Completion of visible contours

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joint work with **Giovanni Bellettini** (Università di Roma "Tor Vergata")
and **Valentina Beorchia** (Università di Trieste)

Outline

- Introductory animation
- The problem
- Apparent contour (with Huffman labelling)
- Visible contour
- Main result and sketch of the proof
- Implementation
- Examples

Related work:
[Karpenko,Hughes] [Carter,Kamada,Saito] [Whitney] [Haefliger] [Ohmoto,Aicardi]

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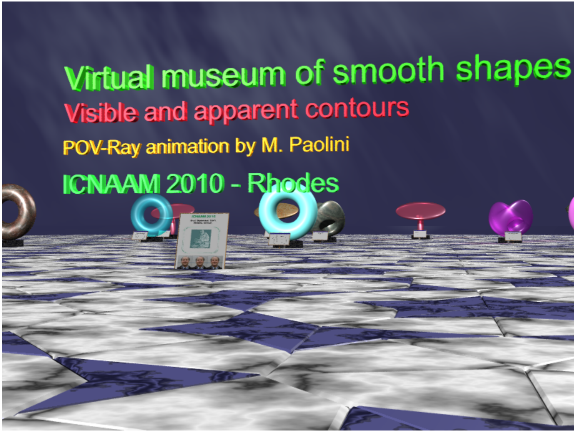
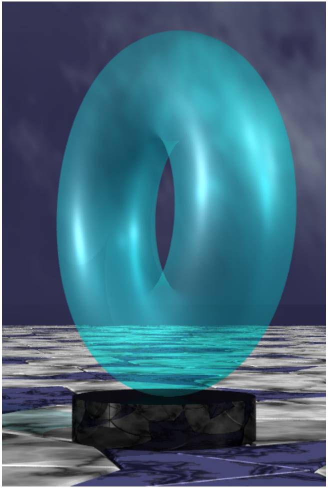
Introductory animation



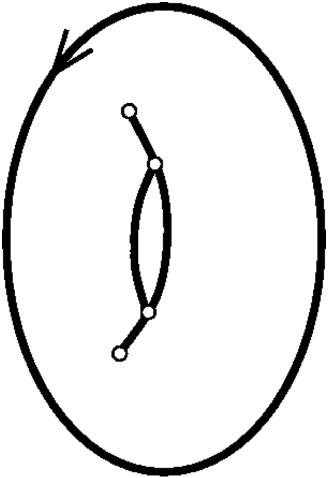
(3 minutes)

The problem

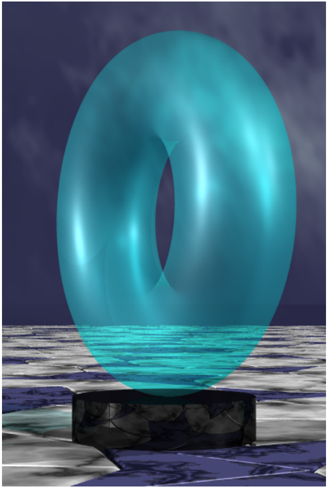
$\Rightarrow$



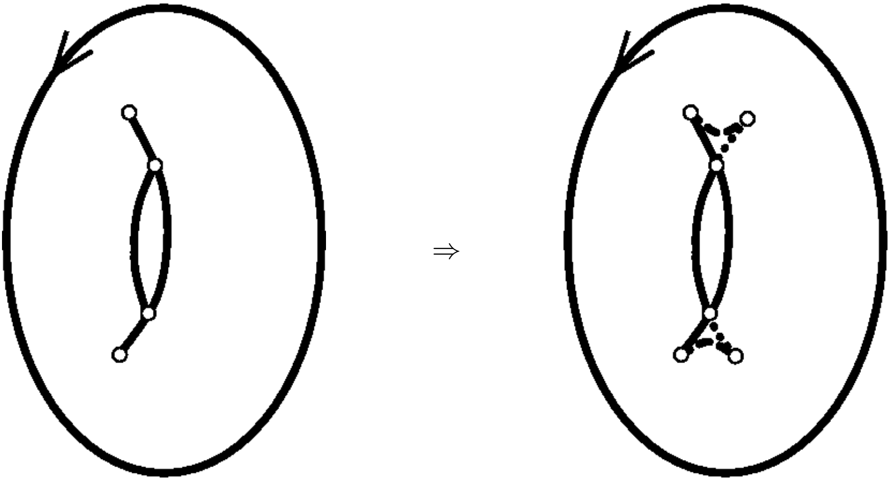
The problem: visible contour ...



