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Concavity principles for nonautonomous elliptic equations and applications

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Abstract. In the study of concavity properties of positive solutions to nonlinear elliptic partial differential equations the diffusion and the nonlinearity are typically independent of the space variable. In this paper we obtain new results aiming to get almost concavity results for a relevant class of anisotropic semilinear elliptic problems with spatially dependent source and diffusion.

Keywords: Approximate convexity principles, anisotropic problems, semilinear elliptic problems

1. Introduction

A rather natural question in the field of nonlinear partial differential equations is whether a positive solution with homogeneous Dirichlet boundary conditions is concave on a given convex domain. Starting from [5] extensive research has been developed in order to deduce symmetry of solutions from the symmetry of the domain, via the so called Alexandroff–Serrin moving plane method. When the symmetry of the domain is dropped, one may wonder if the solutions still inherit some concavity properties from the domain. This was investigated in a series of pioneering papers [3,14,18,19].

When studying concavity properties of solutions, it becomes evident that aiming for concavity is often overly demanding: while it can be achieved for the torsion problem, for example, for suitable perturbation of ellipsoids [8,12,13,16], the first eigenfunctions of the Laplacian are never concave, regardless of the considered bounded domain [10, Remark 3.4]. One may instead search for a strictly increasing function φ that, when composed with the solution u, yields a concave function $\varphi(u)$. In the seminal paper [18] it is shown that the solutions of the torsion problem $-\Delta u = 1$ are such that $\sqrt{u}$ is concave. In [3] the authors show that the positive eigenfunctions of $-\Delta u = \lambda u$ satisfy that $\log u$ is concave. The concavity of solutions to nonlinear equations has been explored in several subsequent papers [7,11–13,15,17,20]

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involving techniques mainly relying on maximum principles applied to suitably defined convexity functions. For instance if $\beta \in (0, 1)$, Ω is convex and u is a positive solution to $-\Delta u = u^\beta$ with Dirichlet boundary data, then $u^{(1-\beta)/2}$ is concave [12].

Most of the cited papers, however, give assumptions on the nonlinearity in the equation in order to have a suitable power u^γ of the solution u to be concave. Recently, for the problem $-\Delta u = f(u)$ (and more generally for quasi-linear problems involving the p-Laplace operator), under suitable assumptions on f the authors of [2] showed concavity of $\int_1^u 1/\sqrt{F(\sigma)}\, d\sigma$, where $F' = f$, thus providing a precise connection on how the concavity of the solution is affected by the nonlinear term f. In [1], these results were then extended to the quasi-linear problem $-\operatorname{div}(\alpha(u)\nabla u) + \frac{1}{2}\alpha'(u)|\nabla u|^2 = f(u)$ related to the so called modified nonlinear Schrödinger equation, under suitable joint hypothesis on α and f. More precisely $\int_\mu^u \sqrt{\alpha(\sigma)/F(\sigma)}\, d\sigma$ turns out to be concave for some positive constant μ.

In cases where the assumptions on the function f which guarantee the concavity of a suitable transformation are not met, some quantitative perturbation results were recently obtained in [4]. These results establish, in essence, a bound on the loss of concavity of u, controlled in the supremum norm in terms of the loss of concavity of f.

In general all the results in the current literature only deal with *autonomous* problems, corresponding to isotropic physical models, namely both the diffusion term in the operator and the nonlinearity *do not* explicitly depend upon the space variable and it is expected that concavity (even up to a transformation) is in general broken due to the x-dependence.

The primary objective of the paper is to establish quantitative perturbation results, which assert that if both the diffusion term in the operator and the nonlinearity exhibit a small variation with respect to the spatial variable, then a suitable transformation $\varphi(u)$ is close to a concave function in the supremum norm, with an error estimate depending precisely on the spatial variation.

Precisely, taking $\Omega \subset \mathbb{R}^n$ a bounded open strictly convex set with smooth boundary, consider the semi-linear problem, for $\beta \in [0, 1)$,

$$\begin{cases} -\sum_{i,j=1}^n \alpha^{ij}(x) D^2_{ij} u = a(x) u^\beta & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with the matrix of coefficients $A = \{\alpha^{ij}\}_{i,j=1}^n$ being symmetric and uniformly elliptic. In the isotropic cases $\alpha^{ij} = \delta_{ij}$ and $a = 1$ this reduces to the already mentioned classical sublinear problem $-\Delta u = u^\beta$ for which a result in [12] establishes concavity of $u^{(1-\beta)/2}$.

As a by product of a general maximum convexity principle (see Theorem 2.3) we prove in Proposition 3.2 that if

$$\|\nabla a\|_{L^\infty(\Omega)} + \max_{i,j \in \{1,\dots,n\}} \|\nabla \alpha^{ij}\|_{L^\infty(\Omega)} < \varepsilon, \quad \varepsilon > 0, \tag{1.2}$$

then there exists a positive constant C and a concave function $w : \Omega \to \mathbb{R}$ such that

$$\|u^{\frac{1-\beta}{2}} - w\|_{L^\infty(\Omega)} \leqslant C\varepsilon,$$

in light also of a Hyers–Ulam theorem (see Proposition 3.5).

Furthermore, as a second example, consider the problem

$$\begin{cases} -\sum_{i,j=1}^{n} \alpha^{ij}(x) D_{ij}^2 u = a(x)u + \varepsilon\varphi(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

with $A = \{\alpha^{ij}\}_{i,j=1}^{n}$ symmetric and uniformly elliptic. If (1.2) holds and φ is non-increasing, then there exists a positive constant C and a concave function $w\colon \Omega \to \mathbb{R}$ such that

$$\| \log u - w \|_{L^\infty(\Omega)} \leqslant C\varepsilon.$$

We obtain this applying Proposition 3.4. We point out that the case $\varphi = 0$ in problem (1.3), which corresponds to $\beta = 1$ in problem (1.1), is out of reach since our general convexity maximum principles fail, precisely since assumption (2.7) is not fulfilled.

