int_{\partial_* M} \boldsymbol{q} \cdot \boldsymbol{n}^M \, d\mathscr{H}^{n-1} = \int_M \mathrm{div}\, \boldsymbol{q}$$

for almost every $M \in \mathcal{M}^\circ$. To expect such a formula, it seems to be necessary to impose the condition $|\mathrm{div}\, \boldsymbol{q}|(\partial_* M) = 0$, which is automatically satisfied when $|\mathrm{div}\, \boldsymbol{q}|$ is absolutely continuous with respect to $\mathscr{L}^n$. On the other hand, it is not so restrictive to require that $\nu(\partial_* M) = 0$: since ν is finite on compact subsets of $\mathrm{int}\, B$, there are "not so many" Borel sets S with $\mathscr{H}^{n-1}(S) < +\infty$ and $\nu(S) > 0$.

Proposition 1.2.9. *The following assertions hold:*

(*a*) *if* $h \in \mathcal{L}^1_{loc,+}(\mathrm{int}\, B)$, $\nu \in \mathfrak{M}(\mathrm{int}\, B)$ *and* $M_1, M_2 \in \mathcal{M}^\circ_{h\nu}$, *then* $(M_1 \cup M_2)_*$, $M_1 \cap M_2$, $(M_1 \setminus M_2)_* \in \mathcal{M}^\circ_{h\nu}$;

(*b*) *if* (h_m), (ν_m) *are sequences in* $\mathcal{L}^1_{loc,+}(\mathrm{int}\, B)$ *and* $\mathfrak{M}(\mathrm{int}\, B)$ *respectively, then there exist* $h \in \mathcal{L}^1_{loc,+}(\mathrm{int}\, B)$ *and* $\nu \in \mathfrak{M}(\mathrm{int}\, B)$ *such that*

$$\mathcal{M}^\circ_{h\nu} \subseteq \bigcap_{m=1}^{\infty} \mathcal{M}^\circ_{h_m \nu_m}.$$

Proof. Assertion (a) is a simple consequence of Proposition 1.1.1.
In order to prove (b), we can take an increasing sequence (K_m) of compact subsets of $\mathrm{int}\, B$ with $\mathrm{int}\, B = \bigcup_{m=1}^{\infty} \mathrm{int}\, K_m$. Setting

$$\forall x \in \mathrm{int}\, B : \quad h(x) = \sum_{m=1}^{\infty} \frac{h_m(x)}{2^m \left(1 + \int_{K_m} h_m \, d\mathscr{L}^n\right)},$$

$$\forall E \in \mathfrak{B}(\mathrm{int}\, B) : \quad \nu(E) = \sum_{m=1}^{\infty} \frac{\nu_m(E)}{2^m (1 + \nu_m(K_m))},$$

it is not difficult to see that h and ν have the required properties. □

Remark 1.2.10. In view of (b) of Proposition 1.2.9, given a countable set of properties such that each of them holds on almost all of $\mathcal{M}^\circ$, there exist $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ and $\nu \in \mathfrak{M}(\operatorname{int} B)$ such that they hold on $\mathcal{M}^\circ_{h\nu}$. The same happens for $\mathcal{N}^\circ$ and $\mathfrak{D}^\circ$.

Next result shows that the measure ν is effective only when it is "quite concentrated".

Proposition 1.2.11. *Let $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ and $\nu \in \mathfrak{M}(\operatorname{int} B)$. Assume that $\nu \ll \mathscr{H}^{n-1}$. Then there exists $k \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ such that $\mathcal{M}^\circ_{k0} \subseteq \mathcal{M}^\circ_{h\nu}$.*

Proof. Let ν_s be the singular part of ν with respect to $\mathscr{L}^n$ and let E be a Borel subset of $\operatorname{int} B$ such that $\mathscr{L}^n(E) = 0$ and $\nu_s(T) = \nu(T \cap E)$ for any $T \in \mathfrak{B}(\operatorname{int} B)$. Set

$$k(x) = \begin{cases} h(x) & \text{if } x \notin E, \\ +\infty & \text{if } x \in E. \end{cases}$$

Then $k \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$. Moreover, if $A \in \mathcal{M}^\circ_{k0}$, we have $\mathscr{H}^{n-1}(E \cap \partial_* A) = 0$. It follows that $\nu_s(\partial_* A) = 0$, hence $\nu(\partial_* A) = 0$. Therefore $A \in \mathcal{M}^\circ_{h\nu}$. □

Definition 1.2.12. *An ordered orthonormal basis $(e_1, \dots, e_n)$ in $\mathbb{R}^n$ will be called* a frame. *A frame $(e_1, \dots, e_n)$ is said to be* positively oriented, *if the determinant of the matrix with columns $e_1, \dots, e_n$ is positive.*
A grid G *is an ordered triple*

$$G = \left(x_0, (e_1, \dots, e_n), \widehat{G}\right),$$

where $x_0 \in \mathbb{R}^n$, $(e_1, \dots, e_n)$ is a positively oriented frame in $\mathbb{R}^n$ and $\widehat{G}$ is a Borel subset of $\mathbb{R}$. If G_1, G_2 are two grids, we write $G_1 \subseteq G_2$ if the first two components agree and $\widehat{G}_1 \subseteq \widehat{G}_2$. A grid G is said to be full, *if $\mathscr{L}^1(\mathbb{R} \setminus \widehat{G}) = 0$.*

Definition 1.2.13. *Let $G = \left(x_0, (e_1, \dots, e_n), \widehat{G}\right)$ be a grid; a subset I of $\mathbb{R}^n$ is said to be* an open n-dimensional G-interval, *if*

$$I = \left\{x \in \mathbb{R}^n : \quad a^{(j)} < (x - x_0) \cdot e_j < b^{(j)} \quad \forall j = 1, \dots, n\right\}$$

for some $a^{(1)}, b^{(1)}, \dots, a^{(n)}, b^{(n)} \in \widehat{G}$.
A subset P of $\mathbb{R}^n$ is said to be a G-figure, *if $P = \left(\bigcup_{I \in \mathcal{F}} I\right)_*$, where $\mathcal{F}$ is a finite family of open n-dimensional G-intervals. We set*

$$\mathcal{I}^\circ_G = \{I : I \textit{ is an open } n\textit{-dimensional } G\textit{-interval with } \operatorname{cl} I \subseteq \operatorname{int} B\},$$

$$\mathcal{M}^\circ_G = \{P : P \textit{ is a } G\textit{-figure with } \operatorname{cl} B \subseteq \operatorname{int} B\},$$

$$\mathfrak{D}^\circ_G = \{(A, C) \in \mathfrak{D} : A, C \in \mathcal{I}^\circ_G\} \cup \{(A, C \cup (\mathbb{R}^n \setminus B)_*) : A, C \in \mathcal{I}^\circ_G\}.$$

Remark 1.2.14. For any grid G, we have $\mathcal{M}^\circ_G \subseteq \mathcal{M}^\circ$ and $\mathfrak{D}^\circ_G \subseteq \mathfrak{D}^\circ$.

Definition 1.2.15. *Let $x_0 \in \mathbb{R}^n$, $(e_1, \dots, e_n)$ be a positively oriented frame in $\mathbb{R}^n$, $E \subseteq \mathbb{R}^n$ and $1 \leqslant j \leqslant n$. For every $s \in \mathbb{R}$ we set*

$$\sigma_{j,s}(E) = \{x \in E : (x - x_0) \cdot e_j = s\}.$$

Proposition 1.2.16. *Let $x_0 \in \mathbb{R}^n$ and $(e_1, \dots, e_n)$ be a positively oriented frame in $\mathbb{R}^n$. Then for every $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ and $\nu \in \mathfrak{M}(\operatorname{int} B)$ there exists a full grid G of the form $G = \left(x_0, (e_1, \dots, e_n), \widehat{G}\right)$ such that $\mathcal{M}^\circ_G \subseteq \mathcal{M}^\circ_{h\nu}$.*

Proof. Let us denote with (K_m) an increasing sequence of compact subsets in $\operatorname{int} B$ such that $\operatorname{int} B = \bigcup\limits_{m=1}^{\infty} \operatorname{int} K_m$. We have

$$\forall m \in \mathbb{N} : \int_{K_m} h \, d\mathscr{L}^n < +\infty.$$

Therefore, setting

$$D_1 = \mathbb{R} \setminus \left(\bigcup_{m \in \mathbb{N}} \bigcup_{j=1}^{n} \left\{ s \in \mathbb{R} : \int_{\sigma_{j,s}(K_m)} h \, d\mathscr{H}^{n-1} = +\infty \right\} \right),$$

by Fubini's Theorem it turns out that D_1 is a Borel set with $\mathscr{L}^1(\mathbb{R} \setminus D_1) = 0$.
On the other hand, we have $\nu(K_m) < +\infty$ for every $m \in \mathbb{N}$, hence $\nu(\sigma_{j,s}(K_m)) \neq 0$ only for s in a countable subset of $\mathbb{R}$. Setting

$$D_2 = \mathbb{R} \setminus \left(\bigcup_{j=1}^{n} \left\{ s \in \mathbb{R} : \nu\Big(\sigma_{j,s}(\operatorname{int} B)\Big) > 0 \right\} \right),$$

we have that D_2 is a Borel set with $\mathscr{L}^1(\mathbb{R} \setminus D_2) = 0$. Now it is easy to see that $\widehat{G} = D_1 \cap D_2$ defines a grid with the required properties. □

Definition 1.2.17. *Let $\mathcal{A} \subseteq \mathcal{N}$. We say that a function $F : \mathcal{A} \to \mathbb{R}$ is* additive *if for every $A_1, A_2 \in \mathcal{A}$ such that $(A_1 \cup A_2)_* \in \mathcal{A}$ and $A_1 \cap A_2 = \varnothing$, we have*

$$F\left((A_1 \cup A_2)_*\right) = F(A_1) + F(A_2).$$

Let $\mathcal{P} \subseteq \mathfrak{D}$. We say that a function $F : \mathcal{P} \to \mathbb{R}$ is biadditive *if the functions*

$$F(\,\cdot\,, C) : \left\{A' \in \mathcal{M} : (A', C) \in \mathcal{P}\right\} \to \mathbb{R},$$

$$F(A, \,\cdot\,) : \left\{C' \in \mathcal{N} : (A, C') \in \mathcal{P}\right\} \to \mathbb{R},$$

are additive for every $(A, C) \in \mathcal{P}$.

Now we are ready to introduce the main character of the chapter.

Definition 1.2.18. *Let $\mathcal{P} \subseteq \mathfrak{D}^\circ$ be a set containing almost all of $\mathfrak{D}^\circ$ and let $I : \mathcal{P} \to \mathbb{R}$. We say that I is* a Cauchy interaction, *if the following properties hold:*

(a) *I is biadditive;*

(b) *there exist* $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$, $\eta \in \mathfrak{M}(\operatorname{int} B \times \operatorname{int} B)$ *and* $\eta_e \in \mathfrak{M}(\operatorname{int} B)$ *such that the inequality*

$$|I(A,C)| \leqslant \begin{cases} \displaystyle\int_{\partial_* A \cap \partial_* C} h \, d\mathscr{H}^{n-1} + \eta(A \times C) & \text{if } C \subseteq B, \\ \displaystyle\int_{\partial_* A \cap \partial_* C} h \, d\mathscr{H}^{n-1} + \eta(A \times (C \cap B)) + \eta_e(A) & \text{otherwise,} \end{cases} \tag{1.1}$$

holds almost everywhere in $\mathfrak{D}^\circ$.

Remark 1.2.19. The dichotomy in the previous definition arise from the thermodynamical intuition that the exterior of the body is considered regardless to its structure, but it can interact with the body, like e.g. a heat reservoir. Of course, one can forget the exterior setting $\eta_e = 0$.

Definition 1.2.20. *A Cauchy interaction I is said to be:*

(a) a body interaction*, if in the previous definition we can choose* $h = 0$*;*

(b) a contact interaction*, if in the previous definition we can choose* $\eta = 0$ *and* $\eta_e = 0$.

1.3 Decomposition of Cauchy interactions

In this section we will show that Cauchy interactions can be decomposed in an essentially unique way into a sum of a body and a contact interaction, in the sense specified below.

Lemma 1.3.1. *Let G be a full grid, K be a compact subset of* $\operatorname{int} B$ *and* $\eta \in \mathfrak{M}(\operatorname{int} B)$. *Then for every* $(A,C) \in \mathfrak{D}^\circ$ *with* $C \subseteq B$ *there exist two sequences* (A_k), (C_k) *in* $\mathcal{M}^\circ_G$ *with* $\operatorname{cl} A_k \cap \operatorname{cl} C_k = \varnothing$ *for every* $k \in \mathbb{N}$ *and*

$$\lim_k \eta((A_k \triangle A) \times K) = 0, \quad \lim_k \eta(K \times (C_k \triangle C)) = 0, \quad \lim_k \eta_e(A_k \triangle A) = 0,$$

where $\triangle$ *denotes the symmetric difference of sets.*

Proof. Given $(A,C) \in \mathfrak{D}^\circ$ such that $C \subseteq B$ and $k \geqslant 1$, let K_1, K_2 be compact subsets of $\operatorname{int} B$ with $K_1 \subseteq A$, $K_2 \subseteq C$ and

$$\eta((A \setminus K_1) \times K) < \frac{1}{k}, \qquad \eta(K \times (C \setminus K_2)) < \frac{1}{k}, \qquad \eta_e(A \setminus K_1) < \frac{1}{k}.$$

Now let $A_k, C_k \in \mathcal{M}^\circ_G$ with $K_1 \subseteq A_k$, $K_2 \subseteq C_k$, $\operatorname{cl} A_k \cap \operatorname{cl} C_k = \varnothing$ and

$$\eta((A_k \setminus K_1) \times K) < \frac{1}{k}, \qquad \eta(K \times (C_k \setminus K_2)) < \frac{1}{k}, \qquad \eta_e(A_k \setminus K_1) < \frac{1}{k}.$$

Therefore we have that $A \setminus A_k \subseteq A \setminus K_1$ and $A_k \setminus A \subseteq A_k \setminus K_1$, hence

$$\eta((A_k \triangle A) \times K) < \frac{2}{k}, \qquad \eta_e(A_k \triangle A) < \frac{2}{k}.$$

The same happens for $\eta(K \times (C_k \triangle C))$. ∎

Now we can prove the decomposition of a Cauchy interaction.

Theorem 1.3.2. *Let I be a Cauchy interaction. Then there exist a body interaction I_b and a contact interaction I_c such that $I = I_b + I_c$ on almost all of $\mathfrak{D}^\circ$.*
Moreover, if there exist a body interaction $\widehat{I}_b$ and a contact interaction $\widehat{I}_c$ with $I = \widehat{I}_b + \widehat{I}_c$ on almost all of $\mathfrak{D}^\circ$, then

$$I_b = \widehat{I}_b, \qquad I_c = \widehat{I}_c$$

on almost all of $\mathfrak{D}^\circ$.
Finally, if I_1, I_2 are two Cauchy interactions that agree, for some full grid G, on $\mathfrak{D}^\circ_G$, then $(I_1)_b = (I_2)_b$ on almost all of $\mathfrak{D}^\circ$.

Proof. Let $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$, $\eta \in \mathfrak{M}(\operatorname{int} B \times \operatorname{int} B)$ and $\eta_e, \nu \in \mathfrak{M}(\operatorname{int} B)$ be such that $\eta \ll \nu \times \nu$, $\eta_e \ll \nu$, the domain of I contains $\mathfrak{D}^\circ_{h\nu}$ and (1.1) holds for every $(A, C) \in \mathfrak{D}^\circ_{h\nu}$, as specified in Remark 1.2.10. Let H be a full grid as in Proposition 1.2.16. For $(A, C) \in \mathfrak{D}^\circ_{h\nu}$ with $C \subseteq B$, there are two compact subsets K_A, K_C of $\operatorname{int} B$ such that $\operatorname{cl} A \subseteq \operatorname{int} K_A$ and $\operatorname{cl} C \subseteq \operatorname{int} K_C$. By Lemma 1.3.1, consider two sequences (A_k), (C_k) in $\mathcal{M}^\circ_H$ such that $\operatorname{cl} A_k \cap \operatorname{cl} C_k = \varnothing$ and

$$\lim_k \eta((A_k \triangle A) \times K_C) = 0, \quad \lim_k \eta(K_A \times (C_k \triangle C)) = 0, \quad \lim_k \eta_e(A_k \triangle A) = 0;$$

without loss of generality, we can require that $A_k \subseteq K_A$ and $C_k \subseteq K_C$. It follows from the biadditivity of I and the properties of normalized subsets that

$$\begin{aligned}
|I(A_k, C_k) - I(A_i, C_i)| = {} & |I((A_k \setminus A_i)_*, C_k) + I(A_k \cap A_i, (C_k \setminus C_i)_*) - \\
& - I(A_i, (C_i \setminus C_k)_*) - I((A_i \setminus A_k)_*, C_i \cap C_k)| \leqslant \\
& \leqslant \eta((A_k \triangle A_i) \times K_C) + \eta(K_A \times (C_k \triangle C_i)) \leqslant \\
& \leqslant \eta((A_k \triangle A) \times K_C) + \eta(K_A \times (C_k \triangle C)) + \\
& + \eta((A_i \triangle A) \times K_C) + \eta(K_A \times (C_i \triangle C)),
\end{aligned}$$

therefore $(I(A_k, C_k))$ is a Cauchy sequence in $\mathbb{R}$. Moreover,

$$\begin{aligned}
|I(A, C) - I(A_k, C_k)| \leqslant {} & |I((A \setminus A_k)_*, C) + I(A \cap A_k, (C \setminus C_k)_*) - \\
& - I((A_k \setminus A)_*, C_k) - I(A_k \cap A, (C_k \setminus C)_*)| \leqslant \\
& \leqslant \int_{\partial_*(A \setminus A_k) \cap \partial_* C} h \, d\mathscr{H}^{n-1} + \int_{\partial_*(A \cap A_k) \cap \partial_*(C \setminus C_k)} h \, d\mathscr{H}^{n-1} + \\
& + \eta((A_k \triangle A) \times K_C) + \eta(K_A \times (C_k \triangle C)) \leqslant \\
& \leqslant 2 \int_{\partial_* A \cap \partial_* C} h \, d\mathscr{H}^{n-1} + \eta((A_k \triangle A) \times K_C) + \eta(K_A \times (C_k \triangle C))
\end{aligned} \tag{1.2}$$

where the last inequality follows from Remark 1.2.6. For every $(A, C) \in \mathfrak{D}^\circ_{h\nu}$ we define

$$I_b(A, C) = \begin{cases} \lim\limits_k I(A_k, C_k) & \text{if } C \subseteq B, \\ \lim\limits_k \left[I(A_k, (C \cap B)_k) + I(A, (\mathbb{R}^n \setminus B)_*) \right] & \text{otherwise.} \end{cases}$$

It is easy to see that I_b does not depend on the chosen sequences. Moreover we have

$$|I_b(A, C)| \leqslant \begin{cases} \eta(A \times C) & \text{if } C \subseteq B, \\ \eta(A \times (C \cap B)) + \eta_e(A) & \text{otherwise,} \end{cases}$$

since $\partial_* A_k \cap \partial_* C_k = \varnothing$ for every $k \in \mathbb{N}$.
We now show the biadditivity of I_b. Let A, A', C be three mutually disjoint subsets of B such that $(A, C), (A', C) \in \mathfrak{D}^\circ_{h\nu}$ and let $(A_k), (A'_k), (C_k)$ three sequences in $\mathcal{M}^\circ_H$ as in Lemma 1.3.1. We can require that $\mathrm{cl}\, A_k \cap \mathrm{cl}\, A'_k = \varnothing$. Since

$$(A \cup A') \triangle (A_k \cup A'_k) \subseteq (A \triangle A_k) \cup (A' \triangle A'_k),$$
$$A \cup A' \subseteq (A \cup A')_* \subseteq A \cup A' \cup (\partial_* A \cap \partial_* A'),$$

it follows that

$$\lim_k \eta\big(((A \cup A')_* \triangle (A_k \cup A'_k)) \times K\big) = 0$$

for every compact subset $K \subseteq \mathrm{int}\, B$. Hence

$$\begin{aligned} I_b((A \cup A')_*, C) &= \lim_k I((A_k \cup A'_k), C_k) = \\ &= \lim_k (I(A_k, C_k) + I(A'_k, C_k)) = I_b(A, C) + I_b(A', C). \end{aligned}$$

The case $C \not\subseteq B$ is similar. In the same way, we can prove the additivity on the second component, therefore I_b is a body interaction. Setting

$$\forall (A, C) \in \mathfrak{D}^\circ_{h\nu} : I_c(A, C) = I(A, C) - I_b(A, C),$$

it follows that I_c is a biadditive function on $\mathfrak{D}^\circ_{h\nu}$; by (1.2) it is a contact interaction.
Now take $h \in \mathcal{L}^1_{loc,+}(\mathrm{int}\, B)$ and $\nu \in \mathfrak{M}(\mathrm{int}\, B)$ such that

$$I = I_b + I_c = \widehat{I}_b + \widehat{I}_c \qquad \text{on } \mathfrak{D}^\circ_{h\nu}.$$

Given $(A, C) \in \mathfrak{D}^\circ_{h\nu}$, let (A_k), (C_k) be two sequences in $\mathcal{M}^\circ_H$ as in Lemma 1.3.1; we have then

$$I_c(A_k, C_k) = \widehat{I}_c(A_k, C_k) = 0.$$

Passing to the limit as $k \to \infty$, it follows

$$\forall (A, C) \in \mathfrak{D}^\circ_{h\nu} : \quad I_b(A, C) = \widehat{I}_b(A, C),$$

and then also $I_c = \widehat{I}_c$ on $\mathfrak{D}^\circ_{h\nu}$.
Finally, if two Cauchy interactions I_1, I_2 agree on $\mathfrak{D}^\circ_G$ for some full grid G, we can choose $h \in \mathcal{L}^1_{loc,+}(\mathrm{int}\, B)$, $\nu \in \mathfrak{M}(\mathrm{int}\, B)$ and the full grid H in the preceding construction such that I_1, I_2 are defined on $\mathfrak{D}^\circ_{h\nu}$ and $H \subseteq G$. Then $(I_1)_b = (I_2)_b$ on $\mathfrak{D}^\circ_{h\nu}$. □

1.4 Body interactions

In this section we will denote by D the set $\{(x, x) : x \in \mathrm{int}\, B\}$.
The following lemma can be checked by a combinatorial technique.

Lemma 1.4.1. *Let $x_0 \in \mathbb{R}^n$ and $(e_1, \ldots, e_n)$ be a positively oriented frame in $\mathbb{R}^n$. Let*

$$J_1 = \Big\{ x \in \mathbb{R}^n : \quad a^{(j)} < (x - x_0) \cdot e_j < b^{(j)} \quad \forall j = 1, \ldots, n \Big\},$$

$$J_2 = \Big\{ x \in \mathbb{R}^n : \quad c^{(j)} < (x - x_0) \cdot e_j < d^{(j)} \quad \forall j = 1, \ldots, n \Big\}$$

be two open n-dimensional G-intervals such that $J_1 \cap J_2 = \varnothing$ and $(J_1 \cup J_2)_*$ is an open n-dimensional G-interval. Then there exists $i \in \{1, \ldots, n\}$ such that:

(*i*) *either* $b^{(i)} = c^{(i)}$ *or* $a^{(i)} = d^{(i)}$;

(*ii*) $a^{(j)} = c^{(j)}$ *and* $b^{(j)} = d^{(j)}$ *for every* $j \neq i$.

Theorem 1.4.2. *Let* $\mu_1 \in \mathfrak{M}(\operatorname{int} B \times \operatorname{int} B)$, $\mu_2 \in \mathfrak{M}(\operatorname{int} B)$, $f \in L^1_{loc}(\operatorname{int} B \times \operatorname{int} B, \mu_1)$ *and* $g \in L^1_{loc}(\operatorname{int} B, \mu_2)$. *Then* f *is* μ_1*-summable on* $A \times (C \cap B)$ *and* g *is* μ_2*-summable on* A *for every* $(A, C) \in \mathfrak{D}^\circ$; *moreover, the formula*

$$I(A,C) = \begin{cases} \displaystyle\int_{A \times C} f\, d\mu_1 & \text{if } C \subseteq B, \\ \displaystyle\int_{A \times (C \cap B)} f\, d\mu_1 + \int_A g\, d\mu_2 & \text{otherwise,} \end{cases}$$

defines a body interaction.

Proof. The summability of f and g is clear. Now let $h = 0$ and $\nu \in \mathfrak{M}(\operatorname{int} B)$ be such that $\mu_1 \ll \nu \times \nu$ and $\mu_2 \ll \nu$, which is possible by Remark 1.2.10. Then I is biadditive on $\mathfrak{D}^\circ_{h\nu}$. Moreover, setting $\eta = |f|\, d\mu_1$ and $\eta_e = |g|\, d\mu_2$, inequality (1.1) is satisfied, hence I is a body interaction. □

The main result of this section is the converse of Theorem 1.4.2.

Theorem 1.4.3. *Let* I *be a body interaction and* $\eta \in \mathfrak{M}(\operatorname{int} B \times \operatorname{int} B)$, $\eta_e \in \mathfrak{M}(\operatorname{int} B)$ *be as in Definition 1.2.18. Then there exist* $\mu \in \mathfrak{M}(\operatorname{int} B \times \operatorname{int} B)$, $\mu_e \in \mathfrak{M}(\operatorname{int} B)$ *and two Borel functions* $b : \operatorname{int} B \times \operatorname{int} B \to \mathbb{R}$, $b_e : \operatorname{int} B \to \mathbb{R}$ *such that*

$$\mu(D) = 0,$$

$$|b(x,y)| = 1 \quad \text{for } \mu\text{-a.e. } (x,y) \in \operatorname{int} B \times \operatorname{int} B,$$

$$|b_e(x)| = 1 \qquad \text{for } \mu_e\text{-a.e. } x \in \operatorname{int} B,$$

$$I(A,C) = \begin{cases} \displaystyle\int_{A \times C} b\, d\mu & \text{if } C \subseteq B, \\ \displaystyle\int_{A \times (C \cap B)} b\, d\mu + \int_A b_e\, d\mu_e & \text{otherwise,} \end{cases} \qquad \text{on almost all of } \mathfrak{D}^\circ.$$

Moreover, we have $\mu \leqslant \eta$ *and* $\mu_e \leqslant \eta_e$.

Proof. Let $\nu \in \mathfrak{M}(\operatorname{int} B)$ such that $\eta \ll \nu \times \nu$ and the domain of I contains $\mathfrak{D}^\circ_{h\nu}$. Let $G = \big(x_0, (e_1, \dots, e_n), \widehat{G}\big)$ be a full grid such that $\mathcal{M}^\circ_G \subseteq \mathcal{M}^\circ_{h\nu}$ and consider the open set $\varOmega = (\operatorname{int} B \times \operatorname{int} B) \setminus D$, the full grid

$$\tilde{G} = \big((x_0, x_0), \big((e_1, 0), \dots, (e_n, 0), (0, e_1), \dots, (0, e_n)\big), \widehat{G}\big)$$

and the set

$$\mathcal{J}_G = \{J \subseteq \mathbb{R}^{2n} : J \text{ is an open } 2n\text{-dimensional } \tilde{G}\text{-interval with } \operatorname{cl} J \subseteq \varOmega\}.$$

Since $\varOmega$ does not contain the pairs (x, x), it is clear that every $J \in \mathcal{J}_G$ is of the form $J = J_1 \times J_2$ with $J_1, J_2 \in \mathcal{I}^\circ_G$, $J_1 \cap J_2 = \varnothing$. By means of this decomposition, we define a function $R : \mathcal{J}_G \to \mathbb{R}$ setting

$$R(J) = I(J_1, J_2).$$

Let $J, J' \in \mathcal{J}_G$ be such that $(J \cup J')_* \in \mathcal{J}_G$; if $J_1, J_2, J_1', J_2' \in \mathcal{I}^\circ_G$ are such that $J = J_1 \times J_2$, $J' = J_1' \times J_2'$, then by Lemma 1.4.1 we have the following alternative:

(i) either $J_1 \cap J_1' = \varnothing$ and $J_2 = J_2'$,

(ii) or $J_2 \cap J_2' = \varnothing$ and $J_1 = J_1'$.

Suppose for instance that (i) holds; it follows

$$R((J \cup J')_*) = I((J_1 \cup J_1')_*, J_2) = I(J_1, J_2) + I(J_1', J_2) = R(J) + R(J').$$

The same happens in the case (ii), hence R is additive. Moreover, $|R(J)| \leqslant \eta(J_1 \times J_2)$ for every $J = J_1 \times J_2$ in $\mathcal{J}_G$, so R is countably additive. By well-known theorems about extensions of additive functions (see e.g. [25, Chap. 12, Sect. 2]), there exists a unique signed measure $\widehat{\mu}$ on $\mathfrak{B}(\varOmega)$ such that

$$\forall J \in \mathcal{J}_G : \widehat{\mu}(J) = R(J),$$

$$\forall E \in \mathfrak{B}(\varOmega) : |\widehat{\mu}|(E) \leqslant \eta(E).$$

We define a measure $\mu \in \mathfrak{M}(\operatorname{int} B \times \operatorname{int} B)$ setting $\mu(E) = |\widehat{\mu}|(E \cap \varOmega)$ and a real function $b : \operatorname{int} B \times \operatorname{int} B \to \mathbb{R}$ as $\dfrac{d\widehat{\mu}}{d\mu}$. Clearly, $|b(x,y)| = 1$ μ-a.e. in $\operatorname{int} B \times \operatorname{int} B$ and

$$I(A, C) = \int_{A \times C} b \, d\mu$$

for every $(A, C) \in \mathfrak{D}^{\circ}_{h\nu}$ with $C \subseteq B$. Modifying the value of b on a μ-negligible set, we can suppose that b is a Borel function on $\operatorname{int} B \times \operatorname{int} B$ as in the assertion.
Now we define an additive function $R_e : \mathcal{I}^{\circ}_G \to \mathbb{R}$ setting $R_e(J) = I(J, (\mathbb{R}^n \setminus B)_*)$. It holds $|R_e(J)| \leqslant \eta_e(J)$, then there exists a signed measure $\widehat{\mu}_e$ on $\mathfrak{B}(\operatorname{int} B)$ such that

$$\forall J \in \mathcal{I}^{\circ}_G : \widehat{\mu}_e(J) = R_e(J),$$

$$\forall A \in \mathfrak{B}(\operatorname{int} B) : |\widehat{\mu}_e|(A) \leqslant \eta_e(A).$$

Setting $\mu_e = |\widehat{\mu}_e|$, we define $b_e = \dfrac{d\widehat{\mu}_e}{d\mu_e}$; as we can suppose that b_e is a Borel function, the proof is complete. □

Theorem 1.4.4. *Let I_1, I_2 be two body interactions and for $j = 1, 2$ let $\mu^{(j)}$, $\mu_e^{(j)}$, $b^{(j)}$, $b_e^{(j)}$ be as in the statement of Theorem 1.4.3. Then $I_1 = I_2$ on almost all of $\mathfrak{D}^{\circ}$ if and only if $\mu^{(1)} = \mu^{(2)}$, $\mu_e^{(1)} = \mu_e^{(2)}$, $b^{(1)}(x) = b^{(2)}(x)$ $\mu^{(1)}$-a.e. in $\operatorname{int} B \times \operatorname{int} B$ and $b_e^{(1)}(x) = b_e^{(2)}(x)$ $\mu_e^{(1)}$-a.e. in $\operatorname{int} B$.*

Proof. Let $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ and $\nu \in \mathfrak{M}(\operatorname{int} B)$ be such that the equality $I_1 = I_2$ holds in $\mathfrak{D}^{\circ}_{h\nu}$. Let G be a full grid such that $\mathcal{M}^{\circ}_G \subseteq \mathcal{M}^{\circ}_{h\nu}$. Then, denoting by $R^{(1)}, R^{(2)}$ the functions on $\mathcal{J}_G$ in the proof of Theorem 1.4.3, we have that $R^{(1)} = R^{(2)}$, hence $\mu^{(1)} = \mu^{(2)}$. In the same way, it follows that $\mu_e^{(1)} = \mu_e^{(2)}$. The remainder of the proof is now easy. □

Chapter 2

Contact interactions and Cauchy fluxes

Since Cauchy's proof of his celebrated Stress Theorem, many attempts have been made in order to generalize his ideas and remove certain additional hypotheses which did not seem to be natural. For instance, Cauchy assumed the exerted traction at a given point on a generic material surface to depend *a priori* only on the point and the normal at the surface at that point (*the Cauchy Postulate*); moreover, he supposed the traction field to depend continuously upon the point itself.

In [22] it was proved that, under suitable conditions, the Cauchy Postulate could be deduced from the balance of linear momentum. Moreover, in [16] it was shown that for a given direction, the linear dependence of the traction upon that direction could be derived for almost all points from the same balance law, thus avoiding the continuity condition. In [15] the notion of Cauchy flux was introduced, which has changed the basic concept in this kind of analysis. The main idea was to replace the exerted traction by the resultant (called *Cauchy flux*) on the material surface, thus specifying properties on the resultant and possibly avoiding those on the traction field. Some years later, it became clear that this approach could be developed in the setting of geometric measure theory [34, 26, 28] and that the whole question refers more to the abstract structure of a balance law than the specific case of the stress. We refer the reader to [31, Chapter III] for an exposition of the basic results in this direction. We also mention [10] as a proof of Cauchy's Stress Theorem based on a variational technique, rather than a measure-theoretic one.

Parallel to these studies, but intimately related, are the investigations on the concepts of subbody and material surface. Apart from their general use in the axiomatic foundation of continuum mechanics [17, 23, 3, 19, 24, 31], the collection $\mathcal{S}$ of all material surfaces of a given continuous body B appears as the natural domain of the Cauchy flux Q and the balance law is classically formulated for any subbody M of B.

Thus, in the case of a scalar flux, a first question is to characterize the functions (Cauchy fluxes) $Q : \mathcal{S} \to \mathbb{R}$ of the form

$$Q(S) = \int_S \boldsymbol{q} \cdot \boldsymbol{n}_S \, d\mathscr{H}^{n-1} \qquad \text{for any material surface } S \text{ of } B$$

under the condition that $\boldsymbol{q}$ (the flux vector) belongs to some functional class. A related problem is to show that the integral form of the balance law

$$\int_{\partial M} \boldsymbol{q} \cdot \boldsymbol{n}^M \, d\mathscr{H}^{n-1} = \int_M b \, d\mathscr{L}^n \qquad \text{for any subbody } M \text{ of } B \tag{2.1}$$

is equivalent to the distributional equation $\operatorname{div} \boldsymbol{q} = b$ (for a detailed study of this point, involving also boundary conditions, see [1]).

The approach has to be generalized, when unbounded $\boldsymbol{q}$'s are considered. In [28] the case where $\boldsymbol{q}$ and $\operatorname{div} \boldsymbol{q}$ are in L^p is treated. In this situation, Q is naturally defined only for *almost all* material surfaces S and also the balance law (2.1) can be formulated only for *almost all* subbodies M.

