# Booklet of Abstracts
# Brescia Trento Nonlinear Days
# Second Edition
# 25th May 2018

## *Recent updates on double phase variational integrals*

Paolo Baroni, Università di Parma

***Abstract:*** I will describe some recent updates in the regularity theory for minimizers of non-autonomous functionals of the Calculus of Variations, so-called double phase functionals. I will try to show in this respect how a unified proof, simpler than all the previous ones, allowed us to catch borderline cases and to detect new, unexpected phenomena.

## *Entire solutions for Liouville systems*

Luca Battaglia, Università di Roma 3

***Abstract:*** I will consider a system of two coupled Liouville equations on the plane $\mathbb{R}^2$. The system admits so-called *scalar* solutions, namely such that the two components $u_1(x), u_2(x)$ coincide. These solutions actually solve a scalar Liouville equation on the plane, hence they are very well known and they have been completely classified. On the other hand, much less is known about non-scalar solutions.
Using bifurcation theory, I will show the existence of some branches of (non-scalar) solutions bifurcating from a scalar solution.
This is a joint work with Francesca Gladiali (Università di Sassari) and Massimo Grossi (Sapienza - Università di Roma).

## *Direct epiperimetric inequalities for the classical and thin obstacle problem*

Maria Colombo, ETH Zürich

***Abstract:*** We study the regularity of the regular and of the singular set of the some free boundary problems in any dimension.

At regular points, one approach to the regularity is given by the epiperimetric inequality of Weiss. In his paper, Weiss uses a contradiction argument and he asks the question if such epiperimetric inequality can be proved in a direct way (namely, exhibiting explicit competitors), which would have significant implications on the regularity of the free boundary in dimension $d > 2$.

We answer positively the question of Weiss, proving at regular points the epiperimetric inequality in a direct way. More significantly we introduce a new tool, which we call logarithmic epiperimetric inequality, which works also at singular points. It allows to study the regularity of the singular set and yields an explicit logarithmic modulus of continuity on the $C^1$ regularity, thus improving the known regularity and providing a fully alternative method.

The talk is based on joint work with Luca Spolaor and Bozhidar Velichkov.

## *A stability result for the Gauss mean value formula*

Giovanni Cupini, Università di Bologna

***Abstract:*** The mean integral of harmonic functions on balls equals the value of these functions at the center. This is the well known Gauss mean value theorem. In 1972 Kuran proved the reverse: if $D$ is a bounded open set containing $x$, such that the mean integral of harmonic functions on $D$ equals the value of these functions at $x$, then $D$ is a ball centered at $x$. Is the Gauss mean value formula stable? That is: if the mean integral of harmonic functions on $D$ is almost equal to the value of these functions at $x$ in $D$, then $D$ is almost a ball with center $x$? In this talk I will discuss recent results on this issue obtained in collaboration with N. Fusco, E. Lanconelli and X. Zhong.

## *New characterization for Sobolev functions on metric measure spaces: a non-local approximation approachTBA*

Simone Di Marino, INDAM - SNS Pisa

***Abstract:*** Around 2001, J. Bourgain, H. Brezis and P. Mironescu, investigated the asymptotic behaviour of a class on nonlocal functionals on a domain $\Omega \subset \mathbb{R}^N$, including those related to the norms of the fractional Sobolev space $W^{s,p}$ as $s \to 1$. In particular they prove that the limit is finite if and only if the function belongs to $W^{1,p}$; moreover the limit is a constant $c = c(p, N)$ times the Sobolev norm. We study this kind of approximation in a general metric measure space $(X, d, \mu)$, under the assumption that the measure is doubling and satisfies a $(1, p)$-Poincaré inequality. Defining

$$F_s(u) = (1 - s) \iint_{X\times X} \frac{|u(x) - u(y)|^p}{\rho(x, y)d(x, y)^{ps}} \, d\mu(x) \, d\mu(y),$$

for suitable $\rho(x, y)$, we prove that $\liminf_{s\to 1^-} F_s(u) < \infty$ if and only if $u \in W^{1,p}$ and moreover there exist $c_1, c_2 > 0$ such that

$$c_1 \int |\nabla u|^p \leq \liminf_{s\to 1^-} F_s(u) \leq \limsup_{s\to 1^-} F_s(u) \leq c_2 \int |\nabla u|^p.$$

The novelties of the proof are the fact that it is significantly simpler than the previous ones, we deal also with the case $p = 1$, and we also study the more complicated non-convex functional of Nguyen, finding the same type of results for $p > 1$. If there is time we will discuss also the $\Gamma$-limit of $F_s$ in the case $\mu$ is only doubling, proving the reflexivity of $W^{1,p}(X, d, \mu)$ when $p > 1$. This is based on a paper in collaboration with M. Squassina.