$\Rightarrow$



The problem: ... apparent contour



Setting and notations

Σ is a closed surface (smooth, compact 2D manifold without boundary), possibly nonconnected, embedded in $\mathbb{R}^3$.

$\Sigma = \partial E$ of some 3D object (smooth bounded set $E \subset \mathbb{R}^3$).

$\pi : \mathbb{R}^3 \to \mathbb{R}^2$: projection of $\mathbb{R}^3$ onto a plane (e.g. $(x_1, x_2, x_3) \to (x_1, x_2)$); it can be a "perspective" projection from some point (*eye*) onto a projection plane placed between the eye and the object E.

A **light ray** is the inverse image $\pi^{-1}(x)$ of some point $x \in \mathbb{R}^2$.

The restriction $\phi = \pi_{|\Sigma}$ is the composition of the embedding $i : \Sigma \to \mathbb{R}^3$ and the projection π.

Note: No selfintersections are allowed!

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Apparent contour (1)

The singular curve $S \subset \Sigma$ is the set of points where the **light ray is tangent** to the surface.

The apparent contour $\Phi = \pi(S)$ is the projection of S.

Σ is in **generic position** with respect to π if the topological structure of Φ is stable under small perturbations of Σ and π.

Generically: Φ is the jump set of the function that counts the number of preimages of π in Σ.

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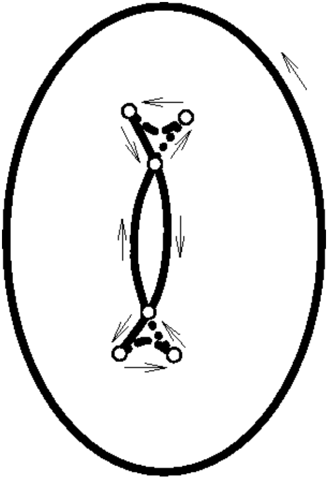
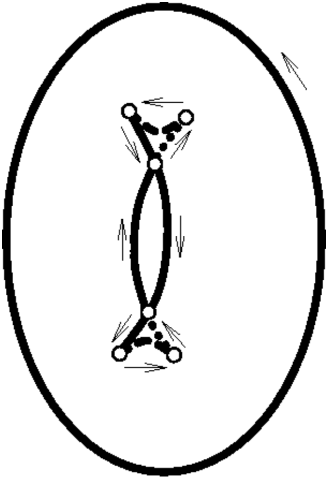
Apparent contour (2)



Roughly the **Apparent Contour** is a *sketch* of the (partially transparent) surface.

1. Oriented plane “graph” possibly with closed arcs
2. Nodes can only be: **crossings** and **cusps**
3. Orientation must be consistent at nodes and at cusps (see Figure)
4. Suitable regularity requirements

Apparent contour (2)



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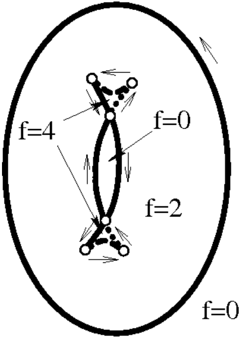
Apparent contour (3)

We define $f : \mathbb{R}^2 \setminus \Phi \to 2\mathbb{N}$ such that

5. $f = 0$ at infinity
6. $f \geq 0$
7. Locally constant on the complement of Φ
8. Jumps of 2 across arcs of Φ
9. The larger value of f lies on the left of arcs of Φ



f is the number of intersections of the light ray with Σ; $\{f = 0\}$ is the "background" of the image.

