# Using covering spaces to model soap films

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# People involved

**Working in our group:**
S. Amato, G. Bellettini, M. P., *Constrained BV functions on covering spaces for minimal networks and Plateau's type problems*, Adv. Calc. Var.
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A. Chambolle, D. Cremers, T. Pock, *A convex approach to minimal partitions*, SIAM Journal on Imaging Sciences, 2012

# Outline

- Motivation
- Covering space
- Knots and links
- The tripod (model for the triple junction)
- The convexification problem
- Other examples

# Motivation

Find a material interface (say soap film) with minimal area and given boundary in 3D

Corresponding evolution problem (Mean Curvature Flow)

Using the BV machinery [De Giorgi], suitable e.g. for relaxation via diffused interface

**Features:** Knotted curves in 3D, triple junctions

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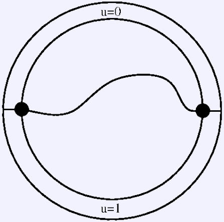
## A "too" simple example




- $\Gamma$ is the equator of the unit sphere
- $\Omega$ is an enlarged sphere
- Force $u = 1$ and $u = 0$ outside the unit sphere
- Minimize the total variation in $BV(\Omega; \{0, 1\})$ subject to the constraint
- The jump set $\Sigma$ of a minimizer is the desired soap film

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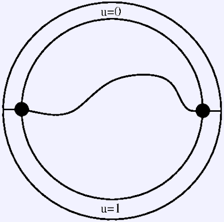


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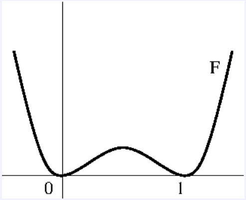
We trick the Plateau problem into a phase-separation problem, the two phases being $\{u = 0\}$ and $\{u = 1\}$

# Allen-Cahn equation: brief review

Reaction/diffusion equation arising in the context of **phase transitions** with a diffused interface:

$$\begin{cases} \epsilon\partial_t u - \epsilon\Delta u + \frac{1}{\epsilon} f(u) = 0 & \text{in } \Omega \\ + \text{ initial and boundary conditions} \end{cases}$$

- $u$: order parameter (phase indicator),
- $\Omega$: domain in $\mathbb{R}^d$, $d = 2, 3$,
- $\epsilon > 0$: small relaxation parameter,
- $f = F'$: derivative of a double equal well potential $F$ (or double-obstacle: deep quench limit [Elliott et al]).

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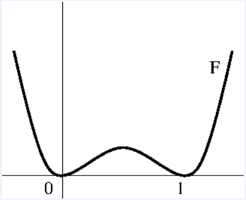


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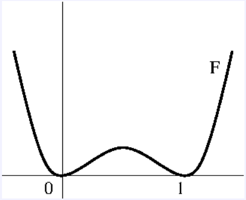


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# Singular limit $\epsilon \to 0$

The transition layer approximates a sharp interface that moves by mean curvature:

$$\boxed{V = -\kappa}$$

[X. Chen, Bronsard-Kohn, Evans-Soner-Souganidis,]
[Barles-Soner-Souganidis, ...]

Optimal $\mathcal{O}(\epsilon^2)$ or quasi-optimal $\mathcal{O}(\epsilon^2|\log \epsilon|)$ error estimate.
[Nochetto-P.-Verdi, Nochetto-Verdi, Bellettini-P.]

A.C. is the gradient flow of

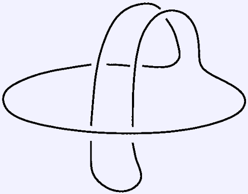
$$\boxed{\mathcal{F}_\epsilon(u) := \frac{\epsilon}{2} \int_\Omega |\nabla u|^2 \, dx + \frac{1}{\epsilon} \int_\Omega F(u) dx}$$

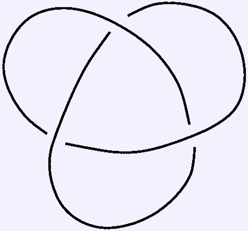
**Convergence as $\epsilon \to 0$**

$\mathcal{F}_\epsilon$ Γ-converges to $c_0 \mathcal{F}$ with $\mathcal{F}(u) := \int_\Omega |Du|$, $c_0$ a suitable constant depending on $F$, $u \in BV(\Omega, \{0, 1\})$

# Motivation 2

This simple $BV$ approach for the circle example does not work in general. The curve $\Gamma$ must be such that it lies in the boundary of a convex (or at least mean-convex) body in $\mathbb{R}^3$