The main purpose of this chapter is to characterize the Cauchy fluxes Q's associated with flux vectors $\boldsymbol{q}$'s in L^1_{loc} with divergence measure (see Corollary 2.3.5 and Theorem 2.5.1). We still call such fluxes *balanced Cauchy fluxes*. This seems to be the highest level of generality in which the integral form of the balance law can be written, provided that the term $b \, d\mathscr{L}^n$ is substituted by a (signed) measure. The equivalence between the integral form of the balance law, formulated for almost all subbodies, and the distributional form (see Theorem 2.5.2) are proved. Moreover, if Q is vector valued and $\boldsymbol{q}$ satisfies a suitable estimate involving momenta, the tensor $\boldsymbol{q}(x)$ is symmetric for a.e. x (see Theorem 2.6.3).

Our second purpose is to show that any suitable set function Q_0, which is only defined for almost all $(n-1)$-dimensional intervals parallel to coordinate subspaces, can be uniquely extended to a balanced Cauchy flux Q defined for almost all material surfaces (see Theorem 2.4.1). Moreover, if the balance law (2.1) is true for almost all n-dimensional intervals, then the distributional form follows (see Theorem 2.5.2). Therefore, at least for the problems we treat here, the choice of the family of subbodies seems not to be so crucial: the behavior of balance laws and Cauchy fluxes on very general objects (such as subsets of finite perimeter and Borel subsets of their boundaries) is determined by that on n-intervals and their faces. Also, notions like, e.g., "almost all n-intervals" are more transparent than, e.g., "almost all subbodies". This alternative approach seems to be more in the spirit of [22, 15] (see e.g. [22, Theorem 4] and [15, Theorem 8]). On the other hand, situations of this kind are typical in classical measure theory, where each set function, defined on n-intervals and satisfying suitable conditions, can be uniquely extended to a measure defined on all Borel subsets.

As there is a strict correspondence between Cauchy fluxes and Cauchy interactions (see Theorem 2.1.8 and Proposition 2.7.2), the results about the integral representation and extension of Cauchy fluxes can be easily extended to contact interactions (Sections 2.7 and 2.8).

2.1 Contact interactions

An oriented surface S in $\mathbb{R}^n$ is a pair $(\widehat{S}, \boldsymbol{n}_S)$, where $\widehat{S}$ is a Borel subset of $\mathbb{R}^n$ and $\boldsymbol{n}_S : \widehat{S} \to \mathbb{R}^n$ is a Borel map such that there exists a normalized set $M \subseteq \mathbb{R}^n$ of finite perimeter with $\widehat{S} \subseteq \partial_* M$ and $\boldsymbol{n}_S = \boldsymbol{n}^M\big|_{\widehat{S}}$. In this case, we say that S is *subordinated* to M. We call $\boldsymbol{n}_S$ *the normal* to the surface S. If S, T are two oriented surfaces, we shall write $S \subseteq T$ if $\widehat{S} \subseteq \widehat{T}$ and $\boldsymbol{n}_T\big|_{\widehat{S}} = \boldsymbol{n}_S$. Two oriented surfaces S and T are said to be *disjoint*, if $\widehat{S} \cap \widehat{T} = \varnothing$. They are said to be *compatible*, if there exists a normalized set $M \subseteq \mathbb{R}^n$ of finite perimeter such that S and T are subordinated to M. If S and T are two compatible oriented surfaces, we denote by $S \cup T$ the oriented surface $(\widehat{S} \cup \widehat{T}, \boldsymbol{n}_{S\cup T})$ such that

$$\boldsymbol{n}_{S\cup T}(x) = \begin{cases} \boldsymbol{n}_S(x) & \text{if } x \in \widehat{S}, \\ \boldsymbol{n}_T(x) & \text{if } x \in \widehat{T}. \end{cases}$$

In the following, we shall sometimes identify $\widehat{S}$ with S and we shall consider expressions like, e.g., "S is compact", "$\mathscr{H}^{n-1}(S)$" instead of "$\widehat{S}$ is compact", "$\mathscr{H}^{n-1}(\widehat{S})$". In the same spirit, if S is an oriented surface and T is a Borel subset of $\widehat{S}$, we shall denote by T also the oriented surface $\left(T, \boldsymbol{n}_S\big|_T\right)$, provided that the reference to S is clear.

Definition 2.1.1. *Let S be an oriented surface. We say that S is* a material surface in the body B, *if S is subordinated to some $A \in \mathcal{M}$.*
We denote by $\mathcal{S}$ the collection of the material surfaces in the body B.

Definition 2.1.2. *For every $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ and $\nu \in \mathfrak{M}(\operatorname{int} B)$ we set*

$$\mathcal{S}_{h\nu} = \{S \in \mathcal{S} : \ S \text{ is subordinated to some } A \in \mathcal{M}^\circ_{h\nu}\}.$$

Given a set $\mathcal{P} \subseteq \mathcal{S}$, we say that $\mathcal{P}$ contains almost all of $\mathcal{S}$, *if there exist $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ and $\nu \in \mathfrak{M}(\operatorname{int} B)$ such that $\mathcal{S}_{h\nu} \subseteq \mathcal{P}$; given a property π, we say that π* holds almost everywhere in $\mathcal{S}$, *if the set*

$$\{S \in \mathcal{S} : \ \pi(S) \text{ is defined and } \pi(S) \text{ holds}\}$$

contains almost all of $\mathcal{S}$.

Definition 2.1.3. *For a grid $G = \left(x_0, (e_1 \dots, e_n), \widehat{G}\right)$ and $1 \leqslant j \leqslant n$, we denote by $\mathcal{S}^j_G$ the family of all the oriented surfaces S with $\boldsymbol{n}_S = e_j$,*

$$\widehat{S} = \left\{x \in \mathbb{R}^n : \quad (x - x_0) \cdot e_j = s, \quad a^{(i)} < (x - x_0) \cdot e_i < b^{(i)} \quad \forall i \neq j\right\},$$

$a^{(1)}, b^{(1)}, \dots, s, \dots, a^{(n)}, b^{(n)} \in \widehat{G}$ and $\operatorname{cl} \widehat{S} \subseteq \operatorname{int} B$. We set also

$$\mathcal{S}_G = \bigcup_{j=1}^n \mathcal{S}^j_G.$$

Remark 2.1.4. Given a positively oriented frame $(e_1, \dots, e_n)$, $x_0 \in \mathbb{R}^n$, $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ and $\nu \in \mathfrak{M}(\operatorname{int} B \times \operatorname{int} B)$, there exists a full grid $G = \left(x_0, (e_1 \dots, e_n), \widehat{G}\right)$ such that $\mathcal{S}_G \subseteq \mathcal{S}_{h\nu}$ (see Proposition 1.2.16).

Definition 2.1.5. *Let $\mathcal{P} \subseteq \mathcal{S}$ be a set containing almost all of $\mathcal{S}$ and let $Q : \mathcal{P} \to \mathbb{R}$. We say that Q is* a (scalar) Cauchy flux, *if the following properties hold:*

(a) if $S, T \in \mathcal{P}$ are compatible and disjoint with $S \cup T \in \mathcal{P}$, then

$$Q(S \cup T) = Q(S) + Q(T);$$

(*b*) *there exists* $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ *such that the inequality*

$$|Q(S)| \leqslant \int_S h \, d\mathscr{H}^{n-1}$$

holds almost everywhere in $\mathcal{S}$.

Lemma 2.1.6. *Let* $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$, $\nu \in \mathfrak{M}(\operatorname{int} B)$, $A \in \mathcal{M}^\circ_{h\nu}$, S *be a material surface subordinated to* A. *Then there exists a sequence* (C_k) *in* $\mathcal{M}^\circ_{h\nu}$ *such that* $A \cap C_k = \varnothing$ *and*

$$\lim_k \mathscr{H}^{n-1}\left((\partial_* A \cap \partial_* C_k) \triangle \widehat{S}\right) = 0.$$

Proof. Let G be a full grid such that $\mathcal{M}^\circ_G \subseteq \mathcal{M}^\circ_{h\nu}$. Since $\mathscr{H}^{n-1}(\widehat{S}) < +\infty$, it follows that for any fixed $k \in \mathbb{N}$ there exists a compact subset of $\widehat{S}$, say K, such that

$$\mathscr{H}^{n-1}(\widehat{S} \setminus K) < \frac{1}{k}.$$

Let (Y_m) be a decreasing sequence in $\mathcal{M}^\circ_G$ such that $K \subseteq Y_m$ and $K = \bigcap_{m=1}^{\infty} \operatorname{cl} Y_m$. It happens that $\mathscr{H}^{n-1}(\partial_* A \cap \operatorname{cl} Y_1) < +\infty$, then there exists an index m_k with

$$\mathscr{H}^{n-1}((\partial_* A \cap \operatorname{cl} Y_{m_k}) \setminus K) < \frac{1}{k}.$$

Set $C_k = (Y_{m_k} \setminus A)_*$; by Proposition 1.1.1 it follows that $C_k \in \mathcal{M}^\circ_{h\nu}$, $A \cap C_k = \varnothing$ and

$$(\partial_* A \cap \partial_* C_k) \setminus \widehat{S} \subseteq (\partial_* A \cap \operatorname{cl} Y_{m_k}) \setminus \widehat{S} \subseteq (\partial_* A \cap \operatorname{cl} Y_{m_k}) \setminus K,$$

$$\widehat{S} \setminus (\partial_* A \cap \partial_* C_k) \subseteq \widehat{S} \setminus K.$$

Then (C_k) is the desired sequence. □

Lemma 2.1.7. *Let* I *be a contact interaction whose domain contains* $\mathfrak{D}^\circ_{h\nu}$. *Let* $A, A' \in \mathcal{M}^\circ_{h\nu}$ *and* S *be a material surface subordinated to* A *and to* A'. *Let* (C_k), (C'_k) *be two sequences in* $\mathcal{M}^\circ_{h\nu}$ *such that*

$$\lim_k \mathscr{H}^{n-1}\left((\partial_* A \cap \partial_* C_k) \triangle \widehat{S}\right) = 0,$$

$$\lim_k \mathscr{H}^{n-1}\left((\partial_* A' \cap \partial_* C'_k) \triangle \widehat{S}\right) = 0.$$

Then we have

$$\lim_k |I(A, C_k) - I(A', C'_k)| = 0.$$

Proof. We want to prove that each element of the decomposition

$$\begin{aligned} I(A, C_k) - I(A', C'_k) \quad &= \quad I((A \setminus A')_*, C_k) + I(A \cap A', (C_k \setminus C'_k)_*) + \\ &- \quad I((A' \setminus A)_*, C'_k) - I(A \cap A', (C' \setminus C)_*) \end{aligned}$$

vanishes as $k \to \infty$. By Proposition 1.1.2 we have that $\mathscr{H}^{n-1}(\partial_*(A \setminus A') \cap \widehat{S}) = 0$, since A and A' share the same unit exterior normal on S. Hence

$$\lim_k \mathscr{H}^{n-1}(\partial_*(A \setminus A') \cap \partial_* C_k) \leqslant \lim_k \mathscr{H}^{n-1}((\partial_* A \cap \partial_* C_k) \setminus \widehat{S}) = 0,$$

$$\lim_k I((A \setminus A')_*, C_k) = 0.$$

On the other hand, by Proposition 1.1.3 we have

$$\begin{aligned}
\mathscr{H}^{n-1}&(\partial_*(A \cap A') \cap \partial_*(C_k \setminus C'_k)) = \\
&= \mathscr{H}^{n-1}((\partial_*(A \cap A') \cap \partial_*(C_k \setminus C'_k)) \setminus \partial_* C'_k) \leqslant \\
&\leqslant \mathscr{H}^{n-1}((\partial_* A \cap \partial_* C_k) \setminus \partial_* C'_k) \leqslant \\
&\leqslant \mathscr{H}^{n-1}((\partial_* A \cap \partial_* C_k) \setminus \widehat{S}) + \mathscr{H}^{n-1}(\widehat{S} \setminus \partial_* C'_k),
\end{aligned}$$

hence

$$\lim_k \mathscr{H}^{n-1}(\partial_*(A \cap A') \cap \partial_*(C_k \setminus C'_k)) = 0,$$

$$\lim_k I(A \cap A', (C_k \setminus C'_k)_*) = 0.$$

In the same way we can show that

$$\lim_k I((A' \setminus A)_*, C'_k) = \lim_k I(A \cap A', (C'_k \setminus C_k)_*) = 0,$$

and the proof is complete. □

The next theorem shows that there is a strict correspondence between contact interactions and Cauchy fluxes. For $(A, C) \in \mathfrak{D}^\circ$, with $\partial_* A \cap \partial_* C$ we will denote also the material surface $\left(\partial_* A \cap \partial_* C, \boldsymbol{n}^A|_{\partial_* A \cap \partial_* C}\right)$.

Theorem 2.1.8. *The following facts hold:*

(*i*) *for every contact interaction I there exists a Cauchy flux Q such that*

$$Q(\partial_* A \cap \partial_* C) = I(A, C)$$

on almost all of $\mathfrak{D}^\circ$ and

$$|Q(S)| \leqslant \int_S \widehat{h}\, d\mathscr{H}^{n-1}$$

for almost all of $\mathcal{S}$, where $\widehat{h} \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ is as in Definition 1.2.18;

(*ii*) *for every Cauchy flux Q there exists a contact interaction I such that*

$$Q(\partial_* A \cap \partial_* C) = I(A, C), \qquad |I(A, C)| \leqslant \int_{\partial_* A \cap \partial_* C} \widehat{h}\, d\mathscr{H}^{n-1}$$

on almost all of $\mathfrak{D}^\circ$, where $\widehat{h} \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ is as in Definition 2.1.5;

(*iii*) *if I_1, I_2 are two contact interactions and Q_1, Q_2 are two Cauchy fluxes with*

$$\forall j = 1, 2 : Q_j(\partial_* A \cap \partial_* C) = I_j(A, C) \quad \textit{on almost all of } \mathfrak{D}^\circ,$$

then we have $Q_1 = Q_2$ on almost all of $\mathcal{S}$ if and only if $I_1 = I_2$ on almost all of $\mathfrak{D}^\circ$.

Proof. (i) Let $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ and $\nu \in \mathfrak{M}(\operatorname{int} B)$ be such that the domain of I contains $\mathfrak{D}^\circ_{h\nu}$. Given a set $S \in \mathcal{S}_{h\nu}$, there exists $A \in \mathcal{M}^\circ_{h\nu}$ such that S is subordinated to A. Let (C_k) be a sequence as in Lemma 2.1.6 and $k, i \in \mathbb{N}$. Then $(A, C_k), (A, C_i) \in \mathfrak{D}^\circ_{h\nu}$ and from Proposition 1.1.3 we have that $\mathscr{H}^{n-1}(\partial_* A \cap \partial_*(C_k \setminus C_i) \cap \partial_* C_i) = 0$. Hence

$$\begin{aligned}\mathscr{H}^{n-1}(\partial_* A \cap \partial_*(C_k \setminus C_i)) &= \mathscr{H}^{n-1}((\partial_* A \cap \partial_*(C_k \setminus C_i)) \setminus \partial_* C_i) \leqslant \\ &\leqslant \mathscr{H}^{n-1}((\partial_* A \cap (C_k \cup \partial_* C_k)) \setminus \partial_* C_i) \leqslant \\ &\leqslant \mathscr{H}^{n-1}((\partial_* A \cap \partial_* C_k) \setminus \widehat{S}) + \mathscr{H}^{n-1}(\widehat{S} \setminus (\partial_* A \cap \partial_* C_i)) \leqslant \\ &\leqslant \mathscr{H}^{n-1}((\partial_* A \cap \partial_* C_k) \triangle \widehat{S}) + \mathscr{H}^{n-1}((\partial_* A \cap \partial_* C_i) \triangle \widehat{S})\end{aligned}$$

and, in the same way,

$$\begin{aligned}&\mathscr{H}^{n-1}(\partial_* A \cap \partial_*(C_i \setminus C_k)) \leqslant \\ &\qquad \leqslant \mathscr{H}^{n-1}((\partial_* A \cap \partial_* C_i) \triangle \widehat{S}) + \mathscr{H}^{n-1}((\partial_* A \cap \partial_* C_k) \triangle \widehat{S}).\end{aligned}$$

Since we have

$$\begin{aligned}|I(A, C_k) - I(A, C_i)| &= |I(A, (C_k \setminus C_i)_*) - I(A, (C_i \setminus C_k)_*)| \leqslant \\ &\leqslant \int_{\partial_* A \cap \partial_*(C_k \setminus C_i)} \widehat{h}\, d\mathscr{H}^{n-1} + \int_{\partial_* A \cap \partial_*(C_i \setminus C_k)} \widehat{h}\, d\mathscr{H}^{n-1},\end{aligned}$$

it follows that $(I(A, C_k))$ is a Cauchy sequence in $\mathbb{R}$. We set $Q(S) = \lim_k I(A, C_k)$; by Lemma 2.1.7, $Q(S)$ does not depend on the set A and on the sequence (C_k). Moreover, we have

$$|Q(S)| \leqslant \int_S \widehat{h}\, d\mathscr{H}^{n-1}$$

on almost all of $\mathcal{S}$.

Now we prove the additivity. Let S, T be two compatible and disjoint surfaces in $\mathcal{S}_{h\nu}$; following Lemma 2.1.6, we can construct two sequences (C_k^S) and (C_k^T) such that $\operatorname{cl} C_k^S \cap \operatorname{cl} C_k^T = \varnothing$ and

$$\lim_k \mathscr{H}^{n-1}((\partial_* A \cap \partial_* C_k^S) \triangle \widehat{S}) = 0,$$

$$\lim_k \mathscr{H}^{n-1}((\partial_* A \cap \partial_* C_k^T) \triangle \widehat{T}) = 0.$$

Moreover we have that $\partial_*(C_k^S \cup C_k^T) = \partial_* C_k^S \cup \partial_* C_k^T$ and

$$\lim_k \mathscr{H}^{n-1}((\partial_* A \cap \partial_*(C_k^S \cup C_k^T)) \triangle (\widehat{S} \cup \widehat{T})) = 0,$$

hence

$$Q(S) + Q(T) = \lim_k (I(A, C_k^S) + I(A, C_k^T)) = \lim_k I(A, C_k^S \cup C_k^T) = Q(S \cup T).$$

Then $Q : \mathcal{S}_{h\nu} \to \mathbb{R}$ is a Cauchy flux.

(ii) Let $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ and $\nu \in \mathfrak{M}(\operatorname{int} B)$ be such that the domain of Q contains $\mathcal{S}_{h\nu}$. For every $(A, C) \in \mathfrak{D}^\circ_{h\nu}$, we set

$$I(A, C) = Q(\partial_* A \cap \partial_* C).$$

First, it is clear that

$$|I(A,C)| \leqslant \int_{\partial_* A \cap \partial_* C} \widehat{h}\, d\mathscr{H}^{n-1}.$$

Now let $(A_1, C), (A_2, C) \in \mathfrak{D}^\circ_{h\nu}$ with $A_1 \cap A_2 = \varnothing$. By Proposition 1.1.3 we observe that $\mathscr{H}^{n-1}(\partial_* A_1 \cap \partial_* A_2 \cap \partial_* C) = 0$ and $\boldsymbol{n}^{A_1}(x) = -\boldsymbol{n}^{A_2}(x)$ for $\mathscr{H}^{n-1}$-a.e. $x \in \partial_* A_1 \cap \partial_* A_2$. Since $A_1 \cap \partial_* A_2 = A_2 \cap \partial_* A_1 = \varnothing$, by Proposition 1.1.2 it follows

$$Q(\partial_*(A_1 \cup A_2) \cap \partial_* C) = Q(\partial_* A_1 \cap \partial_* C) + Q(\partial_* A_2 \cap \partial_* C),$$

hence I is additive on the first component; the additivity on the other component is similar. Then $I : \mathfrak{D}^\circ_{h\nu} \to \mathbb{R}$ is a contact interaction.
(iii) Let $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ and $\nu \in \mathfrak{M}(\operatorname{int} B)$ be such that the domains of I_j and Q_j contain $\mathfrak{D}^\circ_{h\nu}$ and $\mathcal{S}_{h\nu}$ respectively, and

$$\forall (A,C) \in \mathfrak{D}^\circ_{h\nu} : Q_j(\partial_* A \cap \partial_* C) = I_j(A,C),$$

$$\forall S \in \mathcal{S}_{h\nu} : Q_1(S) = Q_2(S).$$

Given $(A,C) \in \mathfrak{D}^\circ_{h\nu}$, we have that $\partial_* A \cap \partial_* C \in \mathcal{S}_{h\nu}$, hence

$$I_1(A,C) = Q_1(\partial_* A \cap \partial_* C) = Q_2(\partial_* A \cap \partial_* C) = I_2(A,C).$$

On the other hand, let $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ and $\nu \in \mathfrak{M}(\operatorname{int} B)$ be such that the domains of I_j and Q_j contain $\mathfrak{D}^\circ_{h\nu}$ and $\mathcal{S}_{h\nu}$ respectively and

$$Q_j(\partial_* A \cap \partial_* C) = I_j(A,C), \qquad I_1(A,C) = I_2(A,C),$$

for every $(A,C) \in \mathfrak{D}^\circ_{h\nu}$. Let $S \in \mathcal{S}_{h\nu}$; then there exists $A \in \mathcal{M}^\circ_{h\nu}$ such that S is subordinated to A. Let (C_k) be a sequence with

$$\lim_k \mathscr{H}^{n-1}((\partial_* A \cap \partial_* C_k) \triangle \widehat{S}) = 0;$$

for $j = 1, 2$ we have that $\partial_* A \cap \partial_* C_k \in \mathcal{S}_{h\nu}$ and $Q_j(S) = \lim_k Q_j(\partial_* A \cap \partial_* C_k)$. Since it happens that $(A, C_k) \in \mathfrak{D}^\circ_{h\nu}$, one has

$$Q_1(S) = \lim_k I_1(A, C_k) = \lim_k I_2(A, C_k) = Q_2(S)$$

and the proof is complete. □

2.2 Balanced Cauchy fluxes: a uniqueness criterion

Now we add a third property to Definition 2.1.5. For a discussion of how it replaces the classical balance, we refer the reader to Section 2.7.

Definition 2.2.1. *Let $\mathcal{P} \subseteq \mathcal{S}$ be a set containing almost all of $\mathcal{S}$ and let $Q : \mathcal{P} \to \mathbb{R}$. We say that Q is* a (scalar) balanced Cauchy flux *on B, if Q is a Cauchy flux such that*

(c) *there exists $\lambda \in \mathfrak{M}(\operatorname{int} B)$ such that the inequality*

$$|Q(\partial_* A)| \leqslant \lambda(A)$$

holds almost everywhere in $\mathcal{M}^\circ$.

Proposition 2.2.2. *Let Q be a balanced Cauchy flux on B, let $x_0 \in \mathbb{R}^n$ and let $(e_1, \dots, e_n)$ be a positively oriented frame in $\mathbb{R}^n$. Then there exist $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$, $\nu \in \mathfrak{M}(\operatorname{int} B)$ and a full grid G of the form $G = \left(x_0, (e_1, \dots, e_n), \widehat{G}\right)$ with the following properties:*

(a) *the domain of Q contains $\mathcal{S}_{h\nu}$; in particular, $Q(S \cup T) = Q(S) + Q(T)$ whenever $S, T \in \mathcal{S}_{h\nu}$ are compatible and disjoint;*

(b) $|Q(S)| \leqslant \int_S h \, d\mathscr{H}^{n-1}$ *for any* $S \in \mathcal{S}_{h\nu}$;

(c) $|Q(\partial_* A)| \leqslant \nu(A)$ *for any* $A \in \mathcal{M}^\circ_{h\nu}$;

(d) $\mathcal{M}^\circ_G \subseteq \mathcal{M}^\circ_{h\nu}$ *and* $\mathcal{S}_G \subseteq \mathcal{S}_{h\nu}$.

Proof. By Proposition 1.2.9 there exist h and ν satisfying (a)-(c). Then, by Proposition 1.2.16 and Remark 2.1.4, there exists a grid G with the required properties. □

Proposition 2.2.3. *Let Q be a balanced Cauchy flux on B and let h, ν be as in (a)-(c) of Proposition 2.2.2. Then the following assertions hold:*

(a) *if (S_k) is an increasing sequence in $\mathcal{S}_{h\nu}$, $S \in \mathcal{S}_{h\nu}$, $S_k \subseteq S$ and*

$$\mathscr{H}^{n-1}\left(S \setminus \left(\bigcup_{k=1}^{\infty} S_k\right)\right) = 0,$$

we have

$$\lim_k Q(S_k) = Q(S);$$

(b) *if (S_k) is a decreasing sequence in $\mathcal{S}_{h\nu}$, $S \in \mathcal{S}_{h\nu}$, $S \subseteq S_k$ and*

$$\mathscr{H}^{n-1}\left(\left(\bigcap_{k=1}^{\infty} S_k\right) \setminus S\right) = 0,$$

we have

$$\lim_k Q(S_k) = Q(S);$$

(c) *if (M_k) is a decreasing sequence in $\mathcal{M}^\circ_{h\nu}$, $M \in \mathcal{M}^\circ_{h\nu}$, $S \in \mathcal{S}_{h\nu}$ is subordinated to M, $S \subseteq M_k$ and*

$$S = \left(\bigcap_{k=1}^{\infty} (M_k \cup \partial_* M_k)\right) \cap (M \cup \partial_* M),$$

we have

$$\lim_k Q(M \cap \partial_* M_k) = \lim_k Q((M \cup \partial_* M) \cap \partial_* M_k) = -Q(S);$$

(d) *for any $S \in \mathcal{S}_{h\nu}$, we have $-S \in \mathcal{S}_{h\nu}$ and $Q(-S) = -Q(S)$.*

Proof. To prove (a), we observe that $S \setminus S_k \in \mathcal{S}_{h\nu}$ and

$$|Q(S) - Q(S_k)| = |Q(S \setminus S_k)| \leqslant \int_{S\setminus S_k} h\, d\mathscr{H}^{n-1}.$$

Then the assertion follows from Lebesgue's Theorem. The proof of (b) is similar.
To prove (c), we set $R_k = \partial_*(M \cap M_k) \setminus [(M \cap \partial_* M_k) \cup S]$. From Proposition 1.1.1 it follows that $\partial_*(M \cap M_k)$ is the disjoint union of $M \cap \partial_* M_k$, S and R_k and that

$$R_k \subseteq [(M_k \cup \partial_* M_k) \cap \partial_* M] \setminus S.$$

Moreover we have

$$|Q(M \cap \partial_* M_k) + Q(S) + Q(R_k)| = |Q(\partial_*(M \cap M_k))| \leqslant \nu(M \cap M_k),$$

$$Q((M \cup \partial_* M) \cap \partial_* M_k) = Q(M \cap \partial_* M_k) + Q(\partial_* M \cap \partial_* M_k),$$

where $\partial_* M \cap \partial_* M_k$ has the orientation induced by $\partial_* M_k$. Since $(M \cap M_k)$ is a decreasing sequence of Borel sets with $\nu(M \cap M_k) < +\infty$ and empty intersection, we have

$$\lim_k |Q(M \cap \partial_* M_k) + Q(S) + Q(R_k)| = 0.$$

On the other hand

$$\begin{aligned} |Q(R_k)| + |Q(\partial_* M \cap \partial_* M_k)| &\leqslant \int_{R_k} h\, d\mathscr{H}^{n-1} + \int_{\partial_* M \cap \partial_* M_k} h\, d\mathscr{H}^{n-1} \leqslant \\ &\leqslant 2 \int_{[(M_k \cup \partial_* M_k) \cap \partial_* M] \setminus S} h\, d\mathscr{H}^{n-1}. \end{aligned}$$

Now, $([(M_k \cup \partial_* M_k) \cap \partial_* M] \setminus S)$ is a decreasing sequence of Borel subsets of $\partial_* M$ with empty intersection. From Lebesgue's Theorem we deduce that

$$\lim_k Q(R_k) = \lim_k Q(\partial_* M \cap \partial_* M_k) = 0$$

and assertion (c) follows.
To prove (d), consider $S \in \mathcal{S}_{h\nu}$ subordinated to $M \in \mathcal{M}^\circ_{h\nu}$. Let G be a full grid as in (d) of Proposition 2.2.2. Since $\mathrm{cl}\, M \subseteq \mathrm{int}\, B$, by an easy variant of [9, Chapter 5, Lemma 1] there exists $Y \in \mathcal{M}^\circ_G$ such that $\mathrm{cl}\, M \subseteq Y$. Then $(Y \setminus M)_* \in \mathcal{M}^\circ_{h\nu}$ by Proposition 1.2.9 and $-S$ is subordinated to $(Y \setminus M)_*$ by Proposition 1.1.1. It follows that $-S \in \mathcal{S}_{h\nu}$.
Now assume that S is compact. Let (Y_k) be a decreasing sequence in $\mathcal{M}^\circ_G$ with $S \subseteq Y_k$, $\mathrm{cl}\, Y_k \subseteq Y$ and

$$S = \bigcap_{k=1}^{\infty} \mathrm{cl}\, Y_k.$$

Since $\mathrm{cl}\, M \subseteq Y$, we have

$$((Y \setminus M)_*) \cup \partial_*((Y \setminus M)_*) = (Y \setminus M) \cup \partial_* Y.$$

Then from assertion (c) we deduce that

$$\lim_k Q(M \cap \partial_* Y_k) = -Q(S),$$

$$\lim_k Q((Y \setminus M) \cap \partial_* Y_k) = -Q(-S).$$

On the other hand, we have

$$|Q(M \cap \partial_* Y_k) + Q((Y \setminus M) \cap \partial_* Y_k)| = |Q(\partial_* Y_k)| \leqslant \nu(Y_k)$$

with $\lim_k \nu(Y_k) = \nu(S) = 0$. It follows $-Q(S) = Q(-S)$.
Finally, consider the general case. Since $\mathscr{H}^{n-1}(S) < +\infty$, from [8, Theorem 2.2.2] it easily follows that

$$S = S_0 \cup \left(\bigcup_{k=1}^{\infty} T_k \right),$$

where $\mathscr{H}^{n-1}(S_0) = 0$ and (T_k) is an increasing sequence of compact sets. If T_k is endowed with the orientation induced by S, from assertion (a) we deduce that $\lim_k Q(\pm T_k) = Q(\pm S)$. On the other hand, we have $Q(-T_k) = -Q(T_k)$ by the previous step. Then assertion (d) follows in its full generality. □

Now we shall prove our main uniqueness criterion for Cauchy fluxes.

Lemma 2.2.4. *Let Q_1, Q_2 be two balanced Cauchy fluxes on B and let $h \in \mathcal{L}^1_{loc,+}(\mathrm{int}\, B)$, $\nu \in \mathfrak{M}(\mathrm{int}\, B)$ be such that*

- (a) *the domains of Q_1 and Q_2 contain $\mathcal{S}_{h\nu}$;*
- (b) $|Q_i(S)| \leqslant \int_S h\, d\mathscr{H}^{n-1}$ *for any $i = 1, 2$ and $S \in \mathcal{S}_{h\nu}$.*

Finally, let $S_0 \in \mathcal{S}_{h\nu}$.
If $Q_1(T) = Q_2(T)$ for every compact material surface $T \subseteq S_0$, then $Q_1(S_0) = Q_2(S_0)$.

Proof. As before, we have

$$S_0 = T_0 \cup \left(\bigcup_{k=1}^{\infty} T_k \right),$$

where $\mathscr{H}^{n-1}(T_0) = 0$ and (T_k), $k \geqslant 1$, is an increasing sequence of compact sets. From (a) of Proposition 2.2.3 we deduce that $\lim_k Q_i(T_k) = Q_i(S_0)$. On the other hand, Q_1 and Q_2 agree on T_k, whence the assertion. □

Theorem 2.2.5. *Let Q_1, Q_2 be two balanced Cauchy fluxes on B and let G be a full grid. Suppose that the domains of Q_1, Q_2 contain $\mathcal{S}_G$ and that $Q_1 = Q_2$ on $\mathcal{S}_G$.*
Then $Q_1 = Q_2$ on almost all of $\mathcal{S}$.

Proof. Without loss of generality, we may assume that G satisfies also (d) of Proposition 2.2.2 for some $h \in \mathcal{L}^1_{loc,+}(\mathrm{int}\, B)$ and $\nu \in \mathfrak{M}(\mathrm{int}\, B)$ satisfying (a)-(c) of the same theorem both for Q_1 and Q_2. The proof will proceed in steps.
I) Let $S \in \mathcal{S}^j_G$ and let T be a compact material surface with $T \subseteq S$. Let (Y_k) be a decreasing sequence in $\mathcal{M}^\circ_G$ with $T \subseteq Y_k$ and

$$T = \bigcap_{k=1}^{\infty} \mathrm{cl}\, Y_k;$$

then $S \cap Y_k \in \mathcal{S}_{h\nu}$. Moreover, by (a), (b) of Proposition 2.2.2 one has $Q_1(S \cap Y_k) = Q_2(S \cap Y_k)$. On the other hand, from Proposition 2.2.3 we deduce that $\lim_k Q_i(S \cap Y_k) = Q_i(T)$, whence $Q_1(T) = Q_2(T)$.
II) If $S \in \mathcal{S}_G^j$ and T is a material surface with $T \subseteq S$, we deduce from the previous step and Lemma 2.2.4 that $Q_1(T) = Q_2(T)$.
III) Consider $Y \in \mathcal{M}_G^\circ$ and a material surface S subordinated to Y. Since S is a disjoint Borel union

$$S = S_0 \cup \left(\bigcup_{k=1}^{m} T_k \right),$$

with $\mathscr{H}^{n-1}(S_0) = 0$ and T_k or $-T_k$ contained in some $S_k \in \mathcal{S}_G^{j_k}$, from the previous step and Theorems 2.2.2 and 2.2.3 we deduce that $Q_1(S) = Q_2(S)$.
IV) Consider $S \in \mathcal{S}_{h\nu}$ with S compact. Let S be subordinated to $M \in \mathcal{M}_{h\nu}^\circ$ and let (Y_k) be a decreasing sequence in $\mathcal{M}_G^\circ$ with $S \subseteq Y_k$ and

$$S = \bigcap_{k=1}^{\infty} \mathrm{cl}\, Y_k.$$

From Proposition 2.2.3 we deduce that

$$\lim_k Q_i(M \cap \partial_* Y_k) = -Q_i(S).$$

On the other hand, by the previous step we have $Q_1(M \cap \partial_* Y_k) = Q_2(M \cap \partial_* Y_k)$, whence $Q_1(S) = Q_2(S)$.
V) Finally, let $S \in \mathcal{S}_{h\nu}$. Combining the previous step with Lemma 2.2.4, we deduce that $Q_1(S) = Q_2(S)$ and the proof is complete. □

2.3 Vector fields with divergence measure

Definition 2.3.1. *Let Ω be an open subset of $\mathbb{R}^n$ and let $\boldsymbol{q} \in \mathcal{L}^1_{loc}(\Omega; \mathbb{R}^n)$. We say that* $\mathrm{div}\, \boldsymbol{q}$ *is* a (local) measure *on Ω, if $\mathrm{div}\, \boldsymbol{q}$ is a distribution on Ω of order* 0*. This means that for every compact subset K of Ω there exists a constant c_K such that*

$$\left| \int_\Omega \boldsymbol{q} \cdot \mathrm{grad}\, f \, d\mathscr{L}^n \right| \leqslant c_K \max_K |f|$$

whenever $f \in C_0^\infty(\Omega)$ and $\mathrm{supt}\, f \subseteq K$.
In such a case, there exist $\mu \in \mathfrak{M}(\Omega)$ and a Borel function $u : \Omega \to \mathbb{R}$ such that $|u(x)| = 1$ for μ-a.e. $x \in \Omega$ and

$$-\int_\Omega \boldsymbol{q} \cdot \mathrm{grad}\, f \, d\mathscr{L}^n = \int_\Omega f u \, d\mu$$

for any Lipschitz function $f : \Omega \to \mathbb{R}$ with compact support. It is well known that μ is uniquely determined, while u is uniquely determined μ-almost everywhere. We put $|\mathrm{div}\, \boldsymbol{q}| = \mu$. Finally, if $M \in \mathfrak{B}(\Omega)$ and $f : M \to \mathbb{R}$ is Borel and μ-summable on M, we set

$$\int_M f \, \mathrm{div}\, \boldsymbol{q} = \int_M f u \, d\mu.$$

Theorem 2.3.2. *Let $\boldsymbol{q} \in \mathcal{L}^1_{loc}(\Omega;\mathbb{R}^n)$ be a vector field with divergence measure. Then there exist a sequence $(\boldsymbol{q}_m)$ in $C^\infty(\Omega;\mathbb{R}^n)$ and $h \in \mathcal{L}^1_{loc,+}(\Omega)$ such that*

$$\forall x \in \Omega : \ h(x) < +\infty \implies \lim_m \boldsymbol{q}_m(x) = \boldsymbol{q}(x), \tag{2.2}$$

$$\forall m \in \mathbb{N}, \forall x \in \Omega : \ \ |\boldsymbol{q}_m(x)| \leqslant h(x) \ \text{ and } |\boldsymbol{q}_m(x)| \leqslant \operatorname*{ess\,sup}_{\Omega} |\boldsymbol{q}|, \tag{2.3}$$

$$\lim_m \int_M f \operatorname{div} \boldsymbol{q}_m \, d\mathscr{L}^n = \int_{M_*} f \operatorname{div} \boldsymbol{q}, \tag{2.4}$$

whenever $f : \Omega \to \mathbb{R}$ is continuous and $M \in \mathfrak{B}(\Omega)$ has compact closure in Ω, provided that $|\operatorname{div} \boldsymbol{q}|(\partial_* M) = 0$.

