Singular Limit of Differential Systems with Memory

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ABSTRACT. We consider differential systems with memory terms, expressed by convolution integrals, which account for the past history of one or more variables. The aim of this work is to analyze the passage to the singular limit when the memory kernel collapses into a Dirac mass. In particular, we focus on the reaction-diffusion equation with memory, and we discuss the convergence of solutions on finite time-intervals. When enough dissipativity is present, we also establish convergence results of the global and the exponential attractors. Nonetheless, the techniques here devised are quite general, and suitable to be applied to a large variety of models.

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1. INTRODUCTION

Many physical phenomena are properly described by (systems of) partial differential equations where the dynamics is influenced by the past history of one or more variables. This amounts to averaging some quantities by means of convolution integrals against a positive summable function, the so-called *memory kernel*. In the limiting situation, when the kernel is a Dirac mass, one recovers the corresponding models without memory. The presence of the memory may render, in some cases, the description of the phenomena more accurate. On the other hand, equations with memory are usually much more difficult to handle than the corresponding ones without memory. Besides, in many situations, the contribution of the past history is not so relevant to significantly affect the results, and so to justify the introduction of further mathematical complications. There are however certain models, such as those describing high-viscosity liquids at low temperatures, or the thermomechanical behavior of polymers (see, e.g., [5, 16–18, 23]), where the past history plays a nontrivial role that has to be taken into account.

On the contrary, if the system keeps a very short memory of the past (which translates into having a rapidly fading memory kernel) a sensible difference between the two descriptions is not expected. Hence, from an heuristic point of view, it is reasonable to believe that if the memory kernel “looks like” a Dirac mass, then the past history is negligible. Clearly, one would like to render this qualitative statement more precise, and somehow to provide quantitative estimates. This is precisely the aim of this work. Namely, we want to show that systems with memory converge in an appropriate sense to the corresponding systems without memory, as the memory kernel converges to the Dirac mass. Incidentally, this fact has a sort of philosophical implication. Indeed, it is not out of the ordinary to hear people say that parabolic equations are unphysical, due to the infinite propagation speed of disturbances, so that in the “real world” the evolution is necessarily hyperbolic. On the other hand, as we will see in a while, it is always possible to view a parabolic equation as the limiting case of an equation with memory of hyperbolic type. Thus, if the memory kernel is very close to a Dirac mass, the parabolic equation provides in fact an accurate description of the phenomenon under consideration (cf. [6, 19]).

Our intention in the present paper is to establish an abstract theory of wide application, and then treat in detail a particular, albeit fairly significant, problem. To better explain our strategy, let us introduce two concrete, and in some sense paradigmatic, examples.

1.1. The reaction-diffusion equation. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. For $u = u(\mathbf{x}, t) : \Omega \times (0, \infty) \to \mathbb{R}$, we consider the equation

$$u_t - \Delta u + \varphi(u) = f, \quad t > 0, \tag{1.1}$$

where, for simplicity, we put all the physical constants equal to one. The functions φ and f are a suitable nonlinearity and a time-independent source term, respectively. Thinking, for instance, of heat propagation processes, Equation (1.1) is obtained assuming the classical Fourier's constitutive law. Appeared in the late 60's, the two famous works [2, 14] suggest that, in certain cases, it is physically more reasonable to take a convolution average of (all or part of) the term $-\Delta u(t)$, in order to account for the past history of u up to time t. This amounts to replacing (1.1) with the equation

$$u_t - \omega\Delta u - (1-\omega)\int_0^\infty k(s)\Delta u(t-s)\,ds + \varphi(u) = f, \quad t > 0. \tag{1.2}$$

Notice that the convolution integral requires the knowledge of the values of u for all past times; this implies that $u = u(\mathbf{x}, t) : \Omega \times \mathbb{R} \to \mathbb{R}$. We point out that $u(t)$ is supposed to be a given datum for $t \le 0$, where it need not fulfill Equation (1.2). Here, $\omega \in [0, 1)$ (for $\omega = 1$ we would fall into the previous case), and the memory kernel $k : [0, \infty) \to \mathbb{R}$ is a continuous nonnegative function, smooth on $(0, \infty)$, vanishing at infinity and satisfying the relation

$$\int_0^\infty k(s)\,ds = 1. \tag{1.3}$$

We refer to the problems with $\omega = 0$ and $\omega > 0$ as the *Gurtin-Pipkin* and the *Coleman-Gurtin* models, respectively. From the physical viewpoint, the presence of the memory accounts for the resistance of the system to a change of state. For instance, the heat equation of Gurtin-Pipkin type is fully hyperbolic. So, in particular, infinite propagation speed of initial disturbances is no longer supported. This matches with the reasonable assumption that if one heats one side of a rod, the effect cannot be instantaneous on the opposite side.

Let now $\omega \in [0, 1)$ be fixed. It is clear that if we (formally) choose $k = \delta_0$ (the Dirac mass at zero), Equation (1.2) turns into Equation (1.1). Hence, for $\varepsilon \in (0, 1]$, let us set

$$k_\varepsilon(s) = \frac{1}{\varepsilon} k\left(\frac{s}{\varepsilon}\right).$$

Then we consider the family of equations

$$u_t - \omega\Delta u - (1 - \omega)\int_0^\infty k_\varepsilon(s)\Delta u(t - s)\, ds + \varphi(u) = f, \quad t > 0. \tag{1.4}$$

Since $k_\varepsilon \to \delta_0$ in the distributional sense, our purpose is to show in what terms we can say that (1.4) converges to the limiting equation (1.1) as $\varepsilon \to 0$.

Remark 1.1. The limit process $\varepsilon \to 0$ is singular, since when we collapse into (1.1) we lose the information on the past history of u. Indeed, (1.1), besides proper boundary conditions, requires only the initial value of u at the initial time $t = 0$. This will be more evident in the next sections, where, following a brilliant intuition of Dafermos [3], we introduce the notion of *extended phase-space*, which is the natural setting to treat equations with memory within the framework of dynamical systems. The extended phase-space is constructed adding a further component to the usual phase-space associated to the corresponding limiting equation, using the past history as an additional variable of the system. The new component is a weighted Banach space, whose weight is determined by the memory kernel.

1.2. The damped wave equation. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, and let $\varepsilon \in (0, 1]$. For $u = u(\mathbf{x}, t) : \Omega \times \mathbb{R} \to \mathbb{R}$, we consider the equation

$$u_{tt} + \alpha u_t - h_\varepsilon(0)\Delta u - \int_0^\infty h'_\varepsilon(s)\Delta u(t - s)\, ds + \varphi(u) = f, \quad t > 0, \tag{1.5}$$

arising, for instance, in the theory of viscoelasticity (see, e.g., [5, 23]). Here, $\alpha \geq 0$ and the memory kernel h_ε is a sufficiently smooth function of the form

$$h_\varepsilon(s) = k_\varepsilon(s) + k_\infty,$$

with $k_\infty > 0$. The functions k_ε, φ and f are as in the former case. A formal integration by parts, recalling that k vanishes at infinity, yields

$$\begin{aligned}\int_0^\infty h'_\varepsilon(s)\Delta u(t - s)\, ds &= \int_0^\infty k'_\varepsilon(s)\Delta u(t - s)\, ds \\ &= -k_\varepsilon(0)\Delta u(t) + \int_0^\infty k_\varepsilon(s)\Delta u_t(t - s)\, ds.\end{aligned}$$

Therefore (1.5) can be rewritten as

$$(1.6)\qquad u_{tt} + \alpha u_t - k_\infty \Delta u - \int_0^\infty k_\varepsilon(s)\Delta u_t(t-s)\,ds + \varphi(u) = f, \quad t > 0.$$

Letting $\varepsilon \to 0$ we (formally) obtain the limiting equation

$$(1.7)\qquad u_{tt} + \alpha u_t - k_\infty \Delta u - \Delta u_t + \varphi(u) = f, \quad t > 0,$$

that is, the (strongly) damped wave equation. Again, it is interesting to specify in what sense we may speak of convergence of (1.6) to (1.7).