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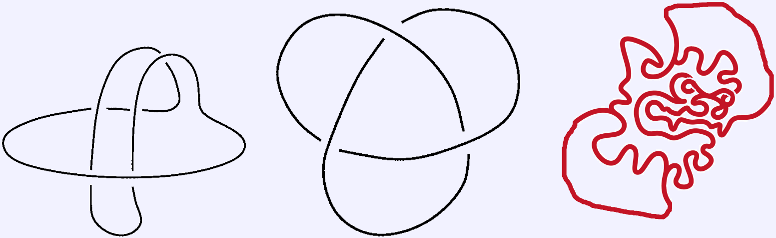

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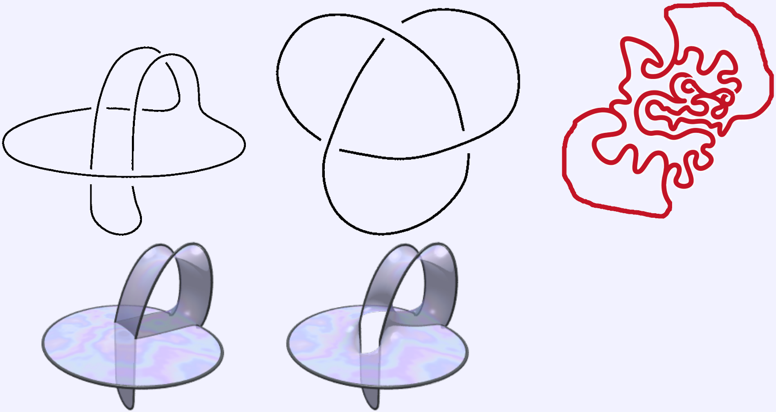


[images courtesy of Emanuele Paolini]

# Motivation 3

Problem: the desired surface $\Sigma$ does not naturally separate two phases near $\Gamma$

**Idea**

Work on a "covering space" $Y$ of $\Omega = \mathbb{R}^3 \setminus \Gamma$ (if regularity of $\partial\Omega$ is needed $\Gamma$ can be replaced by a tubular neighborhood $\Gamma_\epsilon$)

$\Omega = B \setminus \Gamma_\epsilon$, where $B$ is a large ball containing $\Gamma_\epsilon$ or $B = \mathbb{S}^3$
$p : Y \to \Omega$ is a covering (of finite degree $k$)

**Find $u \in BV(Y; \{0,1\})$ such that**

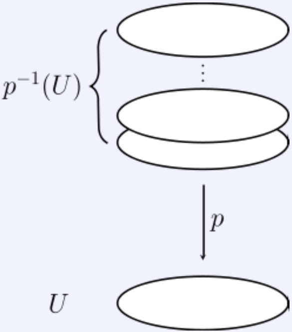
- $u$ is given near the boundary of $\Omega$
- $u$ satisfies the constraint

$$\sum_{y \in p^{-1}(x)} u(y) = 1$$

on the fiber above any $x \in \Omega$

$\Omega$ is path-connected.
$p : Y \to \Omega$ is locally trivial

For any $x \in \Omega$ there is a small neighborhood $U$ such that $p^{-1}(U)$ is topologically the disjoint union of $k$ copies of $U$. $k$ (degree of the covering) is locally constant in $\Omega$, hence constant. It is not finite in general, but for our purposes $k < \infty$



# Covering spaces 2

A covering can be globally trivial: $Y$ is the disjoint union of $k$ copies of $\Omega$, but the interesting case is when $Y$ is connected.

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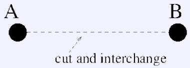
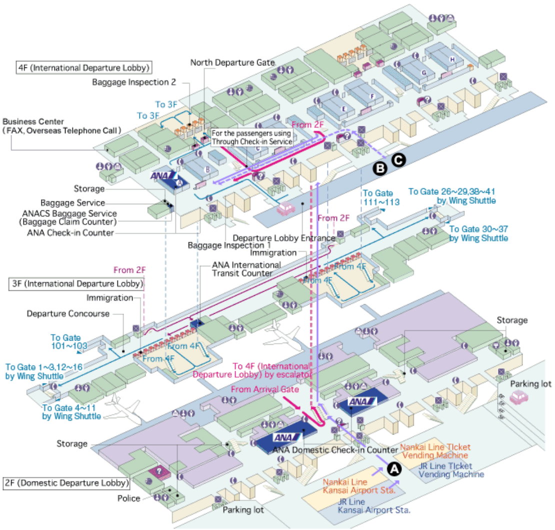
**Example** $(k = 2)$

$\Omega = \mathbb{S}^2 \setminus \{A, B\}$

- Start with two copies of $\Omega$ (deck 1 and deck 2)
- cut along the segment $AB$
- glue deck 1 above with deck 2 below and viceversa