Proof. Let $\rho \in C^\infty_0(\mathbb{R}^n)$ be a nonnegative function with $\int \rho \, d\mathscr{L}^n = 1$ and for any $x \in \mathbb{R}^n$ set $\rho_m(x) = m^n \rho(mx)$. Let also (K_m) be an increasing sequence of compact subsets of Ω with $\Omega = \bigcup_{m=1}^{\infty} \operatorname{int} K_m$ and let $\vartheta_m \in C^\infty_0(\Omega)$ with $0 \leqslant \vartheta_m \leqslant 1$ on Ω and $\vartheta_m = 1$ on K_m. If we set

$$\boldsymbol{q}_m(x) = \int_\Omega \rho_m(x-y)\vartheta_m(y)\boldsymbol{q}(y) \, d\mathscr{L}^n(y),$$

it is well known that $(\boldsymbol{q}_m)$ is a sequence in $C^\infty(\Omega;\mathbb{R}^n)$ converging to $\boldsymbol{q}$ in $L^1_{loc}(\Omega;\mathbb{R}^n)$ and satisfying $|\boldsymbol{q}_m(x)| \leqslant \operatorname*{ess\,sup}_{\Omega} |\boldsymbol{q}|$. According to [4, Theorem IV.9], there exist a subsequence, we still denote by $(\boldsymbol{q}_m)$, and $h \in \mathcal{L}^1_{loc,+}(\Omega)$ satisfying (2.2) and (2.3).
Now let f and M be as in the statement of (2.4). For any sufficiently large m, we have

$$\begin{aligned} \int_M f(x) \operatorname{div} \boldsymbol{q}_m(x) \, d\mathscr{L}^n(x) &= \int_M f(x) \left(\int_\Omega \boldsymbol{q}(y) \cdot (\operatorname{grad} \rho_m)(x-y) \, d\mathscr{L}^n(y) \right) d\mathscr{L}^n(x) = \\ &= \int_M f(x) \left(\int_\Omega \rho_m(x-y) \operatorname{div} \boldsymbol{q}(y) \right) d\mathscr{L}^n(x) = \\ &= \int_\Omega \left(\int_M f(x)\rho_m(x-y) \, d\mathscr{L}^n(x) \right) \operatorname{div} \boldsymbol{q}(y). \end{aligned}$$

Moreover, if K is a compact subset of Ω with $\operatorname{cl} M \subseteq \operatorname{int} K$, we also have

$$\left| \int_M f(x)\rho_m(x-y) \, d\mathscr{L}^n(x) \right| \leqslant \left(\max_K |f| \right) \chi_K(y)$$

eventually as $m \to \infty$. Finally, it is easy to see that the integral on the left hand side is convergent to $f(y)$ on M_* and to 0 on $(\Omega \setminus M)_*$, as $m \to \infty$. From Lebesgue's Theorem, (2.4) follows. □

Theorem 2.3.3. *Let $\boldsymbol{q} \in \mathcal{L}^1_{loc}(\Omega;\mathbb{R}^n)$ and let $\lambda \in \mathfrak{M}(\Omega)$. Assume there exists a full grid G such that, for any open n-dimensional G-interval I with $\operatorname{cl} I \subseteq \Omega$, $\boldsymbol{q}$ is $\mathscr{H}^{n-1}$-summable on $\partial_* I$ and*

$$\left| \int_{\partial_* I} \boldsymbol{q} \cdot \boldsymbol{n}^I \, d\mathscr{H}^{n-1} \right| \leqslant \lambda(I).$$

Then $\boldsymbol{q}$ is a vector field with divergence measure and $|\operatorname{div} \boldsymbol{q}| \leqslant \lambda$.

Proof. Let $G = \left(x_0, (e_1, \ldots, e_n), \widehat{G}\right)$ be a full grid as in the hypothesis. For every $x \in \mathbb{R}^n$ we set

$$|x|_\infty = \max_{1 \leqslant j \leqslant n} |x \cdot e_j|$$

and we define a function $\rho : \mathbb{R}^n \to \mathbb{R}$ by

$$\rho(x) = 2^{-n}(n+1)\,(1 - |x|_\infty)^+ \,.$$

Then ρ is a positive Lipschitz function which is zero outside of $J = \{x \in \mathbb{R}^n : |x|_\infty < 1\}$ and satisfies $\int \rho \, d\mathscr{L}^n = 1$. Set $\rho_m(x) = m^n \rho(mx)$.
Let K_m, ϑ_m be as in the proof of Theorem 2.3.2 and let $\boldsymbol{q}_m : \Omega \to \mathbb{R}^n$ be the function of class C^1 defined by

$$\boldsymbol{q}_m(x) = \int_\Omega \rho_m(x-y)\vartheta_m(y)\boldsymbol{q}(y)\, d\mathscr{L}^n(y).$$

Then $(\boldsymbol{q}_m)$ is convergent to $\boldsymbol{q}$ in $L^1_{loc}(\Omega; \mathbb{R}^n)$ and for any $x \in \Omega$ we have

$$\operatorname{div} \boldsymbol{q}_m(x) = \int_\Omega \vartheta_m(y)\boldsymbol{q}(y) \cdot (\operatorname{grad} \rho_m)(x-y)\, d\mathscr{L}^n(y).$$

We have to show that, for every open set ω with compact closure in Ω, one has

$$\sup \left\{ \left| \int_\Omega \boldsymbol{q} \cdot \operatorname{grad} f \, d\mathscr{L}^n \right| : \quad f \in C^\infty_0(\Omega), \quad \operatorname{supt} f \subseteq \omega, \quad \max_\omega |f| \leqslant 1 \right\} \leqslant \lambda(\omega).$$

Now, if ω is such an open set and $f \in C^\infty_0(\Omega)$ with $\operatorname{supt} f \subseteq \omega$, we have

$$\begin{aligned} \left| \int_\Omega \boldsymbol{q} \cdot \operatorname{grad} f \, d\mathscr{L}^n \right| &= \lim_m \left| \int_\Omega \boldsymbol{q}_m \cdot \operatorname{grad} f \, d\mathscr{L}^n \right| \leqslant \\ &\leqslant \liminf_m \int_\Omega |f| |\operatorname{div} \boldsymbol{q}_m| \, d\mathscr{L}^n \leqslant \\ &\leqslant \left(\max_\omega |f|\right) \liminf_m \int_{\operatorname{supt} f} |\operatorname{div} \boldsymbol{q}_m| \, d\mathscr{L}^n. \end{aligned}$$

Therefore, if we set $C = \operatorname{supt} f$, it is sufficient to show that

$$\liminf_m \int_C |\operatorname{div} \boldsymbol{q}_m| \, d\mathscr{L}^n \leqslant \lambda(\omega). \tag{2.5}$$

For every $x \in \mathbb{R}^n$ and for every $t \in [0, 2^{-n}(n+1)m^n[$, we have

$$\{y \in \mathbb{R}^n : \rho_m(x-y) > t\} = x + J_{m,t}, \qquad J_{m,t} = \frac{2^{-n}(n+1)m^n - t}{2^{-n}(n+1)m^{n+1}}\, J.$$

Assume that m is large enough to ensure that $C + J_{m,0} \subseteq \omega$ and that $\vartheta_m = 1$ on $C + J_{m,0}$. Then for every $x \in C$ it turns out that $x + J_{m,t}$ is an open n-dimensional G-interval with closure in Ω for $\mathscr{L}^1$-a.e. $t \in [0, 2^{-n}(n+1)m^n[$ and by the Coarea Formula [8, Theorem 3.2.12] we deduce that

$$\begin{aligned}
|\operatorname{div} \boldsymbol{q}_m(x)| &= \left| \int_{(x+J_{m,0})} \boldsymbol{q}(y) \cdot \frac{(\operatorname{grad} \rho_m)(x-y)}{|(\operatorname{grad} \rho_m)(x-y)|} \, |(\operatorname{grad} \rho_m)(x-y)| \, d\mathscr{L}^n(y) \right| = \\
&= \left| \int_0^{2^{-n}(n+1)m^n} \int_{\partial_*(x+J_{m,t})} \boldsymbol{q}(y) \cdot \frac{(\operatorname{grad} \rho_m)(x-y)}{|(\operatorname{grad} \rho_m)(x-y)|} \, d\mathscr{H}^{n-1}(y) \, d\mathscr{L}^1(t) \right| \leqslant \\
&\leqslant \int_0^{2^{-n}(n+1)m^n} \lambda\left(x + J_{m,t}\right) \, d\mathscr{L}^1(t) = \\
&= 2^{-n}(n+1)m^n \int_0^1 \lambda\left(x + \frac{s}{m} J\right) \, d\mathscr{L}^1(s).
\end{aligned}$$

From Fubini's Theorem it follows that

$$\begin{aligned}
\int_C |\operatorname{div} \boldsymbol{q}_m(x)| \, d\mathscr{L}^n(x) &\leqslant 2^{-n}(n+1)m^n \int_C \int_0^1 \lambda\left(x + \frac{s}{m} J\right) \, d\mathscr{L}^1(s) \, d\mathscr{L}^n(x) = \\
&= 2^{-n}(n+1)m^n \int_0^1 \int_C \int_\omega \chi_{x+\frac{s}{m}J}(y) \, d\lambda(y) \, d\mathscr{L}^n(x) \, d\mathscr{L}^1(s).
\end{aligned}$$

Again from Fubini's Theorem we conclude that

$$\begin{aligned}
\int_C |\operatorname{div} \boldsymbol{q}_m(x)| \, d\mathscr{L}^n(x) &\leqslant 2^{-n}(n+1)m^n \int_0^1 \int_\omega \int_C \chi_{y+\frac{s}{m}J}(x) \, d\mathscr{L}^n(x) \, d\lambda(y) \, d\mathscr{L}^1(s) \leqslant \\
&\leqslant 2^{-n}(n+1)m^n \int_0^1 \int_\omega \frac{2^n s^n}{m^n} \, d\lambda(y) \, d\mathscr{L}^1(s) = \lambda(\omega).
\end{aligned}$$

Passing to the lower limit as $m \to \infty$, assertion (2.5) follows. □

Now let B be a continuous body and let $\boldsymbol{q} \in \mathcal{L}^1_{loc}(\operatorname{int} B; \mathbb{R}^n)$ be a vector field with divergence measure. Let $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ be as in Theorem 2.3.2 and let $\nu = |\operatorname{div} \boldsymbol{q}|$.

Theorem 2.3.4. *For every $M \in \mathcal{M}^\circ_{h\nu}$ and for every locally Lipschitz function $f : \operatorname{int} B \to \mathbb{R}$ we have that $\boldsymbol{q}$ is $\mathscr{H}^{n-1}$-summable on $\partial_* M$ and*

$$\int_M \boldsymbol{q} \cdot \operatorname{grad} f \, d\mathscr{L}^n = \int_{\partial_* M} f \boldsymbol{q} \cdot \boldsymbol{n}^M \, d\mathscr{H}^{n-1} - \int_M f \operatorname{div} \boldsymbol{q}.$$

Proof. Let $(\boldsymbol{q}_m)$ be as in Theorem 2.3.2. Let also M and f be as in the statement of the theorem. Since $\boldsymbol{q}$ is a Borel map satisfying $|\boldsymbol{q}| \leqslant h$, it is plain that $\boldsymbol{q}$ is $\mathscr{H}^{n-1}$-summable on $\partial_* M$. Moreover, f is Lipschitz continuous in a neighborhood of $\operatorname{cl} M$, so that

$$\int_M \boldsymbol{q}_m \cdot \operatorname{grad} f \, d\mathscr{L}^n = \int_{\partial_* M} f \boldsymbol{q}_m \cdot \boldsymbol{n}^M \, d\mathscr{H}^{n-1} - \int_M f \operatorname{div} \boldsymbol{q}_m \, d\mathscr{L}^n.$$

Now we pass to the limit as $m \to \infty$. We may apply Lebesgue's Theorem to the first two integrals, while the third one passes to the limit by (2.4). Therefore the proof is complete. □

Corollary 2.3.5. *For every $S \in \mathcal{S}_{h\nu}$ the map $\boldsymbol{q}$ is $\mathscr{H}^{n-1}$-summable on S and the formula*

$$Q(S) = \int_S \boldsymbol{q} \cdot \boldsymbol{n}_S \, d\mathscr{H}^{n-1}$$

defines a balanced Cauchy flux $Q : \mathcal{S}_{h\nu} \to \mathbb{R}$ on B.

Proof. Of course Q is well defined on $\mathcal{S}_{h\nu}$, which contains almost all of $\mathcal{S}$, and satisfies properties (a) and (b) of Definition 2.1.5. If $M \in \mathcal{M}^\circ_{h\nu}$ and we apply Theorem 2.3.4 with $f = 1$, we get

$$|Q(\partial_* M)| = \left|\int_{\partial_* M} \boldsymbol{q} \cdot \boldsymbol{n}^M \, d\mathscr{H}^{n-1}\right| = \left|\int_M \operatorname{div} \boldsymbol{q}\right| \leqslant |\operatorname{div} \boldsymbol{q}|(M) = \nu(M).$$

Therefore also (c) of Definition 2.1.5 follows. □

Proposition 2.3.6. *Let $\check{\boldsymbol{q}} \in \mathcal{L}^1_{loc}(\operatorname{int} B; \mathbb{R}^n)$ be another vector field with divergence measure and let G be a full grid. Assume that $\boldsymbol{q}$ and $\check{\boldsymbol{q}}$ are both $\mathscr{H}^{n-1}$-summable on any $S \in \mathcal{S}_G$ and that*

$$\forall S \in \mathcal{S}_G : \int_S \boldsymbol{q} \cdot \boldsymbol{n}_S \, d\mathscr{H}^{n-1} = \int_S \check{\boldsymbol{q}} \cdot \boldsymbol{n}_S \, d\mathscr{H}^{n-1}.$$

Then we have $\boldsymbol{q}(x) = \check{\boldsymbol{q}}(x)$ for $\mathscr{L}^n$-a.e. $x \in \operatorname{int} B$.

Proof. From Fubini's Theorem, we deduce that for any $j = 1, \ldots, n$ and any

$$I = \left\{x \in \mathbb{R}^n : \quad a^{(i)} < (x - x_0) \cdot e_i < b^{(i)} \quad \forall i = 1, \ldots, n\right\} \in \mathcal{I}^\circ_G$$

we have

$$\begin{aligned} \int_I \boldsymbol{q}^{(j)} \, d\mathscr{L}^n &= \int_{a^{(j)}}^{b^{(j)}} \left[\int_{\sigma_{j,s}(I)} \boldsymbol{q}(x) \cdot e_j \, d\mathscr{H}^{n-1}(x)\right] d\mathscr{L}^1(s) = \\ &= \int_{a^{(j)}}^{b^{(j)}} \left[\int_{\sigma_{j,s}(I)} \check{\boldsymbol{q}}(x) \cdot e_j \, d\mathscr{H}^{n-1}(x)\right] d\mathscr{L}^1(s) = \int_I \check{\boldsymbol{q}}^{(j)} \, d\mathscr{L}^n. \end{aligned}$$

Since each open subset of $\operatorname{int} B$ is a countable disjoint union of elements of $\mathcal{I}^\circ_G$, up to an $\mathscr{L}^n$-negligible set, the assertion follows. □

Corollary 2.3.7. *Let $\check{\boldsymbol{q}} \in \mathcal{L}^1_{loc}(\operatorname{int} B; \mathbb{R}^n)$ be another vector field with divergence measure and let Q, $\check{Q}$ be associated with $\boldsymbol{q}$, $\check{\boldsymbol{q}}$, according to Corollary 2.3.5.*
Then we have $\boldsymbol{q}(x) = \check{\boldsymbol{q}}(x)$ for $\mathscr{L}^n$-a.e. $x \in \operatorname{int} B$ if and only if $Q = \check{Q}$ on almost all of $\mathcal{S}$.

Proof. Let $\check{h}$ be associated with $\check{\boldsymbol{q}}$, according to Theorem 2.3.2 and let $\check{\nu} = |\operatorname{div} \check{\boldsymbol{q}}|$.
Assume first that $\boldsymbol{q}(x) = \check{\boldsymbol{q}}(x)$ for $\mathscr{L}^n$-a.e. $x \in \operatorname{int} B$. If we set $\overline{\nu} = \nu + \check{\nu}$ and

$$\overline{h}(x) = \begin{cases} h(x) + \check{h}(x) & \text{if } \boldsymbol{q}(x) = \check{\boldsymbol{q}}(x), \\ +\infty & \text{if } \boldsymbol{q}(x) \neq \check{\boldsymbol{q}}(x), \end{cases}$$

it is readily seen that Q and $\check{Q}$ agree on $\mathcal{S}_{\overline{h}\overline{\nu}}$.
Now assume that $Q = \check{Q}$ on almost all of $\mathcal{S}$. Let $\overline{h} \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ and $\overline{\nu} \in \mathfrak{M}(\operatorname{int} B)$ be such that $\mathcal{S}_{\overline{h}\overline{\nu}} \subseteq \mathcal{S}_{h\nu} \cap \mathcal{S}_{\check{h}\check{\nu}}$ and such that $Q, \check{Q}$ agree on $\mathcal{S}_{\overline{h}\overline{\nu}}$. By Proposition 2.1.4 there exists a full grid G with $\mathcal{S}_G \subseteq \mathcal{S}_{\overline{h}\overline{\nu}}$. From Proposition 2.3.6 we conclude that $\boldsymbol{q}(x) = \check{\boldsymbol{q}}(x)$ for $\mathscr{L}^n$-a.e. $x \in \operatorname{int} B$. □

2.4 An integral representation and extension result

Throughout this section, B will denote a continuous body, $G_0 = \left(x_0, (e_1, \dots, e_n), \widehat{G}_0\right)$ a full grid and $Q_0 : \mathcal{S}_{G_0} \to \mathbb{R}$ a function satisfying the following properties:

(*i*) $Q_0(S) = Q_0(S_1) + Q_0(S_2)$ whenever $S, S_1, S_2 \in \mathcal{S}_{G_0}$, $S_1 \cap S_2 = \varnothing$ and $\mathrm{cl}\, S = \mathrm{cl}\, S_1 \cup \mathrm{cl}\, S_2$;

(*ii*) there exists $h \in \mathcal{L}^1_{loc,+}(\mathrm{int}\, B)$ such that

$$|Q_0(S)| \leqslant \int_S h \, d\mathscr{H}^{n-1}$$

for any $S \in \mathcal{S}_{G_0}$;

(*iii*) there exists $\lambda \in \mathfrak{M}(\mathrm{int}\, B)$ such that

$$\left| \sum_{j=1}^n \left(Q_0(\varphi_j^+(I)) - Q_0(\varphi_j^-(I)) \right) \right| \leqslant \lambda(I)$$

whenever

$$I = \left\{ x \in \mathbb{R}^n : \quad a^{(j)} < (x - x_0) \cdot e_j < b^{(j)} \quad \forall j = 1, \dots, n \right\} \in \mathcal{I}^\circ_{G_0},$$

$$\varphi_j^+(I) = \left\{ x \in \mathbb{R}^n : \quad (x - x_0) \cdot e_j = b^{(j)}, \quad a^{(i)} < (x - x_0) \cdot e_i < b^{(i)} \quad \forall i \neq j \right\},$$

$$\varphi_j^-(I) = \left\{ x \in \mathbb{R}^n : \quad (x - x_0) \cdot e_j = a^{(j)}, \quad a^{(i)} < (x - x_0) \cdot e_i < b^{(i)} \quad \forall i \neq j \right\}.$$

Although the domain of Q_0 is quite restricted, assumptions (*ii*) and (*iii*) provide such a uniform control, that Q_0 can be *uniquely* extended to almost all of $\mathcal{S}$, as the next theorem shows.

Theorem 2.4.1. *There exist a balanced Cauchy flux Q on B, a vector field with divergence measure $\boldsymbol{q} \in \mathcal{L}^1_{loc}(\mathrm{int}\, B; \mathbb{R}^n)$ and a full grid $G \subseteq G_0$ satisfying the following conditions:*

(*a*) *the domain of Q contains $\mathcal{S}_G$, $\boldsymbol{q}$ is $\mathscr{H}^{n-1}$-summable on any $S \in \mathcal{S}_G$ and*

$$\forall S \in \mathcal{S}_G : \ Q(S) = \int_S \boldsymbol{q} \cdot \boldsymbol{n}_S \, d\mathscr{H}^{n-1} = Q_0(S);$$

(*b*) *we have $|\mathrm{div}\, \boldsymbol{q}| \leqslant \lambda$ and*

$$Q(S) = \int_S \boldsymbol{q} \cdot \boldsymbol{n}_S \, d\mathscr{H}^{n-1}$$

on almost all of $\mathcal{S}$.

Moreover, if $\check{Q}$ and $\check{\boldsymbol{q}}$ also satisfy (a) for some full grid $\check{G} \subseteq G_0$, then $\check{Q} = Q$ on almost all of $\mathcal{S}$ and $\check{\boldsymbol{q}}(x) = \boldsymbol{q}(x)$ for $\mathscr{L}^n$-a.e. $x \in \mathrm{int}\, B$.

The section will be devoted to the proof of this result. First of all, let h and λ be as in assumptions (ii) and (iii). By Proposition 1.2.16 and Remark 2.1.4, we may suppose without loss of generality that $\mathcal{M}^\circ_{G_0} \subseteq \mathcal{M}^\circ_{h\lambda}$ and $\mathcal{S}_{G_0} \subseteq \mathcal{S}_{h\lambda}$. Moreover, for any

$$I = \left\{x \in \mathbb{R}^n : \quad a^{(j)} < (x - x_0) \cdot e_j < b^{(j)} \quad \forall j = 1, \dots, n\right\} \in \mathcal{I}^\circ_{G_0}$$

we may set

$$Q_0(\partial_* I) = \sum_{j=1}^n \Big(Q_0(\varphi_j^+(I)) - Q_0(\varphi_j^-(I))\Big),$$

where $\varphi_j^\pm(I)$ are defined as before.

Lemma 2.4.2. *For every $j = 1, \dots, n$, the following assertions hold:*

(a) *if*

$$I = \left\{x \in \mathbb{R}^n : \quad a^{(i)} < (x - x_0) \cdot e_i < b^{(i)} \quad \forall i = 1, \dots, n\right\} \in \mathcal{I}^\circ_{G_0},$$

then the function $\{s \mapsto Q_0(\sigma_{j,s}(I))\}$ is continuous on $\widehat{G}_0 \cap]a^{(j)}, b^{(j)}[$ and any extension to $]a^{(j)}, b^{(j)}[$ is $\mathscr{L}^1$-summable;

(b) *if we define $\mu_j : \mathcal{I}^\circ_{G_0} \to \mathbb{R}$ by*

$$\mu_j(I) = \int_{a^{(j)}}^{b^{(j)}} Q_0(\sigma_{j,s}(I))\, d\mathscr{L}^1(s),$$

we have

$$\mu_j(I) = \mu_j(I_1) + \mu_j(I_2), \qquad |\mu_j(J)| \leqslant \int_J h\, d\mathscr{L}^n$$

whenever $I, I_1, I_2, J \in \mathcal{I}^\circ_{G_0}$, $I_1 \cap I_2 = \varnothing$, $I_1 \cup I_2 \subseteq I$ and $\mathscr{L}^n(I \setminus (I_1 \cup I_2)) = 0$;

(c) *there exists $\boldsymbol{q}^{(j)} \in \mathcal{L}^1_{loc}(\operatorname{int} B; \mathbb{R})$ such that*

$$\mu_j(I) = \int_I \boldsymbol{q}^{(j)}\, d\mathscr{L}^n, \qquad |\boldsymbol{q}^{(j)}(x)| \leqslant h(x)$$

for any $I \in \mathcal{I}^\circ_{G_0}$ and $x \in \operatorname{int} B$.

Proof. (a) We consider, for simplicity, the case $n = 2$ and $j = 1$. Let $r, s \in \widehat{G}_0 \cap \,]a^{(1)}, b^{(1)}[$ with $r < s$. If

$$I_r = \left\{x \in \mathbb{R}^2 : \, r < (x - x_0) \cdot e_1 < s, \, a^{(2)} < (x - x_0) \cdot e_2 < b^{(2)}\right\},$$

we have $I_r \in \mathcal{I}^\circ_{G_0}$, hence $|Q_0(\partial_* I_r)| \leqslant \lambda(I_r)$, and also $\lim_{r \to s^-} \lambda(I_r) = 0$. On the other hand, we have by definition

$$Q_0(\partial_* I_r) = Q_0(\sigma_{1,s}(I)) - Q_0(\sigma_{1,r}(I)) + Q_0(\sigma_{2,b^{(2)}}(I_r)) - Q_0(\sigma_{2,a^{(2)}}(I_r))$$

and by assumption (ii)

$$|Q_0(\sigma_{2,a^{(2)}}(I_r))| \leqslant \int_{\sigma_{2,a^{(2)}}(I_r)} h\, d\mathscr{H}^1(x),$$

$$|Q_0(\sigma_{2,b^{(2)}}(I_r))| \leqslant \int_{\sigma_{2,b^{(2)}}(I_r)} h \, d\mathscr{H}^1(x).$$

It follows

$$\lim_{r \to s^-} Q_0(\sigma_{1,r}(I)) = Q_0(\sigma_{1,s}(I)).$$

The right continuity can be proved in a similar way.
Since G is full, any extension of $\{s \longmapsto Q_0(\sigma_{1,s}(I))\}$ to $]a^{(1)}, b^{(1)}[$ is $\mathscr{L}^1$-measurable. Moreover, Fubini's Theorem and assumption (ii) yield

$$\begin{aligned} \int_{a^{(1)}}^{b^{(1)}} |Q_0(\sigma_{1,s}(I))| \, d\mathscr{L}^1(s) &\leqslant \int_{a^{(1)}}^{b^{(1)}} \left[\int_{\sigma_{1,s}(I)} h \, d\mathscr{H}^1 \right] d\mathscr{L}^1(s) = \\ &= \int_I h \, d\mathscr{L}^2 < +\infty, \end{aligned}$$

whence the $\mathscr{L}^1$-summability of $\{s \longmapsto Q_0(\sigma_{1,s}(I))\}$.
(b) The additivity is evident. If

$$J = \left\{ x \in \mathbb{R}^n : \quad c^{(i)} < (x - x_0) \cdot e_i < d^{(i)} \quad \forall i = 1, \dots, n \right\} \in \mathcal{I}^\circ_{G_0},$$

we also have

$$|\mu_j(J)| = \left| \int_{c^{(j)}}^{d^{(j)}} Q_0(\sigma_{j,s}(J)) \, d\mathscr{L}^1(s) \right| \leqslant \int_J h \, d\mathscr{L}^n.$$

(c) We may define a linear functional $T : C_0^\infty(\operatorname{int} B) \to \mathbb{R}$ by

$$\langle T, f \rangle = \lim \left(\sum_m f(\xi_m) \mu_j(I_m) \right) \qquad \text{as} \sup_m (\operatorname{diam} I_m) \to 0,$$

where each $\{I_m\}$ is a finite disjoint subfamily of $\mathcal{I}^\circ_{G_0}$ whose union contains $\operatorname{supt} f$ up to an $\mathscr{L}^n$-negligible set and $\xi_m \in I_m$. It is readily seen that T is a distribution of order 0 on $\operatorname{int} B$ satisfying

$$\forall f \in C_0^\infty(\operatorname{int} B) : \, |\langle T, f \rangle| \leqslant \int_\Omega |f| h \, d\mathscr{L}^n.$$

Combining the Riesz Representation Theorem with the Radon-Nikodym Theorem, we find $\boldsymbol{q}^{(j)}$ with the required properties. □

Let $\boldsymbol{q} \in \mathcal{L}^1_{loc}(\operatorname{int} B; \mathbb{R}^n)$ be defined by

$$\boldsymbol{q}(x) = \sum_{j=1}^n \boldsymbol{q}^{(j)}(x) \, e_j.$$

Lemma 2.4.3. *There exists a full grid $G_1 \subseteq G_0$ such that $\boldsymbol{q}$ is $\mathscr{H}^{n-1}$-summable on each $S \in \mathcal{S}_{G_1}$ and*

$$\forall S \in \mathcal{S}_{G_1} : \, Q_0(S) = \int_S \boldsymbol{q} \cdot \boldsymbol{n}_S \, d\mathscr{H}^{n-1},$$

$$\forall I \in \mathcal{I}^\circ_{G_1} : \, Q_0(\partial_* I) = \int_{\partial_* I} \boldsymbol{q} \cdot \boldsymbol{n}^I \, d\mathscr{H}^{n-1}.$$

Proof. Let D_2 be a countable and dense subset of $\widehat{G}_0$ and let $G_2 \subseteq G_0$ be the grid such that $\widehat{G}_2 = D_2$. Since $\mathcal{I}^\circ_{G_2}$ is countable, by Fubini's Theorem and well known results on Lebesgue points, there exists a full grid $G_1 \subseteq G_0$ such that $\boldsymbol{q}$ is $\mathscr{H}^{n-1}$-summable on $\sigma_{j,s}(I)$ and

$$\lim_{r\to 0^+} \frac{1}{2r}\int_{s-r}^{s+r}\left|\int_{\sigma_{j,\tau}(I)} q_j \, d\mathscr{H}^{n-1} - \int_{\sigma_{j,s}(I)} q_j \, d\mathscr{H}^{n-1}\right| d\mathscr{L}^1(\tau) = 0$$

for any $I \in \mathcal{I}^\circ_{G_2}$, $1 \leqslant j \leqslant n$ and $s \in \widehat{G}_1$. In particular, $\boldsymbol{q}$ is $\mathscr{H}^{n-1}$-summable on every compact subset of $\sigma_{j,s}(\operatorname{int} B)$.
Now let $S \in \mathcal{S}^j_{G_1}$. First of all, we treat the particular case where

$$\widehat{S} = \left\{x \in \mathbb{R}^n : \quad (x - x_0)\cdot e_j = s, \quad a^{(i)} < (x-x_0)\cdot e_i < b^{(i)} \quad \forall i \neq j\right\}$$

with $s \in \widehat{G}_1$ and $a^{(1)}, b^{(1)}, \dots, a^{(j-1)}, b^{(j-1)}, a^{(j+1)}, b^{(j+1)}, \dots, a^{(n)}, b^{(n)} \in D_2$. Let $a^{(j)}, b^{(j)} \in D_2$ be such that

$$I = \left\{x \in \mathbb{R}^n : \quad a^{(i)} < (x-x_0)\cdot e_i < b^{(i)} \quad \forall i = 1, \dots, n\right\} \in \mathcal{I}^\circ_{G_2}$$

and $a^{(j)} < s < b^{(j)}$. Finally, let (r_m) be a sequence in $\widehat{G}_0$ strictly increasing to s and (s_m) a sequence in $\widehat{G}_0$ strictly decreasing to s. We have

$$\frac{1}{s_m - r_m}\int_{r_m}^{s_m} Q_0\left(\sigma_{j,\tau}(I)\right) d\mathscr{L}^1(\tau) = \frac{1}{s_m - r_m}\int_{r_m}^{s_m}\left[\int_{\sigma_{j,\tau}(I)} q_j \, d\mathscr{H}^{n-1}\right] d\mathscr{L}^1(\tau).$$

Passing to the limit as $m \to \infty$ and taking into account (a) of Lemma 2.4.2, we deduce that

$$Q_0(S) = \int_S q_j \, d\mathscr{H}^{n-1}.$$

Now consider a general $S \in \mathcal{S}^j_{G_1}$. Let

$$\widehat{S} = \left\{x \in \mathbb{R}^n : \quad (x - x_0)\cdot e_j = s, \quad a^{(i)} < (x-x_0)\cdot e_i < b^{(i)} \quad \forall i \neq j\right\}.$$

There exists an increasing sequence (S_m) with

$$\widehat{S}_m = \left\{x \in \mathbb{R}^n : \quad (x - x_0)\cdot e_j = s, \quad a^{(i)}_m < (x-x_0)\cdot e_i < b^{(i)}_m \quad \forall i \neq j\right\}$$

whose union is S with $a^{(i)}_m, b^{(i)}_m \in D_2$. From the previous step we have

$$Q_0(S_m) = \int_{S_m} q_j \, d\mathscr{H}^{n-1}.$$

On the other hand it is easy to pass to the limit at the right hand side, while the left hand side also passes to the limit by assumptions (i) and (ii). Then the assertion follows.
The statement concerning $I \in \mathcal{I}^\circ_{G_1}$ is an obvious consequence. $\square$

Now we can prove Theorem 2.4.1.