In the rest of the paper, we proceed obtaining some maximum principles for concavity functions of solutions of semi-linear equations, which can be viewed as anisotropic counterparts of the results presented in [13, Lemma 1.4] and [12, Lemma 3.1]. We then discuss some applications, precisely problems (1.1) and (1.3). We believe that our techniques could be suitable to investigate other physically relevant anisotropic elliptic problems. To the best of our knowledge this is the first result in the literature providing almost concavity results for anisotropic problems in convex domains.

2. Anisotropic convexity principles

In the rest of the paper, let Ω denote a bounded open convex subset of $\mathbb{R}^n$. Denote furthermore for $x_1, x_3 \in \overline{\Omega}$, $\lambda \in [0, 1]$,

$$x_2 := \lambda x_3 + (1-\lambda)x_1 \in \overline{\Omega} \tag{2.1}$$

and for $s_1, s_3 \in \mathbb{R}$

$$s_2 = \lambda s_3 + (1-\lambda)s_1.$$

For some $u\colon \overline{\Omega} \to \mathbb{R}$, we define the concavity function $\mathcal{C}_u$ as

$$\mathcal{C}_u(x_1, x_3, \lambda) := u(x_2) - \lambda u(x_3) - (1-\lambda)u(x_1). \tag{2.2}$$

For some $g\colon \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, the joint-concavity function $\mathcal{JC}_g$ is defined by

$$\mathcal{JC}_g\big((x_1, s_1), (x_3, s_3), \lambda\big) := g(x_2, s_2) - \lambda g(x_3, s_3) - (1-\lambda)g(x_1, s_1), \tag{2.3}$$

and we will also use the notation

$$\begin{aligned} &\mathcal{JC}_{g(\cdot,u(\cdot))}(x_1, x_3, \lambda) \\ &\quad := g\big(x_2, \lambda u(x_3) + (1-\lambda)u(x_1)\big) - \lambda g\big(x_3, u(x_3)\big) - (1-\lambda)g\big(x_1, u(x_1)\big) \end{aligned} \tag{2.4}$$

when $s_i = u(x_i)$ for $i \in \{1, 2, 3\}$. We define the harmonic concavity function, as in [12], in the following way:

$$\mathcal{HC}_g\big((y_1, s_1), (y_3, s_3), \lambda\big) := \begin{cases} g(y_2, s_2) - \frac{g(y_1,s_1)g(y_3,s_3)}{\lambda g(y_1,s_1)+(1-\lambda)g(y_3,s_3)}, \\ \qquad\qquad \text{if } \lambda g(y_1, s_1) + (1-\lambda)g(y_3, s_3) > 0 \\ g(y_2, s_2), \quad \text{if } g(y_1, s_1) = g(y_3, s_3) = 0. \end{cases} \tag{2.5}$$

It should be noted that such definition is applicable to positive functions g, or functions that can change sign and that meet one of the conditions specified in equation (2.5), at the point $((y_1, s_1), (y_3, s_3), \lambda)$. Notice also that if $g < 0$, none of these conditions are satisfied.

We will also use the notation

$$\mathcal{HC}_{g(\cdot,u(\cdot))}(x_1, x_3, \lambda) = g\big(x_2, \lambda u(x_3) + (1-\lambda)u(x_1)\big) - \frac{g(x_1, u(x_1))g(x_3, u(x_3))}{\lambda g(x_1, u(x_1)) + (1-\lambda)g(x_3, u(x_3))}$$

when $s_i = u(x_i)$. Notice that $\mathcal{C}_u, \mathcal{JC}_g, \mathcal{HC}_g \geqslant 0$ are equivalent to the concavity, joint concavity, respectively harmonic concavity of the functions.

To ensure clarity, we also point out the following definition.

Definition 2.1. We say that the triple (x_1, x_3, λ) is an interior point for $\mathcal{C}_u$ if each of x_1, x_2, x_3 is in Ω with $x_2 = \lambda x_3 + (1-\lambda)x_1$, while we say that the point is on the boundary if at least one x_1, x_2, x_3 belongs to $\partial\Omega$.

Having established our notations, we point out how we obtain our almost-concavity results for transformations of the solutions of (1.1), (1.3). It is obvious that if $u \in C(\overline{\Omega})$, then $\mathcal{C}_u$ achieves a maximum in $\overline{\Omega} \times \overline{\Omega} \times [0, 1]$. We give in this section maximum convexity principles, which cover the case in which $\mathcal{C}_u$ achieves a positive maximum at an interior point in $\Omega \times \Omega \times (0, 1)$. To follow, in Section 3, after noticing that the concavity functions associated to our problems, due to boundary constraints, cannot achieve the positive maximum on the boundary, with a direct applications of the maximum convexity principles we obtain the desired conclusion.

We introduce now the model problem for which we obtain maximum convexity principles.

Problem 1. For all $i, j \in \{1, \dots, n\}$ let the functions

$$a^{ij} : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$$

be such that $a^{ij}(\cdot, \xi) \in C^1(\Omega)$ and $A = [a^{ij}(x, \xi)]_{i,j=1}^n$ is a symmetric positive semidefinite matrix for all $(x, \xi) \in \overline{\Omega} \times \mathbb{R}^n$. Let $b\colon \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be such that $b(x, \cdot, \xi)$ is differentiable in $\mathbb{R} \setminus \{0\}$, for all $(x, \xi) \in \overline{\Omega} \times \mathbb{R}^n$. Consider the equation

$$Lu = 0, \quad Lu = a^{ij}(x, Du)u_{ij} - b(x, u, Du), \tag{2.6}$$

where we use the notation

$$a^{ij}u_{ij} := \sum_{i,j=1}^n a^{ij}u_{ij}.$$

The next result, an anisotropic maximum convexity principle, can be viewed as the anisotropic counterpart of [4, Lemma 2.3], both variations of the classical convexity principle in [13, Lemma 1.4].