We obtain a nontrivial covering of degree 2




Kansai International Airport

## Kansai International Airport

**Terminal 2**

**Important property** If $\gamma : [0,1] \to \Omega$ is a closed curve with $\gamma(0) = \gamma(1) = x$, then after chosing $y \in p^{-1}(x)$ we can "lift" $\gamma$ into $\gamma_\# : [0,1] \to Y$ with $\gamma_\#(0) = y$, continuous and such that $p \circ \gamma_\# = \gamma$.
In general $\gamma_\#(1) \in p^{-1}(x)$ is different from $\gamma_\#(0)$ and independent on continuous deformations of $\gamma$: $\gamma \in \pi_1(\Omega, x)$ acts on the fiber at $x$
$\pi_1(\Omega, x)$ is the fundamental group of $\Omega$ with base point $x$

# Covering spaces 3

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**Trivial example**

If $\Omega$ is simply connected $\Rightarrow$ covering is globally trivial

# Basic example

**Remark**

In this perspective the "point at infinity" of $\mathbb{R}^2$ is important. We prefer to work in $\mathbb{S}^2$. Conversely using $\mathbb{R}^3$ or $\mathbb{S}^3$ is basically equivalent (in dimension 3 a curve cannot be obstructed by a single point)

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## Knot/link $K$, $k = 2$

- Take a Seifert surface $S$ of the knot
- Start with two copies of $\Omega := \mathbb{R}^3 \setminus K$ (deck 1 and deck 2)
- Cut them along $S$ and glue back with a deck exchange

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## Equivalently: Start with $X = \{\text{paths } \gamma \text{ starting at } x\}$

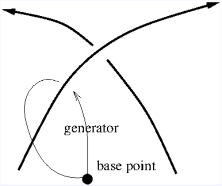
- $\gamma_1 \sim \gamma_2$ if $\gamma_1(1) = \gamma_2(1)$ **and**
- $\gamma_1 \gamma_2^{-1}$ has **even** linking number with $K$

Define $Y = X/\sim$, $p(\gamma) = \gamma(1) \in \Omega$

# Basic example 2

**Wirtinger presentation**

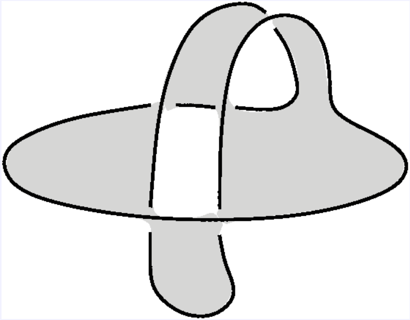
$< a, b, \dots ; r_1, r_2, \cdots >$




In this case $\gamma$ has even linking number iff the corresponding word in the group presentation has even length

$< a, b; aba = bab >$ is a Wirtinger presentation of the trefoil knot;
$< x, y; x^2 = y^3 >$ is **not** a Wirtinger p. of the trefoil knot

There are many ways of constructing a Seifert surface, one is shown in the picture



# The $BV$ setting

Now that we have a "double" ($k = 2$) covering of $\Omega$ we shall consider functions $u \in S_0$ with

$$S_0 := \left\{ u \in BV(Y; \{0,1\}) : \sum_{y \in p^{-1}(x)} u(y) = 1 \right\}$$

Functions in $S_0$ **must** jump when circling $\Gamma$ once along small curves $\Rightarrow$ the jump set must touch $\Gamma$ at all points. We have no control on the topology, but this is not a problem

# The $BV$ setting 2

For $u \in S_0$ we can define the energy

$$\mathcal{F}(u) := \frac{1}{2} \int_Y |Du|$$

Due to the constraint the jump set projects nicely on $\Omega$ and we account the projected jump twice, hence the factor $1/2$
The coarea formula allows to convexify $S_0$ into

$$S := \left\{ u \in BV(Y, [0,1]) : \sum_{p(y)=x} u(y) = 1 \right\}$$

(same energy) and get an essentially equivalent problem

**Warning!**

This is specific to the $k = 2$ case

# Brakke approach in the case $k = 2$

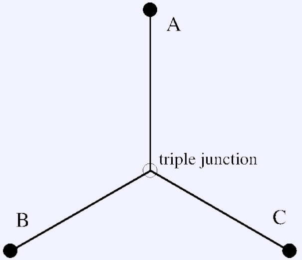
Strictly related: he works in the context of currents on the covering space $Y$, the constraint is imposed by constructing a higher-level covering $W$ of $Y$ made of "pairs" of distinct points on each fiber
This approach shows its power in the case $k > 2$ but makes sense also for $k = 2$
The boundary of a flat chain in $Y$ can be associated (in infinite ways) to a current $T$ in $W$ and we seek to minimize the mass of $T$ (that represents the soap film itself) obtaining the "film mass"