Proof. We have already built a vector field $\boldsymbol{q} \in \mathcal{L}^1_{loc}(\operatorname{int} B; \mathbb{R}^n)$ and we know from Lemma 2.4.3 that

$$\forall S \in \mathcal{S}_{G_1} : Q_0(S) = \int_S \boldsymbol{q} \cdot \boldsymbol{n}_S \, d\mathscr{H}^{n-1},$$

$$\forall I \in \mathcal{I}^\circ_{G_1} : \left| \int_{\partial_* I} \boldsymbol{q} \cdot \boldsymbol{n}^I \, d\mathscr{H}^{n-1} \right| = |Q_0(\partial_* I)| \leqslant \lambda(I).$$

From Theorem 2.3.3 it follows that $\boldsymbol{q}$ has divergence measure with $|\operatorname{div} \boldsymbol{q}| \leqslant \lambda$. Let $\widehat{h}$ be associated with $\boldsymbol{q}$ according to Theorem 2.3.2. By Corollary 2.3.5, $\boldsymbol{q}$ induces a balanced Cauchy flux Q defined on $\mathcal{S}_{\widehat{h}\lambda}$. By Remark 2.1.4 there exists a full grid $G \subseteq G_1$ such that $\mathcal{S}_G \subseteq \mathcal{S}_{\widehat{h}\lambda}$. Then assertions (a) and (b) clearly follow.
Now assume that also $\check{Q}$ and $\check{\boldsymbol{q}}$ satisfy (a) for some $\check{G} \subseteq G_0$. If G' is a full grid with $G' \subseteq G$ and $G' \subseteq \check{G}$, we have

$$Q(S) = \int_S \boldsymbol{q} \cdot \boldsymbol{n}_S \, d\mathscr{H}^{n-1} = Q_0(S) = \check{Q}(S) = \int_S \check{\boldsymbol{q}} \cdot \boldsymbol{n}_S \, d\mathscr{H}^{n-1}$$

for any $S \in \mathcal{S}_{G'}$. From Theorem 2.2.5 and Proposition 2.3.6 we conclude that $\check{Q} = Q$ on almost all of $\mathcal{S}$ and $\check{\boldsymbol{q}}(x) = \boldsymbol{q}(x)$ for $\mathscr{L}^n$-a.e. $x \in \operatorname{int} B$. □

2.5 Integral representation, formulations of the balance law

Let B be a continuous body. The result of the previous section allows us to prove the converse of Corollary 2.3.5, which is one of the main goals of the chapter.

Theorem 2.5.1. *Let Q be a balanced Cauchy flux on B.*
Then there exists a vector field $\boldsymbol{q} \in \mathcal{L}^1_{loc}(\operatorname{int} B; \mathbb{R}^n)$ with divergence measure such that

$$Q(S) = \int_S \boldsymbol{q} \cdot \boldsymbol{n}_S \, d\mathscr{H}^{n-1}$$

on almost all of $\mathcal{S}$. Moreover, $\boldsymbol{q}$ is uniquely determined $\mathscr{L}^n$-almost everywhere.

Proof. Let h, ν and G be as in Proposition 2.2.2. Then the restriction of Q to $\mathcal{S}_G$ satisfies the assumptions (i), (ii) and (iii) considered in the previous section. Let Q', $\boldsymbol{q}$ and $G' \subseteq G$ be as in Theorem 2.4.1. Since $Q'(S) = Q(S)$ for any $S \in \mathcal{S}_{G'}$, we have $Q' = Q$ on almost all of $\mathcal{S}$ by Theorem 2.2.5. Then the integral representation formula follows.
The uniqueness of $\boldsymbol{q}$ is a consequence of Proposition 2.3.7. □

Moreover, we can prove the equivalence between the integral and the distributional formulation of the balance law. For the integral formulation, it turns out that it is enough to consider the elements of $\mathcal{I}^\circ_G$ for some full grid G.

Theorem 2.5.2. *Let $\boldsymbol{q} \in \mathcal{L}^1_{loc}(\operatorname{int} B; \mathbb{R}^n)$, let $\mu \in \mathfrak{M}(\operatorname{int} B)$ and let $u : \operatorname{int} B \to \mathbb{R}$ be a Borel function with $|u(x)| = 1$ for μ-a.e. $x \in \operatorname{int} B$.*
Then the following conditions are equivalent:

(a) *for almost every $M \in \mathcal{M}^\circ$, one has that $\boldsymbol{q}$ is $\mathscr{H}^{n-1}$-summable on $\partial_* M$ and*

$$\int_{\partial_* M} \boldsymbol{q} \cdot \boldsymbol{n}^M \, d\mathscr{H}^{n-1} = \int_M u \, d\mu;$$

(*b*) *there exists a full grid* G *such that, for every* $I \in \mathcal{I}_G^\circ$*, one has that* $\boldsymbol{q}$ *is* $\mathscr{H}^{n-1}$*- summable on* $\partial_* I$ *and*

$$\int_{\partial_* I} \boldsymbol{q} \cdot \boldsymbol{n}^I \, d\mathscr{H}^{n-1} = \int_I u \, d\mu;$$

(*c*) *for every* $f \in C_0^\infty(\operatorname{int} B)$*, one has that*

$$-\int_{\operatorname{int} B} \boldsymbol{q} \cdot \operatorname{grad} f \, d\mathscr{L}^n = \int_{\operatorname{int} B} f u \, d\mu.$$

Proof. $(a) \Longrightarrow (b)$ It follows from Proposition 1.2.16.
$(b) \Longrightarrow (c)$ From Theorem 2.3.3 it follows that $\boldsymbol{q}$ has divergence measure. Let h and ν be as in Theorem 2.3.4 and let $G_1 \subseteq G$ be a full grid with $\mathcal{M}_{G_1}^\circ \subseteq \mathcal{M}_{h\mu}^\circ \cap \mathcal{M}_{h\nu}^\circ$. Then we have

$$\int_I \operatorname{div} \boldsymbol{q} = \int_{\partial_* I} \boldsymbol{q} \cdot \boldsymbol{n}^I \, d\mathscr{H}^{n-1} = \int_I u \, d\mu$$

for any $I \in \mathcal{I}_{G_1}^\circ$. Since each open subset of $\operatorname{int} B$ is a countable disjoint union of elements of $\mathcal{I}_{G_1}^\circ$ up to a set which is both μ- and ν-negligible, we have

$$\forall f \in C_0^\infty(\operatorname{int} B) : \int_{\operatorname{int} B} f \operatorname{div} \boldsymbol{q} = \int_{\operatorname{int} B} f u \, d\mu$$

and the assertion follows.
$(c) \Longrightarrow (a)$ It follows from Theorem 2.3.4. □

2.6 Symmetric flux tensors

In this section we provide further information for vector Cauchy fluxes. We denote by $\operatorname{Lin}(\mathbb{R}^n; \mathbb{R}^N)$ the normed space of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^N$. If $L \in \operatorname{Lin}(\mathbb{R}^n; \mathbb{R}^N)$, then $L^T \in \operatorname{Lin}(\mathbb{R}^N; \mathbb{R}^n)$ will denote its transpose. If $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^N$, then $y \otimes x \in \operatorname{Lin}(\mathbb{R}^n; \mathbb{R}^N)$ is defined by $(y \otimes x)z = (x \cdot z)y$. When $n = N$, we set $x \wedge y = x \otimes y - y \otimes x$.
Let B be a continuous body with $B \subseteq \mathbb{R}^n$.

Definition 2.6.1. *Let* $\mathsf{q} \in \mathcal{L}^1_{loc}(\operatorname{int} B; \operatorname{Lin}(\mathbb{R}^n; \mathbb{R}^N))$*. We say that* $\operatorname{div} \mathsf{q}$ *is* a (local vector) measure *on* $\operatorname{int} B$*, if* $\operatorname{div} \mathsf{q}$ *is a (vector) distribution on* $\operatorname{int} B$ *of order* 0*. This means that for every compact subset* K *of* $\operatorname{int} B$ *there exists a constant* c_K *such that*

$$\left| \int_{\operatorname{int} B} \mathsf{q} \operatorname{grad} f \, d\mathscr{L}^n \right| \leqslant c_K \max_K |f|$$

whenever $f \in C_0^\infty(\operatorname{int} B)$ *and* $\operatorname{supt} f \subseteq K$*.*
In such a case, there exist $\mu \in \mathfrak{M}(\operatorname{int} B)$ *and a Borel map* $\boldsymbol{u} : \operatorname{int} B \to \mathbb{R}^N$ *such that* $|\boldsymbol{u}(x)| = 1$ *for* μ*-a.e.* $x \in \operatorname{int} B$ *and*

$$-\int_{\operatorname{int} B} \mathsf{q} \operatorname{grad} f \, d\mathscr{L}^n = \int_{\operatorname{int} B} f \, \boldsymbol{u} \, d\mu$$

for any Lipschitz function $f : \operatorname{int} B \to \mathbb{R}$ *with compact support. It is well known that* μ *is uniquely determined, while* $\boldsymbol{u}$ *is uniquely determined* μ*-almost everywhere. We put* $|\operatorname{div} \mathsf{q}| = \mu$*. Finally, if* $M \in \mathfrak{B}(\operatorname{int} B)$ *and* $f : M \to \mathbb{R}^N$ *is a Borel map such that* $f \wedge \boldsymbol{u}$ *is* μ*-summable on* M*, we set*

$$\int_M f \wedge \operatorname{div} \mathsf{q} = \int_M f \wedge \boldsymbol{u} \, d\mu.$$

Definition 2.1.5 can be easily adapted to *vector* balanced Cauchy fluxes $Q : \mathcal{D} \to \mathbb{R}^N$. These fluxes are in natural correspondence with tensor fields $\mathsf{q} \in \mathcal{L}^1_{loc}\left(\operatorname{int} B; \operatorname{Lin}\left(\mathbb{R}^n; \mathbb{R}^N\right)\right)$ with divergence measure. We shall not develop such details, as they are straightforward extensions of the results of the previous sections, but we shall study, in the case $n = N$, conditions under which the tensor $\mathsf{q}(x)$ is symmetric for $\mathscr{L}^n$-a.e. $x \in \operatorname{int} B$.

Proposition 2.6.2. *Let* $\mathsf{q} \in \mathcal{L}^1_{loc}\left(\operatorname{int} B; \operatorname{Lin}\left(\mathbb{R}^n; \mathbb{R}^N\right)\right)$ *be a tensor field with divergence measure and let* $\nu = |\operatorname{div} \mathsf{q}|$.
Then there exists $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ *such that, for every* $M \in \mathcal{M}^\circ_{h\nu}$ *and for every locally Lipschitz function* $f : \operatorname{int} B \to \mathbb{R}^N$, *one has that* $\boldsymbol{q}$ *is* $\mathscr{H}^{n-1}$*-summable on* $\partial_* M$ *and*

$$\int_M \left((\operatorname{grad} f)\, \mathsf{q}^T - \mathsf{q}\,(\operatorname{grad} f)^T\right) d\mathscr{L}^n = \int_{\partial_* M} f \wedge \left(\mathsf{q}\boldsymbol{n}^M\right) d\mathscr{H}^{n-1} - \int_M f \wedge \operatorname{div} \mathsf{q},$$

where $\operatorname{grad} f(x) \in \operatorname{Lin}\left(\mathbb{R}^n; \mathbb{R}^N\right)$ *denotes the Fréchet differential of* f *at* x.

Proof. It is sufficient to adapt Theorems 2.3.2 and 2.3.4. □

Theorem 2.6.3. *Let* $\mathsf{q} \in \mathcal{L}^1_{loc}(\operatorname{int} B; \operatorname{Lin}(\mathbb{R}^n; \mathbb{R}^n))$ *be a tensor field with divergence measure. Then the following conditions are equivalent:*

(*a*) *for each* $x_0 \in \mathbb{R}^n$ *and almost every* $M \in \mathcal{M}^\circ$, *one has that* q *is* $\mathscr{H}^{n-1}$*-summable on* $\partial_* M$ *and*

$$\int_{\partial_* M} (x - x_0) \wedge \left(\mathsf{q}(x)\boldsymbol{n}^M(x)\right) d\mathscr{H}^{n-1}(x) = \int_M (x - x_0) \wedge \operatorname{div} \mathsf{q}(x);$$

(*b*) *there exists* $\eta \in \mathfrak{M}(\operatorname{int} B)$ *such that, for each* $x_0 \in \mathbb{R}^n$ *and almost every* $M \in \mathcal{M}^\circ$, *one has that* q *is* $\mathscr{H}^{n-1}$*-summable on* $\partial_* M$ *and*

$$\left|\int_{\partial_* M} (x - x_0) \wedge \left(\mathsf{q}(x)\boldsymbol{n}^M(x)\right) d\mathscr{H}^{n-1}(x)\right| \leqslant \left(\sup_{x \in M} |x - x_0|\right) \eta(M);$$

(*c*) *there exist* $\eta \in \mathfrak{M}(\operatorname{int} B)$ *and a full grid* G *such that, for every* $x_0 \in \mathbb{R}^n$ *and every* $I \in \mathcal{I}_G$, *one has that* q *is* $\mathscr{H}^{n-1}$*-summable on* $\partial_* I$ *and*

$$\left|\int_{\partial_* I} (x - x_0) \wedge \left(\mathsf{q}(x)\boldsymbol{n}^I(x)\right) d\mathscr{H}^{n-1}(x)\right| \leqslant \left(\sup_{x \in I} |x - x_0|\right) \eta(I);$$

(*d*) *for* $\mathscr{L}^n$*-a.e.* $x \in \operatorname{int} B$ *there exists a sequence* (I_m) *of open* n*-dimensional intervals such that* $x \in I_m$, $\lim\limits_m (\operatorname{diam} I_m) = 0$, q *is* $\mathscr{H}^{n-1}$*-summable on* $\partial_* I_m$ *and*

$$\limsup_m \frac{(\operatorname{diam} I_m)^n}{\mathscr{L}^n(I_m)} < +\infty,$$

$$\lim_m \frac{\displaystyle\int_{\partial_* I_m} (\xi - x) \wedge \left(\mathsf{q}(\xi)\boldsymbol{n}^{I_m}(\xi)\right) d\mathscr{H}^{n-1}(\xi)}{\mathscr{L}^n(I_m)} = 0;$$

(*e*) $\mathsf{q}(x)$ *is symmetric for* $\mathscr{L}^n$*-a.e.* $x \in \operatorname{int} B$.

Proof. $(a) \implies (b)$ We have

$$\left| \int_M (x - x_0) \wedge \operatorname{div} \mathsf{q}(x) \right| \leqslant \left(\sup_{x \in M} |x - x_0| \right) |\operatorname{div} \mathsf{q}|(M),$$

whence the assertion.
$(b) \implies (c)$ This follows from Proposition 1.2.16.
$(c) \implies (d)$ Let $x \in \operatorname{int} B$ be such that

$$\limsup_{r \to 0^+} \frac{\eta(B_r(x))}{\mathscr{L}^n(B_r(x))} < +\infty.$$

It is well known that $\mathscr{L}^n$-a.e. $x \in \operatorname{int} B$ has this property. Moreover, let (I_m) be a sequence in $\mathcal{I}_G^\circ$ such that $x \in I_m$, $\lim_m (\operatorname{diam} I_m) = 0$ and

$$\limsup_m \frac{(\operatorname{diam} I_m)^n}{\mathscr{L}^n(I_m)} < +\infty.$$

Since $I_m \subseteq B_{\operatorname{diam} I_m}(x)$ and $(\operatorname{diam} I_m)^n \leqslant c\,\mathscr{L}^n(I_m)$ for some constant c, it is readily seen that

$$\limsup_m \frac{\eta(I_m)}{\mathscr{L}^n(I_m)} < +\infty,$$

whence the assertion.
$(d) \implies (e)$ Let $x \in \operatorname{int} B$ and (I_m) be as in assertion (d). Without loss of generality, we may assume that

$$\limsup_{r \to 0^+} \frac{|\operatorname{div} \mathsf{q}|(B_r(x))}{\mathscr{L}^n(B_r(x))} < +\infty,$$

$$\lim_{r \to 0^+} \frac{1}{\mathscr{L}^n(B_r(x))} \int_{B_r(x)} |\mathsf{q}(\xi) - \mathsf{q}(x)| \, d\mathscr{L}^n(\xi) = 0.$$

Then it is readily seen that

$$\limsup_m \frac{|\operatorname{div} \mathsf{q}|(I_m)}{\mathscr{L}^n(I_m)} < +\infty,$$

$$\lim_m \frac{1}{\mathscr{L}^n(I_m)} \int_{I_m} |\mathsf{q}(\xi) - \mathsf{q}(x)| \, d\mathscr{L}^n(\xi) = 0.$$

On the other hand, by Proposition 2.6.2 we have

$$\int_{I_m} \left(\mathsf{q}(\xi)^T - \mathsf{q}(\xi) \right) d\mathscr{L}^n(\xi) = \int_{\partial_* I_m} (\xi - x) \wedge \left(\mathsf{q}(\xi) \boldsymbol{n}^{I_m}(\xi) \right) d\mathscr{H}^{n-1}(\xi) - \int_{I_m} (\xi - x) \wedge \operatorname{div} \mathsf{q}(\xi),$$

hence

$$\begin{aligned} \left| \int_{I_m} \left(\mathsf{q}(\xi)^T - \mathsf{q}(\xi) \right) d\mathscr{L}^n(\xi) \right| \quad \leqslant \quad & \left| \int_{\partial_* I_m} (\xi - x) \wedge \left(\mathsf{q}(\xi) \boldsymbol{n}^{I_m}(\xi) \right) d\mathscr{H}^{n-1}(\xi) \right| + \\ & + (\operatorname{diam} I_m) \, |\operatorname{div} \mathsf{q}|(I_m). \end{aligned}$$

If we divide both sides by $\mathscr{L}^n(I_m)$ and we pass to the limit as $m \to \infty$, we get $\mathsf{q}(x)^T - \mathsf{q}(x) = 0$, whence the assertion.
$(e) \implies (a)$ It is sufficient to apply Proposition 2.6.2 with $f(x) = x - x_0$. $\square$

2.7 Balanced interactions

In this section we will study the case in which I obeys a balance law, following the ideas in [19]. Choosing for instance a thermodynamical approach, such a balance is expressed by the set-valued version of the First Law of Thermodynamics:

$$\dot{E}(A) = I(A, (\mathbb{R}^n \setminus A)_*),$$

where $\dot{E}(A)$ denotes the rate of change of internal energy in A and $I(A, (\mathbb{R}^n \setminus A)_*)$ the amount of heat transferred into A by its exterior (see [17, 19]). Moreover, the set function $\dot{E}$ is supposed to be Lipschitz with respect to the volume measure $\mathscr{L}^n$.
It is clear that, from a mathematical point of view, $\dot{E}$ can be forgotten and one can express the balance in a concise way by assuming that there exists $K \geqslant 0$ such that

$$|I(A, (\mathbb{R}^n \setminus A)_*)| \leqslant K\mathscr{L}^n(A).$$

In our setting, such a property does not make sense, since $(\mathbb{R}^n \setminus A)_* \notin \mathcal{N}^\circ$. Nevertheless, in view of the other assumptions of [19], such an inequality is in turn equivalent to

$$\exists K \geqslant 0 : |I(A, C)| \leqslant K\mathscr{L}^n(A)$$

whenever $(A, C) \in \mathfrak{D}$ and $\partial_* A \subseteq \partial_* C$ (so that between A and $(\mathbb{R}^n \setminus (A \cup C))_*$ there is no contact interaction).
The purpose of the next Definition 2.7.1 is to generalize and adapt such a condition to our setting. Moreover, we will see in Theorem 2.7.5 that, assuming that condition, also the interaction $I(A, (\mathbb{R}^n \setminus A)_*)$ can be naturally defined and fulfills a property of the form

$$\exists \lambda \in \mathfrak{M}(\operatorname{int} B) : |I(A, (\mathbb{R}^n \setminus A)_*)| \leqslant \lambda(A).$$

Definition 2.7.1. *A Cauchy interaction I is said to be* balanced, *if there exists $\lambda \in \mathfrak{M}(\operatorname{int} B)$ such that*

$$\partial_* A \subseteq \partial_* C \Longrightarrow |I(A, C)| \leqslant \lambda(A) \tag{2.6}$$

on almost all of $\mathfrak{D}^\circ$.

Proposition 2.7.2. *The following properties hold:*

(i) a Cauchy interaction I is balanced if and only if I_b and I_c are both balanced;

(ii) a body interaction I is balanced if and only if $\mu(K \times \operatorname{int} B) < +\infty$ for each compact subset $K \subseteq \operatorname{int} B$, where μ is given by Theorem 1.4.3; if this is the case, one has

$$|I(A, C)| \leqslant \lambda(A)$$

on almost all of $\mathfrak{D}^\circ$;

(iii) a contact interaction I is balanced if and only if the Cauchy flux induced by I is balanced.

Proof. (i) Let $\lambda \in \mathfrak{M}(\operatorname{int} B)$ be as in Definition 2.7.1 and let h, ν as in the proof of Theorem 1.3.2 with $\lambda \leqslant \nu$. Let also H be as in the proof of Theorem 1.3.2. If $A, C \in \mathcal{M}_H^\circ$ and $\operatorname{cl} A \cap \operatorname{cl} C = \varnothing$, let $\widehat{C} \in \mathcal{M}_H^\circ$ be such that $(A \cup C) \cap \widehat{C} = \varnothing$ and $\partial_* A \subseteq \partial_* \widehat{C}$. It follows that $\partial_* A \subseteq \partial_*(C \cup \widehat{C})$, hence

$$|I(A, C)| \leqslant |I(A, (C \cup \widehat{C})_*)| + |I(A, \widehat{C})| \leqslant 2\lambda(A).$$

Let now $(A, C) \in \mathfrak{D}_{h\eta}^\circ$ with $C \subseteq B$ and let (A_k, C_k) be a sequence as in the proof of Theorem 1.3.2 such that $\lim \lambda(A_k \triangle A) = 0$. We have that $|I(A_k, C_k)| \leqslant 2\lambda(A_k)$, then

$$|I_b(A, C)| \leqslant 2\lambda(A). \tag{2.7}$$

If $C \nsubseteq B$, inequality (2.7) still holds, since we can find again a similar $\widehat{C}$. In particular, I_b and I_c are both balanced. The converse is obvious.
(ii) Let I be a balanced body interaction. From (2.7) it follows that

$$|I(A, C)| \leqslant 2\lambda(A)$$

on almost all of $\mathfrak{D}^\circ$. Let $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ and $\nu \in \mathfrak{M}(\operatorname{int} B)$ be such that Theorem 1.4.3 and the preceding inequality hold on $\mathfrak{D}_{h\nu}^\circ$. Let G be a full grid such that $\mathfrak{D}_G^\circ \subseteq \mathfrak{D}_{h\nu}^\circ$. We denote with P the set on which $b = 1$ and with Q a normalized finite union of $2n$-dimensional G-intervals such that $\mu(P \triangle Q) < 1$. Let K be a compact subset of $\operatorname{int} B$ and let $Y \in \mathcal{M}_G^\circ$ be such that $K \subseteq Y$; clearly $\mu(Y \times Y) < +\infty$. Setting $E = (\operatorname{int} B) \setminus Y$, it is enough to prove that $\mu(Y \times E) < +\infty$; we argue by contradiction, supposing $\mu(Y \times E) = +\infty$. For every $m \in \mathbb{N}$ there exists a set $F_m \in \mathcal{M}_G^\circ$ with $\operatorname{cl} F_m \cap \operatorname{cl} Y = \varnothing$ and $\mu(Y \times F_m) > m$. The set $(Y \times F_m) \cap Q$ is a normalized finite union of $2n$-dimensional G-intervals, hence we can find some sets $Y_k \in \mathcal{I}_G^\circ$ and $G_k \in \mathcal{M}_G^\circ$ such that the Y_k's are mutually disjoint and

$$(Y \times F_m) \cap Q = \left(\bigcup_{k=1}^{q} (Y_k \times G_k) \right)_* .$$

In the same way,

$$((Y \times F_m) \setminus Q)_* = \left(\bigcup_{k=1}^{p} (Y'_k \times G'_k) \right)_* ,$$

where $Y'_k \in \mathcal{I}_G^\circ$ are mutually disjoint and $G'_k \in \mathcal{M}_G^\circ$. We have

$$\begin{aligned} 2\lambda(Y) \geqslant 2\lambda\left(\left(\bigcup_{k=1}^{q} Y_k \right)_* \right) \geqslant \left| \sum_{k=1}^{q} I(Y_k, G_k) \right| = \left| \int_{(Y \times F_m) \cap Q} b \, d\mu \right| \geqslant \\ \geqslant \mu((Y \times F_m) \cap Q) - 2. \end{aligned}$$

Acting in the same way, we can prove that

$$2\lambda(Y) \geqslant \mu\big(((Y \times F_m) \setminus Q)_* \big) - 2.$$

Adding the two inequalities we find that

$$4\lambda(Y) \geqslant \mu(Y \times F_m) - 4 \geqslant m - 4;$$

since Y has compact closure in $\operatorname{int} B$, letting $m \to +\infty$ we get the contradiction.
Conversely, suppose that $\mu(K \times \operatorname{int} B) < +\infty$ for every compact subset $K \subseteq \operatorname{int} B$ and consider the measure $\lambda = \mu(\,\cdot\, \times \operatorname{int} B) + \mu_e$; it follows immediately that $\lambda \in \mathfrak{M}(\operatorname{int} B)$ and

$$|I(A,C)| \leqslant \left| \int_{A\times(C\cap \operatorname{int} B)} b\, d\mu \right| + \left| \int_A b_e\, d\mu_e \right| \leqslant \mu(A \times (C \cap B)) + \mu_e(A) \leqslant \lambda(A)$$

on almost all of $\mathfrak{D}^\circ$, hence I is balanced.
(iii) It is obvious. □

Theorem 2.7.3. *Let I_1, I_2 be two balanced Cauchy interactions that agree on $\mathfrak{D}^\circ_G$ for some full grid G. Then $I_1 = I_2$ on almost all of $\mathfrak{D}^\circ$.*

Proof. Let $I_1 = (I_1)_b + (I_1)_c$, $I_2 = (I_2)_b + (I_2)_c$ where $(I_j)_b$ are body interactions and $(I_j)_c$ contact interactions. From Theorem 1.3.2, we have that $(I_1)_b = (I_2)_b$ on almost all of $\mathfrak{D}^\circ$; in particular, there exists a full grid H such that $(I_1)_c = (I_2)_c$ on $\mathcal{I}^\circ_H$. Defining two Cauchy fluxes Q_1, Q_2 by the formula

$$Q_j(\partial_* A \cap \partial_* C) = (I_j)_c(A, C)$$

as in (a) of Theorem 2.1.8, it follows that Q_1 and Q_2 are balanced and agree on $\mathcal{S}_H$. Hence they agree on almost all of $\mathcal{S}$ by Theorem 2.2.5. By (c) of Theorem 2.1.8, it comes that $(I_1)_c = (I_2)_c$ on almost all of $\mathfrak{D}^\circ$. □

Theorem 2.7.4. *Let I be a balanced contact interaction. Then there exists a vector field $\boldsymbol{q} \in \mathcal{L}^1_{loc}(\operatorname{int} B; \mathbb{R}^n)$ with divergence measure such that*

$$I(A,C) = \int_{\partial_* A \cap \partial_* C} \boldsymbol{q} \cdot \boldsymbol{n}^A \, d\mathscr{H}^{n-1}$$

on almost all of $\mathfrak{D}^\circ$.
Moreover, $\boldsymbol{q}$ is uniquely determined $\mathscr{L}^n$-almost everywhere.

Proof. Let Q be a Cauchy flux such that

$$Q(\partial_* A \cap \partial_* C) = I(A, C)$$

on almost all of $\mathfrak{D}^\circ$, as in (a) of Theorem 2.1.8. Since I is balanced, then Q is also balanced. Moreover, Q is uniquely determined on almost all of $\mathcal{S}$.
Now we can apply Theorem 2.5.1 and obtain the assertion. □

For a balanced Cauchy interaction I we can give the following integral representation.

Theorem 2.7.5. *Let I be a balanced Cauchy interaction and let b, b_e, μ, μ_e and $\boldsymbol{q}$ as in Theorems 1.4.3 and 2.7.4. Then there exist $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ and $\nu \in \mathfrak{M}(\operatorname{int} B)$ such that*

$$I(A,C) = \begin{cases} \displaystyle\int_{A\times C} b\, d\mu + \int_{\partial_* A\cap\partial_* C} \boldsymbol{q}\cdot\boldsymbol{n}^A\, d\mathscr{H}^{n-1} & \text{if } C \subseteq B, \\ \displaystyle\int_{A\times(C\cap \operatorname{int} B)} b\, d\mu + \int_A b_e\, d\mu_e + \int_{\partial_* A\cap\partial_* C} \boldsymbol{q}\cdot\boldsymbol{n}^A\, d\mathscr{H}^{n-1} & \text{otherwise,} \end{cases} \tag{2.8}$$

for every $(A, C) \in \mathfrak{D}^\circ_{h\nu}$ *and the same formula allows a natural extension to all*

$$\mathfrak{D}^\circ_{h\nu} \cup \{(A, C) \in \mathcal{M}^\circ_{h\nu} \times \mathcal{N} : (\mathbb{R}^n \setminus C)_* \in \mathcal{M}^\circ_{h\nu}, A \cap C = \varnothing\}.$$

Moreover, there exists $\lambda \in \mathfrak{M}\,(\operatorname{int} B)$ *such that*

$$\forall A \in \mathcal{M}^\circ_{h\nu} : |I(A, (\mathbb{R}^n \setminus A)_*)| \leqslant \lambda(A).$$

Proof. Let $h_0 \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$, $\nu \in \mathfrak{M}\,(\operatorname{int} B)$ and $\lambda \in \mathfrak{M}\,(\operatorname{int} B)$ be such that (2.6) and Theorems 1.4.3 and 2.7.4 hold on $\mathfrak{D}^\circ_{h_0\nu}$. Then it is easy to deduce (2.8). Setting $h = h_0 + |\boldsymbol{q}|$ and remembering that $\mu(K \times \operatorname{int} B) < +\infty$ for every compact subset $K \subseteq \operatorname{int} B$, it is possible to extend the domain of I as stated in the assertion.
Moreover, let G be a full grid with $\mathcal{M}^\circ_G \subseteq \mathcal{M}^\circ_{h\nu}$. For a given $A \in \mathcal{M}^\circ_{h\nu}$, we can find a sequence (Y_k) in $\mathcal{M}^\circ_G$ such that $\operatorname{cl} A \subseteq Y_k$ and $\bigcup\limits_{k=1}^{\infty} Y_k = \operatorname{int} B$. As I is balanced, we have

$$|I(A, (Y_k \setminus A)_* \cup (\mathbb{R}^n \setminus B)_*)| \leqslant \lambda(A),$$

and the left member goes to $|I(A, (\mathbb{R}^n \setminus A)_*)|$ by the Dominated Convergence Theorem. □

Finally, we can state a weak form of the balance equation for a balanced Cauchy interaction.

Theorem 2.7.6. *Let I be a balanced Cauchy interaction and let $\mu, \mu_e, b, b_e, \boldsymbol{q}$ be as in Theorems 1.4.3 and 2.7.4. Then there exist $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$, $\nu \in \mathfrak{M}\,(\operatorname{int} B)$, $\gamma \in \mathfrak{M}\,(\operatorname{int} B)$ and a Borel function $c : \operatorname{int} B \to \mathbb{R}$ such that $|c(x)| = 1$ for γ-a.e. $x \in \operatorname{int} B$ and*

$$\int_A c\, d\gamma = I(A, (\mathbb{R}^n \setminus A)_*) + \int_{A\times A} b\, d\mu$$

for every $A \in \mathcal{M}^\circ_{h\nu}$.
Moreover, γ is uniquely determined and c is uniquely determined γ-a.e.
Finally, one has

$$\int_{\operatorname{int} B} f\, c\, d\gamma = -\int_{\operatorname{int} B} \boldsymbol{q} \cdot \operatorname{grad} f\, d\mathscr{L}^n + \int_{\operatorname{int} B} f\, b_e\, d\mu_e + \\ + \iint_{\operatorname{int} B \times \operatorname{int} B} f(x)\, b(x, y)\, d\mu(x, y)$$

for every $f \in C^\infty_0(\operatorname{int} B)$.

Proof. Let $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ and $\nu \in \mathfrak{M}\,(\operatorname{int} B)$ be as in Theorem 2.7.5; then we can define a function $g : \mathcal{M}^\circ_{h\nu} \to \mathbb{R}$ setting

$$g(A) = \int_A \operatorname{div} \boldsymbol{q} + \int_{A\times \operatorname{int} B} b\, d\mu + \int_A b_e\, d\mu_e.$$

Extending g to a (signed) measure on $\mathfrak{B}\,(\operatorname{int} B)$, we can find $\gamma \in \mathfrak{M}\,(\operatorname{int} B)$ and a Borel function $c : \operatorname{int} B \to \mathbb{R}$ such that $|c(x)| = 1$ for γ-a.e. $x \in \operatorname{int} B$ and

$$\int_A c\, d\gamma = g(A)$$

for every $A \in \mathcal{M}^\circ_{h\nu}$. The measure γ is clearly unique and the function c is uniquely determined γ-a.e.
The last assertion follows from the Gauss-Green Theorem. □

2.8 Extension of balanced interactions

Although the domain of a Cauchy interaction is quite large, in this section we will prove that each function defined only on $\mathfrak{D}^\circ_G$, for some full grid G, and satisfying suitable conditions, can be uniquely extended to almost all of $\mathfrak{D}^\circ$.
Let $G_0 = (x_0, (e_1, \dots, e_n), \widehat{G}_0)$ denote a full grid and $I_0 : \mathfrak{D}^\circ_{G_0} \to \mathbb{R}$ a map satisfying the following properties:

(a) I_0 is biadditive;

(b) there exist $h \in \mathcal{L}^1_{loc,+}(\mathrm{int}\, B)$, $\eta \in \mathfrak{M}(\mathrm{int}\, B \times \mathrm{int}\, B)$ and $\eta_e \in \mathfrak{M}(\mathrm{int}\, B)$ such that

$$|I_0(A,C)| \leqslant \begin{cases} \displaystyle\int_{\partial_* A \cap \partial_* C} h \, d\mathscr{H}^{n-1} + \eta(A \times C) & \text{if } C \subseteq B, \\ \displaystyle\int_{\partial_* A \cap \partial_* C} h \, d\mathscr{H}^{n-1} + \eta(A \times (C \cap B)) + \eta_e(A) & \text{otherwise,} \end{cases}$$

for every $(A, C) \in \mathfrak{D}^\circ_{G_0}$.