In this work, we investigate in detail the case of the reaction-diffusion equation. The damped wave equation, as well as other physically relevant models with memory, will be analyzed in forthcoming papers. Nonetheless, we want to establish some general results, suitable to treat a quite vast class of models with memory.

2. Preliminaries

Let us consider Equation (1.4). For the sake of simplicity, we shall restrict to the physically relevant case of space-dimension $n = 3$. In order to carry out our analysis, and to exploit the machinery of the theory of dynamical systems, we need to associate with the equation a strongly continuous semigroup of operators (cf. Remark 1.1). This can be done, along the line of [3], by introducing the so-called *integrated past history* of u, i.e., the auxiliary variable

$$(2.1)\qquad \eta^t(\mathbf{x}, s) = \int_0^s u(\mathbf{x}, t-y)\,dy, \quad s > 0,\ t > 0.$$

Keeping in mind the hypotheses on k, and setting

$$(2.2)\qquad \mu(s) = -(1-\omega)k'(s),$$

a formal integration by part yields

$$(1-\omega)\int_0^\infty k_\varepsilon(s)\Delta u(t-s)\,ds = \int_0^\infty \mu_\varepsilon(s)\Delta\eta^t(s)\,ds,$$

where

$$\mu_\varepsilon(s) = \frac{1}{\varepsilon^2}\mu\left(\frac{s}{\varepsilon}\right).$$

Hence (1.4) turns into

$$(2.3)\qquad u_t - \omega\Delta u - \int_0^\infty \mu_\varepsilon(s)\Delta\eta(s)\,ds + \varphi(u) = f, \quad t > 0.$$

At this point, a further equation ruling the evolution of η is needed. Differentiation of equality (2.1) leads to

$$\eta_t^t(s) = -\eta_s^t(s) + u(t), \quad t > 0. \tag{2.4}$$

The translation of (1.4) into the system (2.3)–(2.4), endowed with appropriate initial and boundary conditions, will allow us to provide a description of the solutions in terms of a strongly continuous semigroup of operators (or dynamical system) $S_\varepsilon(t)$, acting on a proper (extended) phase-space (see [10, 11, 20]). Notice that, since u is supposed to be known for $t \le 0$, the initial condition for η is given by

$$\eta^0(s) = \int_0^s u(-y)\, dy.$$

Besides, from (2.1) we also get the boundary condition

$$\eta^t(0) = \lim_{s\to 0} \eta^t(s) = 0, \quad \forall\, t \ge 0. \tag{2.5}$$

Of course, one might think that the link between the original equation (1.4) and system (2.3)–(2.4) along with the boundary condition (2.5) is only formal. It is not so. Indeed, once appropriate initial and boundary conditions are given, it is possible to show that (1.4) and (2.3)–(2.5) are completely equivalent (as a matter of fact, it is actually true that the latter generalizes the former). The relationship between the two descriptions is discussed in detail in the review paper [13], to which we address the reader.

Remark 2.1. This approach is valid for partial differential equations with memory of the first order in time. A similar argument applies *verbatim* to the damped wave equation with memory, and, in general, to partial differential equations with memory of the second order in time (see [13, 22] for the details). The only difference is the choice of the auxiliary variable, being this time $\eta^t(\mathbf{x}, s) = u(\mathbf{x}, t) - u(\mathbf{x}, t-s)$, that is, the difference between the function at time t and the past history of the function up to time t.

In the next sections, we introduce the proper functional setting to define the semigroups $S_\varepsilon(t)$, along with the semigroup $S_0(t)$ related to the limiting equation (1.1). Our first task is to show that $S_\varepsilon(t)$ and $S_0(t)$ are *close within* ε on every time-interval $[0, T]$, $T > 0$. This is done in Section 5, where we provide results both for the Coleman-Gurtin and the Gurtin-Pipkin cases. Then we study in detail the asymptotic properties of the more dissipative Coleman-Gurtin model, and we establish stability results which are independent of ε. In particular, we prove the existence of global attractors $\mathcal{A}_\varepsilon$ (in Section 7) and exponential attractors $\mathcal{E}_\varepsilon$ (in Section 9), which are continuous (in an appropriate sense) with respect to the singular limit $\varepsilon \to 0$. In order to accomplish this plan, we shall make use of an abstract result (discussed in the Appendix) on the convergence of exponential attractors for systems with memory to the exponential attractors of the corresponding limiting equations.

3. Notation and Basic Tools

3.1. The functional setting. Given a Banach space $\mathcal{H}$, we denote by $B_{\mathcal{H}}(R)$ the closed ball in $\mathcal{H}$ of radius $R \geq 0$ centered at zero.

Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain. The symbols $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ stand for the norm and the inner product on $L^2(\Omega)$, respectively. Let $A = -\Delta$ be the Laplace operator on $L^2(\Omega)$ with domain $\mathcal{D}(A) = H_0^1(\Omega) \cap H^2(\Omega)$. We introduce the hierarchy of Hilbert spaces

$$H^r = \mathcal{D}(A^{r/2}), \quad r \in \mathbb{R},$$

endowed with the inner products

$$\langle u_1, u_2 \rangle_{H^r} = \langle A^{r/2} u_1, A^{r/2} u_2 \rangle.$$

It is well known that $H^{r_1} \Subset H^{r_2}$ for $r_1 > r_2$.

Next, let $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, with $\mathbb{R}^+ = (0, \infty)$, fulfill the following conditions:

$$\mu(s) \geq 0, \quad \forall\, s \in \mathbb{R}^+, \tag{3.1}$$

$$\mu'(s) + \delta\mu(s) \leq 0, \quad \forall\, s \in \mathbb{R}^+, \text{ for some } \delta > 0. \tag{3.2}$$

Notice that μ is decreasing, and the Gronwall Lemma entails the exponential decay

$$\mu(s) \leq \mu(s_0) e^{-\delta(s-s_0)}, \quad \forall\, s \geq s_0 > 0. \tag{3.3}$$

For any given $\varepsilon \in (0, 1]$, we define the function

$$\mu_\varepsilon(s) = \frac{1}{\varepsilon^2} \mu\left(\frac{s}{\varepsilon}\right),$$

and we consider the weighted Hilbert spaces

$$\mathcal{M}_\varepsilon^r = L^2_{\mu_\varepsilon}(\mathbb{R}^+, H^{r+1}), \quad r \in \mathbb{R},$$

endowed with the inner products

$$\langle \eta_1, \eta_2 \rangle_{\mathcal{M}_\varepsilon^r} = \int_0^\infty \mu_\varepsilon(s) \langle A^{(1+r)/2} \eta_1(s), A^{(1+r)/2} \eta_2(s) \rangle \, ds.$$

The embeddings $\mathcal{M}_\varepsilon^{r_1} \subset \mathcal{M}_\varepsilon^{r_2}$ are clearly continuous for $r_1 > r_2$. Unfortunately, they are not compact (cf. [22]). To bypass this obstacle, we need to construct further spaces.