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open convex set. Let $u \in C^2(\Omega)$ be a solution of Problem 1. Assume that $\mathcal{C}_u$ achieves a positive interior maximum at $(x_1, x_3, \lambda) \in \Omega \times \Omega \times (0,1)$. If there is some $\sigma > 0$ such that for all x on the segment $[x_1, x_3]$ and s on the segment $[u(x_1), u(x_3)]$ it holds that*

$$\frac{\partial b}{\partial s}\big(x, s, Du(x_1)\big) \geqslant \sigma, \tag{2.7}$$

then

$$\mathcal{C}_u(x_1, x_3, \lambda) \leqslant -\frac{\mathcal{J}\mathcal{C}_{b(\cdot,u(\cdot),\xi)}(x_1, x_3, \lambda)}{\sigma} + \frac{C\varepsilon(D(u(x_1))}{\sigma},$$

where

$$\varepsilon\big(Du(x_1)\big) := \max_{i,j\in\{1,\dots,n\}} \sup_{x\in[x_1,x_3]} \big|D_x a^{ij}\big(x, Du(x_1)\big)\big| \tag{2.8}$$

and

$$C := n^2 \max_{i,j\in\{1,\dots,n\}} \max_{k\in\{1,3\}} \big|u_{ij}(x_k)\big| \operatorname{diam}(\Omega) > 0. \tag{2.9}$$

Proof. Notice that if $x_1 = x_3$, then the inequality trivially holds. We may hence assume that x_1, x_2, x_3 are distinct. Since $\mathcal{C}_u$ achieves a maximum at (x_1, x_3, λ), recalling (2.1) and (2.2), we get that

$$(D_{x_1}\mathcal{C}_u)(x_1, x_3, \lambda) = (D_{x_3}\mathcal{C}_u)(x_1, x_3, \lambda) = 0$$

hence

$$(1-\lambda)Du(x_2) - (1-\lambda)Du(x_1) = \lambda Du(x_2) - \lambda Du(x_3) = 0.$$

Let us set

$$\xi := Du(x_1) = Du(x_2) = Du(x_3),$$

and consider the auxiliary function $\bar{\varphi}\colon \mathbb{R}^n \to \mathbb{R}$ defined as

$$\bar{\varphi}(v) := \mathcal{C}_u(x_1 + v, x_3 + v, \lambda) = u(x_2 + v) - \lambda u(x_3 + v) - (1-\lambda)u(x_1 + v).$$

Since $\bar{\varphi}$ has a local maximum at $v = 0$, we get that

$$\nabla_v \bar{\varphi}(0) = 0 \quad \text{and} \quad \big[D_v^2 \bar{\varphi}(0)\big] \leqslant 0.$$

We recall that if A and B are two $n \times n$ real symmetric positive semidefinite matrices, then $\mathrm{Tr}(AB) \geqslant 0$ (see [12, Lemma A.1]). Since $A = [a^{ij}(x_2, \xi)]_{i,j=1}^{n}$ is positive semidefinite, it follows that

$$a^{ij}(x_2, \xi)\big(u_{ij}(x_2) - \lambda u_{ij}(x_3) - (1-\lambda)u_{ij}(x_1)\big) \leqslant 0.$$

Denote

$$\begin{aligned} e_1 &= \big(a^{ij}(x_2, \xi) - a^{ij}(x_1, \xi)\big)u_{ij}(x_1), \\ e_3 &= \big(a^{ij}(x_2, \xi) - a^{ij}(x_3, \xi)\big)u_{ij}(x_3), \end{aligned} \tag{2.10}$$

and using the equation (2.6), we have

$$\begin{aligned} b\big(x_2, u(x_2), \xi\big) &= a^{ij}(x_2, \xi)u_{ij}(x_2) \leqslant \lambda a^{ij}(x_2, \xi)u_{ij}(x_3) + (1-\lambda)a^{ij}(x_2, \xi)u_{ij}(x_1) \\ &= \lambda a^{ij}(x_3, \xi)u_{ij}(x_3) + \lambda e_3 + (1-\lambda)a^{ij}(x_1, \xi)u_{ij}(x_1) + (1-\lambda)e_1 \\ &= \lambda b\big(x_3, u(x_3), \xi\big) + (1-\lambda)b\big(x_1, u(x_1), \xi\big) + (1-\lambda)e_1 + \lambda e_3. \end{aligned}$$

So we get in turn

$$\begin{aligned} &b\big(x_2, u(x_2), \xi\big) - b\big(x_2, \lambda u(x_3) + (1-\lambda)u(x_1), \xi\big) \\ &\quad \leqslant \lambda b\big(x_3, u(x_3), \xi\big) + (1-\lambda)b\big(x_1, u(x_1), \xi\big) \\ &\qquad - b\big(x_2, \lambda u(x_3) + (1-\lambda)u(x_1), \xi\big) + (1-\lambda)e_1 + \lambda e_3. \end{aligned}$$

Using Lagrange's theorem, we can estimate

$$\max\big\{|e_1|, |e_3|\big\} \leqslant C\varepsilon(\xi), \tag{2.11}$$

so we get that

$$(1-\lambda)e_1 + \lambda e_3 \leqslant (1-\lambda)|e_1| + \lambda|e_3| \leqslant C\varepsilon(\xi).$$

Then we can apply Lagrange's theorem to obtain that there exists $\bar{s}$ on the segment $[u(x_2), \lambda u(x_3) + (1-\lambda)u(x_1)]$, thus on the segment $[u(x_1), u(x_3)]$, such that

$$\begin{aligned} \sigma \mathcal{C}_u(x_1, x_3, \lambda) &\leqslant \frac{\partial b}{\partial s}(x_2, \bar{s}, \xi)\big(u(x_2) - \lambda u(x_3) - (1-\lambda)u(x_1)\big) \\ &\leqslant -\mathcal{J}\mathcal{C}_{b(\cdot, u(\cdot), \xi)}(x_1, x_3, \lambda) + C\varepsilon(\xi), \end{aligned}$$

concluding the proof of the Theorem. □

We have now the second anisotropic approximate convexity principle, counterpart of [4, Lemma 2.9], both variations of the classical Convexity Principle in [12, Lemma 3.1].