Lemma 2.8.1. *There exist a full grid $G \subseteq G_0$ and two functions $(I_0)_b, (I_0)_c : \mathfrak{D}_G \to \mathbb{R}$ satisfying properties (a) and (b) for every $(A, C) \in \mathfrak{D}^\circ_G$ with $h = 0$ and $\eta = 0, \eta_e = 0$ respectively, such that $I_0 = (I_0)_b + (I_0)_c$ on $\mathfrak{D}^\circ_G$.*
Moreover, if $\check{G}$, $(\check{I}_0)_b$ and $(\check{I}_0)_c$ have the same properties, then $(\check{I}_0)_b = (I_0)_b$ and $(\check{I}_0)_c = (I_0)_c$ on $\mathfrak{D}^\circ_G \cap \mathfrak{D}_{\check{G}}$.

Proof. Let G be a full grid such that $G \subseteq G_0$ and $\int_{\partial_* A} h \, d\mathscr{H}^{n-1} < +\infty$, $\eta((\partial_* A) \times \mathrm{int}\, B) = \eta((\mathrm{int}\, B) \times \partial_* A) = \eta_e(\partial_* A) = 0$ for every $A \in \mathcal{I}^\circ_G$. Let $(A, C) \in \mathfrak{D}^\circ_G$; then

$$A = \left\{ x \in \mathbb{R}^n : \quad a^{(j)} < (x - x_0) \cdot e_j < b^{(j)} \quad \forall j = 1, \dots, n \right\},$$

$$C = \left\{ x \in \mathbb{R}^n : \quad c^{(j)} < (x - x_0) \cdot e_j < d^{(j)} \quad \forall j = 1, \dots, n \right\},$$

for some $a^{(j)}, b^{(j)}, c^{(j)}, d^{(j)} \in G$. If $\partial_* A \cap \partial_* C = \varnothing$, then we set $(I_0)_b(A, C) = I_0(A, C)$ and $(I_0)_c(A, C) = 0$. Elsewhere, denote by i the index in $\{1, \dots, n\}$ such that

$$\partial_* A \cap \partial_* C \subseteq \{x \in \mathbb{R}^n : \quad x \cdot e_i = 0\}$$

and suppose that $b^{(i)} \leqslant c^{(i)}$. Let (s_k) be a sequence in G such that $s_k \downarrow c^{(i)}$ as $k \to \infty$. We set

$$C_k = C \cap \{x \in \mathbb{R}^n : \quad x \cdot e_i > s_k\}.$$

Then it is clear that $(A, C_k) \in \mathfrak{D}^\circ_G$ for every $k \in \mathbb{N}$, $(I_0(A, C_k))$ is a Cauchy sequence in $\mathbb{R}$ and $|I_0(A, C_k)| \leqslant \eta(A \times C_k)$. Moreover,

$$\begin{aligned} |I_0(A,C) - I_0(A, C_k)| \leqslant |I_0(A, (C \setminus C_k)_*)| \leqslant \\ \leqslant \int_{\partial_* A \cap \partial_* (C \setminus C_k)} h \, d\mathscr{H}^{n-1} + \eta(A \times (C \setminus C_k)). \end{aligned}$$

We define

$$(I_0)_b(A,C) = \begin{cases} \lim\limits_k I_0(A, C_k) & \text{if } C \subseteq B, \\ I_0(A, (\mathbb{R}^n \setminus B)_*) + \lim\limits_k I_0(A, (C \cap B)_k) & \text{otherwise,} \end{cases}$$

and also

$$(I_0)_c(A,C) = I_0(A,C) - (I_0)_b(A,C).$$

Then $(I_0)_b$ and $(I_0)_c$ satisfy (a) and (b) with $h = 0$ and $\eta = 0, \eta_e = 0$ respectively. The remainder of the proof is now easy. □

Lemma 2.8.2. *Let $I_0 : \mathfrak{D}^\circ_{G_0} \to \mathbb{R}$ be a map satisfying properties (a) and (b) with $h = 0$. Then there exists a body interaction I such that:*

(i) its domain contains $\mathfrak{D}^\circ_{G_0}$;

(ii) it coincides with I_0 on $\mathfrak{D}^\circ_{G_0}$.

Moreover, if another body interaction $\check{I}$ shares properties (i) and (ii), then $\check{I} = I$ on almost all of $\mathfrak{D}^\circ$.

Proof. Following the proof of Theorem 1.4.3, we find a measure $\mu \in \mathfrak{M}(\operatorname{int} B \times \operatorname{int} B)$ and a function $b \in L^1_{loc}(\operatorname{int} B \times \operatorname{int} B, \mu)$ such that

$$I_0(A,C) = \int_{A \times C} b \, d\mu$$

for every $(A,C) \in \mathfrak{D}^\circ_{G_0}$ with $C \subseteq B$. In the same way we find a measure $\mu_e \in \mathfrak{M}(\operatorname{int} B)$ and a function $b_e \in L^1_{loc}(\operatorname{int} B, \mu_e)$ with

$$I_0(A, (\mathbb{R}^n \setminus B)_*) = \int_A b_e \, d\mu_e.$$

Defining, whenever possible,

$$I(A,C) = \begin{cases} \displaystyle\int_{A \times C} b \, d\mu & \text{if } C \subseteq B, \\ \displaystyle\int_{A \times (C \cap B)} b \, d\mu + \int_A b_e \, d\mu_e & \text{otherwise,} \end{cases}$$

we have that the domain of I contains $\mathfrak{D}^\circ_{G_0}$, I is a body interaction by Theorem 1.4.2 and

$$I_0(A,C) = I(A,C) \quad \text{for every } (A,C) \in \mathfrak{D}^\circ_{G_0}.$$

If $\check{I}$ is another body interaction that extends I_0, it is obvious that $\check{I}(A,C) = I(A,C)$ for every $(A,C) \in \mathfrak{D}^\circ_{G_0}$; then by Theorem 1.3.2 we have that $\check{I} = I$ on almost all of $\mathfrak{D}^\circ$. □

Now we require the map I_0 to satisfy also the following balance property:

(c) there exists $\lambda \in \mathfrak{M}(\operatorname{int} B)$ such that

$$\left|\sum_{j=1}^{k} I_0(A, C^{(j)})\right| \leqslant \lambda(A)$$

whenever $(A, C^{(j)}) \in \mathfrak{D}^\circ_{G_0}$ for every $j = 1, \dots, k$, the sets $C^{(j)}$ are mutually disjoint and $\partial_* A \subseteq \partial_* \left(\bigcup_{j=1}^{k} C^{(j)}\right)$.

Lemma 2.8.3. *Consider the full grid G and the maps $(I_0)_b$ and $(I_0)_c$ of Lemma 2.8.1; consider also the extension I_b of $(I_0)_b$, as stated in Lemma 2.8.2. Then the following facts hold:*

(i) *there exist a balanced contact interaction I_c and a full grid $H \subseteq G$ such that the domain of I_c contains $\mathfrak{D}^\circ_H$ and $I_c = (I_0)_c$ on $\mathfrak{D}^\circ_H$; moreover, if $\check{H}$ and $\check{I}_c$ have the same properties of H and I_c, then $\check{I}_c = I_c$ on almost all of $\mathfrak{D}^\circ$;*

(ii) *I_b is balanced.*

Proof. (i) First of all, we will prove that $(I_0)_c$ satisfies property (c). In fact, for $j = 1, \dots, k$ let $(A, C^{(j)}) \in \mathfrak{D}^\circ_G$ be such that the sets $C^{(j)}$ are mutually disjoint and $\partial_* A \subseteq \partial_* \left(\bigcup_{j=1}^{k} C^{(j)}\right)$; then consider the sequences $(C^{(j)}_k)$ as in the proof of Lemma 2.8.1. We have that

$$(I_0)_c(A, C^{(j)}) = \lim_k I_0(A, (C^{(j)} \setminus C^{(j)}_k)_*)$$

and $\partial_* A \subseteq \partial_* \left(\bigcup_{j=1}^{k} (C^{(j)} \setminus C^{(j)}_k)_*\right)$; hence

$$\left|\sum_{j=1}^{k} (I_0)_c(A, C_j)\right| = \lim_k \left|\sum_{j=1}^{k} I_0(A, (C^{(j)} \setminus C^{(j)}_k)_*)\right| \leqslant \lambda(A).$$

Now let $S \in \mathcal{S}_G$; then there exists $(A, C) \in \mathfrak{D}^\circ_G$ such that $(\partial_* A \cap \partial_* C, \boldsymbol{n}^A|_{\partial_* A \cap \partial_* C}) = (S, \boldsymbol{n}_S)$. If $(\widehat{A}, \widehat{C})$ has the same property, by biadditivity of $(I_0)_c$ and properties (a) and (b) it is easy to prove that

$$(I_0)_c(A, C) = (I_0)_c(A \cap \widehat{A}, C \cap \widehat{C}) = (I_0)_c(\widehat{A}, \widehat{C}).$$

This allows us to define the map

$$\begin{aligned} Q_0 :\ & \mathcal{S}_G \longrightarrow \mathbb{R} \\ & S \mapsto (I_0)_c(A, C), \end{aligned}$$

which happens to satisfy (i), (ii) and (iii) of Section 2.4.
Combining Theorem 2.4.1 with Theorem 2.1.8 and Proposition 2.7.2, it results that there exist a balanced contact interaction I_c and a full grid $H \subseteq G$ such that the domain of I_c contains $\mathfrak{D}^\circ_H$ and

$$I_c(A, C) = (I_0)_c(A, C) \quad \text{for every } (A, C) \in \mathfrak{D}^\circ_H.$$

(ii) This is easily proved nothing that, by difference, also $(I_0)_b$ satisfies property (c). ∎

We summarize Lemmas 2.8.1, 2.8.2 and 2.8.3 in the following statement.

Theorem 2.8.4. *There exists a full grid $G \subseteq G_0$ and a balanced Cauchy interaction I such that the domain of I contains $\mathfrak{D}^\circ_G$ and $I = I_0$ on $\mathfrak{D}^\circ_G$.*
Moreover, if $\check{G}$ and $\check{I}$ have the same properties of G and I, then $\check{I} = I$ on almost all of $\mathfrak{D}^\circ$.

2.9 Exploiting the whole body

Throughout this chapter, we have presented the weakest approach which allows to obtain the equivalence between the integral and the distributional versions of the balance law. For this reason, only the topological interior of B plays a true role and compact subsets of $\operatorname{int} B$ are so much recalled.
However, one could be interested in a different setting, where the whole B is involved (let us point out that $\operatorname{int} B$ may even be empty). The purpose of this section is to show how this case can be reduced to the former approach.
Let us first prove another property of the density of a balanced Cauchy flux. In a few words, when the flux concentrates around a subbody, then the density vanishes almost everywhere outside that subbody.

Theorem 2.9.1. *Let Q be a balanced Cauchy flux such that there exists $M \in \mathcal{M}$ with*

$$Q(S) = Q(S \cap M) \tag{2.9}$$

on almost all of $\mathcal{S}$. Let $\boldsymbol{q} \in L^1_{loc}(\operatorname{int} B; \mathbb{R}^n)$ be the density associated with Q. Then $\boldsymbol{q}(x) = 0$ for a.e. $x \in \operatorname{int} B \setminus M$.

Proof. Let $G = \big(x_0, (e_1 \dots, e_n), \widehat{G}\big)$ be a full grid such that the integral representation of Q holds on $\mathcal{S}_G$. Taking into account (2.9) and Fubini's Theorem, for any

$$I = \Big\{x \in \mathbb{R}^n : \quad a^{(i)} < (x - x_0) \cdot e_i < b^{(i)} \quad \forall i = 1, \dots, n\Big\} \in \mathcal{I}^\circ_G$$

and any $j = 1, \dots, n$ one has

$$\begin{aligned} \int_{I \setminus M} \boldsymbol{q}^{(j)} \, d\mathscr{L}^n &= \int_{a^{(j)}}^{b^{(j)}} \left[\int_{\sigma_{j,s}(I) \setminus M} \boldsymbol{q}(x) \cdot e_j \, d\mathscr{H}^{n-1}(x) \right] d\mathscr{L}^1(s) = \\ &= \int_{a^{(j)}}^{b^{(j)}} Q(\sigma_{j,s}(I) \setminus M) \, d\mathscr{H}^{n-1}(x) = 0. \end{aligned}$$

Take now $x \notin M$ such that x is a Lebesgue point for the functions $\boldsymbol{q}$ and $\chi_M \boldsymbol{q}$ and consider a sequence of cubes $(J_k) \subseteq \mathcal{I}^\circ_G$ with $x \in J_k$ and $\operatorname{diam} J_k \to 0$ as $k \to +\infty$. It follows

$$\boldsymbol{q}(x) = \lim_{k \to 0} \frac{\int_{J_k} \boldsymbol{q} \, d\mathscr{L}^n}{\mathscr{L}^n(J_k)} = \lim_{k \to 0} \frac{\int_{J_k \cap M} \boldsymbol{q} \, d\mathscr{L}^n}{\mathscr{L}^n(J_k)} = \lim_{k \to 0} \frac{\int_{J_k} \chi_M \boldsymbol{q} \, d\mathscr{L}^n}{\mathscr{L}^n(J_k)} = 0$$

and the proof is complete. □

Let $\mathfrak{M}_f\,(B)$ be the collection of (positive) Borel measures ν on $\mathfrak{B}\,(B)$ with $\nu(B) < +\infty$ and $\mathcal{L}^1_+\,(B)$ be the set of Borel functions $h : B \to [0, +\infty]$ with $\int_B h \, d\mathscr{L}^n < +\infty$. We denote by $\overline{\mathcal{M}}$ the family of normalized subsets A of B with finite perimeter. Moreover, we set

$$\overline{\mathcal{N}} = \left\{C \subseteq \mathbb{R}^n : \, C \in \overline{\mathcal{M}} \text{ or } (\mathbb{R}^n \setminus C)_* \in \overline{\mathcal{M}}\right\},$$
$$\overline{\mathfrak{D}} = \left\{(A, C) \in \overline{\mathcal{M}} \times \overline{\mathcal{N}} : \, A \cap C = \varnothing\right\}.$$

Given $h \in \mathcal{L}^1_+\,(B)$ and $\nu \in \mathfrak{M}_f\,(B)$, we set

$$\overline{\mathcal{M}}_{h\nu} = \left\{A \in \overline{\mathcal{M}} : \int_{B \cap \partial_* A} h \, d\mathscr{H}^{n-1} < +\infty, \, \nu(B \cap \partial_* A) = 0\right\},$$
$$\overline{\mathcal{N}}_{h\nu} = \left\{C \in \overline{\mathcal{N}} : C \in \overline{\mathcal{M}}_{h\nu} \text{ or } (\mathbb{R}^n \setminus C)_* \in \overline{\mathcal{M}}_{h\nu}\right\},$$
$$\overline{\mathfrak{D}}_{h\nu} = \overline{\mathfrak{D}} \cap \left(\overline{\mathcal{M}}_{h\nu} \times \overline{\mathcal{N}}_{h\nu}\right),$$
$$\overline{\mathcal{S}}_{h\nu} = \left\{S \in \mathcal{S} : \, S \subseteq B \text{ and } S \text{ is subordinated to some } A \in \overline{\mathcal{M}}_{h\nu}\right\}.$$

We will say that a property π *holds almost everywhere* in $\overline{\mathcal{M}}$, if there are $h \in \mathcal{L}^1_+\,(B)$ and $\nu \in \mathfrak{M}_f\,(B)$ such that π holds on $\overline{\mathcal{M}}_{h\nu}$. The same for $\overline{\mathcal{N}}$, $\overline{\mathfrak{D}}$ and $\overline{\mathcal{S}}$.
Let now $\mathcal{P}$ be a set containing almost all of $\overline{\mathcal{S}}$ and consider a function $Q : \mathcal{P} \to \mathbb{R}$ such that:

(a) if S, T are compatible and disjoint with $S \cup T \in \mathcal{P}$, then

$$Q(S \cup T) = Q(S) + Q(T);$$

(b) there exist $h \in \mathcal{L}^1_+\,(B)$ with

$$|Q(S)| \leqslant \int_S h \, d\mathscr{H}^{n-1}$$

almost everywhere in $\overline{\mathcal{S}}$;

(c) there exists $\nu \in \mathfrak{M}_f\,(B)$ with

$$|Q(B \cap \partial_* A)| \leqslant \nu(A)$$

almost everywhere in $\overline{\mathcal{M}}$.

Theorem 2.9.2. *There exists an essentially unique function $\boldsymbol{q} \in L^1(\mathbb{R}^n; \mathbb{R}^n)$ with divergence measure such that $\boldsymbol{q} = 0$ a.e. in $\mathbb{R}^n \setminus B$, the total variation of* $\operatorname{div} \boldsymbol{q}$ *is bounded on $\mathbb{R}^n$ and*

$$Q(S) = \int_S \boldsymbol{q} \cdot \boldsymbol{n}_S \, d\mathscr{H}^{n-1}$$

almost everywhere in $\overline{\mathcal{S}}$.

Proof. Modify h and ν in a way such that $\mathcal{P} \subseteq \overline{\mathcal{S}}_{h\nu}$ and (b), (c) holds on $\overline{\mathcal{S}}_{h\nu}$. Given $R > 0$ such that $\operatorname{cl} B \subseteq B_R\,(0)$, we set $\widehat{B} = B_R\,(0)$ and refer the sets $\mathcal{M}^\circ$, $\mathcal{N}^\circ$, $\mathfrak{D}^\circ$ and $\mathcal{S}$ to the body $\widehat{B}$. Then consider the function $\widehat{h} \in \mathcal{L}^1_{loc,+}(\operatorname{int} \widehat{B})$ which extends h to zero outside B and the measure $\widehat{\nu} \in \mathfrak{M}(\operatorname{int} \widehat{B})$ defined by $\widehat{\nu}(E) = \nu(E \cap B)$. It can be verified that the function $\widehat{Q} : \mathcal{S}_{\widehat{h}\widehat{\nu}} \to \mathbb{R}$ defined by

$$\widehat{Q}(S) = Q(S \cap B)$$

is a balanced Cauchy flux on $\widehat{B}$. Then there exists $\widehat{\boldsymbol{q}} \in \mathcal{L}^1_{loc}(\widehat{B}; \mathbb{R}^n)$ with divergence measure such that

$$\widehat{Q}(S) = \int_S \widehat{\boldsymbol{q}} \cdot \boldsymbol{n}_S \, d\mathscr{H}^{n-1}$$

on almost all of $\mathcal{S}$. In particular, one has

$$Q(S) = \int_S \widehat{\boldsymbol{q}} \cdot \boldsymbol{n}_S \, d\mathscr{H}^{n-1}$$

for almost every $S \in \overline{\mathcal{S}}$. Moreover, taking into account Theorem 2.9.1, one has that $\widehat{\boldsymbol{q}} = 0$ for a.e. $x \in \widehat{B} \setminus B$. If $\boldsymbol{q}$ is the extension of $\widehat{\boldsymbol{q}}$ to $\mathbb{R}^n$ with value 0 outside $\widehat{B}$, then $\boldsymbol{q} \in L^1(\mathbb{R}^n; \mathbb{R}^n)$, the total variation of $\operatorname{div} \boldsymbol{q}$ is bounded on $\mathbb{R}^n$ and $\boldsymbol{q} = 0$ a.e. in $\mathbb{R}^n \setminus B$. Finally, such a $\boldsymbol{q}$ is unique $\mathscr{L}^n$-a.e. by Proposition 2.3.6. □

Now consider a set $\mathcal{P} \subseteq \overline{\mathfrak{D}}$ containing almost all of $\overline{\mathfrak{D}}$ and a function $I : \mathcal{P} \to \mathbb{R}$ such that:

(a) I is biadditive;

(b) there exist $h \in \mathcal{L}^1_+(B)$, $\eta \in \mathfrak{M}_f(B \times B)$ and $\eta_e \in \mathfrak{M}_f(B)$ with

$$|I(A, C)| \leqslant \begin{cases} \displaystyle\int_{B \cap \partial_* A \cap \partial_* C} h \, d\mathscr{H}^{n-1} + \eta(A \times C) & \text{if } C \subseteq B, \\ \displaystyle\int_{B \cap \partial_* A \cap \partial_* C} h \, d\mathscr{H}^{n-1} + \eta(A \times (C \cap B)) + \eta_e(A) & \text{otherwise,} \end{cases}$$

almost everywhere in $\overline{\mathfrak{D}}$;

(c) there exists $\nu \in \mathfrak{M}_f(B)$ with

$$\partial_* A \subseteq \partial_* C \Rightarrow |I(A, C)| \leqslant \nu(A)$$

almost everywhere in $\overline{\mathfrak{D}}$.

Theorem 2.9.3. *There exist $\mu \in \mathfrak{M}_f(B \times B)$, $\mu_e \in \mathfrak{M}_f(B)$, two Borel functions $b : B \times B \to \mathbb{R}$, $b_e : B \to \mathbb{R}$ and a field $\boldsymbol{q} \in \mathcal{L}^1_{loc}(\mathbb{R}^n; \mathbb{R}^n)$ with divergence measure, such that $\boldsymbol{q} = 0$ a.e. in $\mathbb{R}^n \setminus B$, the total variation of $\operatorname{div} \boldsymbol{q}$ is bounded on $\mathbb{R}^n$ and the formula*

$$I(A, C) = \begin{cases} \displaystyle\int_{A \times C} b \, d\mu + \int_{\partial_* A \cap \partial_* C} \boldsymbol{q} \cdot \boldsymbol{n}^A \, d\mathscr{H}^{n-1} & \textit{if } C \subseteq B, \\ \displaystyle\int_{A \times (C \cap \operatorname{int} B)} b \, d\mu + \int_A b_e \, d\mu_e + \int_{\partial_* A \cap \partial_* C} \boldsymbol{q} \cdot \boldsymbol{n}^A \, d\mathscr{H}^{n-1} & \textit{otherwise,} \end{cases}$$

holds almost everywhere in $\overline{\mathfrak{D}}$.

Proof. Following the idea of the preceding theorem, we define a function $\widehat{I} : \mathfrak{D}^{\circ}_{\hat{h}\hat{\nu}} \to \mathbb{R}$ setting

$$\widehat{I}(A, C) = \begin{cases} I(A \cap B, C \cap B) & \text{if } C \subseteq \widehat{B}, \\ I(A \cap B, C \cap B) + I(A \cap B, (\mathbb{R}^n \setminus B)_*) & \text{otherwise.} \end{cases}$$

It can be verified that $\widehat{I}$ is a balanced Cauchy interaction on $\widehat{B}$ and gives the integral representation for I. $\square$

Chapter 3

Balance laws with inequalities: the case of entropy

In this chapter we reconsider the classical formulation of the Second Law of Thermodynamics in the general framework of fields with divergence measure. To our knowledge, the first attempt to extend the Second Law of Thermodynamics to the framework of Geometric Measure Theory is [19]. Here, with respect to Chapter 1, a more general form of the balance law is considered, in the sense that we survey balance laws with inequalities, as well as superadditive entropy production functions. Two main improvements have been made: first of all, we obtain the existence of temperatures and the Clausius-Duhem inequality with flux vector having divergence measure. Second, and perhaps more important, we state a weaker form of the usual axioms which constitute the statement of the Second Law of Thermodynamics, in order to have the validity of the Clausius-Duhem inequality on almost all subbodies. Indeed, the validity of the Second Law on a particularly simple class of multi-intervals is sufficient to extend the result on almost all subbodies of finite perimeter, which is a very wide class. In this framework it appears that the family of multi-intervals seems to be preferable to that of simple n-intervals, since the lack of additivity of some involved quantities (in particular the entropy production) does not allow natural extensions from the latter to the former.
After some recalls in Section 3.1, we state the entropy inequality in Section 3.2 and get the temperature functions in Section 3.3. Finally, Section 3.4 contains the extension result and Section 3.5 some technical proofs.

3.1 Further definitions

Definition 3.1.1. *For any grid G, we set*

$$\begin{aligned}\mathfrak{P}_G = &\{(A,C) \in \mathfrak{D} : A, C \in \mathcal{M}_G^\circ,\ A \cap C = \varnothing\} \cup \\ &\cup \{(A, C \cup (\mathbb{R}^n \setminus B)_*) : A, C \in \mathcal{M}_G^\circ,\ A \cap C = \varnothing\}.\end{aligned}$$

Now we give the definitions concerning interactions and superadditive functions defined on subbodies.

Definition 3.1.2. *Let $\mathcal{A} \subseteq \mathcal{N}^\circ$. A function $F : \mathcal{A} \longrightarrow \mathbb{R}$ is* superadditive *if*

$$F((A_1 \cup A_2)_*) \geqslant F(A_1) + F(A_2)$$

whenever $A_1, A_2, (A_1 \cup A_2)_* \in \mathcal{A}$ *and* $A_1 \cap A_2 = \varnothing$.

Clearly, if the above relation holds with the equal sign, F is additive.
If we take $\mathscr{D} = \mathfrak{P}_G$ for some grid G and require I to satisfy Definitions 1.2.18 and 2.7.1 only on $\mathfrak{P}_G$ for suitable h, η, λ, we shall call the function I *a G-interaction*. This means that we have an interaction defined only on special subbodies, the G-figures. In Theorem 2.8.4, it is proved that if G is a full grid, then a G-interaction can be extended in an essentially unique way to a balanced Cauchy interaction.

Remark 3.1.3. When I is balanced, it is possible to enlarge the domain of I to the set

$$\mathfrak{D}^\circ_{h\nu} \cup \{(A, C) : \, A \in \mathcal{M}^\circ_{h\nu}, \, C \text{ is normalized}, \, (\mathbb{R}^n \setminus C)_* \in \mathcal{M}^\circ_{h\nu}, \, A \cap C = \varnothing\}$$

for suitable h and ν (cf. Theorem 2.7.5). In particular the interaction between a subbody A and its exterior, $I(A, (\mathbb{R}^n \setminus A)_*)$, is defined for almost every $A \in \mathcal{M}^\circ$.

3.2 Balance of entropy

During a thermodynamic process, an amount of entropy (as well as other quantities, such as heat) transfers from a subbody to another, so that we can model this transfer by means of a Cauchy interaction. In view of a balance law, we call *entropy transfer* a balanced Cauchy interaction $M(A, C)$ which can be interpreted as the amount of entropy that the subbody C transfers to A. It is therefore a biadditive function. The balance of entropy can be expressed by the Second Law of Thermodynamics:

$$\dot{S}(A) \geqslant M(A, A^e),$$

where A^e denotes the exterior of the subbody A and $\dot{S}$ is the rate of change of internal entropy. It means that an extra amount of entropy can be produced in the process. Hence we introduce an entropy production term, that is, a real-valued function $\varGamma$ defined on almost all of $\mathcal{M}^\circ$. In this way, the balance law of entropy becomes

$$\dot{S}(A) = M(A, A^e) + \varGamma(A)$$

and a local form for regular functions is deduced. For example, if all expressions make sense, one can write

$$\dot{S}(A) = \int_A \dot{s}\, d\mathscr{L}^n, \quad M(A, A^e) = \int_{\partial_* A} \boldsymbol{q} \cdot \boldsymbol{n}^A \, d\mathscr{H}^{n-1}, \quad \varGamma(A) = \int_A \gamma \, d\mathscr{L}^n$$

and get the usual differential expression

$$\dot{s} = \operatorname{div} \boldsymbol{q} + \gamma.$$

As in Section 2.7, we do not assume the existence of $\dot{S}$, but we express the balance by supposing that M is balanced and $\varGamma$ is dominated by a Radon measure. The usual interpretation of the principle of increase of entropy implies $\varGamma \geqslant 0$; moreover, using the biadditivity of M, it is not hard to see that

$$\dot{S}(A \cup C) - \dot{S}(A) - \dot{S}(C) = -\big(M(A, C) + M(C, A)\big) + \varGamma(A \cup C) - \varGamma(A) - \varGamma(C).$$

If the terms $M(A,C)$ and $M(C,A)$ are surface integrals, then their sum vanishes and

$$\dot{S}(A \cup C) - \dot{S}(A) - \dot{S}(C) = \Gamma(A \cup C) - \Gamma(A) - \Gamma(C).$$

In [30, Appendix G4], Gurtin and Williams called the left-hand side *binding entropy*; since in some cases it may be strictly positive (see again [30] and [19, Section 6]), we will suppose in the sequel that Γ is superadditive. Moreover, we want Γ to be possibly singular with respect to volume measure. We collect the properties of Γ in the following definition.

Definition 3.2.1. *An* entropy production *is a real-valued function Γ defined on almost all of $\mathcal{M}^\circ$ such that*

(*a*) *$\Gamma \geqslant 0$ on almost all of $\mathcal{M}^\circ$;*

(*b*) *there exists $\eta \in \mathfrak{M}(\operatorname{int} B)$ with*

$$\Gamma(A) \leqslant \eta(A)$$

for almost every $A \in \mathcal{M}^\circ$;

(*c*) *Γ is superadditive.*

In our setting, the balance of entropy will be expressed by assuming that such a function exists.
We remark that an entropy production is always monotone, thanks to (c). Of course this definition extends to every situation in which a balance law holds with an extra production term. The following theorem states the main property of an entropy production.

Theorem 3.2.2. *Let Γ be an entropy production. Then there exists a real Borel function $g : \operatorname{int} B \to [0, +\infty)$ such that:*

(i) for almost every $A \in \mathcal{M}^\circ$,

$$\int_A g \, d\eta \leqslant \Gamma(A);$$

(ii) g is maximal, in the sense that if G is a full grid such that Γ is defined on $\mathcal{M}_G^\circ$ and $f : \operatorname{int} B \to [0, +\infty)$ is a Borel function satisfying

$$\forall P \in \mathcal{M}_G^\circ : \quad \int_P f \, d\eta \leqslant \Gamma(P),$$

then $f(x) \leqslant g(x)$ for η-almost all $x \in \operatorname{int} B$.

Moreover, such a g is uniquely determined η-a.e.

This is a plain consequence of Theorem 3.5.7, which we postpone to the last section of this chapter. In particular, the maximal density g produces a maximal measure

$$\gamma(A) = \int_A g \, d\eta$$

with $\gamma(A) \leqslant \Gamma(A)$ on almost all of $\mathcal{M}^\circ$.

Definition 3.2.3. *Such a measure γ is called* the optimal entropy production.

Now we recall the integral representation of the entropy transfer M: by Theorem 2.7.5 we have

$$M(A,C)=\begin{cases}\displaystyle\int\limits_{A\times C} k\,d\alpha+\int\limits_{\partial_* A\cap\partial_* C}\boldsymbol{j}\cdot\boldsymbol{n}^A\,d\mathscr{H}^{n-1} & \text{if } C\subseteq B,\\ \displaystyle\int\limits_{A\times(C\cap\operatorname{int}B)} k\,d\alpha+\int\limits_{A} k_e\,d\alpha_e+\int\limits_{\partial_* A\cap\partial_* C}\boldsymbol{j}\cdot\boldsymbol{n}^A\,d\mathscr{H}^{n-1} & \text{otherwise,}\end{cases}\tag{3.1}$$

on almost all of $\mathfrak{D}^\circ$, where $\alpha\in\mathfrak{M}(\operatorname{int}B\times\operatorname{int}B)$, $\alpha_e\in\mathfrak{M}(\operatorname{int}B)$, $\boldsymbol{j}\in\mathcal{L}^1_{loc}(\operatorname{int}B;\mathbb{R}^n)$ has divergence measure, the Borel functions $k:\operatorname{int}B\times\operatorname{int}B\to\mathbb{R}$ and $k_e:\operatorname{int}B\to\mathbb{R}$ are such that $|k|=1$ α-a.e. and $|k_e|=1$ α_e-a.e. respectively. .
In (3.1), k represents the density of the bulk entropy transfer, while $\boldsymbol{j}$ is the density of the entropy flux transmitted by contact. Finally, k_e takes into account possible bulk entropy exchanges with the exterior of the body.
For almost every $A\in\mathcal{M}^\circ$ we define

$$M(A,A):=\int_{A\times A} k\,d\alpha,$$

$$\begin{aligned}M(A,\mathbb{R}^n):=M(A,(\mathbb{R}^n\setminus A)_*)+M(A,A)=\\ =\int_{A\times\operatorname{int}B} k\,d\alpha+\int_A k_e\,d\alpha_e+\int_{\partial_* A}\boldsymbol{j}\cdot\boldsymbol{n}_{\partial_* A}\,d\mathscr{H}^{n-1}.\end{aligned}$$

Notice that, in order to speak of $(\mathbb{R}^n\setminus A)_*$ as a subbody, we repose on Remark 3.1.3.
Now combine the entropy production and the entropy transfer: clearly $\gamma+M(\,\cdot\,,\mathbb{R}^n)$ is additive and bounded by a Radon measure on almost all of $\mathcal{M}^\circ$, so it can be represented by a measure $\sigma\in\mathfrak{M}(\operatorname{int}B)$ and a Borel function u with $|u(x)|=1$ for σ-a.e. $x\in\operatorname{int}B$, i.e.

$$\gamma(A)+M(A,\mathbb{R}^n)=\int_A u\,d\sigma$$

on almost all of $\mathcal{M}^\circ$. Since γ is positive, it is obvious that $\displaystyle\int_A u\,d\sigma\geqslant M(A,\mathbb{R}^n)$, hence

$$\int_A u\,d\sigma\geqslant\int_A\operatorname{div}\boldsymbol{j}+\int_{A\times\operatorname{int}B} k\,d\alpha+\int_A k_e\,d\alpha_e\tag{3.2}$$

on almost all of $\mathcal{M}^\circ$. This is the measure-theoretic version of *the entropy inequality.*

Remark 3.2.4. Clearly we have

$$\varGamma(A)+M(A,(\mathbb{R}^n\setminus A)_*)\geqslant\gamma(A)+M(A,(\mathbb{R}^n\setminus A)_*).$$

In the classical context, the quantity $\varGamma(A)+M(A,(\mathbb{R}^n\setminus A)_*)$ is interpreted as *the rate of change of the internal entropy* of the subbody A; hence we can look at $\gamma(A)+M(A,(\mathbb{R}^n\setminus A)_*)$ as the *optimal* rate of change of internal entropy of A, according to Definition 3.2.3.

3.3 Existence of temperatures

In this section we suppose that, beyond the entropy transfer, there exists a balanced Cauchy interaction H which we refer to as *the heat transfer*; by the representation theorem 2.7.5, there exist $b, b_e, \mu, \mu_e, \boldsymbol{q}$, with meanings analogous to the case of entropy, such that

$$H(A,C) = \begin{cases} \displaystyle\int_{A\times C} b\,d\mu + \int_{\partial_* A\cap\partial_* C} \boldsymbol{q}\cdot\boldsymbol{n}^A\,d\mathscr{H}^{n-1} & \text{if } C \subseteq B, \\ \displaystyle\int_{A\times(C\cap\operatorname{int} B)} b\,d\mu + \int_A b_e\,d\mu_e + \int_{\partial_* A\cap\partial_* C} \boldsymbol{q}\cdot\boldsymbol{n}^A\,d\mathscr{H}^{n-1} & \text{otherwise,} \end{cases}$$

on almost all of $\mathfrak{D}^\circ$.