Let T_ε be the linear operator on $\mathcal{M}^0_\varepsilon$ with domain

$$\mathcal{D}(T_\varepsilon) = \{\eta \in \mathcal{M}^0_\varepsilon \mid \eta_s \in \mathcal{M}^0_\varepsilon,\ \eta(0) = 0\}$$

defined by

$$T_\varepsilon \eta = -\eta_s, \quad \eta \in \mathcal{D}(T_\varepsilon).$$

Here, η_s denotes the distributional derivative of η with respect to the internal variable s (indeed, T_ε is the infinitesimal generator of the right-translation semigroup on $\mathcal{M}^0_\varepsilon$). It is worth noting that, on account of (3.2), there holds (cf. [13])

$$\langle T_\varepsilon \eta, \eta \rangle_{\mathcal{M}^0_\varepsilon} \leq -\frac{\delta}{2\varepsilon} \|\eta\|^2_{\mathcal{M}^0_\varepsilon}, \quad \forall\, \eta \in \mathcal{D}(T_\varepsilon). \tag{3.4}$$

Then, we introduce the spaces

$$\mathcal{L}^r_\varepsilon = \{\eta \in \mathcal{M}^r_\varepsilon \mid \eta \in \mathcal{D}(T_\varepsilon),\ \sup_{x \geq 1} x \mathbb{T}^\varepsilon_\eta(x) < \infty\}.$$

Here, $\mathbb{T}^\varepsilon_\eta$ is the *tail function* of η, given by

$$\mathbb{T}^\varepsilon_\eta(x) = \varepsilon \int_{(0,1/x)\cup(x,\infty)} \mu_\varepsilon(s) \|A^{1/2}\eta(s)\|^2\, ds, \quad x \geq 1.$$

It is readily seen (cf. [8]) that $\mathcal{L}^r_\varepsilon$ is a Banach space endowed with the norm

$$\|\eta\|^2_{\mathcal{L}^r_\varepsilon} = \|\eta\|^2_{\mathcal{M}^r_\varepsilon} + \varepsilon \|T_\varepsilon \eta\|^2_{\mathcal{M}^0_\varepsilon} + \sup_{x \geq 1} x \mathbb{T}^\varepsilon_\eta(x).$$

On account of an immediate generalization of a compactness result [22, Lemma 5.5] (see also [8]), we have the following result.

Lemma 3.1. *Let $\mathcal{K} \subset \mathcal{M}^0_\varepsilon$ satisfy, for some $r > 0$,*

$$\sup_{\eta \in \mathcal{K}} \big[\|\eta\|_{\mathcal{M}^r_\varepsilon} + \|\eta_s\|_{\mathcal{M}^{-1}_\varepsilon}\big] < \infty \quad \text{and} \quad \lim_{x\to\infty} \big[\sup_{\eta \in \mathcal{K}} \mathbb{T}^\varepsilon_\eta(x)\big] = 0.$$

Then $\mathcal{K}$ is relatively compact in $\mathcal{M}^0_\varepsilon$. As a consequence, $\mathcal{L}^r_\varepsilon \Subset \mathcal{M}^0_\varepsilon$ for every $r > 0$.

Finally, for $\varepsilon \in [0,1]$, we define the product Banach spaces

$$\mathcal{H}^r_\varepsilon = \begin{cases} H^r \times \mathcal{M}^r_\varepsilon, & \text{if } \varepsilon > 0, \\ H^r, & \text{if } \varepsilon = 0, \end{cases}$$

$$\mathcal{Z}^r_\varepsilon = \begin{cases} H^r \times \mathcal{L}^r_\varepsilon, & \text{if } \varepsilon > 0, \\ H^r, & \text{if } \varepsilon = 0, \end{cases}$$

normed by

$$\|(u,\eta)\|^2_{\mathcal{H}^r_\varepsilon} = \|u\|^2_{H^r} + \|\eta\|^2_{\mathcal{M}^r_\varepsilon},$$
$$\|(u,\eta)\|^2_{\mathcal{Z}^r_\varepsilon} = \|u\|^2_{H^r} + \|\eta\|^2_{\mathcal{L}^r_\varepsilon}.$$

Remark 3.2. When $\varepsilon = 0$, we agree to interpret the pair (u,η) just as u. Accordingly, the norms reduce to the first summands only.

In particular, $\mathcal{H}^0_\varepsilon$ will be the extended phase-space on which we shall construct the dynamical system associated with our problem. Due to Lemma 3.1, $\mathcal{Z}^r_\varepsilon \Subset \mathcal{H}^0_\varepsilon$ for every $r > 0$. We shall also make use of the *lifting map* $\mathbb{L}_\varepsilon : \mathcal{H}^0_0 \to \mathcal{H}^0_\varepsilon$, and of the *projection maps* $\mathbb{P} : \mathcal{H}^0_\varepsilon \to \mathcal{H}^0_0$ and $\mathbb{Q}_\varepsilon : \mathcal{H}^0_\varepsilon \to \mathcal{M}^0_\varepsilon$, given by

$$\mathbb{L}_\varepsilon u = \begin{cases} (u,0), & \text{if } \varepsilon > 0, \\ u, & \text{if } \varepsilon = 0, \end{cases}$$

and

$$\mathbb{P}(u,\eta) = u, \quad \mathbb{Q}_\varepsilon(u,\eta) = \eta.$$

In view of the remark above, if $\varepsilon = 0$, then $\mathbb{P}$ and $\mathbb{Q}_0$ are the identity and the null map, respectively.

3.2. The representation formula. Assume that u is a given function belonging to $L^1(0,T;H^1)$ for every $T > 0$. Then, for every $\eta_0 \in \mathcal{M}^0_\varepsilon$, the Cauchy problem

$$\begin{cases} \eta_t = T_\varepsilon \eta + u, & t > 0, \\ \eta^0 = \eta_0, \end{cases}$$

has a unique solution $\eta \in C([0,\infty),\mathcal{M}^0_\varepsilon)$ which has the explicit representation formula (see [13])

$$\eta^t(s) = \begin{cases} \displaystyle\int_0^s u(t-y)\,dy, & 0 < s \le t, \\ \eta_0(s-t) + \displaystyle\int_0^t u(t-y)\,dy, & s > t. \end{cases} \tag{3.5}$$

We now establish some results that will be needed in the sequel.

Lemma 3.3. *Let $\eta_0 \in \mathcal{D}(T_\varepsilon)$, and assume that $\|A^{1/2}u(t)\| \le \rho$, for some $\rho > 0$ and every $t \ge 0$. Then*

$$\varepsilon\|T_\varepsilon \eta^t\|^2_{\mathcal{M}^0_\varepsilon} \le e^{-\delta t}\varepsilon\|T_\varepsilon\eta_0\|^2_{\mathcal{M}^0_\varepsilon} + \|\mu\|_{L^1}\rho^2, \quad \forall\, t \ge 0.$$

Proof. Since $\eta_0(0) = 0$, we can express $T_\varepsilon\eta^t$ through the representation formula (3.5), so getting

$$T_\varepsilon\eta^t(s) = \begin{cases} -u(t-s), & 0 < s \le t, \\ T_\varepsilon\eta_0(s-t), & s > t. \end{cases}$$

Consequently,

$$\varepsilon\|T_\varepsilon\eta^t\|^2_{\mathcal{M}^0_\varepsilon} \le \rho^2 \int_0^t \varepsilon\mu_\varepsilon(s)\,ds + \varepsilon\int_t^\infty \mu_\varepsilon(s)\|A^{1/2}T_\varepsilon\eta_0(s-t)\|^2\,ds.$$

Observe that

$$\int_0^t \varepsilon\mu_\varepsilon(s)\,ds = \int_0^{t/\varepsilon} \mu(s)\,ds \le \int_0^\infty \mu(s)\,ds < \infty.$$

Moreover, since $\varepsilon \le 1$, appealing to (3.3) we obtain

$$\varepsilon\int_t^\infty \mu_\varepsilon(s)\|A^{1/2}T_\varepsilon\eta_0(s-t)\|^2\,ds \le e^{-\delta t}\varepsilon\|T_\varepsilon\eta_0\|^2_{\mathcal{M}^0_\varepsilon},$$

as claimed. ◻

Lemma 3.4. *Let $\eta_0 \in \mathcal{D}(T_\varepsilon)$, and assume that $\|A^{1/2}u(t)\| \le \rho$, for some $\rho > 0$ and every $t \ge 0$. Then*

$$\sup_{x\ge1} x\mathbb{T}^\varepsilon_{\eta^t}(x) \le \sup_{x\ge1} x\mathbb{T}^\varepsilon_{\eta_0}(x)\Psi(t) + \Pi\rho^2, \qquad \forall\, t \ge 0,$$

where $\Psi(t) = 2(t+2)e^{-\delta t}$ and $\Pi > 0$ is a given constant.