Note: f can be uniquely recovered from Φ and requirement $f = 0$ at infinity.

Positivity of f is not guaranteed, hence this must be viewed as a further constraint on Φ.

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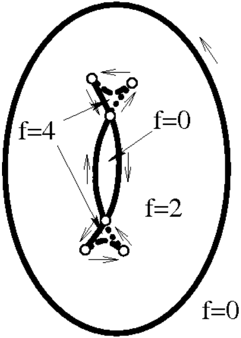

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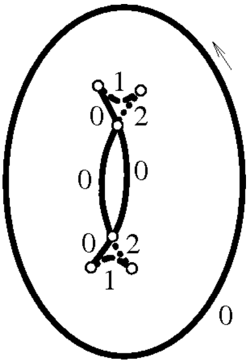
Huffman labelling

Finally we need a **labelling** d of the arcs that takes into account the *depth* information that is lost after projection of Σ. Function d counts the number of times that the light ray crosses the surface (transversally) in front of the singular curve.



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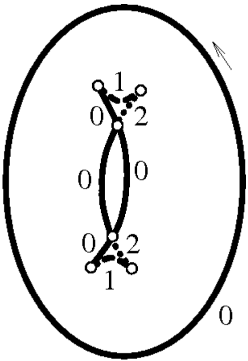

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Apparent contour with Huffman labelling

We say that a drawing Φ is an **Apparent contour with Huffman labelling** (in short **Consistent contour**) if requirements 1 – 12 are satisfied. In short:

- Topological structure: plane “graph” with only “crossings” and “cusps” with consistent orientation (1 – 3);
- Nonnegative f (this implies a nonlocal of constraint on the orientations) (6);
- Equipped with a Huffman labelling (10 – 12);
- Smoothness requirements (4).

Theorem: recovering the shape

We have a crucial result: [Bellettini,Beorchia,P.]

Theorem

A 2D drawing Φ is the apparent contour of some closed surface $\Sigma = \partial E$ embedded in $\mathbb{R}^3$ if and only if Φ is a **Consistent contour***. Moreover the 3D shape can be reconstructed from Φ up to a monotone deformation of the depth coordinate (distance from the projection plane).*

Hence, reconstruction of the 3D structure from a sketch Γ is reconducted to the construction of a **Consistent contour** extending Γ. This is a topological (eventually combinatorial) problem.

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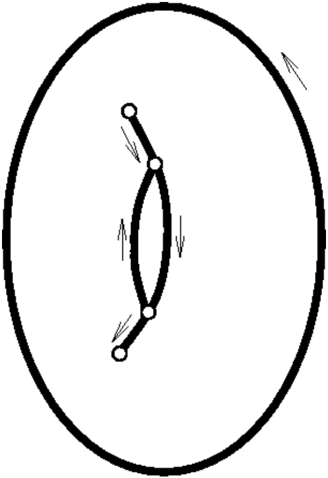
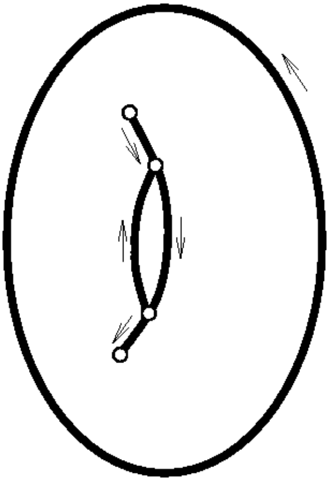
Visible contour



What we actually have is the **visible part** Γ of the apparent contour of our shape (arcs with Huffman label 0).