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open convex set. Let $u \in C^2(\Omega)$ be a solution of Problem 1. Assume that $\mathcal{C}_u$ achieves a positive interior maximum at $(x_1, x_3, \lambda) \in \Omega \times \Omega \times (0,1)$, and additionally that there is some $\nu, \sigma > 0$ such that for all x on the segment $[x_1, x_3]$ and s on the segment $[u(x_1), u(x_3)]$ it holds that*

$$b\big(x, s, Du(x_1)\big) \geqslant \nu \tag{2.12}$$

and

$$\frac{\partial b}{\partial s}\big(x, s, Du(x_1)\big) \geqslant \sigma. \tag{2.13}$$

If b is jointly concave (i.e. $\mathcal{JC}_b \geqslant 0$), then

$$\mathcal{C}_u(x_1, x_3, \lambda) \leqslant \frac{1}{\sigma}\left[C\varepsilon\big(Du(x_1)\big) + \frac{C^2\varepsilon^2(Du(x_1))}{\nu}\right],$$

otherwise

$$\begin{aligned}\mathcal{C}_u(x_1, x_3, \lambda) \leqslant \frac{1}{\sigma}\bigg[&-\mathcal{HC}_{b(\cdot,u(\cdot),\xi)}(x_1, x_3, \lambda) + C\varepsilon\big(Du(x_1)\big)\left(1 - \frac{\mathcal{JC}_{b(\cdot,u(\cdot),\xi)}(x_1, x_3, \lambda)}{\nu}\right)\\ &+ \frac{C^2\varepsilon^2(Du(x_1))}{\nu}\bigg],\end{aligned}$$

where notations (2.8) and (2.9) are in place.

Proof. As in Theorem 2.2, we denote by ξ the common value of Du at the points x_1, x_2, x_3. Let us also define the $2n \times 2n$ matrices

$$C := \big[D^2\mathcal{C}_u(x_1, x_3, \lambda)\big] = \begin{bmatrix} D^2_{x_1}\mathcal{C}_u(x_1, x_3, \lambda) & D^2_{x_1,x_3}\mathcal{C}_u(x_1, x_3, \lambda) \\ D^2_{x_1,x_3}\mathcal{C}_u(x_1, x_3, \lambda) & D^2_{x_3}\mathcal{C}_u(x_1, x_3, \lambda)\end{bmatrix}$$

(which is negative semidefinite since (x_1, x_3, λ) is a maximum for $\mathcal{C}_u$ in the interior), and

$$B := \begin{bmatrix} s^2a^{ij}(x_2, \xi) & sta^{ij}(x_2, \xi) \\ sta^{ij}(x_2, \xi) & t^2a^{ij}(x_2, \xi)\end{bmatrix}$$

for $s, t \in \mathbb{R}$. The matrix B is positive semidefinite by hypothesis, therefore the trace of BC is non-negative. That is, denoting

$$\begin{aligned}&\alpha := \operatorname{Tr}\big(a^{ij}(x_2, \xi)D^2_{x_1}\mathcal{C}_u\big), \qquad \beta := \operatorname{Tr}\big(a^{ij}(x_2, \xi)D^2_{x_1,x_3}\mathcal{C}_u\big),\\ &\gamma := \operatorname{Tr}\big(a^{ij}(x_2, \xi)D^2_{x_3}\mathcal{C}_u\big),\end{aligned}$$

we have that

$$\alpha s^2 + 2\beta st + \gamma t^2 \leqslant 0,$$

i.e.

$$\alpha, \gamma \leqslant 0, \qquad \beta^2 \leqslant \alpha\gamma. \tag{2.14}$$

Then we obtain

$$\begin{aligned}
\alpha &= (1-\lambda)^2 a^{ij}(x_2, \xi) u_{ij}(x_2) - (1-\lambda) a^{ij}(x_2, \xi) u_{ij}(x_1), \\
\gamma &= \lambda^2 a^{ij}(x_2, \xi) u_{ij}(x_2) - \lambda a^{ij}(x_2, \xi) u_{ij}(x_3), \\
\beta &= \lambda(1-\lambda) a^{ij}(x_2, \xi) u_{ij}(x_2).
\end{aligned}$$

Denote for $k \in \{1, 2, 3\}$

$$Q_k = a^{ij}\big(x_k, Du(x_k)\big) u_{ij}(x_k)$$

and use once more the notations in (2.10). Then we have that

$$\begin{aligned}
\alpha &= (1-\lambda)^2 Q_2 - (1-\lambda)(Q_1 + e_1), \\
\gamma &= \lambda^2 Q_2 - \lambda(Q_3 + e_3), \\
\beta &= \lambda(1-\lambda) Q_2.
\end{aligned}$$

Using (2.14), we obtain

$$Q_2 \leqslant \frac{1}{1-\lambda}(Q_1 + e_1), \qquad Q_2 \leqslant \frac{1}{\lambda}(Q_3 + e_3), \tag{2.15}$$

and

$$Q_2\big((1-\lambda)Q_3 + \lambda Q_1\big) \leqslant Q_1 Q_3 + e_3\big(-(1-\lambda)Q_2 + Q_1 + e_1\big) + e_1(-\lambda Q_2 + Q_3 + e_3) - e_1 e_3.$$

Recalling that $b > 0$, hence $(1-\lambda)Q_3 + \lambda Q_1 > 0$, then

$$Q_2 \leqslant \frac{Q_1 Q_3}{(1-\lambda)Q_3 + \lambda Q_1} + \frac{e_3(-(1-\lambda)Q_2 + Q_1 + e_1) + e_1(-\lambda Q_2 + Q_3 + e_3) - e_1 e_3}{(1-\lambda)Q_3 + \lambda Q_1}.$$