Proof. Defining $\eta_0(s) = 0$ for $s < 0$, from (3.5), we get at once the inequality

$$\|A^{1/2}\eta^t(s)\|^2 \le 2\rho^2 s^2 + 2\|A^{1/2}\eta_0(s-t)\|^2.$$

Fix now $x \ge 1$ and $t \ge 0$. Then

$$\begin{aligned} x\mathbb{T}^\varepsilon_{\eta^t}(x) \le 2\rho^2\varepsilon x &\int_{(0,1/x)\cup(x,\infty)} s^2\mu_\varepsilon(s)\,ds \\ &+ 2\varepsilon x\int_{\min\{1/x,t\}}^{1/x} \mu_\varepsilon(s)\|A^{1/2}\eta_0(s-t)\|^2\,ds \\ &+ 2\varepsilon x\int_{\max\{x,t\}}^\infty \mu_\varepsilon(s)\|A^{1/2}\eta_0(s-t)\|^2\,ds. \end{aligned}$$

Concerning the first term of the right-hand side, notice that

$$x\int_0^{1/x} s^2\mu_\varepsilon(s)\,ds = \varepsilon x\int_0^{1/(\varepsilon x)} s^2\mu(s)\,ds \le \int_0^{1/(\varepsilon x)} s\mu(s)\,ds \le \int_0^\infty s\mu(s)\,ds,$$

and

$$x\int_x^\infty s^2\mu_\varepsilon(s)\,ds = \varepsilon x\int_{x/\varepsilon}^\infty s^2\mu(s)\,ds \le \varepsilon^2 \sup_{y\ge 0}\Big[y\int_y^\infty s^2\mu(s)\,ds\Big].$$

Hence, setting

$$\Pi = 2\int_0^\infty s\mu(s)\,ds + 2\sup_{y\ge 0}\Big[y\int_y^\infty s^2\mu(s)\,ds\Big],$$

which is certainly finite due to (3.3), we learn that

$$2\rho^2\varepsilon x\int_{(0,1/x)\cup(x,\infty)} s^2\mu_\varepsilon(s)\,ds \le \Pi\rho^2.$$

We now estimate the remaining terms. Exploiting (3.3), and recalling that $\varepsilon \le 1$, we have

$$\begin{aligned}
2\varepsilon x&\int_{\min\{1/x,t\}}^{1/x} \mu_\varepsilon(s)\|A^{1/2}\eta_0(s-t)\|^2\,ds\\
&\le 2\varepsilon x e^{-\delta t}\int_{\min\{0,1/x-t\}}^{1/x-t} \mu_\varepsilon(s)\|A^{1/2}\eta_0(s)\|^2\,ds\\
&\le 2e^{-\delta t} x\mathbb{T}^\varepsilon_{\eta_0}(x),
\end{aligned}$$

and

$$\begin{aligned}
2\varepsilon x&\int_{\max\{x,t\}}^\infty \mu_\varepsilon(s)\|A^{1/2}\eta_0(s-t)\|^2\,ds\\
&\le 2\varepsilon x e^{-\delta t}\int_{\max\{x-t,0\}}^\infty \mu_\varepsilon(s)\|A^{1/2}\eta_0(s)\|^2\,ds.
\end{aligned}$$

If $t > x-1$, then

$$2\varepsilon x e^{-\delta t}\int_{\max\{x-t,0\}}^\infty \mu_\varepsilon(s)\|A^{1/2}\eta_0(s)\|^2\,ds < 2(t+1)e^{-\delta t}\mathbb{T}^\varepsilon_{\eta_0}(1).$$

Conversely, if $t \le x - 1$,

$$
\begin{aligned}
2\varepsilon x e^{-\delta t} \int_{\max\{x-t,0\}}^{\infty} &\mu_\varepsilon(s)\|A^{1/2}\eta_0(s)\|^2\,ds \\
&\le 2x e^{-\delta t}\mathbb{T}^{\varepsilon}_{\eta_0}(x-t) \\
&\le 2e^{-\delta t}(x-t)\mathbb{T}^{\varepsilon}_{\eta_0}(x-t) + 2te^{-\delta t}\mathbb{T}^{\varepsilon}_{\eta_0}(1).
\end{aligned}
$$

In either case we conclude that

$$
2\varepsilon x e^{-\delta t} \int_{\max\{x-t,0\}}^{\infty} \mu_\varepsilon(s)\|A^{1/2}\eta_0(s)\|^2\,ds \le 2(t+1)e^{-\delta t}\sup_{x\ge 1} x\mathbb{T}^{\varepsilon}_{\eta_0}(x).
$$

Adding the inequalities above we get the thesis. □

Clearly, if we only require that $\|A^{1/2}u(t)\| \le \rho$ for every $t \in [0,T]$, then the results above hold on $[0,T]$. Hence, a straightforward consequence of (3.5), Lemma 3.3 and Lemma 3.4 is

Corollary 3.5. *If $\eta_0 \in \mathcal{D}(T_\varepsilon)$ and $u \in L^\infty(0,T;H^1)$ for every $T > 0$, then $\eta^t \in \mathcal{D}(T_\varepsilon)$ for all $t \ge 0$.*

We will also make use of a weakened version of Lemma 3.4 on finite-time intervals.

Lemma 3.6. *Let $\eta_0 \in \mathcal{D}(T_\varepsilon)$ and $T > 0$. Assume that*

$$
\int_0^T \|A^{1/2}u(y)\|^2\,dy \le \rho,
$$

for some $\rho > 0$. Then

$$
\sup_{x\ge 1} x\mathbb{T}^{\varepsilon}_{\eta^t}(x) \le \Xi(\rho + \sup_{x\ge 1} x\mathbb{T}^{\varepsilon}_{\eta_0}(x)), \qquad \forall\, t \in [0,T],
$$

where $\Xi = \Xi(T) > 0$ is a given constant.