1. Oriented plane graph possibly with closed arcs
2. Nodes can only be: **T-junctions** and **terminal points**
3. Orientation must be consistent across T-junctions (see Figure)
4. Suitable regularity requirements

Visible contour



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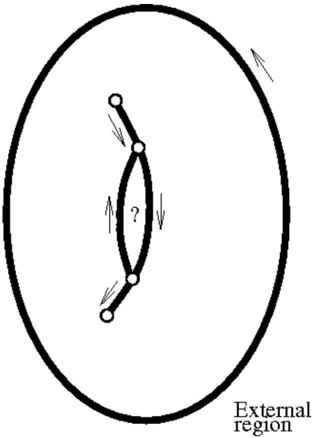
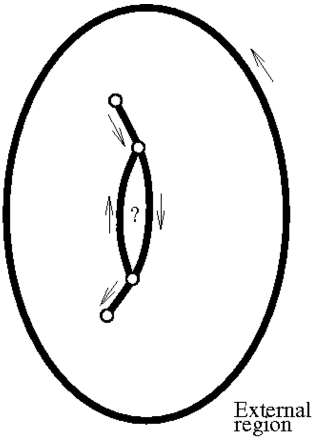
Visible contour and the external region



We call **regions** the connected components of $\mathbb{R}^2 \setminus \Gamma$. The *unbounded* region will be called **external region**. If Γ is the visible contour of some shape we have

5. Arcs of Γ cannot have the external region on their left side;
6. In particular **terminal points** cannot be adjacent to the external region.

Visible contour and the external region



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5. Arcs of Γ cannot have the external region on their left side;
6. In particular **terminal points** cannot be adjacent to the external region.

Admissible visible contour

We say that a drawing Γ is an **Admissible visible contour** if requirements 1 – 6 above are satisfied. In short:

- Topological structure: plane "graph" with only "T-junctions" and "terminal points" with consistent orientation (1 – 3);
- Compatible orientation with respect to the external region; (5 – 6);
- Smoothness requirements (4).

Main result

Given a **Consistent visible contour** Γ, we can extend it to a **Consistent apparent contour** Φ.

Sharpness: a drawing Γ is the visible contour of some Σ if and only if it is a consistent visible contour.

Nonuniqueness: the reconstruction is highly nonunique, even in topological sense.

Remark: The reconstructed apparent contour will have $f = 0$ (background) in the external region. It is possible to force $f = 0$ in some internal region, if feasible.

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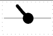
Morse description

We need a way to describe the topological structure of the **visible contour** Γ.

Morse description

record all "events" occurring while a "sweep line" traverses the image from top to bottom.

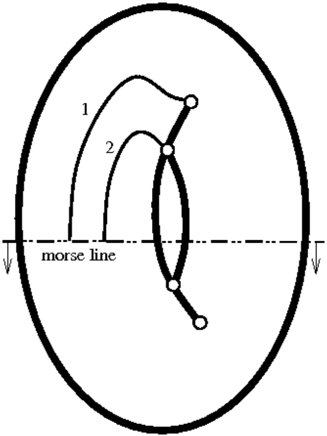
We can assume that all events are "generic", in finite number and occur at different "critical" times.

	local maximum		local minimum)
	terminal point		terminal point
	NorthEast T-junction		NW T-junction
	SE T-junction		SW T-junction
	transv. intersection		

Proof

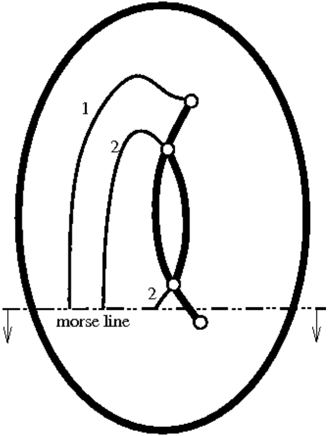
Idea: build a partial completion behind a sweeping line (traversing the drawing downwards) and then provide a mechanism to continue this partial contour past each Morse event.