Denoting

$$\zeta(x_1, x_3, \lambda) := \frac{e_3(-(1-\lambda)Q_2 + Q_1 + e_1) + e_1(-\lambda Q_2 + Q_3 + e_3) - e_1 e_3}{(1-\lambda)Q_3 + \lambda Q_1},$$

we use the equation (2.6) and get that

$$\begin{aligned}
&b\big(x_2, u(x_2), \xi\big) - b\big(x_2, (1-\lambda)u(x_1) + \lambda u(x_3), \xi\big) \\
&\quad \leqslant \frac{b(x_1, u(x_1), \xi) b(x_3, u(x_3), \xi)}{(1-\lambda) b(x_3, u(x_3), \xi) + \lambda b(x_1, u(x_1), \xi)} - b\big(x_2, (1-\lambda)u(x_1) + \lambda u(x_3), \xi\big) + \zeta(x_1, x_3, \lambda).
\end{aligned}$$

According to (2.2), (2.5) and using the Lagrange theorem, we have that

$$\partial_s b(x_2, \bar{s}, \xi)\mathcal{C}_u(x_1, x_3, \lambda) \leqslant -\mathcal{HC}_{b(\cdot,u(\cdot),\xi)}(x_1, x_3, \lambda) + \zeta(x_1, x_3, \lambda), \tag{2.16}$$

for some $\bar{s}$ on the segment $[u(x_2), \lambda u(x_3)+(1-\lambda)u(x_1)]$. To estimate $\zeta(x_1, x_3, \lambda)$, we use (2.11) together with (2.15), which give that

$$\begin{aligned}
\zeta(x_1, x_3, \lambda) &\leqslant \frac{|e_3|(-(1-\lambda)Q_2 + Q_1 + e_1) + |e_1|(-\lambda Q_2 + Q_3 + e_3) - e_1 e_3}{(1-\lambda)Q_3 + \lambda Q_1} \\
&= C\frac{\varepsilon(\xi)(\lambda Q_1 + (1-\lambda)Q_3 + (1-\lambda)Q_1 + \lambda Q_3 - Q_2) + (|e_3|e_1 + |e_1|e_3 - e_1 e_3)}{(1-\lambda)Q_3 + \lambda Q_1} \\
&\leqslant C\varepsilon(\xi)\left(1 + \frac{(1-\lambda)Q_1 + \lambda Q_3 - Q_2}{(1-\lambda)Q_3 + \lambda Q_1}\right) + \frac{C^2\varepsilon^2(\xi)}{\nu},
\end{aligned} \tag{2.17}$$

using also (2.12) and that

$$|e_3|e_1 + |e_1|e_3| - e_1 e_3 \leqslant |e_1 e_3| \leqslant C^2\varepsilon^2(\xi).$$

Now, using again the equation satisfied by u, notice that

$$\begin{aligned}
&(1-\lambda)Q_1 + \lambda Q_3 - Q_2 \\
&\quad = (1-\lambda)b\big(x_1, u(x_1), \xi\big) + \lambda b\big(x_3, u(x_3), \xi\big) - b\big(x_2, (1-\lambda)u(x_1) + \lambda u(x_3), \xi\big) \\
&\qquad + b\big(x_2, (1-\lambda)u(x_1) + \lambda u(x_3), \xi\big) - b\big(x_2, u(x_2), \xi\big) \\
&\quad = -\mathcal{JC}_{b(\cdot,u(\cdot),\xi)}(x_1, x_3, \lambda) - \partial_s b(x_2, s, \xi)\mathcal{C}_u(x_1, x_3, \lambda),
\end{aligned}$$

according to (2.4) and to Lagrange's theorem. Since $\mathcal{C}_u(x_1, x_3, \lambda) \geqslant 0$, thanks to (2.13) the second term is non-positive, so

$$(1-\lambda)Q_1 + \lambda Q_3 - Q_2 \leqslant -\mathcal{JC}_{b(\cdot,u(\cdot),\xi)}(x_1, x_3, \lambda).$$

Therefore, plugging this into (2.17) and (2.16), we have reached

$$\begin{aligned}
&\partial_s b(x_2, \bar{s}, \xi)\mathcal{C}_u(x_1, x_3, \lambda) \\
&\quad \leqslant -\mathcal{HC}_{b(\cdot,u(\cdot),\xi)}(x_1, x_3, \lambda) + C\varepsilon(\xi)\left(1 - \frac{\mathcal{JC}_{b(\cdot,u(\cdot),\xi)}(x_1, x_3, \lambda)}{(1-\lambda)Q_3 + \lambda Q_1}\right) + \frac{C^2\varepsilon^2(\xi)}{\nu}.
\end{aligned}$$

We point out that $\mathcal{HC}_b \geqslant \mathcal{JC}_b$, hence if $\mathcal{JC}_b \geqslant 0$, i.e. b is jointly concave, then b is also harmonic concave and in that case,

$$\partial_s b(x_2, s, \xi)\mathcal{C}_u(x_1, x_3, \lambda) \leqslant \varepsilon(\xi) + \frac{\varepsilon^2(\xi)}{\nu}.$$

Otherwise, if $\mathcal{JC}_b \leqslant 0$, using also (2.12), we get that

$$\mathcal{C}_u(x_1, x_3, \lambda) \leqslant \frac{1}{\sigma}\left[-\mathcal{HC}_{b(\cdot,u(\cdot),\xi)}(x_1, x_3, \lambda) + C\varepsilon(\xi)\left(1 - \frac{\mathcal{JC}_{b(\cdot,u(\cdot),\xi)}(x_1, x_3, \lambda)}{\nu}\right) + \frac{C^2\varepsilon^2(\xi)}{\nu}\right].$$

This concludes the proof. □

3. Application to semi-linear equations

We investigate two applications of our general maximum convexity principles. We point out that the characteristics of these applications, particularly the boundary conditions, drive the convexity function C_u of the solution to attain a positive maximum within the interior of the domain. Then we readily apply Theorems 2.2, 2.3 to obtain the estimates on the loss of concavity of a transformed of the solution u.

It is worth mentioning that, in the classical case, the concavity of the solution depends on the (harmonic)-concavity of the nonlinearity. In both our subsequent applications, Problem 2 and 3, already a direct use of the maximum convexity principles in Theorems 2.2, 2.3 provides this connection. We give a bound on the convexity of the nonlinearity in terms of the spatial variation, to emphasize the role of the introduced anisotropy, see also subsequent Remark 3.3.