## *Local minimality results for the Mumford-Shah functional via monotonicity*

Alessandro Giacomini, Università di Brescia

***Abstract:*** Let $\Omega \subset \mathbb{R}^2$ be an open bounded set, and let

$$MS(u) := \int_\Omega |\nabla u|^2 \, dx + \alpha \mathcal{H}^1(J_u) + \beta \int_\Omega |u-g|^2 \, dx$$

denote the Mumford-Shah functional on $SBV(\Omega)$, with $\alpha, \beta > 0$ and $g \in L^\infty(\Omega)$.

We prove that if $\Omega$ is sufficiently regular, then the solution of

$$\begin{cases} -\Delta u + \beta u = \beta g \\ u \in H^1(\Omega) \end{cases}$$

is a local minimizer for $MS$ in $SBV(\Omega)$ with respect to the $L^1$-topology. This is obtained by employing a boundary monotonicity formula for a suitable notion of *quasi-minimizers* of the Mumford-Shah energy, in the spirit of that proposed recently by Bucur and Luckhaus (ARMA 2013). This is a joint work with D. Bucur and I. Fragalà.

## *On the best constant in the Sobolev inequality on bounded domains*

Hynek Kovařík, Università di Brescia

***Abstract:*** Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be an open bounded domain and let $V : \Omega \to \mathbb{R}$ be a continuous potential function. We are interested in the variational problem

$$S_\varepsilon(\Omega) := \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 + \varepsilon \int_\Omega V \, |u|^2}{\left(\int_\Omega |u|^q\right)^{2/q}} \qquad q = \frac{2n}{n-2} \, ,$$

where $\varepsilon > 0$. It is well-known that

$$S_0(\Omega) = S_0 = \pi n(n-2) \left( \frac{\Gamma(n/2)}{\Gamma(n)} \right)^{2/n}$$

is the best Sobolev constant in $\mathbb{R}^n$ which depends only on $n$ and not on $\Omega$. In [BN] Brezis and Nirenberg proved that while $S_\varepsilon(\Omega) = S_0$ for $\varepsilon$ small enough if $n = 3$, for $n \geq 4$ the constant $S_\varepsilon(\Omega)$ is unstable in the following sense; as soon as $V(x) < 0$ for some $x \in \Omega$, then $S_\varepsilon(\Omega) < S_0$ for any $\varepsilon > 0$.

In this talk I will present a two-term asymptotic expansion of $S_\varepsilon(\Omega)$ as $\varepsilon \to 0$ in the case $n \geq 4$. This is a joint work with Rupert Frank (Munich).

### References

[BN] H. Brezis, L. Nirenberg: Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Comm. Pure Appl. Math.* **36** (1983) 437–477.

# *Sobolev-type inequalities on Cartan-Hadamard manifolds under curvature bounds, and nonlinear diffusions*

Matteo Muratori, Politecnico di Milano

***Abstract:*** It is well known that the standard Sobolev inequality not only holds in the Euclidean space, but also on any *Cartan-Hadamard* manifold, namely a complete and simply connected Riemannian manifold with everywhere nonpositive sectional curvatures. Moreover, the optimal constant is exactly the same as in the Euclidean framework. On the other hand, the Poincaré inequality *fails* in the Euclidean space but is satisfied in the hyperbolic space: equivalently, the Laplace-Beltrami operator has a *spectral gap*, which is explicitly determined by the curvature and the spatial dimension. Actually the Poincaré inequality holds on any Cartan-Hadamard manifold having sectional curvatures bounded from above by a negative constant: this is a celebrated result that goes back to H.P. McKean (1970). The main motivation for our work came from the fact that there was basically no result available in between, that is when bounds on sectional curvatures are still negative but allowed to go to zero at spatial infinity. What we prove here is that *power-type* decays of the curvatures give rise, in the *radial* setting, to Sobolev-type inequalities that interpolate between Poincaré ($p = 2$) and Sobolev ($p = 2^*$). There is a threshold at the inverse-quadratic decay, where the type of inequality that is valid depends on the multiplicative constant of the rate. For curvatures that decay with a faster power, only the Euclidean Sobolev inequality can hold. We also show, by explicit examples, that our results are optimal with respect to the dependence on $p$. We then discuss some consequences regarding *smoothing effects* for certain (radial) nonlinear diffusions of porous medium type: the $L^\infty$ bounds that follow from our inequalities are optimal with respect to previous pointwise estimates established by completely different techniques. Finally, we prove a theorem that implies the *failure* of the above inequalities in the *nonradial* setting under reverse curvature bounds: this shows that, for nonradial functions, in general one cannot expect more than the Euclidean Sobolev inequality (at least if curvatures have power-decay rates).

This is a joint work with Alberto Roncoroni (Università degli Studi di Pavia).