We have to manage each Morse event while taking into account all the **dangling** invisible ($d > 0$) arcs.

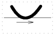


Example case: North-East T-junction

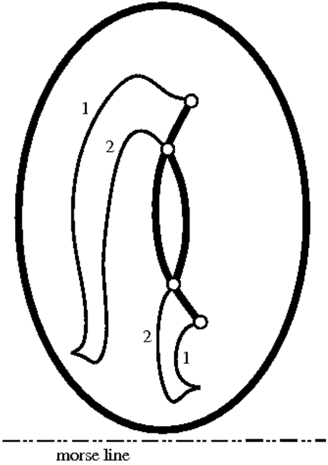
Morse event:  (crossing a North-East T-junction) produced the result shown in figure (this is an easy case)



Tricky case: Minimum above external region

Morse event:  (crossing a 'right'-minimum). If we are above the external region we need to "close" all dangling invisible arcs contained in the involved region. This can be done by adding cusps and joining arcs pairwise in local minima.

Recall: All constraints for the labelling *d* must be met!



Resulting morse description

The outcome of the procedure is itself a **Morse description** of the reconstructed apparent contour (with a different set of Morse events) augmented with information about orientation and the labelling.

Implementation

We implemented the completion procedure in a software code that takes the *morse description* of the visible contour in input and produces the *morse description* of the recovered apparent contour.

Morse description as ASCII text:

Morse event	ASCII representation	Morse event	ASCII representation
	|		
	^		U
	, (comma)		‘ or ’
	\’		‘/
	/.		.\

Example

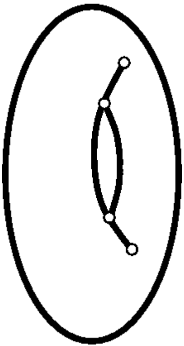
Input description:

```
morse {
    ^     ;
|   ,   | ;
| /.    | ;
| |e|   | ;
| \'    | ;
|   `   | ;
    U     ;
}
```

Outcome:

```
morse {
^10 ;
|d0 ^l1 |u0 ;
|d0 |d1 >0+ |u0 ;
|d0 |d1 ^r2 |u0 |u0 ;
|d0 |d1 |u2 Xu0d0 |u0 ;
|d0 |d1 |u2 Xd2u0 |u0 ;
|d0 |d1 |u2 |d2 >1- |u0 ;
|d0 <1+ |u2 |d2 |u1 |u0 ;
|d0 Ur2 |d2 |u1 |u0 ;
|d0 |d2 >2- |u0 ;
|d0 Ur2 |u0 ;
Ur0 ;
}
```

Visible contour:



Example

Input description:

```
morse {
    ^     ;
|   ,   | ;
| /.   | ;
| |e|  | ;
| \'   | ;
|   `  | ;
    U     ;
}
```

Outcome:

```
morse {
^l0 ;
|d0 ^l1 |u0 ;
|d0 |d1 >0+ |u0 ;
|d0 |d1 ^r2 |u0 |u0 ;
|d0 |d1 |u2 Xu0d0 |u0 ;
|d0 |d1 |u2 Xd2u0 |u0 ;
|d0 |d1 |u2 |d2 >1- |u0 ;
|d0 <1+ |u2 |d2 |u1 |u0 ;
|d0 Ur2 |d2 |u1 |u0 ;
|d0 |d2 >2- |u0 ;
|d0 Ur2 |u0 ;
Ur0 ;
}
```

Example

Input description:

```
morse {
    ^     ;
|   ,   | ;
| /.    | ;
| |e|   | ;
| \'    | ;
|   `   | ;
    U     ;
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```

Outcome:

```
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^l0 ;
|d0 ^l1 |u0 ;
|d0 |d1 >0+ |u0 ;
|d0 |d1 ^r2 |u0 |u0 ;
|d0 |d1 |u2 Xu0d0 |u0 ;
|d0 |d1 |u2 Xd2u0 |u0 ;
|d0 |d1 |u2 |d2 >1- |u0 ;
|d0 <1+ |u2 |d2 |u1 |u0 ;
|d0 Ur2 |d2 |u1 |u0 ;
|d0 |d2 >2- |u0 ;
|d0 Ur2 |u0 ;
Ur0 ;
}
```
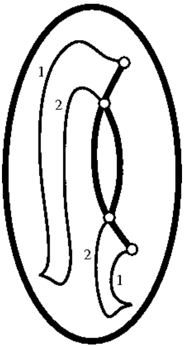

Interpretation:



Using “appcontour” software

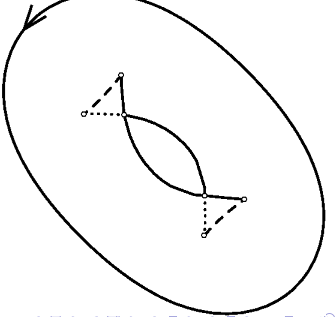
[Pasquarelli,P.]

appcontour is a software for topological management of apparent contours; it can read the output of our completion software and extracts topological information up to smooth deformations of $\mathbb{R}^2$.

3D info:

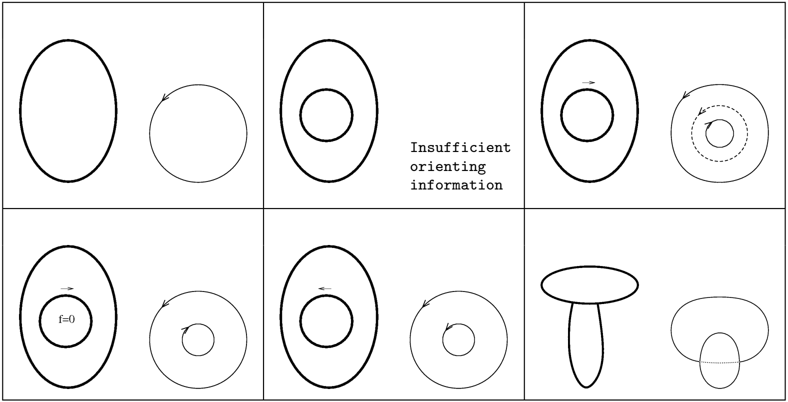
```
[...]
Properties of the 3D surface:
Connected comp.:    1
Total Euler ch.:    0
[...]
```

Graphic picture



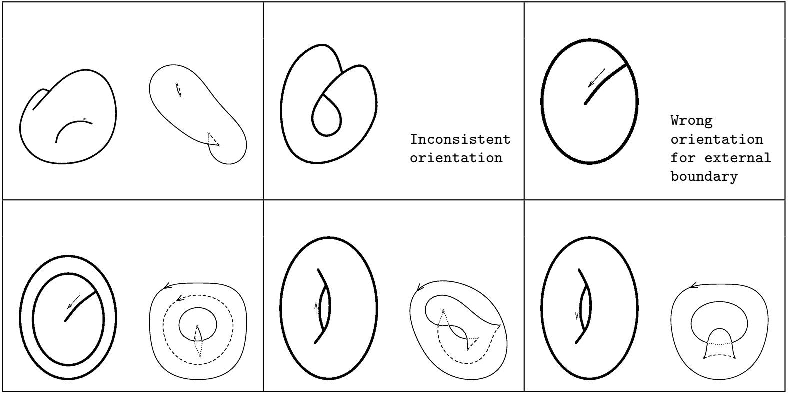
Examples (1)

On the left: input visible contour (to be coded manually with its morse description)
On the right: picture automatically obtained by our completion code feeded into the `appcontour` software



Examples (2)

A few more complex examples...



Thanks

THANK YOU FOR YOUR ATTENTION!