In this section, let $\Omega \subset \mathbb{R}^n$ be a bounded open strongly convex set with C^1 boundary.

Problem 2. Let

$$a : \overline{\Omega} \to (0, +\infty)$$

and for all $i, j \in \{1, \ldots, n\}$ let the functions

$$\alpha^{ij} : \overline{\Omega} \to (0, +\infty)$$

be such that there exists $\zeta > 0$ such that

$$\sum_{i,j=1}^{n} \alpha^{ij}(x) p_i p_j \geqslant \zeta |p|^2, \quad \text{for all } p \in \mathbb{R}^n,$$

and $a, \alpha^{ij}(\cdot) \in C^1(\Omega)$. Consider the equation

$$\begin{cases} -\sum_{i,j=1}^{n} \alpha^{ij}(x) D^2_{ij} u = a(x) u^\beta, & \beta \in [0, 1) \quad \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We recall the following property of convex sets [13].

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^n$ be bounded strongly convex set with C^1 boundary. Then there exist $r_0 > 0$ such, that for every $\rho \in (0, r_o]$, the set*

$$\Omega_\rho := \{x \in \Omega : d(x, \partial\Omega) > \rho\}.$$

is convex with C^1 boundary.

We recall once more that, when the coefficients α^{ij} do not depend on x and when $a(x) = 1$, the power function u^α, for some $\alpha := \alpha(\beta)$, is concave. We want to understand the impact of introducing a dependency on x in the equation. We are able to obtain a precise quantitative result about the extent to which the transformed solution deviates from concavity.

Proposition 3.2. *Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a solution of Problem 2, assuming additionally that*

$$\|\nabla a\|_{L^\infty(\Omega)} + \max_{i,j\in\{1,\dots,n\}} \left\|\nabla \alpha^{ij}\right\|_{L^\infty(\Omega)} < \varepsilon$$

for some $\varepsilon > 0$. Then

$$\max_{\overline{\Omega}\times\overline{\Omega}\times[0,1]} \mathcal{C}_{-u^{\frac{1-\beta}{2}}}(x, y, t) \leqslant C\varepsilon,$$

for some $C > 0$.

Proof. Let

$$v := -u^{\frac{1-\beta}{2}}.$$

We point out that $v < 0$, $v \in C^2(\Omega)$ and we focus on deriving the equation satisfied by v. We have that $u = (-v)^{\frac{2}{1-\beta}}$, and

$$D_i u = -\frac{2}{1-\beta}(-v)^{\frac{1+\beta}{1-\beta}} D_i v$$

where $D_i = \frac{\partial}{\partial x_i}$, and

$$D^2_{ij} u = \frac{2(1+\beta)}{(1-\beta)^2}(-v)^{\frac{2\beta}{1-\beta}} D_i v D_j v - \frac{2}{1-\beta}(-v)^{\frac{1+\beta}{1-\beta}} D^2_{ij} v$$

where $D^2_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$. This gives that

$$\sum_{i,j=1}^n \alpha^{ij}(x) D^2_{ij} u = \frac{2(1+\beta)}{(1-\beta)^2}(-v)^{\frac{2\beta}{1-\beta}} \sum_{i,j=1}^n \alpha^{ij}(x) D_i v D_j v - \frac{2}{1-\beta}(-v)^{\frac{1+\beta}{1-\beta}} \sum_{i,j=1}^n \alpha^{ij}(x) D^2_{ij} v.$$

Thus we obtain

$$-\frac{2(1+\beta)}{(1-\beta)^2}(-v)^{\frac{2\beta}{1-\beta}} \sum_{i,j=1}^n \alpha^{ij}(x) D_i v D_j v + \frac{2}{1-\beta}(-v)^{\frac{1+\beta}{1-\beta}} \sum_{i,j=1}^n \alpha^{ij}(x) D^2_{ij} v = a(x)(-v)^{\frac{2\beta}{1-\beta}}.$$

Dividing by $\frac{2}{1-\beta}(-v)^{\frac{1+\beta}{1-\beta}}$ yields

$$\sum_{i,j=1}^n \alpha^{ij}(x) D^2_{ij} v = (-v)^{-1}\left(\frac{a(x)(1-\beta)}{2} + \frac{1+\beta}{1-\beta}\sum_{i,j=1}^n \alpha^{ij}(x) D_i v D_j v\right), \tag{3.1}$$

that is

$$\sum_{i,j=1}^{n} \alpha^{ij}(x) D^2_{ij} v - b(x, v, Dv) = 0,$$

where

$$b(x, s, \xi) := (-s)^{-1} f_\xi(x), \tag{3.2}$$

and for any $\xi \in \mathbb{R}^n$, $f_\xi : \Omega \to \mathbb{R}$ is

$$f_\xi(x) := \frac{a(x)(1-\beta)}{2} + \frac{1+\beta}{1-\beta} \sum_{i,j=1}^{n} \alpha^{ij}(x)\xi_i \xi_j.$$

If $\mathcal{C}_{-u^{(1-\beta)/2}} \leqslant 0$ in $\overline{\Omega} \times \overline{\Omega} \times [0, 1]$, then there is nothing to prove. Otherwise, from [4, Corollary 3.2], we have that $\mathcal{C}_{-u^{(1-\beta)/2}}$, cannot achieve any positive maximum on the boundary, i.e. the positive maximum of $\mathcal{C}_v$ is attained at some point $(x_1, x_3, \lambda) \in \Omega \times \Omega \times (0, 1)$. Recalling Proposition 3.1, let

$$\rho := \min\{r_0, d(x_1, \Omega), d(x_3, \Omega)\},$$

then $(x_1, x_3, \lambda) \in \overline{\Omega}_\rho \times \overline{\Omega}_\rho \times (0, 1)$ and define

$$m_\rho := \|v\|_{C(\overline{\Omega}_\rho)}, \qquad M_\rho := \|Dv\|_{C(\overline{\Omega}_\rho)}.$$