Proof. The argument is, with minor changes, the same of the previous proof. From (3.5), we have now the inequality

$$
\|A^{1/2}\eta^t(s)\|^2 \le 2\rho s + 2\|A^{1/2}\eta_0(s-t)\|^2.
$$

But, reasoning as before,

$$
\varepsilon x \int_{(0,1/x)\cup(x,\infty)} s\mu_\varepsilon(s)\,ds \le \int_0^\infty \mu(s)\,ds + \varepsilon^2 \sup_{y\ge 0}\Big[y\int_y^\infty s\mu(s)\,ds\Big].
$$

The details are left to the reader. □

3.3. Hausdorff distances and fractal dimension. Let $\mathcal{H}$ be a Banach space. Given $\mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{H}$, we denote by

$$\mathrm{dist}_{\mathcal{H}}(\mathcal{B}_1, \mathcal{B}_2) = \sup_{z_1 \in \mathcal{B}_1} \inf_{z_2 \in \mathcal{B}_2} \|z_1 - z_2\|_{\mathcal{H}}$$

the *Hausdorff semidistance* in $\mathcal{H}$ from $\mathcal{B}_1$ to $\mathcal{B}_2$, and by

$$\mathrm{dist}_{\mathcal{H}}^{\mathrm{sym}}(\mathcal{B}_1, \mathcal{B}_2) = \max\{\mathrm{dist}_{\mathcal{H}}(\mathcal{B}_1, \mathcal{B}_2), \mathrm{dist}_{\mathcal{H}}(\mathcal{B}_2, \mathcal{B}_1)\}$$

the *symmetric Hausdorff distance* in $\mathcal{H}$ between $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. Also, given a relatively compact set $\mathcal{B} \subset \mathcal{H}$, we denote by

$$\dim_{\mathcal{H}}[\mathcal{B}] = \limsup_{r \to 0} \frac{\ln \mathcal{N}_r(\mathcal{B}, \mathcal{H})}{\ln(1/r)}$$

the *fractal dimension* in $\mathcal{H}$ of $\mathcal{B}$. The (finite) number $\mathcal{N}_r(\mathcal{B}, \mathcal{H})$ is the minimum number of r-balls of $\mathcal{H}$ necessary to cover $\mathcal{B}$. We address the reader, for instance, to the treatise [24] for more details on these definitions and the related applications.

3.4. Some useful lemmas. We recall three technical results that will be needed in the course of this investigation. The first two are a generalized version of the Gronwall Lemma (see [21, Appendix]) and the uniform Gronwall Lemma [24, Lemma III.1.1], respectively, whereas the latter is the so-called *transitivity property* of exponential attraction, devised in [4, Theorem 5.1].

Lemma 3.7. *Let Φ be a nonnegative absolutely continuous function on $[0, \infty)$ which satisfies, for some $\nu > 0$ and $0 \le \sigma < 1$, the differential inequality*

$$\frac{d}{dt}\Phi + \nu\Phi \le g(1 + \Phi^{\sigma}),$$

where g is nonnegative function satisfying

$$\sup_{t \ge 0} \int_t^{t+1} g(y)\, dy < \infty.$$

Then there exists $C = C(\sigma, \nu, g)$ such that

$$\Phi(t) \le \frac{1}{1 - \sigma}\Phi(0)e^{-\nu t} + C, \quad \forall\, t \ge 0.$$

Lemma 3.8. *Let Φ be a nonnegative absolutely continuous function on $[0, \infty)$ which satisfies, for some nonnegative function g, the differential inequality*

$$\frac{d}{dt}\Phi \le g(1 + \Phi).$$

Assume also that $\sup_{t\geq 0}\int_t^{t+1}\Phi(y)\,dy \leq c_0$ *and* $\sup_{t\geq 0}\int_t^{t+1} g(y)\,dy \leq c_1$, *for some* $c_0, c_1 \geq 0$. *Then*

$$\Phi(t+1) \leq (c_0 + c_1)e^{c_1}, \quad \forall\, t \geq 0.$$

Lemma 3.9. *Let* $S(t)$ *be a strongly continuous semigroup on a Banach space* $\mathcal{H}$. *Let* $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{H}$ *be such that*

$$\mathrm{dist}_{\mathcal{H}}(S(t)\mathcal{B}_0, \mathcal{B}_1) \leq J_1 e^{-\vartheta_1 t},$$

$$\mathrm{dist}_{\mathcal{H}}(S(t)\mathcal{B}_1, \mathcal{B}_2) \leq J_2 e^{-\vartheta_2 t},$$

for some $\vartheta_1, \vartheta_2 > 0$ *and* $J_1, J_2 \geq 0$. *Assume also that, for all* $z_1, z_2 \in \mathcal{H}$, *there holds*

$$\|S(t)z_1 - S(t)z_2\|_{\mathcal{H}} \leq e^{\vartheta_0 t}\|z_1 - z_2\|_{\mathcal{H}},$$

for some $\vartheta_0 \geq 0$. *Then it follows that*

$$\mathrm{dist}_{\mathcal{H}}(S(t)\mathcal{B}_0, \mathcal{B}_2) \leq (J_1 + J_2)e^{-\vartheta t},$$

where $\vartheta = \vartheta_1\vartheta_2/(\vartheta_0 + \vartheta_1 + \vartheta_2)$.

3.5. A word of warning. Throughout the paper, we will denote by $c \geq 0$ a generic constant. All the quantities appearing in the sequel, and c in particular, are understood to be *independent* of $\varepsilon \in [0,1]$. Further dependencies of c will be specified on occurrence. Besides, we will diffusely make use (without explicit mention) of the Young, the Hölder and the Poincaré inequalities, as well as of the Sobolev embeddings.

4. THE REACTION-DIFFUSION EQUATION WITH MEMORY

Let $\omega \in [0,1)$ be fixed. For $\varepsilon \in (0,1]$, we consider the family of Equations (1.4), along with the limiting equation (1.1). In view of the preceding discussion, we translate (1.4) into the system (2.3)–(2.4) with the boundary condition (2.5). We also take the Dirichlet boundary condition for u, i.e.,

$$u(t) = 0, \quad \text{on } \partial\Omega.$$

The equality above holds for all $t \in \mathbb{R}$ when $\varepsilon > 0$, and for $t \geq 0$ when $\varepsilon = 0$. Concerning the memory kernel μ, we assume hereafter (3.1)–(3.2). Notice that, due to the normalization condition (1.3) and the position (2.2), we have

$$\int_0^\infty s\mu_\varepsilon(s)\,ds = 1 - \omega, \quad \forall\, \varepsilon \in (0,1]. \tag{4.1}$$

Indeed, (4.1) is obtained integrating by parts and observing that k (being summable and decreasing) satisfies

$$\lim_{s\to\infty} sk(s) = 0.$$

Conditions on φ and f. For both the Coleman-Gurtin and the Gurtin-Pipkin models, let $\varphi \in C^1(\mathbb{R})$, with

$$\varphi(0) = 0,$$

be such that

$$\liminf_{|x|\to\infty} \varphi'(x) > -\omega\lambda_1, \tag{4.2}$$

where λ_1 is the first eigenvalue of A. In particular, φ' is bounded below, i.e.,

$$\inf_{x\in\mathbb{R}} \varphi'(x) \geq -\ell, \quad \forall\, x \in \mathbb{R}, \tag{4.3}$$

for some $\ell \geq 0$. Moreover,

$$|\varphi'(x)| \leq c(1 + |x|^\gamma), \quad \forall\, x \in \mathbb{R},\ \gamma \leq 4. \tag{4.4}$$

Notice that the physically significant case of the derivative of the double-well potential, namely, $\varphi(x) = x^3 - x$, is an allowed nonlinearity. Finally, we assume

$$f \in H^0 \quad \text{independent of time.}$$

Remark 4.1. In view of the results that we have in mind, other conditions are possible for φ. For instance, one could replace (4.2) and (4.4) with

$$a_1|x|^p - c \leq x\varphi(x) \leq a_2|x|^p + c, \quad \forall\, x \in \mathbb{R},$$

for some $p \geq 2$ and some $a_2 \geq a_1 \geq 0$.

We have now all the ingredients to introduce the following problems, depending on $\varepsilon \in [0, 1]$.

***Problem* $\mathbf{P}_\varepsilon$ ($\varepsilon > 0$).** *Let $\omega \in [0, 1)$. Given $(u_0, \eta_0) \in \mathcal{H}^0_\varepsilon$, find $(u, \eta) \in C([0,\infty), \mathcal{H}^0_\varepsilon)$ solution to*

$$\begin{cases} u_t + \omega A u + \displaystyle\int_0^\infty \mu_\varepsilon(s) A\eta(s)\, ds + \varphi(u) = f, \\ \eta_t = T_\varepsilon \eta + u, \end{cases}$$

for $t > 0$, satisfying the initial conditions $u(0) = u_0$ and $\eta^0 = \eta_0$.