Let $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$, $\nu \in \mathfrak{M}(\operatorname{int} B)$ be such that the representation formulas for M and H hold on $\mathfrak{D}^\circ_{h\nu}$. We assume the following statement:

$$\begin{aligned}&\text{given } (A,C) \in \mathfrak{D}^\circ_{h\nu}, \text{ suppose that } H(\widehat{A},\widehat{C}) = 0 \text{ for every } (\widehat{A},\widehat{C}) \in \mathfrak{D}^\circ_{h\nu} \\ &\text{such that } \widehat{A} \subseteq A,\ \widehat{C} \subseteq C; \text{ then } M(A,C) = 0.\end{aligned} \tag{3.3}$$

Hence if there is no heat transfer between some parts of two subbodies, no entropy can be transferred between them. Assumption (3.3) can be considered as a part of the Second Law of Thermodynamics. It expresses the link between heat and entropy, leading to the construction of temperature.

The goal of this section is to prove that the measures associated to the representation (3.1) are absolutely continuous with respect to those associated with H; in this way, we will find three functions that will play the rôle of the temperature in a generalization of the classical Clausius-Duhem inequality. This is not trivial, since we need a way to associate a measure of $A \times C$ to the surface integrals.

From now on, we will use the notation $E^* = E \cup \partial_* E$; this is *the measure-theoretic closure* of E. First of all, we state a slightly different representation theorem for balanced Cauchy interactions in terms of the measure-theoretic closed sets E^*.

Theorem 3.3.1. *Let H be an arbitrary balanced Cauchy interaction and let $\boldsymbol{q}$, b, b_e, μ, μ_e be as in Theorem 2.7.5. Then the formula*

$$H(A,C) = \begin{cases} \displaystyle\int_{A^*\times C^*} b\,d\mu + \int_{\partial_* A^*\cap\partial_* C^*} \boldsymbol{q}\cdot\boldsymbol{n}^A\,d\mathscr{H}^{n-1} & \textit{if } C \subseteq B, \\ \displaystyle\int_{A^*\times(C\cap\operatorname{int} B)^*} b\,d\mu + \int_{A^*} b_e\,d\mu_e + \int_{\partial_* A^*\cap\partial_* C^*} \boldsymbol{q}\cdot\boldsymbol{n}^A\,d\mathscr{H}^{n-1} & \textit{otherwise,} \end{cases}$$

holds on almost all of $\mathfrak{D}^\circ$.

Proof. Clearly, for every $A \in \mathfrak{B}(\operatorname{int} B)$ we have that A^* is the disjoint union of A_* and $\partial_* A$. Since we can choose $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ and $\nu \in \mathfrak{M}(\operatorname{int} B)$ with

$$\mu(\partial_* A \times \operatorname{int} B) = \mu(\operatorname{int} B \times \partial_* C) = \mu_e(\partial_* A \cup \partial_* C) = 0$$

for every $(A,C) \in \mathfrak{D}^\circ_{h\nu}$, it is clear that the volume integrals involving $\partial_* A$ and $\partial_* C$ vanish. Notice finally that $\partial_* A = \partial_* A^*$. □

Let now $(A, C) \in \mathfrak{D}^\circ_{h\nu}$ with $C \subseteq B$ and let $f : \operatorname{int} B \to \operatorname{int} B \times \operatorname{int} B$ be such that $f(x) = (x, x)$; we define two real-valued measures m_1, m_2 on $\mathfrak{B}\,(A \times C)$ setting

$$m_1(E) = \sqrt{2} \int_E \boldsymbol{q}\left(f^{-1}(x,y)\right) \cdot \boldsymbol{n}^A \left(f^{-1}(x,y)\right) d\mathscr{H}^{n-1}(x,y),$$
$$m_2(E) = \sqrt{2} \int_E \boldsymbol{j}\left(f^{-1}(x,y)\right) \cdot \boldsymbol{n}^A \left(f^{-1}(x,y)\right) d\mathscr{H}^{n-1}(x,y).$$

The previous definitions allow to express the contact parts associated to H and M, denoted by H_c and M_c respectively, in terms of the measures m_i: by a change of variables it results that for every $(\widehat{A}, \widehat{C}) \in \mathfrak{D}^\circ_{h\nu}$ with $\widehat{A} \subseteq A$ and $\widehat{C} \subseteq C$, we have

$$H_c(\widehat{A}, \widehat{C}) = m_1(\widehat{A}^* \times \widehat{C}^*), \quad M_c(\widehat{A}, \widehat{C}) = m_2(\widehat{A}^* \times \widehat{C}^*).$$

Details are omitted for the sake of brevity.
Let p_1, p_2 the densities of m_1, m_2, respectively. Combining this fact with Theorem 3.3.1, it follows that

$$H(\widehat{A}, \widehat{C}) = \int_{\widehat{A}^* \times \widehat{C}^*} b \, d\mu + \int_{\widehat{A}^* \times \widehat{C}^*} p_1 \, d|m_1|,$$
$$M(\widehat{A}, \widehat{C}) = \int_{\widehat{A}^* \times \widehat{C}^*} k \, d\alpha + \int_{\widehat{A}^* \times \widehat{C}^*} p_2 \, d|m_2|,$$

for every $(\widehat{A}, \widehat{C}) \in \mathfrak{D}^\circ_{h\nu}$ with $\widehat{A} \subseteq A$, $\widehat{C} \subseteq C$. Moreover, the supports of $|m_1|$ and $|m_2|$ lie in $(\partial_* A \cap \partial_* C) \times (\partial_* A \cap \partial_* C)$, while those of μ and α lie in $A \times C \setminus \big((\partial_* A \cap \partial_* C) \times (\partial_* A \cap \partial_* C)\big)$. By (3.3), we have that $\alpha + |m_2| \ll \mu + |m_1|$; since the supports are disjoint, it means that

$$\alpha \ll \mu, \quad |m_2| \ll |m_1|$$

and there exist two Borel functions $\theta_{A,C}$, $d_{A,C} : A \times C \to \mathbb{R}$ such that

$$M(\widehat{A}, \widehat{C}) = \int_{\widehat{A}^* \times \widehat{C}^*} \theta_{A,C} \, b \, d\mu + \int_{\widehat{A}^* \times \widehat{C}^*} d_{A,C} \, p_1 \, d|m_1|$$

for every $(\widehat{A}, \widehat{C}) \in \mathfrak{D}^\circ_{h\nu}$ with $\widehat{A} \subseteq A$, $\widehat{C} \subseteq C$. Our choice of having fixed A and C is not restrictive: it is readily seen that for $(A_1, C_1), (A_2, C_2) \in \mathfrak{D}^\circ_{h\nu}$ one has

$$\theta_{A_1 \cap A_2, C_1 \cap C_2} = \theta_{A_1, C_1}\big|_{(A_1 \cap A_2) \times (C_1 \cap C_2)} = \theta_{A_2, C_2}\big|_{(A_1 \cap A_2) \times (C_1 \cap C_2)} \quad \mu\text{-a.e.}$$

$$d_{A_1 \cap A_2, C_1 \cap C_2} = d_{A_1, C_1}\big|_{(A_1 \cap A_2) \times (C_1 \cap C_2)} = d_{A_2, C_2}\big|_{(A_1 \cap A_2) \times (C_1 \cap C_2)} \quad |m_1|\text{-a.e.}$$

Since $\theta_{A,C}$ and $d_{A,C}$ agree on intersections, they can be extended in a unique (almost everywhere) way to $\theta, d : \operatorname{int} B \times \operatorname{int} B \to \mathbb{R}$.
We are now in position to introduce the notion of the reciprocal of the *contact temperature,* setting $\frac{1}{T} = d \circ f$; we have that

$$M_c(A, C) = \int_{\partial_* A \cap \partial_* C} \frac{1}{T} \, \boldsymbol{q} \cdot \boldsymbol{n}^A \, d\mathscr{H}^{n-1}$$

on almost all of $\mathfrak{D}^\circ$.(1)

(1) Notice that we introduce $\frac{1}{T}$ and not T. In fact, when $\frac{1}{T}$ vanishes, the usual temperature T does not exist.

On the other hand, if $(A, (\mathbb{R}^n \setminus B)_*) \in \mathfrak{D}^\circ_{h\nu}$ we can repeat in an easier manner the same procedure, getting a Borel function $\theta_e : \operatorname{int} B \to \mathbb{R}$ such that

$$M(A, (\mathbb{R}^n \setminus B)_*) = \int_{A^*} \theta_e \, b_e \, d\mu_e.$$

Finally we have

$$M(A, C) = \int_{\partial_* A \cap \partial_* C} \frac{1}{T} \boldsymbol{q} \cdot \boldsymbol{n}^A \, d\mathscr{H}^{n-1} + \int_{A^* \times C^*} \theta \, b \, d\mu + \int_{A^*} \theta_e \, b_e \, d\mu_e \tag{3.4}$$

on almost all of $\mathfrak{D}^\circ$. The meaning of the integrands in (3.4) is that of reciprocal of absolute temperatures related to interactions: T is the usual one, since it is associated to the contact part, while the others, associated to bulk and external bulk heat transfer, appear in some applications, such as the theory of radiative transfer.

We can now restate the entropy inequality of the previous section, involving the densities of the heat interaction:

$$\begin{aligned} \int_A u \, d\sigma \geqslant \int_{\partial_* A} \frac{1}{T}(x) \, \boldsymbol{q}(x) \cdot \boldsymbol{n}^A(x) \, d\mathscr{H}^{n-1}(x) + \\ + \int_{A \times \operatorname{int} B} \theta(x,y) \, b(x,y) \, d\mu(x,y) + \int_A \theta_e(x) \, b_e(x) \, d\mu_e(x) \end{aligned} \tag{3.5}$$

on almost all of $\mathcal{M}^\circ$, which is a generalization of the integral form of the Clausius-Duhem inequality.

3.4 A weaker form of the Second Law

The aim of this section is to introduce the Second Law of Thermodynamics in a sort of "discrete" context and deduce all results proved up to here.

In Section 2.8, a notion of balanced Cauchy interaction defined only on almost all n-intervals is introduced, and an extension theorem is proved. In that context, the choice between multi-intervals or n-intervals felt on the second class, since the biadditivity bears immediately an extension to the first. Conversely, here the entropy production is only superadditive and there is no natural way to extend it. By virtue of this fact, the class of almost all multi-intervals seems to be the simplest natural class of subbodies which gives enough information in order to extend the notions to almost all normalized sets of finite perimeter, at least in the framework of Thermodynamics.

Let G be a full grid and consider two G-interactions $M, H : \mathfrak{P}_G \to \mathbb{R}$. These functions represent the entropy transfer and the heat transfer, respectively. In Chapter 2 it is proved that M and H can be extended in an essentially unique way to balanced Cauchy interactions. Then consider a function $\Gamma : \mathcal{M}^\circ_G \to \mathbb{R}$, which takes the place of the entropy production of Section 3.2. Our restatement of the Second Law of Thermodynamics can be given in the following way.

Axiom 3.4.1. (Second Law of Thermodynamics) *For a full grid G there exist two G-interactions M, H and a function $\Gamma : \mathfrak{P}_G \to \mathbb{R}$ such that:*

(i) $\Gamma \geqslant 0$*;*

(ii) *there exists* $\eta \in \mathfrak{M}\,(\operatorname{int} B)$ *with*

$$\Gamma(P) \leqslant \eta(P)$$

for every $P \in \mathcal{M}_G^\circ$*;*

(iii) Γ *is superadditive;*

(iv) *given* $(A, C) \in \mathfrak{P}_G$*, we have* $M(A, C) = 0$ *whenever* $H(\widehat{A}, \widehat{C}) = 0$ *for every* $(\widehat{A}, \widehat{C}) \in \mathfrak{P}_G$ *such that* $\widehat{A} \subseteq A$*,* $\widehat{C} \subseteq C$*.*

Clearly, this is a more general version of the Second Law of Thermodynamics, since each requirement involves only G-figures. The following theorem shows that this axiom is sufficient to prove the validity of the Clausius-Duhem inequality in its general form. This is the main result of the chapter.

Theorem 3.4.2. *If Axiom 3.4.1 holds, then* (3.5) *holds on almost all of* $\mathcal{M}^\circ$*.*

Proof. Theorem 3.5.7 below allows to find a unique optimal entropy production γ such that (3.2) holds on almost all of $\mathcal{M}^\circ$. Moreover, by means of (iv) of Axiom 3.4.1 we can again prove that $\alpha \ll \mu$ and $|m_1| \ll |m_2|$ in Section 3.3. □

Once more we stress the fact that testing the validity of the Second Law only on almost all multi-intervals, we obtain the same results as testing it on almost all of $\mathcal{M}^\circ$ and $\mathfrak{D}^\circ$.

3.5 Superadditive functions

This section is essentially devoted to the proof of Theorem 3.2.2. First we need some preliminary material.
Let $\mathrm{C}(x, r)$ denote the closed cube with center x and edge $2r$. Given a full grid G, we denote by $\mathscr{C}_G$ the set of all closed cubes $\mathrm{C}(x, r)$ such that $x \in \operatorname{int} B$, $r > 0$ and $\operatorname{int} \mathrm{C}(x, r) \in \mathcal{I}_G^\circ$.

Remark 3.5.1. (Geometric property) For any $r > 0$ there exists a maximal number of closed cubes with edges greater than r which do not contain the center of each other and such that they all intersect a given cube with edge r.

Definition 3.5.2. *A subfamily* $\mathscr{C}$ *of* $\mathscr{C}_G$ *is* fine *with respect to a set* $E \subseteq \operatorname{int} B$*, if every* $x \in E$ *is the center of an element* $C \in \mathscr{C}$ *and*

$$\inf\{r : \mathrm{C}(x, r) \in \mathscr{C}\} = 0$$

for every $x \in E$*.*

The following, usually known as *Besicovitch Theorem*, is a standard tool in Measure Theory and strongly relies on Remark 3.5.1. For a proof the reader is referred to [8, 2.8.15].

Theorem 3.5.3. *Let* $\eta \in \mathfrak{M}\,(\operatorname{int} B)$*,* $E \in \mathfrak{B}\,(\operatorname{int} B)$ *with* $\eta(E) < +\infty$ *and* $\mathscr{C}$ *be a subfamily of* $\mathscr{C}_G$ *which is fine with respect to* E*. Then for every open set* $A \subseteq \operatorname{int} B$ *with* $E \subseteq A$*, there exists a countable disjoint subfamily* $\mathscr{G}$ *of* $\mathscr{C}$ *such that*

$$\left(\bigcup_{\mathscr{G}} C\right) \subseteq A, \quad \eta\left(E \setminus \bigcup_{\mathscr{G}} C\right) = 0.$$

The previous result applies also to the case of open normalized n-intervals and normalized unions, by choosing a suitable full grid G, as stated below.

Corollary 3.5.4. *Let $\eta \in \mathfrak{M}(\operatorname{int} B)$ and $E \in \mathfrak{B}(\operatorname{int} B)$ with $\eta(E) < +\infty$. Let G be a full grid such that $\mathcal{I}_G^\circ \subseteq \mathcal{M}_{0\eta}^\circ$*(2) *and $\mathscr{C}$ be a subfamily of $\mathscr{C}_G$ which is fine with respect to E. Then for every open normalized set $A \in \operatorname{int} B$ with $E \subseteq A$, there exists a countable disjoint subfamily $(I_k) \subseteq \mathcal{I}_G^\circ$ such that*

$$\left(\bigcup_{k\in\mathbb{N}} I_k\right)_* \subseteq A, \quad \eta\left(E \setminus \left(\bigcup_{k\in\mathbb{N}} I_k\right)_*\right) = 0.$$

Proof. Let $\mathscr{G} = \{C_k : k \in \mathbb{N}\}$ be a countable subfamily of $\mathscr{C}$ as in Theorem 3.5.3 and set $I_k = \operatorname{int} C_k$. Then the family (I_k) is contained in $\mathcal{I}_G^\circ$ and, keeping into account that A is normalized, one has

$$\left(\bigcup_{k\in\mathbb{N}} I_k\right)_* \subseteq A.$$

Moreover, $\eta \in \mathcal{M}_{0\eta}^\circ$ implies that

$$\eta\left(E \setminus \left(\bigcup_{k\in\mathbb{N}} I_k\right)_*\right) = \eta\left(E \setminus \bigcup_{k\in\mathbb{N}} I_k\right) = 0$$

and the proof is complete. □

Let now $\mathcal{A} \subseteq \mathcal{M}^\circ$ and $\Gamma : \mathcal{A} \to [0, +\infty)$ be a superadditive function. We suppose that there exist a full grid G and a measure $\eta \in \mathfrak{M}(\operatorname{int} B)$ such that $\mathcal{M}_G^\circ \subseteq \mathcal{A}$, $\mathcal{M}_G^\circ \subseteq \mathcal{M}_{0\eta}^\circ$ and $\Gamma \leqslant \eta$ on $\mathcal{M}_G^\circ$. Moreover, we suppose that $\mathcal{A}$ is closed by finite intersection. Note that if $\mathcal{A} = \mathcal{M}_{h\nu}^\circ$ for suitable h and ν, as in Section 3.2, or $\mathcal{A} = \mathcal{M}_G^\circ$ as in Section 3.4, then in any case it satisfies the previous requirement.
Now we set

$$g(x) = \inf_{\rho>0} \sup\left(\{0\} \cup \left\{\frac{\Gamma(I)}{\eta(I)} : I \in \mathscr{C}_G,\, x \in I,\, \operatorname{diam} I < \rho,\, \eta(I) > 0\right\}\right). \tag{3.6}$$

Lemma 3.5.5. *The function $g : \operatorname{int} B \to [0, +\infty)$ is Borel and bounded.*

Proof. As $\Gamma \leqslant \eta$, it is clear that $0 \leqslant g(x) \leqslant 1$, hence it is bounded. Let us define, for $\rho > 0$,

$$\delta_\rho(x) = \sup\left(\{0\} \cup \left\{\frac{\Gamma(I)}{\eta(I)} : I \in \mathscr{C}_G,\, x \in I,\, \operatorname{diam} I < \rho,\, \eta(I) > 0\right\}\right).$$

Since we have $g(x) = \inf_{\rho>0} \delta_\rho(x)$, it is sufficient to prove that each δ_ρ is a Borel function (see [8, 2.2.15]). Let $c \in \mathbb{R}$ and $x \in \operatorname{int} B$; if $\delta_\rho(x) > c$, then there exists $I \in \mathscr{C}_G$ such that $x \in I$, $\operatorname{diam} I < \rho$, $\eta(I) > 0$ and $\Gamma(I) > c\eta(I)$. It means that $\delta_\rho(y) > 0$ for every $y \in I$, which is an open set, and so $\delta_\rho\,(]c, +\infty[)$ is an open set; in particular, it is Borel. □

(2) Clearly, 0η means $h = 0$, $\eta \in \mathfrak{M}(\operatorname{int} B)$.

For $A \subseteq \mathbb{R}^n$ an $\rho > 0$, we denote with $\mathcal{N}_\rho(A)$ the set

$$\{x \in \mathbb{R}^n : d(x, A) < \rho\}.$$

Lemma 3.5.6. *Let $c > 0$ and K be a compact subset of $\operatorname{int} B$ with $K \subseteq \{x : g(x) > c\}$. Then for every G-figure $P \in \mathcal{M}_G^\circ$ such that $K \subseteq P$, we have*

$$c\eta(K) \leqslant \Gamma(P).$$

Proof. Let

$$\mathscr{C} = \mathscr{C}_G \cap \left\{ \operatorname{cl} I : I \in \mathcal{I}_G^\circ,\, I \cap K \neq \varnothing,\, I \subseteq P,\, \eta(I) > 0,\, \frac{\Gamma(I)}{\eta(I)} > c \right\};$$

then $\mathscr{C}$ is fine with respect to K. Applying Corollary 3.5.4, we find a countable disjoint subfamily $\{I_k : k \in \mathbb{N}\} \subseteq \mathcal{I}_G^\circ$ such that

$$\left(\bigcup_{k\in\mathbb{N}} I_k \right)_* \subseteq P, \quad \eta\left(K \setminus \left(\bigcup_{k\in\mathbb{N}} I_k \right)_* \right) = 0.$$

For every $l \in \mathbb{N}$ we have

$$\Gamma(P) \geqslant \Gamma\left(\left(\bigcup_{k=1}^{l} I_k \right)_* \right) \geqslant \sum_{k=1}^{l} \Gamma(I_k) \geqslant c \sum_{k=1}^{l} \eta(I_k) = c\eta\left(\bigcup_{k=1}^{l} I_k \right).$$

As $l \to \infty$, it follows that

$$\Gamma(P) \geqslant c\eta\left(\bigcup_{k\in\mathbb{N}} I_k \right) = c\eta\left(\left(\bigcup_{k\in\mathbb{N}} I_k \right)_* \right) \geqslant c\eta(K),$$

where we used the fact that $\eta(\partial_* I_k) = 0$ for every $k \in \mathbb{N}$. □

Now we are ready to prove the main theorem.

Theorem 3.5.7. *Let $\Gamma : \mathcal{A} \to [0, +\infty)$ be a superadditive function. Suppose that there exist a full grid G with $\mathcal{M}_G^\circ \subseteq \mathcal{A}$ and a measure $\eta \in \mathfrak{M}(\operatorname{int} B)$ with $\mathcal{M}_G^\circ \subseteq \mathcal{M}_{0\eta}^\circ$ and $\Gamma \leqslant \eta$ on $\mathcal{M}_G^\circ$. Then there exists a Borel function $g : \operatorname{int} B \to [0, 1]$ such that:*

(i)

$$\int_E g \, d\eta \leqslant \Gamma(E)$$

for every $E \in \mathcal{A}$;

(ii) if $f : \operatorname{int} B \to [0, +\infty)$ is a Borel function satisfying

$$\forall P \in \mathcal{M}_G^\circ : \quad \int_P f \, d\eta \leqslant \Gamma(P),$$

then $f(x) \leqslant g(x)$ for η-almost all $x \in \operatorname{int} B$.

Moreover, such a g is uniquely determined η-a.e.

Proof. Let g be defined as in (3.6). Let $0 < t < 1$ and $\varepsilon > 0$. For $h \in \mathbb{N}$ consider the sets

$$E_h = \{x \in E : t^{h+1} < g(x) \leqslant t^h\};$$

since η is a regular measure, there exists a sequence (K_h) of compact sets in $\operatorname{int} B$ such that $K_h \subseteq E_h$ and $\eta(E_h \setminus K_h) \leqslant \varepsilon 2^{-h-1}$ for every $h \in \mathbb{N}$; in particular, (K_h) is a disjoint sequence. Then we have

$$\int_E g\, d\eta = \sum_{h\in\mathbb{N}} \int_{E_h} g\, d\eta \leqslant \sum_{h\in\mathbb{N}} t^h \eta(E_h) \leqslant \sum_{h=0}^{l} t^h \eta(E_h) + \varepsilon \leqslant \sum_{h=0}^{l} t^h \eta(K_h) + 2\varepsilon$$

for a suitable $l \in \mathbb{N}$.
By Lemma 3.5.6, for every $h \in \mathbb{N}$ there exists a set $P_h \in \mathcal{M}_G^\circ$ such that $K_h \subseteq P_h$ and $t^{h+1}\eta(K_h) \leqslant \Gamma(P_h)$. Setting $d(A, B) = \inf\{|x - y| : x \in A,\, y \in B\}$, let

$$\rho = \frac{1}{2} \min\{d(K_h, K_j) : 0 \leqslant h < j \leqslant l\};$$

since the sets K_h are compact and pairwise disjoint, we have $\rho > 0$. Without loss of generality, we may assume that $P_h \subseteq \mathcal{N}_\rho(K_h)$, so that (P_h) is a disjoint sequence. Moreover, by substituting P_h with $P_h \cap E$, which again belongs to $\mathcal{A}$, we may also assume that $P_h \subseteq E$. Hence we can continue the inequalities:

$$\sum_{h=0}^{l} t^h \eta(K_h) + 2\varepsilon \leqslant t^{-1} \sum_{h=0}^{l} \Gamma(P_h) + 2\varepsilon \leqslant t^{-1}\Gamma(E) + 2\varepsilon.$$

Letting $\varepsilon \to 0^+$ and $t \to 1^-$, (i) is proved.
Now let f be as in the statement; let $x \in \operatorname{int} B$, $\rho > 0$ and $I \in \mathcal{I}_G^\circ$ be such that $x \in I$, $\eta(I) > 0$ and $\operatorname{diam} I < \rho$. Since

$$\frac{1}{\eta(I)} \int_I f\, d\eta \leqslant \frac{\Gamma(I)}{\eta(I)},$$

taking the supremum limit as $\rho \to 0^+$ we conclude the proof. □

Chapter 4

An alternative approach: the Cauchy power

The following chapter presents a different way to state and prove the version of Cauchy's Stress Theorem given in Chapter 2. The main idea is to assume a weak balance of the mechanical power instead of a balance of the stress. We show that the two approaches are equivalent, so that we can apply the results of Chapter 2 also in this case. The great advantage of this technique is that we are able to generalize the method to the case of a body which is an abstract differentiable manifold and no longer a normalized subset of a Euclidean space. In a few words, denoting with B the body, if $B \subset \mathbb{R}^n$ one can test the mechanical power with constant fields in order to obtain the usual balance of stresses; on the contrary, if B is a manifold the existence of the constant fields could be impossible. In this case, a theory founded on a balance of the power seems to be more general and can give further results.

4.1 The Cauchy power

In the classical case, the existence of the stress tensor T is assumed and one can define the power of the stress of a subbody M on a vector field $\boldsymbol{v}$ setting

$$P(M, \boldsymbol{v}) = \int_{\partial M} \mathsf{T}\boldsymbol{n} \cdot \boldsymbol{v} \, d\mathscr{H}^{n-1}.$$

In particular, it is clear that P is linear in the second argument and the inequality

$$|P(M, \boldsymbol{v})| \leqslant \int_{\partial M} |\boldsymbol{v}| h \, d\mathscr{H}^{n-1}$$

holds with $h = \|\mathsf{T}\|$. Moreover, supposing $\mathsf{T}, \boldsymbol{v}$ and ∂M smooth, one can apply Gauss-Green Theorem, obtaining

$$P(M, \boldsymbol{v}) = \int_M [(\operatorname{div} \mathsf{T}) \cdot \boldsymbol{v} + \mathsf{T} : \operatorname{grad} \boldsymbol{v}] \, d\mathscr{L}^n,$$

from which one deduces that P is additive in the first argument and

$$|P(M, \boldsymbol{v})| \leqslant \|\boldsymbol{v}\|_\infty \eta(M) + \|\operatorname{grad} \boldsymbol{v}\|_\infty \int_M h \, d\mathscr{L}^n,$$

exactly with $\eta = \|\operatorname{div} \mathsf{T}\|\mathscr{L}^n$ and $h = \|\mathsf{T}\|$.
In the spirit of [5], we aim to generalize this matter to the case where the stress tensor T can have as divergence a measure not necessarily absolutely continuous with respect to the Lebesgue measure. We consider the last inequality as an assumption for some general η, h and deduce the existence of the stress tensor, in the sense specified below. At this level of generality, we suppose the velocity field having values in $\mathbb{R}^N$, while the dimension of the body is n.

Let B denote a bounded normalized subset of $\mathbb{R}^n$ with finite perimeter.

Definition 4.1.1. A Cauchy power *on B is a function*

$$P : \mathcal{D} \times C_c^\infty\big(\operatorname{int} B; \mathbb{R}^N\big) \to \mathbb{R},$$

where $\mathcal{D}$ contains almost all of $\mathcal{M}^\circ$ and the following properties hold:

(*a*) *$P(\,\cdot\,, \boldsymbol{v})$ is additive for every $\boldsymbol{v} \in C_c^\infty\big(\operatorname{int} B; \mathbb{R}^N\big)$;*

(*b*) *$P(M, \cdot\,)$ is linear for almost every $M \in \mathcal{M}^\circ$;*

(*c*) *there exists $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ such that*

$$|P(M, \boldsymbol{v})| \leqslant \int_{\partial_* M} |\boldsymbol{v}| h \, d\mathscr{H}^{n-1}$$

for every $\boldsymbol{v} \in C_c^\infty\big(\operatorname{int} B; \mathbb{R}^N\big)$ and almost every $M \in \mathcal{M}^\circ$.

Remark 4.1.2. Taking into account property (c), it is easy to see that the definition of $P(M, \boldsymbol{v})$ depends only on the values of $\boldsymbol{v}$ on the measure-theoretic boundary of M, that is: if $\boldsymbol{v}_1(x) = \boldsymbol{v}_2(x)$ for $\mathscr{H}^{n-1}$-a.e. $x \in \partial_* M$, then $P(M, \boldsymbol{v}_1) = P(M, \boldsymbol{v}_2)$.

First of all, we state a representation theorem about general Cauchy fluxes.

Proposition 4.1.3. *Let Q be a Cauchy flux. Then for almost every $M \in \mathcal{M}^\circ$ there exists a unique (up to $\mathscr{H}^{n-1}$-negligible sets) Borel function $\boldsymbol{t}_{Q,M} : \partial_* M \to \mathbb{R}^N$ such that*

$$Q(S) = \int_S \boldsymbol{t}_{Q,M} \, d\mathscr{H}^{n-1}$$

for every material surface S subordinated to M.
Moreover, if h is a function satisfying (b) of Definition 2.1.5, then $|\boldsymbol{t}_{Q,M}| \leqslant h$.

Proof. Let h and ν be such that Q is defined and (a), (b), of Definition 2.1.5 hold on $\mathcal{M}^\circ_{h\nu}$. Let $M \in \mathcal{M}^\circ_{h\nu}$; then Q is additive on $\mathfrak{B}(\partial_* M)$ and for any material surface $S \subseteq \partial_* M$ we have

$$|Q(S)| \leqslant \int_{\partial_* M} h \, d\mathscr{H}^{n-1},$$

hence Q is a finite vector measure on $\partial_* M$. By the Radon-Nikodym Theorem there exists a function $\boldsymbol{t}_{Q,M} : \partial_* M \to \mathbb{R}^N$ such that $|\boldsymbol{t}_{Q,M}| \leqslant h$ and

$$Q(S) = \int_S \boldsymbol{t}_{Q,M} \, d\mathscr{H}^{n-1},$$

and the proof is complete. □

Now we introduce a slightly different version of the Cauchy flux, which connects very well with Definition 4.1.1, as we will see in Theorems 4.1.7 and 4.1.8.

Definition 4.1.4. *Let $\mathcal{D} \subseteq \mathcal{S}$ be a set containing almost all of $\mathcal{S}$ and let $Q : \mathcal{D} \to \mathbb{R}^N$. We say that Q is* an equilibrated (vector) Cauchy flux *on B, if Q is a Cauchy flux (see Definition 2.1.5) and further the following property holds:*

(c) *the equality*

$$Q(-S) = -Q(S)$$

holds almost everywhere in $\mathcal{S}$.

Definition 4.1.5. *We say that an equilibrated Cauchy flux Q and a Cauchy power P are* associated *if the formula*

$$P(M, \boldsymbol{v}) = \int_{\partial_* M} \boldsymbol{v} \cdot \boldsymbol{t}_{Q,M} \, d\mathscr{H}^{n-1}$$

holds for every $\boldsymbol{v} \in C_c^\infty\big(\operatorname{int} B; \mathbb{R}^N\big)$ and for almost every $M \in \mathcal{M}^\circ$.

In the following theorems, we prove the essential one-to-one correspondence between Cauchy powers and equilibrated Cauchy fluxes.