Notice that

$$f_\xi(x) \geqslant \frac{1+\beta}{1-\beta} \zeta |\xi|^2 + \frac{1-\beta}{2} \min_{\overline{\Omega}_\rho} a(x) \geqslant \frac{1-\beta}{2} \min_{\overline{\Omega}_\rho} a(x) > 0.$$

We have that for all $x \in \overline{\Omega}_\rho$, $s \in [-m_\rho, 0)$, $\xi \in \overline{B}_{M_\rho}$

$$b(x, s, \xi) \geqslant \frac{1-\beta}{2m_\rho} \min_{\overline{\Omega}_\rho} a(x) := \nu > 0, \qquad \partial_s b(x, s, \xi) \geqslant \frac{1-\beta}{2m_\rho^2} \min_{\overline{\Omega}_\rho} a(x) := \sigma > 0.$$

For clarity, we point out that we have $[x_1, x_3] \subset \overline{\Omega}_\rho$, $[v(x_1), v(x_3)] \subset [-m_\rho, 0)$ and $Dv(x_1) \in \overline{B}_{M_\rho}$, thus the hypothesis (2.12), (2.13) in Theorem 2.3 are fulfilled. Denote for all $\xi \in \overline{B}_{M_\rho}$,

$$\mathfrak{m} := \min_{\overline{\Omega}_\rho} f_\xi(x), \qquad \mathfrak{M} = \max_{\overline{\Omega}_\rho} f_\xi(x)$$

and remark that, for some $\bar{x}, \tilde{x} \in \overline{\Omega}_\rho$ and $\bar{z}$ on the segment $[\bar{x}, \tilde{x}]$ lying in $\overline{\Omega}_\rho$,

$$\begin{aligned}\mathfrak{M} - \mathfrak{m} &= f_\xi(\bar{x}) - f_\xi(\tilde{x}) \leqslant |\nabla f_\xi(\bar{z})| \operatorname{diam}(\Omega_\rho) \\ &\leqslant \varepsilon\left(\frac{1-\beta}{2} + \frac{1+\beta}{1-\beta} n^2 M_\rho^2\right) := \varepsilon \mathfrak{C}.\end{aligned}$$

Let $\upsilon_\rho := \min_{\overline{\Omega}_\rho}(-v) > 0$, there holds[1]

$$
\begin{aligned}
&\mathcal{HC}_{b(\cdot,v(\cdot),\xi)}(x_1,x_3,\lambda)\\
&\quad\geqslant \frac{1}{\lambda(-v)(x_3)+(1-\lambda)(-v)(x_1)}\\
&\quad\quad\times\left(f_\xi\big((1-\lambda)x_1+\lambda x_3\big)-f_\xi(x_1)f_\xi(x_3)\frac{\lambda(-v)(x_3)+(1-\lambda)(-v)(x_1)}{\lambda f_\xi(x_1)(-v)(x_3)+(1-\lambda)f_\xi(x_3)(-v)(x_1)}\right)\\
&\quad\geqslant \frac{1}{\lambda(-v)(x_3)+(1-\lambda)(-v)(x_1)}\left(\mathfrak{m}-\frac{f_\xi(x_1)f_\xi(x_3)}{\mathfrak{m}}\right)\\
&\quad\geqslant \frac{1}{\lambda(-v)(x_3)+(1-\lambda)(-v)(x_1)}\left(\mathfrak{m}-\frac{\mathfrak{M}^2}{\mathfrak{m}}\right)\\
&\quad\geqslant \frac{1}{\upsilon_\rho}\frac{\mathfrak{m}^2-\mathfrak{M}^2}{\mathfrak{m}}=-\varepsilon\frac{\mathfrak{C}}{\upsilon_\rho}\frac{\mathfrak{M}+\mathfrak{m}}{\mathfrak{m}}.
\end{aligned}
$$

Also we have that

$$
\mathcal{JC}_{b(\cdot,(-v)(\cdot),\xi)}(x_1,x_3,\lambda)\geqslant\frac{\mathfrak{m}}{m_\rho}-\frac{\mathfrak{M}}{\upsilon_\rho}:=-\alpha.
$$

According to Theorem 2.3, we have that for all $(x,y,t)\in\overline{\Omega}\times\overline{\Omega}\times[0,1]$,

$$
\begin{aligned}
\max_{\overline{\Omega}\times\overline{\Omega}\times[0,1]}\mathcal{C}_{-u^{\frac{1-\beta}{2}}}(x,y,t)&=\mathcal{C}_{-u^{\frac{1-\beta}{2}}}(x_1,x_3,\lambda)\\
&=\mathcal{C}_v(x_1,x_3,\lambda)\leqslant\frac{1}{\sigma}\left[\frac{C}{\upsilon_\rho}\frac{\mathfrak{M}+\mathfrak{m}}{\mathfrak{m}}\varepsilon+C\varepsilon\left(1-\frac{\alpha}{\nu}\right)+\frac{C^2\varepsilon}{\nu}\right]:=C_\rho\varepsilon,
\end{aligned}
$$

when ε is small enough. □

Remark 3.3. We point out the difference with what is obtained for the autonomous model case $-\Delta u=u^\beta$. There, the transformation $u^{(1-\beta)/2}$ is concave, since the right hand side of (3.1), the transformed equation, is harmonic concave. In our case, we control the "loss of concavity" by the variation of the introduced anisotropy, and this is the best one can hope for: the function b defined in (3.2) is never harmonic concave, jointly in the two variables (x,s). Indeed, it is known that a positive function b is harmonic concave if and only if $B=1/b$ is convex. However, even in the plane, the hessian of the function $B(x,s)=sg(x)$ is negatively defined, hence B is nor convex, nor concave, unless g is constant.