***Problem* $\mathbf{P}_0$.** *Given $u_0 \in \mathcal{H}^0_0$, find $u \in C([0,\infty), \mathcal{H}^0_0)$ solution to*

$$u_t + Au + \varphi(u) = f,$$

for $t > 0$, satisfying the initial condition $u(0) = u_0$.

The Problems $\mathbf{P}_\varepsilon$ (for $\varepsilon \geq 0$) reformulate (1.1) and (2.3)–(2.5) in the correct functional setting. The particular choice of the phase-spaces $\mathcal{H}^0_\varepsilon$ accounts for the Dirichlet boundary condition (cf. [10, 11, 20]). Clearly, different boundary conditions could be considered in the same fashion, such as Neumann's, upon properly redefining H^r.

Existence and uniqueness of solutions is ensured by the following theorem.

Theorem 4.2. *For every $\varepsilon \geq 0$, Problem $\mathbf{P}_\varepsilon$ defines a strongly continuous semigroup (or dynamical system) $S_\varepsilon(t)$ on the phase-space $\mathcal{H}^0_\varepsilon$.*

Proof. We prove the result for $\varepsilon > 0$, the other case being well known. Setting

$$\varphi_0(x) = \varphi(x) + \ell x, \quad x \in \mathbb{R},$$

so that, by (4.3), $\varphi_0' \geq 0$, we rewrite our system as the ordinary differential equation on $\mathcal{H}^0_\varepsilon$

$$\frac{d}{dt}(u, \eta) + \mathbb{A}(u, \eta) = (\ell u + f, 0),$$

where $\mathbb{A}$ is the (nonlinear) operator on $\mathcal{H}^0_\varepsilon$ with domain

$$\mathcal{D}(\mathbb{A}) = \Big\{(u, \eta) \in \mathcal{H}^0_\varepsilon \mid u \in H^1,\ \eta \in \mathcal{D}(T_\varepsilon), \\ \omega u + \int_0^\infty \mu_\varepsilon(s)\eta(s)ds + A^{-1}\varphi_0(u) \in H^2\Big\},$$

defined by

$$\mathbb{A}(u, \eta) = \Big(A\Big[\omega u + \int_0^\infty \mu_\varepsilon(s)\eta(s)\, ds + A^{-1}\varphi_0(u)\Big], -T_\varepsilon\eta - u\Big).$$

Appealing to maximal monotone operator theory (see [1, Théorème 3.4]), and subsequently applying a standard fixed point argument to deal with the term ℓu, we get the thesis if we show that $\mathbb{A}$ is maximal monotone. According to [1, Proposition 2.2], this is true provided that

$$\langle \mathbb{A}z_1 - \mathbb{A}z_2, z_1 - z_2\rangle_{\mathcal{H}^0_\varepsilon} \geq 0, \quad \forall\, z_1, z_2 \in \mathcal{D}(\mathbb{A}),$$

and range$(\mathbb{I} + \mathbb{A}) = \mathcal{H}^0_\varepsilon$, where $\mathbb{I}$ is the identity map on $\mathcal{H}^0_\varepsilon$. The first condition follows directly from (3.4). Concerning the second, selecting $(\hat{u}, \hat{\eta}) \in \mathcal{H}^0_\varepsilon$, we have to solve in $\mathcal{D}(\mathbb{A})$ the elliptic problem

$$u + A\Big[\omega u + \int_0^\infty \mu_\varepsilon(s)\eta(s)\, ds + A^{-1}\varphi_0(u)\Big] = \hat{u}, \tag{4.5}$$

$$\eta - T_\varepsilon\eta - u = \hat{\eta}. \tag{4.6}$$

Integration of (4.6) entails

$$\eta(s) = (1 - e^{-s})u + \int_0^s e^{\sigma - s}\hat{\eta}(\sigma)\,d\sigma. \tag{4.7}$$

Thus, Equation (4.5) turns into

$$u + dAu + \varphi_0(u) = w, \tag{4.8}$$

where

$$w = \hat{u} - \int_0^\infty \mu_\varepsilon(s)\int_0^s [e^{\sigma - s}A\hat{\eta}(\sigma)\,d\sigma]\,ds,$$
$$d = \omega + \int_0^\infty \mu_\varepsilon(s)(1 - e^{-s})\,ds > 0.$$

Notice now that $w \in H^{-1}$. Indeed, by (3.2),

$$\begin{aligned}\int_0^\infty \mu_\varepsilon(s)\int_0^s \Big[e^{\sigma - s}\|A^{1/2}\hat{\eta}(\sigma)\|\,d\sigma\Big]\,ds \\ &= \int_0^\infty e^{\sigma}\|A^{1/2}\hat{\eta}(\sigma)\|\Big[\int_\sigma^\infty \mu_\varepsilon(s)e^{-s}\,ds\Big]\,d\sigma \\ &\le \int_0^\infty \mu_\varepsilon(\sigma)\|A^{1/2}\hat{\eta}(\sigma)\|\,d\sigma \le \Big(\int_0^\infty \mu_\varepsilon(\sigma)\,d\sigma\Big)^{1/2}\|\hat{\eta}\|_{\mathcal{M}_\varepsilon^0}.\end{aligned}$$

Thus, by standard arguments, Equation (4.8) has a (unique) solution $u \in H^1$. From (4.7) it is then easy to see that $\eta \in \mathcal{M}_\varepsilon^0$ and $\eta(0) = 0$; besides, from (4.6) we read that $T_\varepsilon\eta \in \mathcal{M}_\varepsilon^0$, and so $\eta \in \mathcal{D}(T_\varepsilon)$. Finally, by comparison in (4.5),

$$\omega u + \int_0^\infty \mu_\varepsilon(s)\eta(s)\,ds + A^{-1}\varphi_0(u) \in H^2.$$

Hence $(u, \eta) \in \mathcal{D}(\mathbb{A})$.

Incidentally, observe that the result holds the same replacing (3.2) with the milder condition $\mu'(s) \le 0$. ◻

Remark 4.3. It is important to point out that with this method we obtain weak solutions in the sense of maximal monotone operator theory (cf. [1, Definition 3.1]). In fact, it is not even necessary to restrict to the case $\gamma \le 4$ (one has just to suitably modify the domain of $\mathbb{A}$). It is understood that, whenever we perform multiplications, we are assuming to deal with a sequence of strong solutions approximating the weak solution.

Remark 4.4. On account of the hypotheses on φ, when $\omega > 0$ we also have an integral control on the gradient of the first component of the solution, that is, $u \in L^2(0, T; H^1)$ for every $T > 0$.

For further use, we detail the continuous dependence estimate (cf. [11]).

Theorem 4.5. *There exists $\kappa_0 > 0$ such that,*

$$\|S_\varepsilon(t)z_1 - S_\varepsilon(t)z_2\|_{\mathcal{H}^0_\varepsilon} + \omega\|\mathbb{P}S_\varepsilon(t)z_1 - \mathbb{P}S_\varepsilon(t)z_2\|_{L^2(0,t;H^1)} \leq e^{\kappa_0 t}\|z_1 - z_2\|_{\mathcal{H}^0_\varepsilon}, \tag{4.9}$$

for every $t \geq 0$ and every $z_1, z_2 \in \mathcal{H}^0_\varepsilon$.