Lemma 4.1.6. *Let Q be an equilibrated Cauchy flux defined on $\mathcal{M}^\circ_{h\nu}$. Then for almost every $M, N \in \mathcal{M}^\circ$ with $M \cap N = \varnothing$ and every $\boldsymbol{v} \in C_c^\infty\big(\operatorname{int} B; \mathbb{R}^N\big)$ one has*

$$\int_{\partial_* M \cap \partial_* N} \boldsymbol{v} \cdot \boldsymbol{t}_{Q,M} \, d\mathscr{H}^{n-1} = -\int_{\partial_* M \cap \partial_* N} \boldsymbol{v} \cdot \boldsymbol{t}_{Q,N} \, d\mathscr{H}^{n-1},$$
$$\int_{\partial_* M \setminus \partial_* N} \boldsymbol{v} \cdot \boldsymbol{t}_{Q,M} \, d\mathscr{H}^{n-1} = \int_{\partial_* M \setminus \partial_* N} \boldsymbol{v} \cdot \boldsymbol{t}_{Q,(M \cup N)_*} \, d\mathscr{H}^{n-1}.$$

Proof. We drop the subscript Q from $\boldsymbol{t}$. Let h and ν be such that Proposition 4.1.3 holds on $\mathcal{M}^\circ_{h\nu}$. Let $\boldsymbol{v} \in C_c^\infty\big(\operatorname{int} B; \mathbb{R}^N\big)$; then for every $i = 1, \ldots, N$ there exists a sequence $(E_{i,h})$ of Borel sets in $\operatorname{int} B$ such that

$$\boldsymbol{e}_i \cdot \boldsymbol{v}(x) = \sum_{h=1}^{\infty} \frac{1}{h} \chi_{E_{i,h}}$$

(see [7]). Given $M, N \in \mathcal{M}^\circ_{h\nu}$ with $M \cap N = \varnothing$, by Propositions 1.1.2 and 1.1.3 we can show that

$$\mathscr{H}^{n-1}(\partial_* M \cap \partial_* N \cap \partial_*(M \cup N)) = 0$$

and that $\boldsymbol{n}^M(x) = -\boldsymbol{n}^N(x)$ for $\mathscr{H}^{n-1}$-a.e. $x \in \partial_* M \cap \partial_* N$. Denoting by $\boldsymbol{v}_i$ and $\boldsymbol{t}_{i,M}$ the components along $\boldsymbol{e}_i$ of $\boldsymbol{v}$ and $\boldsymbol{t}_M$ respectively, we have

$$\begin{aligned}
\int_{\partial_* M \cap \partial_* N} \boldsymbol{v}_i \boldsymbol{t}_{i,M} \, d\mathscr{H}^{n-1} &= \sum_{h=1}^{\infty} \frac{1}{h} \int_{\partial_* M \cap \partial_* N \cap E_{i,h}} \boldsymbol{t}_{i,M} = \\
&= \sum_{h=1}^{\infty} \frac{1}{h} Q(\partial_* M \cap \partial_* N \cap E_{i,h}, \boldsymbol{n}^M) \cdot \boldsymbol{e}_i = \\
&= -\sum_{h=1}^{\infty} \frac{1}{h} Q(\partial_* M \cap \partial_* N \cap E_{i,h}, \boldsymbol{n}^N) \cdot \boldsymbol{e}_i = \\
&= -\int_{\partial_* M \cap \partial_* N} \boldsymbol{v}_i \boldsymbol{t}_{i,N} \, d\mathscr{H}^{n-1},
\end{aligned}$$

and the first formula is proved.
Now take $\widehat{S} \subseteq \partial_* M \setminus \partial_* N$; we have that the material surface $(\widehat{S}, \boldsymbol{n}^M)$ is subordinated to M and also to $(M \cup N)_*$ up to a set of zero $\mathscr{H}^{n-1}$-measure, hence it is in the domain of Q and

$$\int_{\widehat{S}} \boldsymbol{t}_M \, d\mathscr{H}^{n-1} = \int_{\widehat{S}} \boldsymbol{t}_{(M\cup N)_*} \, d\mathscr{H}^{n-1}.$$

Then we can prove the other formula using the same technique as above. □

Theorem 4.1.7. *Let Q be an equilibrated Cauchy flux. Then there exists a Cauchy power P associated to Q. Moreover, if $\widehat{P}$ is another Cauchy power associated to Q, then $\widehat{P} = P$ on almost all of $\mathcal{M}^\circ$.*

Proof. Let $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ and $\nu \in \mathfrak{M}(\operatorname{int} B)$ be such that Definition 4.1.4 holds and $\boldsymbol{t}_{Q,M}$ exists for every $M \in \mathcal{M}^\circ_{h\nu}$. We will show that the function

$$P(M, \boldsymbol{v}) = \int_{\partial_* M} \boldsymbol{v} \cdot \boldsymbol{t}_{Q,M} \, d\mathscr{H}^{n-1}$$

defined on $\mathcal{M}^\circ_{h\nu} \times C^\infty_c(\operatorname{int} B; \mathbb{R}^N)$ is a Cauchy power. Linearity on the second argument is obvious as well as the fact that

$$|P(M, \boldsymbol{v})| \leqslant \int_{\partial_* M} |\boldsymbol{v}| h \, d\mathscr{H}^{n-1}.$$

Now take two disjoint subsets $M, N \in \mathcal{M}^\circ_{h\nu}$ and $\boldsymbol{v} \in C^\infty_c(\operatorname{int} B; \mathbb{R}^N)$; taking into account that by Proposition 1.1.2 one has $\partial_*(M \cup N) = \partial_* M \triangle \partial_* N$ up to a set of zero $\mathscr{H}^{n-1}$-measure, applying Lemma 4.1.6 we get

$$\begin{aligned}
P((M\cup N)_*, \boldsymbol{v}) &= \int_{\partial_*(M\cup N)} \boldsymbol{v} \cdot \boldsymbol{t}_{Q,(M\cup N)_*} \, d\mathscr{H}^{n-1} = \\
&= \int_{\partial_* M \setminus \partial_* N} \boldsymbol{v} \cdot \boldsymbol{t}_{Q,M} \, d\mathscr{H}^{n-1} + \int_{\partial_* N \setminus \partial_* M} \boldsymbol{v} \cdot \boldsymbol{t}_{Q,N} \, d\mathscr{H}^{n-1} = \\
&= \int_{\partial_* M} \boldsymbol{v} \cdot \boldsymbol{t}_{Q,M} \, d\mathscr{H}^{n-1} + \int_{\partial_* N} \boldsymbol{v} \cdot \boldsymbol{t}_{Q,N} \, d\mathscr{H}^{n-1} = \\
&= P(M, \boldsymbol{v}) + P(N, \boldsymbol{v}),
\end{aligned}$$

showing that P is additive. Then P is a Cauchy power.
Finally, if $\widehat{P}$ is another Cauchy power associated to Q, then for almost every $M \in \mathcal{M}^\circ$ we have

$$\widehat{P}(M, \boldsymbol{e}_i) = \int_{\partial_* M} \boldsymbol{e}_i \cdot \boldsymbol{t}_{Q,M} \, d\mathscr{H}^{n-1} = P(M, \boldsymbol{e}_i),$$

hence $\widehat{P} = P$ on almost all of $\mathcal{M}^\circ$. □

Theorem 4.1.8. *Let P be a Cauchy power on B. Then there exists an equilibrated Cauchy flux Q associated to P. Moreover, if $\widehat{Q}$ is another equilibrated Cauchy flux associated to P, then $\widehat{Q} = Q$ on almost all of $\mathcal{S}$.*

Proof. Let $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ and $\nu \in \mathfrak{M}(\operatorname{int} B)$ be such that the domain of P contains $\mathcal{M}^\circ_{h\nu}$ and Definition 4.1.1 holds on $\mathcal{M}^\circ_{h\nu}$. Let us fix $M \in \mathcal{M}^\circ_{h\nu}$; then we have that the function $P(M,\,\cdot\,) : C^\infty_c(\operatorname{int} B; \mathbb{R}^n) \to \mathbb{R}$ is linear and

$$|P(M,\boldsymbol{v})| \leqslant \|\boldsymbol{v}\|_\infty \int_{\partial_* M} h \, d\mathscr{H}^{n-1},$$

hence $P(M,\,\cdot\,)$ is a vector distribution of order zero. By the Riesz Representation Theorem, there exists a unique $\mu \in \mathfrak{M}(\operatorname{int} B)$ and a μ-essentially unique Borel function $\boldsymbol{c}_M : \operatorname{int} B \to \mathbb{R}^N$ such that $|\boldsymbol{c}_M| = 1$ μ-almost everywhere in $\operatorname{int} B$ and

$$P(M,\boldsymbol{v}) = \int_{\operatorname{int} B} \boldsymbol{c}_M \cdot \boldsymbol{v} \, d\mu.$$

Moreover, since

$$|P(M,\boldsymbol{v})| \leqslant \int_{\partial_* M} |\boldsymbol{v}| h \, d\mathscr{H}^{n-1},$$

we have that $\mu \ll \mathscr{H}^{n-1} \llcorner \partial_* M$, hence

$$P(M,\boldsymbol{v}) = \int_{\partial_* M} \boldsymbol{b}_M \cdot \boldsymbol{v} \, d\mathscr{H}^{n-1}$$

for an $\mathscr{H}^{n-1}$-essentially unique Borel function $\boldsymbol{b}_M : \partial_* M \to \mathbb{R}^N$ with $|\boldsymbol{b}_M| \leqslant h$.
Let now S be a material surface in $\mathcal{S}_{h\nu}$ such that S is subordinated to M_1 and M_2, with $M_1, M_2 \in \mathcal{M}^\circ_{h\nu}$; then $\widehat{S} \subseteq \partial_* M_1 \cap \partial_* M_2$ and $\boldsymbol{n}^{M_1} = \boldsymbol{n}^{M_2}$ on $\widehat{S}$. Moreover, we have that the sets $\widehat{S} \setminus \partial_*(M_1 \cap M_2)$ and $\widehat{S} \cap \partial_*(M_1 \setminus M_2)$ are $\mathscr{H}^{n-1}$-negligible (see Proposition 1.1.3). Suppose the set $\widehat{S}$ to be compact and fix an element $\boldsymbol{a}$ of $\mathbb{R}^N$. Then there exists a sequence $(\boldsymbol{v}_h) \in C^\infty_c\big(\operatorname{int} B; \mathbb{R}^N\big)$ such that $|\boldsymbol{v}_h| \leqslant |\boldsymbol{a}|$ and $\boldsymbol{v}_h \longrightarrow \chi_{\widehat{S}} \boldsymbol{a}$ pointwise. Since P is additive, it follows

$$\begin{aligned}\int_{\partial_* M_1} \boldsymbol{v}_h \cdot \boldsymbol{b}_{M_1} \, d\mathscr{H}^{n-1} = & \int_{\partial_*(M_1 \setminus M_2)} \boldsymbol{v}_h \cdot \boldsymbol{b}_{M_1 \setminus M_2} \, d\mathscr{H}^{n-1} + \\ & + \int_{\partial_*(M_1 \cap M_2)} \boldsymbol{v}_h \cdot \boldsymbol{b}_{M_1 \cap M_2} \, d\mathscr{H}^{n-1},\end{aligned}$$

and by the Dominated Convergence Theorem we have that

$$\boldsymbol{a} \cdot \int_{\widehat{S}} \boldsymbol{b}_{M_1} \, d\mathscr{H}^{n-1} = \boldsymbol{a} \cdot \int_{\widehat{S}} \boldsymbol{b}_{M_1 \cap M_2} \, d\mathscr{H}^{n-1}.$$

Due to the symmetry of M_1 and M_2 and the arbitrariness of $\boldsymbol{a}$, we have that

$$\int_{\widehat{S}} \boldsymbol{b}_{M_1} \, d\mathscr{H}^{n-1} = \int_{\widehat{S}} \boldsymbol{b}_{M_2} \, d\mathscr{H}^{n-1}.$$

If $\widehat{S}$ is not compact, we can find a sequence (S_h) in $\mathcal{S}_{h\nu}$ such that

$$\widehat{S} = \bigcup_{h \in \mathbb{N}} \widehat{S}_h \cup N \quad \text{with } \mathscr{H}^{n-1}(N) = 0,$$

then

$$\int_{\widehat{S}} \boldsymbol{b}_{M_1}\, d\mathscr{H}^{n-1} = \lim_h \int_{\widehat{S}_h} \boldsymbol{b}_{M_1}\, d\mathscr{H}^{n-1} = \lim_h \int_{\widehat{S}_h} \boldsymbol{b}_{M_2}\, d\mathscr{H}^{n-1} = \int_{\widehat{S}} \boldsymbol{b}_{M_2}\, d\mathscr{H}^{n-1}.$$

Hence we can define a function $Q : \mathcal{S}_{h\nu} \to \mathbb{R}^N$ as

$$Q(S) = \int_{\widehat{S}} \boldsymbol{b}_M\, d\mathscr{H}^{n-1},$$

where S is subordinated to M. It is clear that (a) and (b) of Definition 4.1.4 hold; we are going to prove that $Q(-S) = -Q(S)$. So let S be a material surface and let $M, N \in \mathcal{M}^{\circ}_{h\nu}$ be such that S and $-S$ are subordinated to M and N, respectively. Then, again from Proposition 1.1.3 it follows that $\widehat{S} \setminus \partial_*(M \cup N)$ is $\mathscr{H}^{n-1}$-negligible. Let $\boldsymbol{a} \in \mathbb{R}^N$; if $\widehat{S}$ is compact, there exists a sequence $(\boldsymbol{v}_h)$ in $C^\infty_c\big(\operatorname{int} B; \mathbb{R}^N\big)$ such that $\boldsymbol{v}_h \longrightarrow \chi_{\widehat{S}}\boldsymbol{a}$ pointwise. Since $P((M \cup N)_*, \boldsymbol{v}_h) = P(M, \boldsymbol{v}_h) + P(N, \boldsymbol{v}_h)$, we have

$$\int_{\partial_*(M\cup N)} \boldsymbol{v}_h \cdot \boldsymbol{b}_{(M\cup N)_*}\, d\mathscr{H}^{n-1} = \int_{\partial_* M} \boldsymbol{v}_h \cdot \boldsymbol{b}_M\, d\mathscr{H}^{n-1} + \int_{\partial_* N} \boldsymbol{v}_h \cdot \boldsymbol{b}_N\, d\mathscr{H}^{n-1};$$

applying the Dominated Convergence Theorem, we obtain

$$Q(-S) = -Q(S).$$

If $\widehat{S}$ is not compact, we can apply the same technique as above. Moreover, we have immediately that $\boldsymbol{b}_M = \boldsymbol{t}_{Q,M}$, hence Q is an equilibrated Cauchy flux associated to P.
Finally, if $\widehat{Q}$ is another equilibrated Cauchy flux associated to P, for almost every $M \in \mathcal{M}^{\circ}$ it follows that

$$\int_{\partial_* M} \boldsymbol{v} \cdot \boldsymbol{t}_{Q,M}\, d\mathscr{H}^{n-1} = P(M, \boldsymbol{v}) = \int_{\partial_* M} \boldsymbol{v} \cdot \boldsymbol{t}_{\widehat{Q},M}\, d\mathscr{H}^{n-1}$$

for every $\boldsymbol{v} \in C^\infty_c\big(\operatorname{int} B; \mathbb{R}^N\big)$, hence $\widehat{Q} = Q$ on almost all of $\mathcal{S}$. $\square$

We now add a crucial assumption to the Cauchy power, in order to obtain an object which can be related with a balanced Cauchy flux.

Definition 4.1.9. *A Cauchy power P is said to be* balanced *if there exist $\eta \in \mathfrak{M}(\operatorname{int} B)$ and $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ such that*

$$|P(M, \boldsymbol{v})| \leqslant \|\boldsymbol{v}\|_\infty \eta(M) + \|\operatorname{grad} \boldsymbol{v}\|_\infty \int_M h\, d\mathscr{L}^n$$

for every $\boldsymbol{v} \in C^\infty_c\big(\operatorname{int} B; \mathbb{R}^N\big)$ and almost every $M \in \mathcal{M}^{\circ}$.

We recall that a Cauchy flux Q is said to be *balanced*, if there exists $\eta \in \mathfrak{M}(\operatorname{int} B)$ such that the inequality

$$|Q(\partial_* M)| \leqslant \eta(M) \tag{4.1}$$

holds almost everywhere in $\mathcal{M}^{\circ}$. In particular, since as proved in Chapter 2 there is a global integral representation, a balanced Cauchy flux satisfies $Q(-S) = -Q(S)$ on almost all of $\mathcal{S}$.

Theorem 4.1.10. *A Cauchy power is balanced if and only if the associated Cauchy flux is balanced.*

Proof. Suppose that a Cauchy power P is balanced and consider the Cauchy flux associated to P; then, for every $\boldsymbol{a} \in \mathbb{R}^N$ we have

$$|Q(\partial_* M) \cdot \boldsymbol{a}| \leqslant \left| \int_{\partial_* M} \boldsymbol{a} \cdot \boldsymbol{t}_{Q,M} \, d\mathscr{H}^{n-1} \right| = |P(M, \boldsymbol{a})| \leqslant \|\boldsymbol{a}\| \eta(M),$$

on almost all of $\mathcal{M}^\circ$, hence in particular Q is balanced.
On the other hand, supposing that Q is balanced, by Theorem 2.5.1 one can deduce that there exists a tensor $\mathsf{q} \in L^1_{loc}(\operatorname{int} B; \operatorname{Lin}(\mathbb{R}^n; \mathbb{R}^N))$ with divergence measure such that $|\operatorname{div} \mathsf{q}| \leqslant \eta$ and $\boldsymbol{t}_{Q,M} = \mathsf{q}\boldsymbol{n}^M$ on almost all of $\mathcal{M}^\circ$. Denoting with P the Cauchy power associated with Q and setting $h(x) = \|\mathsf{q}(x)\|$ we have that

$$\begin{aligned} |P(M, \boldsymbol{v})| &= \left| \int_{\partial_* M} \mathsf{q}\boldsymbol{n}^M \cdot \boldsymbol{v} \, d\mathscr{H}^{n-1} \right| = \left| \int_M \boldsymbol{v} \cdot \operatorname{div} \mathsf{q} + \int_M \mathsf{q} : \operatorname{grad} \boldsymbol{v} \, d\mathscr{L}^n \right| \leqslant \\ &\leqslant \|\boldsymbol{v}\|_\infty \eta(M) + \|\operatorname{grad} \boldsymbol{v}\|_\infty \int_M h \, d\mathscr{L}^n \end{aligned}$$

on almost all of $\mathcal{M}^\circ$, hence P is balanced. □

Corollary 4.1.11. *Let P be a balanced Cauchy power and let η be as in Definition 4.1.9. Then there exists $\mathsf{q} \in L^1_{loc}(\operatorname{int} B; \operatorname{Lin}(\mathbb{R}^n; \mathbb{R}^N))$ with divergence measure such that $|\operatorname{div} \mathsf{q}| \leqslant \eta$ and*

$$P(M, \boldsymbol{v}) = \int_{\partial_* M} \mathsf{q}\boldsymbol{n}^M \cdot \boldsymbol{v} \, d\mathscr{H}^{n-1}$$

for every $\boldsymbol{v} \in C^\infty_c(\operatorname{int} B; \mathbb{R}^N)$ and for almost every $M \in \mathcal{M}^\circ$. Moreover, q is uniquely determined $\mathscr{L}^n$-almost everywhere.

Proof. The balanced Cauchy power P is associated to a Cauchy flux that is balanced by Theorem 4.1.10. The conclusion follows by Theorem 2.5.1. □

Finally, we give an extension formula for balanced Cauchy powers, which states that the behavior of a Cauchy power on almost all n-intervals distinguishes it on almost all of $\mathcal{M}^\circ$.

Corollary 4.1.12. *Let G be a full grid and $P : \mathcal{I}^\circ_G \times C^\infty_c(\operatorname{int} B; \mathbb{R}^N) \to \mathbb{R}$ a function which satisfies the following assumptions:*

(*a*) *for every $\boldsymbol{v} \in C^\infty_c(\operatorname{int} B; \mathbb{R}^N)$ and every finite disjoint family $\{I_k : k \in \Lambda\} \subseteq \mathcal{I}^\circ_G$ it holds*

$$\Big(\bigcup_{k \in \Lambda} I_k \Big)_* \in \mathcal{I}^\circ_G \quad \Rightarrow \quad P\bigg(\Big(\bigcup_{k \in \Lambda} I_k \Big)_*, \boldsymbol{v} \bigg) = \sum_{k \in \Lambda} P(I_k, \boldsymbol{v});$$

(*b*) *$P(I, \cdot)$ is linear for every $I \in \mathcal{I}^\circ_G$;*

(*c*) *there exists $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ such that*

$$|P(I, \boldsymbol{v})| \leqslant \int_{\partial I} |\boldsymbol{v}| h \, d\mathscr{H}^{n-1}$$

for every $\boldsymbol{v} \in C^\infty_c(\operatorname{int} B; \mathbb{R}^N)$ and $I \in \mathcal{I}^\circ_G$;

(d) *there exist* $\eta \in \mathfrak{M}(\operatorname{int} B)$ *and* $\tilde{h} \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ *such that*

$$|P(I,\boldsymbol{v})| \leqslant \|\boldsymbol{v}\|_\infty \eta(I) + \|\operatorname{grad}\boldsymbol{v}\|_\infty \int_I \tilde{h}\, d\mathscr{L}^n$$

for every $\boldsymbol{v} \in C_c^\infty(\operatorname{int} B;\mathbb{R}^N)$ *and* $I \in \mathcal{I}_G^\circ$.

Then there exists a full grid $H \subseteq G$ *and a Cauchy power* $\widetilde{P}$ *such that the domain of* $\widetilde{P}$ *contains* $\mathcal{I}_H^\circ$ *and* $\widetilde{P}(I) = P(I)$ *for every* $I \in \mathcal{I}_H^\circ$.
Moreover, if $\widehat{P}$ *has the same properties of* $\widetilde{P}$, *then* $\widehat{P} = \widetilde{P}$ *on almost all of* $\mathcal{M}^\circ$.

Proof. Let $S \in \mathcal{S}_G$ and let $I \in \mathcal{I}_G^\circ$ be such that S is subordinated to I. Given $\boldsymbol{a} \in \mathbb{R}^N$, there exists a sequence $(\boldsymbol{v}_k)$ in $C_c^\infty(\operatorname{int} B;\mathbb{R}^N)$ such that $|\boldsymbol{v}_k| \leqslant \boldsymbol{a}$ and $\boldsymbol{v}_k \to \chi_S \boldsymbol{a}$ pointwise. Define the component of $Q : \mathcal{S}_G \to \mathbb{R}^N$ with respect to $\boldsymbol{a}$ as

$$\boldsymbol{a} \cdot Q(S) = \lim_k P(I,\boldsymbol{v}_k).$$

Then we state some facts.
1) Q does not depend on the choice of the sequence $\boldsymbol{v}_k$, since

$$\left|\lim_k P(I,\boldsymbol{v}_k - \tilde{\boldsymbol{v}}_k)\right| \leqslant \lim_k \int_{\partial_* I} |\boldsymbol{v}_k - \tilde{\boldsymbol{v}}_k| h\, d\mathscr{H}^{n-1} = 0.$$

2) If $\mathscr{H}^{n-1}(S \triangle T) = 0$, then it is easy to check that $Q(S) = Q(T)$.
3) $Q(S)$ does not depend on the set I: if S is subordinated to I_1, I_2, let $\{J_i : i \in \Lambda\}$ be a finite disjoint family in $\mathcal{I}_G^\circ$ such that $(\bigcup_{i\in\Lambda} J_i)_* = (I_1 \triangle I_2)_*$; then $\mathscr{H}^{n-1}(S \triangle (\bigcup_{i\in\Lambda} J_i)_*) = 0$ and

$$\lim_k P(I_1,\boldsymbol{v}_k) = \lim_k P(I_1 \cap I_2,\boldsymbol{v}_k) = \lim_k P(I_2,\boldsymbol{v}_k)$$

by the symmetry of I_1 and I_2.
4) $Q(-S) = -Q(S)$, since P is additive.

Now we prove that Q satisfies (i), (ii) and (iii) of Section 2.4 with $G_0 = G$.
(i) Let $S, S_1, S_2 \in \mathcal{S}_G$ with $S_1 \cap S_2 = \varnothing$ and $\operatorname{cl} S = \operatorname{cl} S_1 \cup \operatorname{cl} S_2$. Then there exist $I_1, I_2 \in \mathcal{I}_G^\circ$ such that $I_1 \cap I_2 = \varnothing$, $(I_1 \cup I_2)_* \in \mathcal{I}_G^\circ$ and S_j is subordinated to I_j, S is subordinated to $(I_1 \cup I_2)_*$. Given $\boldsymbol{a} \in \mathbb{R}^N$, let $\boldsymbol{v}_k^{(j)} \to \chi_{S_j}\boldsymbol{a}$; then we have that $\boldsymbol{v}_k^{(1)} + \boldsymbol{v}_k^{(2)} \to \chi_{S_1 \cup S_2}\boldsymbol{a}$. Since the symmetric difference between $S_1 \cup S_2$ and S is $\mathscr{H}^{n-1}$-negligible, it follows that

$$\begin{aligned}\boldsymbol{a} \cdot Q(S) &= \lim_k P(I,\boldsymbol{v}_k^{(1)} + \boldsymbol{v}_k^{(2)}) = \\ &= \lim_k \left(P(I,\boldsymbol{v}_k^{(1)}) + P(I,\boldsymbol{v}_k^{(2)})\right) = \boldsymbol{a} \cdot Q(S_1) + \boldsymbol{a} \cdot Q(S_2).\end{aligned}$$

(ii) It is obvious.
(iii) Let $I \in \mathcal{I}_G^\circ$; using the notation of Section 2.4, we start showing that

$$\boldsymbol{a} \cdot \sum_{j=1}^N \left(Q(\varphi_j^+(I)) - Q(\varphi_j^-(I))\right) = P(I,\boldsymbol{a})$$

for every $\boldsymbol{a} \in \mathbb{R}^N$. The surfaces $\varphi_j^+(I)$ and $-\varphi_j^-(I)$ are subordinated to I, so let $(\boldsymbol{v}_k^{(j)+}), (\boldsymbol{v}_k^{(j)-})$ be sequences as in the definition of Q relative to $\varphi_j^+(I), -\varphi_j^-(I)$ respectively. Since

$$\sum_{j=1}^{N} \left(\boldsymbol{v}_k^{(j)+} + \boldsymbol{v}_k^{(j)-}\right) \to \chi_{\partial_* I}\boldsymbol{a},$$

it follows that

$$\boldsymbol{a} \cdot \sum_{j=1}^{N} \left(Q(\varphi_j^+(I)) + Q(-\varphi_j^-(I))\right) = \lim_k P\left(I, \sum_{j=1}^{N} \left(\boldsymbol{v}_k^{(j)+} + \boldsymbol{v}_k^{(j)-}\right)\right) = P(I, \boldsymbol{a}).$$

Then for every $\boldsymbol{a} \in \mathbb{R}^N$ we have that

$$\left|\boldsymbol{a} \cdot \sum_{j=1}^{N} \left(Q(\varphi_j^+(I)) - Q(\varphi_j^-(I))\right)\right| = |P(I, \boldsymbol{a})| \leqslant \eta(I)$$

and (iii) is proved. Finally, we can apply Theorem 2.4.1. $\square$

4.2 The case B manifold

Let us suppose now that B is an n-dimensional differentiable manifold (second countable, Hausdorff, paracompact). We will denote by $\{U_i, \varphi_i\}$ an atlas for the manifold. It is known that every such manifold can be endowed with a Riemannian structure, i.e. a smooth $(0, 2)$ tensor field. The beginning of this section is devoted to introduce some topics which are related, but independent, with the Riemannian structure. We begin with a simple but fundamental proposition .

Proposition 4.2.1. *If g_1, g_2 are two Riemannian metrics on B, then there exist two strictly positive continuous functions C_1, C_2 on B such that*

$$\forall x \in B, \forall v \in T_x B: \quad C_1(x)\langle g_2(x)v, v\rangle \leqslant \langle g_1(x)v, v\rangle \leqslant C_2(x)\langle g_2(x)v, v\rangle. \tag{4.2}$$

Proof. The property clearly holds for every fixed x, since $T_x B$ is finite-dimensional; we now prove that one can choose the constants in a continuous way. Let $x \in B$; it is not restrictive to suppose the trivialization of TB, i.e. $TB \simeq \mathbb{R}^n \times \mathbb{R}^n$, in a suitable neighborhood U of x. Being the function $v \mapsto \langle g_1(x)v, v\rangle$ a norm, we have

$$\sup_{v \in \mathbb{R}^n \setminus \{0\}} \frac{\langle g_1(x)v, v\rangle}{\langle g_2(x)v, v\rangle} = \max_{|v|=1} \frac{\langle g_1(x)v, v\rangle}{\langle g_2(x)v, v\rangle},$$

where $|\cdot|$ denotes the Euclidean norm. Define

$$C_1(y) = \min_{|v|=1} \frac{\langle g_1(y)v, v\rangle}{\langle g_2(y)v, v\rangle}, \qquad C_2(y) = \max_{|v|=1} \frac{\langle g_1(y)v, v\rangle}{\langle g_2(y)v, v\rangle};$$

then C_1, C_2 satisfies (4.2) for every $y \in U$. Now consider C_2 and let $(x_k) \subseteq U$ with $x_k \to x$; by the continuity of g_1, g_2 one has, for every $v \in \mathbb{R}^n \setminus \{0\}$,

$$C_2(x) \geqslant \frac{\langle g_1(x)v, v\rangle}{\langle g_2(x)v, v\rangle} = \lim_k \frac{\langle g_1(x_k)v, v\rangle}{\langle g_2(x_k)v, v\rangle}.$$

Taking the supremum on v, one gets

$$C_2(x) \geqslant \lim_k C_2(x_k).$$

On the other hand, if w achieves the maximum for $C_2(x)$, then

$$C_2(x) = \frac{\langle g_1(x)w, w\rangle}{\langle g_2(x)w, w\rangle} \leqslant \lim_k \frac{\langle g_1(x_k)w, w\rangle}{\langle g_2(x_k)w, w\rangle} \leqslant \liminf_k C_2(x_k),$$

hence C_2 is continuous in x. The same holds for C_1. □

Definition 4.2.2. *If $M \subseteq B$ and $x \in B$, we say that x is* a point of density for M in B *if, given a chart (U, φ) around x, one has that $\varphi(x) \in (\varphi(U \cap M))_*$.*

We show now that this definition is independent of the chosen chart.

Lemma 4.2.3. *If A, B are two open subsets of $\mathbb{R}^n$, $\varphi : A \to B$ is a diffeomorphism and $K \subseteq A$ is compact, then there exist $c_1, c_2 > 0$ such that for every $M \subseteq K$:*

- (a) $c_1 \operatorname{diam} M \leqslant \operatorname{diam} \varphi(M) \leqslant c_2 \operatorname{diam} M$*;*
- (b) $c_1 \mathscr{H}^k(M) \leqslant \mathscr{H}^k(\varphi(M)) \leqslant c_2 \mathscr{H}^k(M)$ *for every $k = 0, \dots, n$;*
- (c) *if $x \in M_*$, then $\varphi(x) \in (\varphi(M))_*$.*

Proof. (a) and (b) are well-known results since φ is bilipschitz. In particular, for $k = n$ one has

$$c_1 \mathscr{L}^n(M) \leqslant \mathscr{L}^n(\varphi(M)) \leqslant c_2 \mathscr{L}^n(M)$$

In order to prove (c), we recall that in the definition of $x \in M_*$ (see Section 1.1), one can replace the balls $B_r(x)$ with sets I_r such that $x \in \operatorname{cl} I_r$, $\operatorname{diam} I_r \to 0$ as $r \to 0$ and

$$\limsup_{r \to 0} \frac{(\operatorname{diam} I_r)^n}{\mathscr{L}^n(I_r)} \leqslant k.$$

Let us choose $I_r = \varphi(B_r(x))$; then they have the required properties and if $x \in M_*$ one has

$$\lim_{r\to 0} \frac{\mathscr{L}^n(I_r \setminus \varphi(M))}{\mathscr{L}^n(I_r)} = \lim_{r\to 0} \frac{\mathscr{L}^n(\varphi(B_r(x) \setminus M))}{\mathscr{L}^n(I_r)} \leqslant \lim_{r\to 0} \frac{c_2}{c_1 \omega_n} \frac{\mathscr{L}^n(B_r(x) \setminus M)}{r^n} = 0,$$

where ω_n denotes the volume of the unit ball. Hence $\varphi(x) \in (\varphi(M))_*$. □

Definition 4.2.4. *Let $M \subseteq B$. We denote by M_* the set of all points of density for M in B. If $M = M_*$, we shall say that M is* normalized.

Note that the whole manifold B is normalized, since B is open and for every open set $A \subseteq B$ one has $A \subseteq A_*$.

Definition 4.2.5. *We define the measure-theoretic boundary of M as*

$$\partial_* M = B \setminus (M_* \cup (B \setminus M)_*).$$

Fix now a Riemannian structure g on B; then we can introduce on B the $(n-1)$-Hausdorff measure $\mathscr{H}_g^{n-1}$.

Lemma 4.2.6. *Let g_1, g_2 be two Riemannian structures on B. Then*

(a) *for every $M \subseteq B$ we have $\mathscr{H}_{g_1}^{n-1}(M) = 0$ if and only if $\mathscr{H}_{g_2}^{n-1}(M) = 0$;*

(b) *for every $M \subseteq B$ with compact closure, we have $\mathscr{H}_{g_1}^{n-1}(M) < +\infty$ if and only if $\mathscr{H}_{g_2}^{n-1}(M) < +\infty$.*

Proof. (a) Let $\mathscr{H}_{g_1}^{n-1}(M) = 0$ and let $\{V_j : j \in J\}$ be an open cover of M such that each V_j has compact closure in some coordinate neighborhood. Since B is second countable, by Lindelöf's Theorem we can extract a countable subcover $\{V_{j_k} : k \in \mathbb{N}\}$. We have $\mathscr{H}_{g_1}^{n-1}(M \cap V_{j_k}) = 0$, hence $\mathscr{H}_{g_2}^{n-1}(M \cap V_{j_k}) = 0$ by (b) of Lemma 4.2.3. It follows $\mathscr{H}_{g_2}^{n-1}(M) = 0$. □

Hence, the fact that $\int_M h \, d\mathscr{H}^{n-1}$ or $\mathscr{H}^{n-1}(M)$ vanish, is independent of the Riemannian structure. In the same way, when M has compact closure in B the fact that $\int_M h \, d\mathscr{H}^{n-1}$ or $\mathscr{H}^{n-1}(M)$ are finite is independent of the Riemannian structure.

Definition 4.2.7. *Let $M \subseteq B$ be a set with compact closure. We say that M has* finite perimeter*, if $\mathscr{H}^{n-1}(\partial_* M) < +\infty$.*

Note that this makes sense, since $\partial_* M$ has compact closure in B. Since the definitions of $\mathcal{M}^\circ, \mathcal{M}_{h\nu}^\circ$ and *almost all* are given in terms of sets with compact closure, they extend naturally to the case of the manifold B.
Now we are ready to give the main definition of this section. We denote by $\mathscr{X}_c(B)$ the set of all smooth vector fields on B with compact support.

Definition 4.2.8. *Let $\mathcal{D} \subseteq \mathcal{M}^\circ$ be a set containing almost all of $\mathcal{M}^\circ$ and take a function $P : \mathcal{D} \times \mathscr{X}_c(B) \to \mathbb{R}$. We say that P is* a Cauchy power *on B, if the following properties hold:*

(a) *$P(\,\cdot\,, \boldsymbol{v})$ is additive for every $\boldsymbol{v} \in \mathscr{X}_c(B)$;*

(b) *$P(M, \cdot\,)$ is linear for almost every $M \in \mathcal{D}$;*

(c) *there exists $h \in \mathcal{L}^1_{loc,+}(B)$ such that*

$$|P(M, \boldsymbol{v})| \leqslant \int_{\partial_* M} |\boldsymbol{v}| h \, d\mathscr{H}^{n-1}$$

for every $\boldsymbol{v} \in \mathscr{X}_c(B)$ and almost every $M \in \mathcal{M}^\circ$.

It is clear that the existence of such an h as in (c) is independent of the Riemannian structure.

Definition 4.2.9. *A Cauchy power P is said to be* balanced *if, given a Riemannian structure on B, there exist $\eta \in \mathfrak{M}(B)$ and $h \in \mathcal{L}^1_{loc,+}(B)$ such that*

$$|P(M, \boldsymbol{v})| \leqslant \|\boldsymbol{v}\|_\infty \eta(M) + \mathrm{Lip}(\boldsymbol{v}) \int_M h \, d\mathscr{H}^n$$

for every $\boldsymbol{v} \in \mathscr{X}_c(B)$ and almost every $M \in \mathcal{M}^\circ$, where $\mathrm{Lip}(\boldsymbol{v})$ denotes the Lipschitz constant of $\boldsymbol{v}$ in the Riemannian structure induced on TB.

Again, the balance of P does not depend on the Riemannian structure.

We recall now some notions about the integration of differential forms over sets with finite perimeter. Let ω be a differential form of degree n on B of class C^∞. Let $M \in \mathcal{M}^\circ$ be such that $M \subseteq U$ for some chart (U, φ); then we define

$$\int_M \omega = \int_{\varphi(M)} \varphi^* \omega \, d\mathscr{L}^n,$$

where $\varphi^*\omega$ denotes the representation of ω in $\varphi(U)$. This definition is independent of φ. For a differential form ω of degree $n-1$, by the Gauss-Green formula on sets with finite perimeter we have

$$\int_M d^*\omega = \int_{\varphi(M)} \varphi^*(d^*\omega) \, d\mathscr{L}^n = \int_{\partial_*(\varphi(M))} \varphi^*\omega \, d\mathscr{H}^{n-1} = \int_{\varphi(\partial_* M)} \varphi^*\omega \, d\mathscr{H}^{n-1}.$$

Setting

$$\int_{\partial M} \omega = \int_M d^*\omega,$$

fixed a Riemannian structure g it results that $\int_{\partial M} \omega$ can be represented by an integral with support in $\partial_* M$, i.e.

$$\int_{\partial M} \omega = \int_{\partial_* M} \omega \, d\mathscr{H}^{n-1}_g,$$

independently of the Riemannian structure g.
For a general $M \in \mathcal{M}^\circ$, we take a finite open cover $\{V_j : j = 1, \ldots, k\}$ and a smooth partition of unity $\{\theta_j : j = 1, \ldots, k\}$ such that each V_j has compact closure in some coordinate neighborhood and $\theta_j(x) = 0$ if $x \notin V_j$. Then we set

$$\int_{\partial M} \omega = \sum_{j=1}^{k} \int_M \theta_j d^*\omega.$$

The definition does not depend on the cover and the partition of unity. Note that the previous definitions make sense also in the case when the coefficients of ω are L^1_{loc}.
Now we recall some notations. Let $\boldsymbol{a}$ be an m-vector and Q a p-form with $p > m$; we define a $(p-m)$-form $\boldsymbol{a} \,\lrcorner\, Q$ by

$$\langle \boldsymbol{a} \,\lrcorner\, Q, \xi \rangle = \langle Q, \xi \wedge \boldsymbol{a} \rangle$$

for every $(p-m)$-vector ξ. In the same way, if $\mathcal{Q}$ is a differential form of degree p and $\boldsymbol{v}$ an m-vector field, we define a differential form $\boldsymbol{v} \,\lrcorner\, \mathcal{Q}$ of degree $p-m$ by

$$(\boldsymbol{v} \,\lrcorner\, \mathcal{Q})(x) = \boldsymbol{v}(x) \,\lrcorner\, \mathcal{Q}(x)$$

for every $x \in B$ (see [8, p. 351]).
The following theorem states the representation formula for a balanced Cauchy power on a manifold.