Problem 3. Let

$$
a:\overline{\Omega}\to(0,+\infty)
$$

[1]We thank Marco Gallo for pointing out a preliminary version of this estimate.

and for all $i, j \in \{1, \dots, n\}$ let the functions

$$\alpha^{ij} : \overline{\Omega} \to (0, +\infty)$$

be such that there exists $\zeta > 0$ such that

$$\sum_{i,j=1}^{n} \alpha^{ij}(x) p_i p_j \geqslant \zeta |p|^2, \quad \text{for all } p \in \mathbb{R}^n,$$

and $a, \alpha^{ij}(\cdot) \in C^1(\Omega)$. Let $\varphi \colon (0, +\infty) \to (0, +\infty)$ be such that $\varphi \in C^1(0, +\infty)$ and $\varphi'(t) \leqslant 0$. Consider, for $\varepsilon > 0$, the problem

$$\begin{cases} -\sum_{i,j=1}^{n} \alpha^{ij}(x) D^2_{ij} u = a(x)u + \varepsilon\varphi(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This can be considered as a perturbation of the nonautonomous version of the eigenvalue problem for second order elliptic operators. We remark that the condition on φ can be loosened to accommodate other perturbations, in particular one can require $\varphi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ be such that, $e^s\varphi(e^{-s}) - \varphi'(e^{-s}) \geqslant \gamma > 0$, for all $s \geqslant -R$, for some $R > 0$, e.g. $\varphi(t) = t^\beta$, $\beta \in [0, 1)$ can also be considered.

We have the following result.

Proposition 3.4. *Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of Problem 3. Assume that*

$$\|\nabla a\|_{L^\infty(\Omega)} + \max_{i,j\in\{1,\dots,n\}} \left\|\nabla \alpha^{ij}\right\|_{L^\infty(\Omega)} < \varepsilon.$$

Then

$$\max_{\overline{\Omega}\times\overline{\Omega}\times[0,1]} \mathcal{C}_{-\log u}(x, y, t) \leqslant C\varepsilon,$$

for some $C > 0$.

Proof. Letting $u = e^{-v}$ we have $v = -\log u$. We notice that $v \in C^2(\Omega)$ and that by a direct calculation we obtain

$$\sum_{i,j=1}^{n} \alpha^{ij}(x) D^2_{ij} v = b(x, v, Dv),$$

where

$$b(x, s, \xi) := \sum_{i,j=1}^{n} \alpha^{ij}(x)\xi_i\xi_j + a(x) + \varepsilon e^s \varphi\big(e^{-s}\big).$$

If $\mathcal{C}_v \leqslant 0$ in $\overline{\Omega} \times \overline{\Omega} \times [0,1]$, then there is nothing to prove. Otherwise, using [10, Lemma 3.11], we have that $\mathcal{C}_v$, cannot achieve any positive maximum on the boundary, i.e. the maximum of $\mathcal{C}_v$ is attained at some point $(x_1, x_3, \lambda) \in \Omega \times \Omega \times (0,1)$. Recalling Proposition 3.1, let

$$\rho := \min\{r_0, d(x_1, \Omega), d(x_3, \Omega)\},$$

then $(x_1, x_3, \lambda) \in \overline{\Omega}_\rho \times \overline{\Omega}_\rho \times (0,1)$ and define

$$m_\rho := \|v\|_{C(\overline{\Omega}_\rho)}, M_\rho := \|Dv\|_{C(\overline{\Omega}_\rho)}.$$

Notice that for all $x \in \overline{\Omega}_\rho$, $s \in [-m_\rho, m_\rho]$, $\xi \in \overline{B}_{M_\rho}$, using the hypothesis on φ,

$$\partial_s b(x,s,\xi) \geqslant \varepsilon\big(e^s\varphi\big(e^{-s}\big) - \varphi'\big(e^{-s}\big)\big) \geqslant \varepsilon e^{-m_\rho}\varphi\big(e^{m_\rho}\big) := \sigma.$$

For clarity, we observe that we have $[x_1, x_3] \subset \overline{\Omega}_\rho$, $[v(x_1), v(x_3)] \subset [-m_\rho, m_\rho]$ and $Dv(x_1) \in \overline{B}_{M_\rho}$, thus the hypothesis (2.7) in Theorem 2.2 are fulfilled. We have that

$$\big|\mathcal{J}\mathcal{C}_{b(\cdot,v(\cdot),\xi)}(x_1,x_3,\lambda)\big| \leqslant C\big(\big\|\nabla\alpha^{ij}\big\|_{L^\infty(\Omega_\rho)} + \|\nabla a\|_{L^\infty(\Omega_\rho)} + \varepsilon\big) \leqslant C\varepsilon.$$

Applying Theorem 2.2 yields the conclusion. $\square$

Finally, we recall the following [9, Theorem 2]

Proposition 3.5 (Hyers–Ulam)**.** *Let X be a space of finite dimension and $D \subset X$ convex. Assume that $f : D \to \mathbb{R}$ is δ-convex, i.e. for all $(x,y,t) \in D \times D \times [0,1]$*

$$\mathcal{C}_f(x,y,t) \leqslant \delta.$$

Then there exists a convex function $g : D \to \mathbb{R}$ such that $\|f - g\|_{L^\infty(D)} \leqslant \delta k_n$, where $k_n > 0$ depends only on $n = \dim(X)$.

By using this result, based upon the estimates of Propositions 3.2 and 3.4 we obtain the approximate concavity results stated in the introduction for the transformations $u^{(1-\beta)/2}$ in the case $\beta \in (0,1)$ and $\log u$ for the case $\beta = 1$.

Remark 3.6. The constant C appearing in the conclusions of Proposition 3.2 and 3.4 is related to the C introduced in formula (2.9) which depends on n, $\operatorname{diam}(\Omega_\rho)$ and on the supremum norms of the second order derivatives of the transformation v on Ω_ρ and hence (since u is bounded away from 0 on Ω_ρ) on the supremum norms of $D^2_{ij}u$ on Ω_ρ. By the classical Schauder estimates for second order linear elliptic operators (see [6, Theorem 6.2]), in turn C depends on n, $\operatorname{diam}(\Omega_\rho)$, the ellipticity constant ζ, $\|\alpha^{ij}\|_{C^{0,\alpha}(\Omega_\rho)}$, $\|a\|_{C^{0,\alpha}(\Omega_\rho)}$ and $\|u\|_{C^{0,\alpha}(\Omega_\rho)}$ for some $\alpha \in (0,1)$.

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