5. The Singular Limit on Finite Time-Intervals

In this section, we provide a precise quantitative estimate of the closeness of the semigroups $S_\varepsilon(t)$ and $S_0(t)$, as ε tends to zero, on finite time-intervals. For both the Coleman-Gurtin and the Gurtin-Pipkin models, the core of our theory can be summarized as follows. If we take initial data $z = (u_0, \eta_0)$ in a bounded subset of $\mathcal{H}^1_\varepsilon$, then the first component of the solution $S_\varepsilon(t)z$ tends to $S_0(t)u_0$ in the H^0-norm on every time-interval $[0, T]$, whereas the second component goes to zero in the history-space $\mathcal{M}^0_\varepsilon$ on every time-interval $[\tau, T]$, with $\tau > 0$. Nonetheless, as it is to be expected, the results are stronger when $\omega > 0$. Therefore, we will analyze the two models separately.

5.1. The Coleman-Gurtin case. Let $\omega \in (0, 1)$ be fixed. Then we have the following result.

Theorem 5.1. *For every $R \geq 0$ there exist $K_R \geq 0$ such that, for any $z = (u_0, \eta_0) \in B_{\mathcal{H}^1_\varepsilon}(R)$ and every $t \geq 0$, there hold*

$$\|\mathbb{P}S_\varepsilon(t)z - S_0(t)\mathbb{P}z\|_{H^0} \leq K_R h(t)\sqrt[8]{\varepsilon}, \tag{5.1}$$

$$\|\mathbb{P}S_\varepsilon(t)z - S_0(t)\mathbb{P}z\|_{L^2(0,t;H^1)} \leq K_R h(t)\sqrt[8]{\varepsilon}, \tag{5.2}$$

$$\|\mathbb{Q}_\varepsilon S_\varepsilon(t)z\|_{\mathcal{M}^0_\varepsilon} \leq \|\eta_0\|_{\mathcal{M}^0_\varepsilon} e^{-\delta t/(4\varepsilon)} + K_R\sqrt{\varepsilon}, \tag{5.3}$$

where

$$h(t) = (1 + t)^{3/4} e^{\ell t},$$

with ℓ given by (4.3).

Theorem 5.2. *If in addition u_0 belongs to a bounded subset of H^2, then the term $\sqrt[8]{\varepsilon}$ above can be replaced by $\sqrt[4]{\varepsilon}$ times a constant depending on the H^2-bound of u_0.*

Remark 5.3. Collecting (5.1) and (5.3), we obtain the estimate

$$\|S_\varepsilon(t)z - \mathbb{L}_\varepsilon S_0(t)\mathbb{P}z\|_{\mathcal{H}^0_\varepsilon} \leq \|\eta_0\|_{\mathcal{M}^0_\varepsilon} e^{-\delta t/(4\varepsilon)} + K_R h(t)\sqrt[8]{\varepsilon}. \tag{5.4}$$

Let us make a few comments on these results. As shown in (5.4), the convergence of the solution $S_\varepsilon(t)z$ to $\mathbb{L}_\varepsilon S_0(t)\mathbb{P}z$ in $\mathcal{H}^0_\varepsilon$ occurs only on time-intervals of the form $[\tau, T]$, with $T > \tau > 0$. This is naturally due to the presence of the initial history η_0. It is then apparent that, if we are given initial data of the form $(u_0, 0)$, the limiting process can be controlled on the *whole* time-interval $[0, T]$. We stress that the convergence of the solutions as ε tends to zero is *uniform* with respect to initial data belonging to bounded subset of $\mathcal{H}^1_\varepsilon$.

The proofs of the theorems require several steps. Along this subsection, the generic constant $c \geq 0$ may depend on R. We need to anticipate a result from the subsequent Section 6 (cf. (6.4)–(6.5) and Remark 6.9). Namely, for all $\varepsilon \in [0, 1]$,

$$\sup_{\|z\|_{\mathcal{H}^1_\varepsilon} \leq R} \left[\|S_\varepsilon(t)z\|_{\mathcal{H}^1_\varepsilon} + \frac{1}{1+t} \int_0^t \|A\mathbb{P}S_\varepsilon(y)z\|^2 dy \right] \leq c, \qquad \forall\, t \geq 0. \tag{5.5}$$

Let then $\varepsilon \in (0, 1]$ be fixed. Given $z = (u_0, \eta_0) \in B_{\mathcal{H}^1_\varepsilon}(R)$, we denote

$$(\hat{u}(t), \hat{\eta}^t) = S_\varepsilon(t)z \quad \text{and} \quad u(t) = S_0(t)u_0.$$

The main point is now to reconstruct the "missing" component η^t corresponding to $u(t)$, in order to perform an appropriate comparison with $S_\varepsilon(t)z$. Hence, let η^t be the solution at time t of the Cauchy problem in $\mathcal{M}^0_\varepsilon$

$$\begin{cases} \eta_t = T_\varepsilon \eta + u, & t > 0, \\ \eta^0 = \eta_0. \end{cases}$$

We first estimate the norms of $\hat{\eta}^t$ and η^t in $\mathcal{M}^0_\varepsilon$ in terms of ε. Incidentally, this will entail the relation (5.3).

Lemma 5.4. *There holds*

$$\max\{\|\hat{\eta}^t\|^2_{\mathcal{M}^0_\varepsilon}, \|\eta^t\|^2_{\mathcal{M}^0_\varepsilon}\} \leq \|\eta_0\|^2_{\mathcal{M}^0_\varepsilon} e^{-\delta t/(2\varepsilon)} + c\varepsilon, \qquad \forall\, t \geq 0.$$

Proof. We write the proof for η (the other one being the same). So, multiplying the equation for η times η in $\mathcal{M}^0_\varepsilon$, and exploiting (3.4) and (5.5), we get

$$\begin{aligned} \frac{d}{dt}\|\eta\|^2_{\mathcal{M}^0_\varepsilon} + \frac{\delta}{\varepsilon}\|\eta\|^2_{\mathcal{M}^0_\varepsilon} &\leq c \int_0^\infty \mu_\varepsilon(s)\|A^{1/2}\eta(s)\|\, ds \\ &\leq c\left(\int_0^\infty \mu_\varepsilon(s)\, ds\right)^{1/2} \|\eta\|_{\mathcal{M}^0_\varepsilon} \\ &= \frac{c}{\sqrt{\varepsilon}}\|\eta\|_{\mathcal{M}^0_\varepsilon} \leq \frac{\delta}{2\varepsilon}\|\eta\|^2_{\mathcal{M}^0_\varepsilon} + c. \end{aligned}$$

The assertion then follows from the Gronwall Lemma. ◻

The next step is the control of the difference between $\hat{u}(t)$ and $u(t)$. Preliminarily notice that

$$\int_{\sqrt{\varepsilon}}^{\infty} s\mu_\varepsilon(s)\,ds \le c\varepsilon, \qquad \forall\,\varepsilon > 0. \tag{5.6}$$

This easily follows from (3.3) (actually, this estimate is rather gross, but enough for our scopes). Consequently, we deduce

$$\int_{\sqrt{\varepsilon}}^{\infty} \mu_\varepsilon(s)\,ds \le c\sqrt{\varepsilon}, \qquad \forall\,\varepsilon > 0. \tag{5.7}$$

Lemma 5.5. *For every $t \ge 0$, there holds*

$$\|\hat{u}(t) - u(t)\|^2 + \int_0^t \|A^{1/2}\hat{u}(y) - A^{1/2}u(y)\|^2\,dy \le c(1+t)^{3/2} e^{2\ell t}\sqrt[4]{\varepsilon}.$$

Proof. Set

$$\bar{u}(t) = \hat{u}(t) - u(t) \quad \text{and} \quad \bar{\eta}^t = \hat{\eta}^t - \eta^t.$$