Theorem 4.2.10. *Let P be a balanced Cauchy power on B. Then there exists a differential form $\mathcal{Q}$ of degree n and of class L^1_{loc} on B such that*

$$P(M, \boldsymbol{v}) = \int_{\partial_* M} \boldsymbol{v} \,\lrcorner\, \mathcal{Q} \tag{4.3}$$

for every $\boldsymbol{v} \in \mathscr{X}_c(B)$ and almost every $M \in \mathcal{M}^\circ$. Moreover, $\partial \mathcal{Q}$ is a distribution representable by integration.

Proof. Let $x \in B$ and (U, φ) be a chart with $\varphi(U) = B_r(\varphi(x))$ for a suitable $r > 0$. Then U is a normalized set with finite perimeter. We can define a function $G : \mathcal{D} \times C_c^\infty(\varphi(U); \mathbb{R}^n) \to \mathbb{R}^n$ setting

$$G(A, \boldsymbol{v}) = P(\varphi^{-1}(A), (d\varphi)^{-1}\boldsymbol{v}),$$

where $\mathcal{D}$ contains almost all of $\mathcal{M}^\circ(\varphi(U))$. Indeed, $\varphi^{-1}(A) \in \mathcal{M}^\circ$ and $(d\varphi)^{-1}\boldsymbol{v} \in \mathscr{X}_c(B)$ (provided that it is extended with zero value outside U). We claim that such a function G is a balanced Cauchy power on $\varphi(U)$. Additivity and linearity are obvious, while (c) follows by the fact that there exists $\tilde{h} \in \mathcal{L}^1_{loc,+}(\varphi(U))$ with

$$|G(A, \boldsymbol{v})| \leqslant \int_{\partial_* \varphi^{-1}(A)} |(d\varphi)^{-1}\boldsymbol{v}| h \, d\mathscr{H}^{n-1} \leqslant \int_{\partial_* A} \tilde{h} |\boldsymbol{v}| \, d\mathscr{H}^{n-1}$$

for every $\boldsymbol{v} \in C_c^\infty(\varphi(U); \mathbb{R}^n)$ and almost every $A \in \mathcal{M}^\circ$. Moreover, keeping into account that P is balanced, one can prove that G is balanced. Then we can apply Corollary 4.1.11 to G with $n = N$, getting an essentially unique function $\mathsf{q}^U \in L^1_{loc}(\varphi(U); \mathrm{Lin}(\mathbb{R}^n; \mathbb{R}^n))$ with divergence measure such that

$$G(A, \boldsymbol{v}) = \int_{\partial_* A} \mathsf{q}^U \boldsymbol{n}^A \cdot \boldsymbol{v} \, d\mathscr{H}^{n-1}$$

for every $\boldsymbol{v} \in C_c^\infty(\varphi(U); \mathbb{R}^n)$ and for almost every $A \in \mathcal{M}^\circ(\varphi(U))$. Moreover, it is not hard to prove that if $U \cap V \neq \varnothing$, then $\mathsf{q}^U = \mathsf{q}^V$ on $\varphi(U) \cap \varphi(V)$. Hence the function $\{\boldsymbol{v} \to \left(\mathsf{q}^U\right)^t \boldsymbol{v}\}$ is a differential form of degree $n-1$ on $\varphi(U)$. Pulling back the functions q^U on B, one obtains the function $\mathcal{Q}$ which satisfies (4.3). Moreover, since q^U has divergence measure in $\mathbb{R}^n$, then $\partial \mathcal{Q}$ can be represented by integration on B. $\square$

Appendix A

Continuity of Cauchy fluxes and interactions

Let us imagine this situation: we would like to know what is the distribution of heat in a body. We have already proved that it is enough to measure the heat flux on almost all the rectangles, but it is impossible to measure exactly the flux on a rectangle when the lack of continuity makes the experimental errors decisive. On the other hand, also the notion of "surface" is not completely clear: from a physical point of view, it is regarded as an "infinitely thin" object.
In order to avoid these problems, we introduce new concepts of subbody, Cauchy interaction and Cauchy flux. The main idea is to average the set function on a suitable thickening of the boundary of a subbody. An n-dimensional interval is replaced by a sort of truncated pyramid, obtained as a convolution of its characteristic function, while a rectangle of dimension $n-1$ is replaced by a parallelepiped, which is associated with the same normal to the surface. These new definitions of interaction and flux turn out be continuous with respect to the natural topology on the subbodies.

A.1 Continuous Cauchy interactions

Fix for simplicity the canonical frame in $\mathbb{R}^n$. Let us set, for $\sigma > 0$,

$$\rho_\sigma(x) = \begin{cases} \frac{1}{\sigma^n} & \text{if } x \in \left]-\frac{\sigma}{2}, \frac{\sigma}{2}\right[^n \\ 0 & \text{elsewhere} \end{cases}$$

and for every $I \in \mathcal{I}^\circ$ define the function

$$\gamma_{I,\sigma}(x) = \int_I \rho_\sigma(x-y)\, d\mathscr{L}^n(y).$$

Clearly, if $I, J \in \mathcal{I}^\circ$ are such that $I \cap J = \varnothing$ and $(I \cup J)_* \in \mathcal{I}^\circ$, then we have

$$\gamma_{(I\cup J)_*,\sigma} = \gamma_{I,\sigma} + \gamma_{J,\sigma}.$$

We say that $\gamma_{I,\sigma}, \gamma_{J,\sigma}$ are *disjoint*, if $\gamma_{I,\sigma} + \gamma_{J,\sigma} \leqslant 1$. It follows that $\gamma_{I,\sigma}, \gamma_{J,\sigma}$ are disjoint if and only if $I \cap J = \varnothing$. Moreover, we denote with I_σ the set where $\gamma_{I,\sigma} > 0$ and with $\operatorname{bd} I_\sigma$ the set where $0 < \gamma_{I,\sigma} < 1$. Note that if $I \cap J = \varnothing$, then $I_\sigma \cap J_\sigma = \operatorname{bd} I_\sigma \cap \operatorname{bd} J_\sigma$.

Now set

$$\mathscr{D} = \{(\gamma_{I,\sigma}, \gamma_{J,\sigma}) : I, J \in \mathcal{I}^\circ, \sigma > 0, I_\sigma, J_\sigma \subseteq \operatorname{int} B, \gamma_{I,\sigma} + \gamma_{J,\sigma} \leqslant 1\}.$$

We want to introduce a real-valued function on $\mathscr{D}$ which will replace the Cauchy interaction. In order to simplify the notation, we suppose the interaction due to the exterior of the body to vanish.

Definition A.1.1. *Let $\mathscr{I} : \mathscr{D} \to \mathbb{R}$. We say that $\mathscr{I}$ is a* continuous balanced Cauchy interaction *(or simply a* continuous interaction*) if the following properties hold:*

(a) (biadditivity)*: for every disjoint $\gamma_{I_1,\sigma}, \gamma_{I_2,\sigma}, \gamma_{J,\sigma}$ such that $(I_1 \cup I_2)_* \in \mathcal{I}^\circ$ we have*

$$\mathscr{I}(\gamma_{I_1,\sigma} + \gamma_{I_2,\sigma}, \gamma_{J,\sigma}) = \mathscr{I}(\gamma_{I_1,\sigma}, \gamma_{J,\sigma}) + \mathscr{I}(\gamma_{I_2,\sigma}, \gamma_{J,\sigma});$$
$$\mathscr{I}(\gamma_{J,\sigma}, \gamma_{I_1,\sigma} + \gamma_{I_2,\sigma}) = \mathscr{I}(\gamma_{J,\sigma}, \gamma_{I_1,\sigma}) + \mathscr{I}(\gamma_{J,\sigma}, \gamma_{I_2,\sigma});$$

(b) *there exist $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ and $\eta \in \mathfrak{M}(\operatorname{int} B \times \operatorname{int} B)$ such that the inequality*

$$|\mathscr{I}(\gamma_{I,\sigma}, \gamma_{J,\sigma})| \leqslant \int_{\operatorname{bd} I_\sigma \cap \operatorname{bd} J_\sigma} h \, d\mathscr{L}^n + \eta(I_\sigma \times J_\sigma) \tag{A.1}$$

holds on $\mathscr{D}$.

(c) *there exists $\lambda \in \mathfrak{M}(\operatorname{int} B)$ such that*

$$\left| \sum_{j=1}^{k} \mathscr{I}(\gamma_{I,\sigma}, \gamma_{J_j,\sigma}) \right| \leqslant \lambda(I_\sigma)$$

whenever $(\gamma_{I,\sigma}, \gamma_{J_j,\sigma}) \in \mathscr{D}$, $\gamma_{J_j,\sigma}$ are mutually disjoint and $\operatorname{bd} I_\sigma \subseteq \left(\bigcup_{j=1}^{k} \operatorname{bd} J_j\right)_*$.

Theorem A.1.2. *Given $\mu \in \mathfrak{M}(\operatorname{int} B \times \operatorname{int} B)$ with $\mu(K \times \operatorname{int} B) < +\infty$ for every compact subset K of $\operatorname{int} B$, a Borel map $b : \operatorname{int} B \to \mathbb{R}$ with $|b(x)| = 1$ for a.e. $x \in \operatorname{int} B$ and a vector field $\boldsymbol{q} \in \mathcal{L}^1_{loc}(\operatorname{int} B; \mathbb{R}^n)$ with divergence measure, the function $\mathscr{I} : \mathscr{D} \to \mathbb{R}$ defined by*

$$\mathscr{I}(\gamma_{I,\sigma}, \gamma_{J,\sigma}) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \gamma_{I,\sigma}(x) \gamma_{J,\sigma}(y) \, b(x,y) \, d\mu(x,y) + \sigma \int_{\operatorname{bd} I_\sigma \cap \operatorname{bd} J_\sigma} \boldsymbol{q} \cdot \operatorname{grad} \gamma_{I,\sigma} \, d\mathscr{L}^n. \tag{A.2}$$

is a continuous interaction.

Proof. Take $I_1, I_2, J \in \mathcal{I}^\circ$ disjoint; we want to prove the biadditivity of $\mathscr{I}$. The first integral is obviously biadditive; for the second, we note that

$$\operatorname{grad} \gamma_{(I_1 \cup I_2)_*,\sigma} = \operatorname{grad} \gamma_{I_1,\sigma} + \operatorname{grad} \gamma_{I_2,\sigma}$$

and that $\operatorname{grad} \gamma_{I_1,\sigma}(x) = 0$ when $x \notin I_{1_\sigma}$. Property (b) follows by the estimate

$$|\sigma \boldsymbol{q} \cdot \operatorname{grad} \gamma_{I,\sigma}| \leqslant |\boldsymbol{q}| \, |\sigma \operatorname{grad} \gamma_{I,\sigma}| \leqslant 2n|\boldsymbol{q}|$$

setting $h = 2n|\boldsymbol{q}|$. Finally, property (c) follows from the fact that $\operatorname{div} \boldsymbol{q}$ is a measure and $\mu(\cdot \times \operatorname{int} B)$ is finite on compact sets. ∎

Theorem A.1.3. *Let $\mathscr{I}$ be a continuous interaction. Then there exist $b, \mu, \boldsymbol{q}$ such that* (A.2) *holds on $\mathscr{D}$. Moreover, μ is uniquely determined, b is uniquely determined μ-a.e. and $\boldsymbol{q}$ is uniquely determined $\mathscr{L}^n$-a.e.*

Proof. Taking into account that $\gamma_{I,\sigma}(x) \to \chi_I(x)$ (where χ_I denotes the characteristic function of the set I) and $\sigma \operatorname{grad} \gamma_{I,\sigma}(x) \to \boldsymbol{n}^I(x)$ as $\sigma \to 0^+$, one obtains that the function

$$Q(I,J) = \lim_{\sigma \to 0^+} \mathscr{I}(\gamma_{I,\sigma}, \gamma_{J,\sigma})$$

induces a balanced Cauchy flux on $\operatorname{int} B$. Then the proof is easily completed. □

A.2 Continuous Cauchy fluxes

We fix an orthonormal frame $(e_1, \dots, e_n)$ and a point $x_0 \in \mathbb{R}^n$. We remind that $\mathcal{I}^\circ$ denotes the set of all open n-dimensional intervals I such that $\operatorname{cl} I \subseteq \operatorname{int} B$.

Definition A.2.1. *We set*

$$\mathscr{C} = \left\{ I = (\widehat{I}, \boldsymbol{n}_I) : \widehat{I} \in \mathcal{I}^\circ,\, \boldsymbol{n}_I \in \{e_1, \dots e_n\} \right\}.$$

We refer to the elements of $\mathscr{C}$ as the material intervals *and call the vector $\boldsymbol{n}_I$* the normal *to the material interval I.*

Definition A.2.2. *A finite disjoint family $\{I_k = (\widehat{I}_k, \boldsymbol{n}_{I_k}) :\, k \in \Lambda\} \subseteq \mathscr{C}$ is said* to surround *an open n-dimensional interval $I \in \mathcal{I}^\circ$, if $I \cap \widehat{I}_k = \varnothing$,*

$$\left(\bigcup_{k \in \Lambda} \widehat{I}_k \cup I \right)_* \in \mathcal{I}^\circ, \quad \partial I \subseteq \bigcup_{k \in \Lambda} \partial \widehat{I}_k,$$

$\boldsymbol{n}^I(x) = \boldsymbol{n}_{I_k}$ for every $x \in \partial \widehat{I}_k \cap \partial I$.

Now we define a function which will replace the concept of balanced Cauchy flux.

Definition A.2.3. *Let $\mathcal{Q} : \mathscr{C} \to \mathbb{R}$. We say that $\mathcal{Q}$ is a* continuous balanced Cauchy flux *(or simply a* continuous flux*) if the following properties hold:*

(*a*) (additivity)*: for every $I, J \in \mathscr{C}$ such that $\widehat{I} \cap \widehat{J} = \varnothing$, $(\widehat{I} \cup \widehat{J})_* \in \mathcal{I}^\circ$ and $\boldsymbol{n}_I = \boldsymbol{n}_J$ we have*

$$\mathcal{Q}((\widehat{I} \cup \widehat{J})_*, \boldsymbol{n}_I) = \mathcal{Q}(I) + \mathcal{Q}(J);$$

(*b*) *there exists $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$ such that*

$$|\mathcal{Q}(I)| \leqslant \int_{\widehat{I}} h \, d\mathscr{L}^n$$

for every $I \in \mathscr{C}$;

(c) *there exists* $\eta \in \mathfrak{M}(\operatorname{int} B)$ *such that*

$$\sum_{k=1}^{n} |\mathcal{Q}(I_k) - \mathcal{Q}(J_k)| \leqslant \eta(I)$$

for every $I \in \mathcal{I}^\circ$ *and for every family* $\{J_k\} \cup \{I_k\}$ *which surrounds* I *with* $\boldsymbol{n}_{I_k} = -\boldsymbol{n}_{J_k}$ *for every* k.

The function introduced above can be considered as the continuous version of a balanced Cauchy flux, in the sense specified below.

Theorem A.2.4. *Let* $\boldsymbol{q} \in \mathcal{L}^1_{loc}(\operatorname{int} B; \mathbb{R}^n)$ *be a vector field with divergence measure. Then the function* $\mathcal{Q} : \mathscr{C} \to \mathbb{R}$ *defined by*

$$\mathcal{Q}(\widehat{I}, e_k) = \int_{\widehat{I}} \boldsymbol{q} \cdot e_k \, d\mathscr{L}^n.$$

is a continuous balanced Cauchy flux.
Moreover, $\mathcal{Q}$ *is continuous with respect to the topology of* $\mathscr{C}$ *induced by the natural one on* $\mathbb{R}^n \times \mathbb{R}$.

Proof. It is easily checked. □

We can prove also the converse of the preceding theorem.

Theorem A.2.5. *Let* $\mathcal{Q} : \mathscr{C} \to \mathbb{R}$ *a continuous balanced Cauchy flux. Then there exists a vector field* $\boldsymbol{q} \in \mathcal{L}^1_{loc}(\operatorname{int} B; \mathbb{R}^n)$ *with divergence measure such that*

$$\forall (\widehat{I}, e_k) \in \mathscr{C} : \ \mathcal{Q}(\widehat{I}, e_k) = \int_{\widehat{I}} \boldsymbol{q} \cdot e_k \, d\mathscr{L}^n.$$

Moreover, $\boldsymbol{q}$ *is uniquely determined* $\mathscr{L}^n$*-a.e.*

Proof. We follow the proof of Lemma 2.4.2. Define, for $j = 1, \dots, n$, a linear operator $T_j : C_0^\infty(\operatorname{int} B) \to \mathbb{R}$ as

$$\langle T_j, f \rangle = \lim_{\sup_m (\operatorname{diam} \widehat{I}_m) \to 0} \left(\sum_m f(\xi_m) \mathcal{Q}(I_m) \right),$$

where each $\{I_m\}$ is a finite disjoint subfamily of $\mathscr{C}$ of material intervals which share the same normal e_j, such that $\mathscr{L}^n\left((\operatorname{supt} f) \setminus \bigcup_m \widehat{I}_m\right) = 0$ and $\xi_m \in \widehat{I}_m$. Hence T_j is a distribution of order 0 on $\operatorname{int} B$ and

$$\forall f \in C_0^\infty(\operatorname{int} B) : \ |\langle T_j, f \rangle| \leqslant \int_{\operatorname{int} B} |f| h \, d\mathscr{L}^n.$$

Then there exists $\boldsymbol{q}^{(j)} \in \mathcal{L}^1_{loc}(\operatorname{int} B; \mathbb{R})$ such that $|\boldsymbol{q}^{(j)}| \leqslant h$ and

$$\mathcal{Q}(I) = \int_{\widehat{I}} \boldsymbol{q}^{(j)} \, d\mathscr{L}^n$$

for every $I \in \mathscr{C}$ of the form $(\widehat{I}, e_j)$. Defining

$$Q(S) = \int_{\widehat{S}} \boldsymbol{q}^{(j)}\, d\mathscr{H}^{n-1}$$

for every $S \in \mathcal{S}_G$ with $S = (\widehat{S}, e_j)$ and applying Theorem 2.4.1, we obtain the result. $\square$

It can be verified that, as in the case of balanced Cauchy fluxes and interactions, also in this case there is a link between continuous fluxes and continuous interactions, which is summarized in the following:

$$\mathscr{I}(\gamma_{I,\sigma}, \gamma_{J,\sigma}) = \mathcal{Q}(I_\sigma \cap J_\sigma, \boldsymbol{n}^I\big|_{I_\sigma \cap J_\sigma}).$$

Appendix B

Extension of a Cauchy interaction to the boundary of the body

Usually the formulation of a problem of heat conduction or stress distribution is given in a local form: the classical balance law has a local version and one assigns to the body some data. Then, in order to solve the problem, the local version is dropped and one finds weak solutions of the distributional equation. The approach introduced in this dissertation bypasses this procedure: a Cauchy interaction is naturally associated with measures and the problem can be directly stated by means of measures. Moreover, in this way less regularity is required.

The Cauchy interaction between two subbodies, as was remarked in Chapter 1, requires the boundary of the first subbody not to meet the boundary of the continuous body B; in fact, the definition of the subbodies was given with the condition $\operatorname{cl} A \subseteq \operatorname{int} B$. Now, in order to set out the problem of finding an interaction given some boundary conditions, we need to extend the definition of a Cauchy interaction also for those subbodies such that $\partial_* A \cap \partial B \neq \varnothing$. This is not a trivial fact; we need to assume some regularity on the tensor flux density q and the body B, namely $\mathsf{q} \in L^\infty(B; \mathbb{R}^n)$ and B with Lipschitz boundary, in order to apply a result due to Anzellotti [2] about traces of the normal component of functions with divergence measure.

B.1 Trace of a tensor field with divergence measure

Requiring some regularity, we can introduce a notion of trace of the normal component of an essentially bounded vector field with divergence measure, in the following way.

Theorem B.1.1 (Anzellotti). *Let Ω be an open bounded set with locally Lipschitz boundary. Then there exists a linear operator*

$$\gamma : \{\mathsf{q} \in L^\infty(\Omega; \mathbb{R}^n) : \operatorname{div} \mathsf{q} \text{ is a signed bounded measure on } \Omega\} \to L^\infty(\partial\Omega)$$

such that:

(a) $\|\gamma(\mathsf{q})\|_{\infty,\partial\Omega} \leqslant \|\mathsf{q}\|_{\infty,\Omega}$;

(b) *if* $\mathsf{q} \in C^1(\operatorname{cl}\Omega; \mathbb{R}^N)$, *then* $\gamma(\mathsf{q}) = \mathsf{q} \cdot \boldsymbol{n}_{\partial\Omega}$ *on* $\partial\Omega$.

Proof. See [2]. $\square$

By means of the preceding theorem, the trace of the normal component of a tensor field $\mathsf{q} \in L^\infty\left(\Omega; \mathrm{Lin}\left(\mathbb{R}^n; \mathbb{R}^N\right)\right)$ with divergence measure can be easily defined. In fact, denoting with $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_N)$ the canonical basis of $\mathbb{R}^N$, we have that $\mathsf{q}\boldsymbol{e}_i \in L^\infty(\Omega; \mathbb{R}^n)$ and we can define a linear operator

$$\Gamma : \left\{\mathsf{q} \in L^\infty\left(\Omega; \mathrm{Lin}\left(\mathbb{R}^n; \mathbb{R}^N\right)\right) : \ \mathrm{div}\,\mathsf{q} \text{ is a bounded measure on } \Omega\right\} \to L^\infty\left(\partial\Omega; \mathbb{R}^N\right)$$

$$\Gamma(\mathsf{q}) = \left(\gamma(\mathsf{q}\boldsymbol{e}_1), \dots, \gamma(\mathsf{q}\boldsymbol{e}_N)\right).$$

Now we introduce the notions of "almost all of $\mathfrak{D}$" and "almost all of $\mathcal{M}$" by removing the condition that the closure of the subbodies be contained in $\mathrm{int}\, B$.

Definition B.1.2. *We say that:*

(i) *a set $\mathcal{P}$* contains almost all of $\mathcal{M}$, *if there exist $h \in \mathcal{L}^1_{loc,+}(\mathbb{R}^n)$ and $\nu \in \mathfrak{M}(\mathbb{R}^n)$ such that*

$$\mathcal{M}_{h\eta} = \left\{A \in \mathcal{M} : \int_{\partial_* A} h\, d\mathscr{H}^{n-1} < +\infty, \quad \nu(\partial_* A) = 0\right\} \subseteq \mathcal{P};$$

(ii) *a set $\mathcal{P}$* contains almost all of $\mathfrak{D}$, *if there exist $h \in \mathcal{L}^1_{loc,+}(\mathbb{R}^n)$ and $\nu \in \mathfrak{M}(\mathbb{R}^n)$ such that*

$$\mathfrak{D}_{h\eta} = \left\{(A, C) \in \mathfrak{D} : \int_{\partial_* A \cup \partial_* C} h\, d\mathscr{H}^{n-1} < +\infty, \quad \nu(\partial_* A) = \nu(\partial_* C) = 0\right\} \subseteq \mathcal{P}.$$

Note that the function h and the measure ν are defined on $\mathbb{R}^n$ and on $\mathfrak{B}(\mathbb{R}^n)$ respectively; this means in particular that $\int_A h\, d\mathscr{H}^{n-1} < +\infty$ and $\nu(A) < +\infty$ whenever A is a bounded set.

Let B denote the body and $\mathcal{P}$ a set containing almost all of $\mathfrak{D}^\circ$. The notion of Cauchy interaction can be extended to the vectorial case $I : \mathcal{P} \to \mathbb{R}^N$ (cf. Section 2.6), by requiring that:

(a) I is biadditive;

(b) there exist $\widehat{h} \in \mathcal{L}^1_{loc,+}(\mathrm{int}\, B)$, $\eta \in \mathfrak{M}(\mathrm{int}\, B \times \mathrm{int}\, B)$, $\eta_e \in \mathfrak{M}(\mathrm{int}\, B)$ such that the inequality

$$|I(A, C)| \leqslant \begin{cases} \displaystyle\int_{\partial_* A \cap \partial_* C} \widehat{h}\, d\mathscr{H}^{n-1} + \eta(A \times C) & \text{if } C \subseteq B, \\ \displaystyle\int_{\partial_* A \cap \partial_* C} \widehat{h}\, d\mathscr{H}^{n-1} + \eta(A \times (C \cap B)) + \eta_e(A) & \text{otherwise,} \end{cases}$$

holds almost everywhere in $\mathfrak{D}^\circ$;

(c) there exists $\lambda \in \mathfrak{M}(\mathrm{int}\, B)$ with

$$\partial_* A \subseteq \partial_* C \Longrightarrow |I(A, C)| \leqslant \lambda(A),$$

for almost all of $\mathfrak{D}^\circ$.

The results stated in Chapters 1 and 2 hold true also in this case, with obvious adaptations. In particular, by Theorem 2.7.5 there exist $h \in \mathcal{L}^1_{loc,+}(\operatorname{int} B)$, $\nu \in \mathfrak{M}(\operatorname{int} B)$, $\boldsymbol{b} : \operatorname{int} B \times \operatorname{int} B \to \mathbb{R}^N$, $\boldsymbol{b}_e : \operatorname{int} B \to \mathbb{R}^N$, $\mu \in \mathfrak{M}(\operatorname{int} B \times \operatorname{int} B)$, $\mu_e \in \mathfrak{M}(\operatorname{int} B)$ and $\mathsf{q} \in \mathcal{L}^1_{loc}\left(\operatorname{int} B; \operatorname{Lin}\left(\mathbb{R}^n; \mathbb{R}^N\right)\right)$ such that $|\boldsymbol{b}| = 1$ μ-a.e. in $\operatorname{int} B \times \operatorname{int} B$, $|\boldsymbol{b}_e| = 1$ μ_e-a.e. in $\operatorname{int} B$ and

$$I(A,C) = \begin{cases} \displaystyle\int_{A\times C} \boldsymbol{b}\, d\mu + \int_{\partial_* A \cap \partial_* C} \mathsf{q}\boldsymbol{n}_{\partial_* A\cap\partial_* C}\, d\mathscr{H}^{n-1} & \text{if } C \subseteq B, \\ \displaystyle\int_{A\times(C\cap B)} \boldsymbol{b}\, d\mu + \int_A \boldsymbol{b}_e\, d\mu_e + \int_{\partial_* A\cap\partial_* C} \mathsf{q}\boldsymbol{n}_{\partial_* A\cap\partial_* C}\, d\mathscr{H}^{n-1} & \text{otherwise,} \end{cases} \tag{B.1}$$

for every $(A,C) \in \mathfrak{D}^\circ_{h\nu} \cup \{(A,C) \in \mathcal{M}^\circ_{h\nu} \times \mathcal{N} : (\mathbb{R}^n \setminus C)_* \in \mathcal{M}^\circ_{h\nu}, A \cap C = \varnothing\}$. Moreover, we have $\mu \leqslant \eta$ and $|\operatorname{div} \mathsf{q}| \leqslant \lambda$.
Suppose now that $\operatorname{int} B$ has locally Lipschitz boundary; in particular, since B is normalized, it turns out that it is open. Further, suppose that $\widehat{h} \in \mathcal{L}^1_{loc,+}(\mathbb{R}^n)$, $\eta \in \mathfrak{M}(\mathbb{R}^n \times \mathbb{R}^n)$, $\eta_e \in \mathfrak{M}(\mathbb{R}^n)$ and $\lambda \in \mathfrak{M}(\mathbb{R}^n)$; in particular it follows that $\mu(B \times B) < +\infty$ and $\int_B |\operatorname{div} \mathsf{q}| < +\infty$. Finally, assume that $\mathsf{q} \in L^\infty\left(B; \operatorname{Lin}\left(\mathbb{R}^n; \mathbb{R}^N\right)\right)$.

Remark B.1.3. Following Šilhavý (cf. [28, Proposition 8.1]), we note that the flux tensor satisfies the essential boundedness if and only if

$$\left| \int_{\partial_* A\cap\partial_* C} \mathsf{q}\boldsymbol{n}^A\, d\mathscr{H}^{n-1} \right| \leqslant K\mathscr{H}^{n-1}(\partial_* A \cap \partial_* C)$$

on almost all of $\mathfrak{D}^\circ$, i.e. the assumption (ii) in Definition 1.2.18 is strengthened by a Lipschitz continuity with respect to the measure $\mathscr{H}^{n-1}$.

It follows that every component of q is in the conditions of Theorem B.1.1 and there exists a linear operator Γ as explained above. We define a new function $\overline{I} : \mathfrak{D}_{h\nu} \to \mathbb{R}$ by setting

$$\overline{I}(A,C) = \begin{cases} \displaystyle\int_{A\times C} \boldsymbol{b}\, d\mu + \int_{\partial_* A\cap\partial_* C\cap B} \mathsf{q}\boldsymbol{n}_{\partial_* A\cap\partial_* C}\, d\mathscr{H}^{n-1} + \\ \displaystyle + \int_{\partial_* A\cap\partial_* C\cap\partial B} \Gamma(\mathsf{q})\, d\mathscr{H}^{n-1} & \text{if } C \subseteq B, \\ \displaystyle\int_{A\times(C\cap B)} \boldsymbol{b}\, d\mu + \int_A \boldsymbol{b}_e\, d\mu_e + \int_{\partial_* A\cap\partial_* C\cap B} \mathsf{q}\boldsymbol{n}_{\partial_* A\cap\partial_* C}\, d\mathscr{H}^{n-1} + \\ \displaystyle + \int_{\partial_* A\cap\partial_* C\cap\partial B} \Gamma(\mathsf{q})\, d\mathscr{H}^{n-1} & \text{otherwise.} \end{cases}$$

The function $\overline{I}$ is clearly biadditive; this explains the following definition.

Definition B.1.4. *Let I be a balanced Cauchy interaction as in the preceding. Then we say that I is* extendable to the boundary *and $\overline{I}$ is the extension of I.*

B.2 Statement of a boundary value problem

In the representation of a Cauchy interaction, the double integral

$$\int_{A\times C} \boldsymbol{b}(x,y)\, d\mu(x,y)$$

deals with the body action of the subbody C on A. The hypotheses on the function $\boldsymbol{b}$ are that it is Borel and $|\boldsymbol{b}| = 1$ μ-a.e. in $B \times B$. Suppose now to assume that

$$\int_{A\times A} \boldsymbol{b}\, d\mu = 0 \qquad \text{for every Borel subset } A \subseteq B. \tag{B.2}$$

This is equivalent to assume the *law of action and reaction* (cf. [23, p. 76])

$$\int_{A\times C} \boldsymbol{b}\, d\mu = -\int_{C\times A} \boldsymbol{b}\, d\mu \qquad \text{for almost every } (A, C) \in \mathfrak{D}^\circ.$$

With this new assumption, it is easy to check that the set function

$$\Big\{ A \mapsto I(A, (\mathbb{R}^n \setminus A)_*) \Big\}$$

can be extended to a vector-valued measure, i.e. there exist a unique measure $\psi \in \mathfrak{M}(\mathbb{R}^n)$ and a unique (up to sets of zero ψ measure) Borel function $\boldsymbol{c} : B \to \mathbb{R}^N$ such that $|\boldsymbol{c}| = 1$ ψ-a.e. in B and

$$\int_A \boldsymbol{c}\, d\psi = I(A, (\mathbb{R}^n \setminus A)_*)$$

for almost every $A \in \mathcal{M}$ (cf. Theorem 2.7.6). We remark that no constitutive equations are given at this point: the goal of this section is to show that a problem with boundary conditions can be naturally stated in the framework of Cauchy interactions. One has to introduce further hypotheses, such as constitutive equations, in order to solve this problem.

Take an open bounded set B with locally Lipschitz boundary (the body) and suppose to assign two kinds of data: the density of the stress, given as a vector measure defined on the body, and some constraint on the boundary, given as an essentially bounded function on the boundary. Let us denote with $\boldsymbol{c}\psi$ the density of the stress, where $\psi \in \mathfrak{M}(\mathbb{R}^n)$ and $\boldsymbol{c} : \mathbb{R}^n \to \mathbb{R}^N$ is a Borel function such that $|\boldsymbol{c}| = 1$ ψ-a.e. Moreover, let us denote with $g \in L^\infty\left(\partial B; \mathbb{R}^N\right)$ the boundary datum. The problem consists in finding a Cauchy interaction I extendable to the boundary such that

$$\begin{cases} I(A, (\mathbb{R}^n \setminus A)_*) = \displaystyle\int_A \boldsymbol{c}\, d\psi \\ \mathit{\Gamma}(\mathsf{q}) = g \end{cases}$$

for almost every $A \in \mathcal{M}$. Due to the arbitrariness of A, this problem can be stated by using measures:

$$\begin{cases} \operatorname{div} \mathsf{q} + \boldsymbol{b}\mu(\,\cdot\, \times B) + \boldsymbol{b}_e \mu_e = \boldsymbol{c}\psi & \text{on } \mathfrak{B}(B), \\ \mathit{\Gamma}(\mathsf{q}) = g & \text{on } \partial B. \end{cases}$$

The study of suitable constitutive relations and the solving of the problem go beyond the purposes of this dissertation. As the only remark, we note that the function g and the measure $\boldsymbol{c}\psi$ cannot be completely independent. In fact, by Theorem B.1.1 it follows that

$$\int_{\partial B} \mathit{\Gamma}(\mathsf{q})\, d\mathscr{H}^{n-1} = \int_B \operatorname{div} \mathsf{q},$$

hence

$$\int_B \boldsymbol{c}\, d\psi = \int_{B\times B} \boldsymbol{b}\, d\mu + \int_B \boldsymbol{b}_e\, d\mu_e + \int_{\partial B} g\, d\mathscr{H}^{n-1}.$$

In particular, in the case in which $\boldsymbol{b} = 0$ and $\boldsymbol{b}_e = 0$ (i.e. when the volume stresses vanish), the data have to satisfy

$$\int_B \boldsymbol{c}\, d\psi = \int_{\partial B} g\, d\mathscr{H}^{n-1}.$$

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