# The tripod

Model problem to tackle triple junctions:

$$\Omega := \mathbb{S}^2 \setminus \{A, B, C\}$$

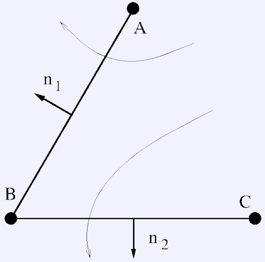
where $A$, $B$, $C$ are the vertices of an equilateral triangle. The solution that we would like to see is the "tripod" To guarantee that $\Gamma$ touches all three points coverings of degree 2 do not work...




This is the solution of the Steiner problem (angles of 120 degrees)

# A covering for the tripod

For $k = 3$ a natural construction by cut & paste is obtained by cutting three copies of $\mathbb{S}^3$ along two of the three sides, say $AB$ and $BC$ and glue them again according to given permutations $\sigma_1$ and $\sigma_2$ of the three strata




The only working choice turns out to be

$$\sigma_1 = (1,2,3), \qquad \sigma_2 = \sigma_1^{-1} = (1,3,2)$$

and again minimize $\mathcal{F}(u) := \frac{1}{2}\int_Y |Du|$ on the domain

$$S_0 := \left\{ u \in BV(Y;\{0,1\}) : \sum_{p(y)=x} u(y) = 1 \right\}$$

[show video]

# Convexification is tricky

$$S := \left\{ u \in BV(Y;[0,1]) : \sum_{p(y)=x} u(y) = 1 \right\}$$

If we minimize $\int_Y |Du|$, the possibility of mixing all three "phases" allows to obtain a lower energy than for the nonconvex problem

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## Brakke: "film mass"

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**Brakke: "film mass"**

**Chambolle et al.: "local convex envelope"**

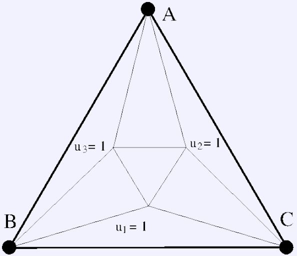
$\Rightarrow$ the same energy (extension of $\mathcal{F}$ to the convexified domain $S$) in the case of the tripod

# Trivializing the tripod example

Since this example is very special, we can restate it in a more usual setting by restricting $\Omega$ to $\Omega_T$, the triangle $ABC$, thus obtaining a globally trivial 3-covering with restrictions of $u$ on the three decks that we can denote

$$u_1, u_2, u_3 \in BV(\Omega_T, [0,1]), \qquad \sum_i u_i = 1$$

**Faulty formulation**




Mixture of all three phases inside the small triangle leads to total variation smaller than that of the tripod

# Identifying the convexified

The Brakke's "film mass" turns out to be

$$\mathcal{F}(u) = sup_{\varphi \in K} \sum_i \int_{\Omega_T} u_i \, \mathrm{div} \varphi_i \; dx$$

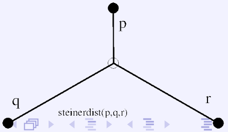
where

$$K = \{\varphi_i \in [C_c^\infty(\Omega_T)]^3, i = 1,2,3 : |\varphi_i(x) - \varphi_j(x)|_2 \leq 2 \, \forall i \neq j, x \in \Omega\}$$

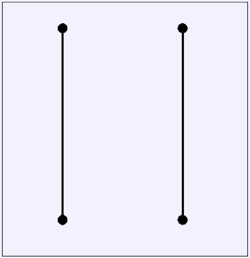
In the case $u_1 \in W^{1,1}$ a lengthy computation shows that this can be computed as

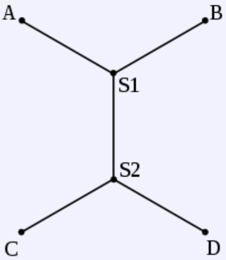
$$\mathcal{F}(u) = \frac{1}{3} \int_{\Omega_T} \mathrm{steinerdist}(\nabla(u_1 - u_2), \nabla(u_2 - u_3), \nabla(u_3 - u_1)) \; dx$$

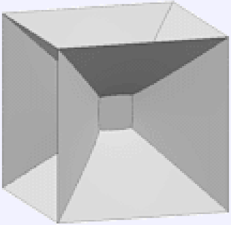
consistent with the so-called local convex envelope of Chambolle et al.



# Other examples





"blow" the central "square"

THANK YOU FOR YOUR PATIENCE!