Then we have the system

$$\begin{cases} \bar{u}_t + \omega A\bar{u} + \int_0^\infty \mu_\varepsilon(s)A\bar{\eta}(s)\,ds - (1-\omega)Au + \varphi(\hat{u}) - \varphi(u) = 0, \\ \bar{\eta}_t = T_\varepsilon\bar{\eta} + \bar{u}, \\ (\bar{u}(0), \bar{\eta}^0) = (0,0). \end{cases}$$

We multiply the first equation by $\bar{u}$ in H^0, and the second by $\bar{\eta}$ in $\mathcal{M}_\varepsilon^0$. Taking (3.4) into account and adding the results, we end up with

$$\begin{aligned} \frac{d}{dt}(\|\bar{u}\|^2 + \|\bar{\eta}\|^2_{\mathcal{M}_\varepsilon^0}) + 2\omega\|A^{1/2}\bar{u}\|^2 &\le -2\langle\varphi(\hat{u}) - \varphi(u), \bar{u}\rangle \\ &\quad - 2\int_0^\infty \mu_\varepsilon(s)\langle A^{1/2}\eta(s), A^{1/2}\bar{u}\rangle\,ds + 2(1-\omega)\langle A^{1/2}u, A^{1/2}\bar{u}\rangle. \end{aligned}$$

By virtue of (4.3), we readily get

$$-2\langle\varphi(\hat{u}) - \varphi(u), \bar{u}\rangle \le 2\ell\|\bar{u}\|^2.$$

Exploiting the normalization property (4.1) and the representation formula (3.5) for η, we have the equality

$$-2\int_0^\infty \mu_\varepsilon(s)\langle A^{1/2}\eta(s), A^{1/2}\bar{u}\rangle\,ds + 2(1-\omega)\langle A^{1/2}u, A^{1/2}\bar{u}\rangle = 2\sum_{j=1}^5 I_j,$$

where we set

$$I_1(t) = \int_{\sqrt{\varepsilon}}^{\infty} s\mu_\varepsilon(s)\langle A^{1/2}u(t), A^{1/2}\bar{u}(t)\rangle\, ds,$$
$$I_2(t) = -\int_{\sqrt{\varepsilon}}^{\infty} \mu_\varepsilon(s)\langle A^{1/2}\eta^t(s), A^{1/2}\bar{u}(t)\rangle\, ds,$$
$$I_3(t) = -\int_{\min\{\sqrt{\varepsilon},t\}}^{\sqrt{\varepsilon}} \mu_\varepsilon(s)\langle A^{1/2}\eta_0(s-t), A^{1/2}\bar{u}(t)\rangle\, ds,$$
$$I_4(t) = \int_{\min\{\sqrt{\varepsilon},t\}}^{\sqrt{\varepsilon}} (s-t)\mu_\varepsilon(s)\langle A^{1/2}u(t), A^{1/2}\bar{u}(t)\rangle\, ds,$$
$$I_5(t) = \int_0^{\sqrt{\varepsilon}} \mu_\varepsilon(s)\left[\int_0^{\min\{s,t\}} \langle A^{1/2}u(t) - A^{1/2}u(t-y), A^{1/2}\bar{u}(t)\rangle\, dy\right] ds.$$

Hence, the differential inequality above turns into

$$\frac{d}{dt}(\|\bar{u}\|^2 + \|\bar{\eta}\|^2_{\mathcal{M}^0_\varepsilon}) + 2\omega\|A^{1/2}\bar{u}\|^2 \leq 2\ell\|\bar{u}\|^2 + 2\sum_{j=1}^{5} I_j.$$

We now have to estimate the terms I_j.

- From (5.5) and (5.6),
$$I_1(t) \leq c\varepsilon.$$

- From (5.5), (5.7) and Lemma 5.4,
$$\begin{aligned} I_2(t) &\leq c\int_{\sqrt{\varepsilon}}^{\infty} \mu_\varepsilon(s)\|A^{1/2}\eta^t(s)\|\, ds \\ &\leq c\int_{\sqrt{\varepsilon}}^{\infty} \mu_\varepsilon(s)\|A^{1/2}\eta^t(s)\|^2\, ds + c\int_{\sqrt{\varepsilon}}^{\infty} \mu_\varepsilon(s)\, ds \\ &\leq c\|\eta^t\|^2_{\mathcal{M}^0_\varepsilon} + c\sqrt{\varepsilon} \leq ce^{-\delta t/(2\varepsilon)} + c\sqrt{\varepsilon}. \end{aligned}$$

- We assume $t < \sqrt{\varepsilon}$, otherwise $I_3(t) = 0$. From (3.3) and (5.5),
$$\begin{aligned} I_3(t) &\leq c\int_t^{\sqrt{\varepsilon}} \mu_\varepsilon(s)\|A^{1/2}\eta_0(s-t)\|\, ds \\ &\leq ce^{-\delta t/\varepsilon}\int_0^{\infty} \mu_\varepsilon(s)\|A^{1/2}\eta_0(s)\|\, ds \\ &\leq ce^{-\delta t/\varepsilon}\left(\int_0^{\infty} \mu_\varepsilon(s)\, ds\right)^{1/2}\|\eta_0\|_{\mathcal{M}^0_\varepsilon} \leq \frac{c}{\sqrt{\varepsilon}}e^{-\delta t/\varepsilon}, \end{aligned}$$

  which then holds for all $t \geq 0$.

- Reasoning as in the previous case, and using (4.1), we have

$$I_4(t) \le ce^{-\delta t/\varepsilon} \int_0^\infty s\mu_\varepsilon(s)\, ds = ce^{-\delta t/\varepsilon}.$$

- From the Agmon inequality,

$$\|\varphi(u)\|^2 \le c(1 + ||u||_{L^6}^6 \, ||u||_{L^\infty}^4) \le c(1 + \|A^{1/2}u\|^8 \, \|Au\|^2) \le c(1 + \|Au\|^2).$$

Hence, on account of (5.5) for $\varepsilon = 0$, it is apparent from the equation that

$$\int_0^t ||u_t(w)||^2 \, dw \le c(1+t),$$

which implies, for $y \in [0, t]$,

$$\|u(t) - u(t-y)\| \le \int_{t-y}^t \|u_t(w)\|\, dw \le c(1+t)^{1/2}\sqrt{y}.$$

Therefore, due to (4.1),

$$\begin{aligned}
I_5(t) &= \int_0^{\sqrt{\varepsilon}} \mu_\varepsilon(s)\Big[\int_0^{\min\{s,t\}} \langle u(t) - u(t-y), A\bar{u}(t)\rangle\, dy\Big]\, ds \\
&\le \|A\bar{u}(t)\| \int_0^{\sqrt{\varepsilon}} \mu_\varepsilon(s)\Big[\int_0^{\min\{s,t\}} \|u(t) - u(t-y)\|\, dy\Big]\, ds \\
&\le c(1+t)^{1/2}\sqrt[4]{\varepsilon}\, \|A\bar{u}(t)\| \int_0^{\sqrt{\varepsilon}} s\mu_\varepsilon(s)\, ds \\
&\le c(1+t)^{1/2}\sqrt[4]{\varepsilon}\, \|A\bar{u}(t)\|.
\end{aligned}$$

Collecting all the estimates above, we finally obtain

$$\frac{d}{dt}(\|\bar{u}\|^2 + ||\bar{\eta}||^2_{\mathcal{M}_\varepsilon^0}) + 2\omega\|A^{1/2}\bar{u}\|^2 \le 2\ell\|\bar{u}\|^2 + g_1 + g_2,$$

where we put

$$\begin{aligned}
g_1(t) &= c\sqrt{\varepsilon} + \frac{c}{\sqrt{\varepsilon}} e^{-\delta t/(2\varepsilon)}, \\
g_2(t) &= c(1+t)^{1/2}\sqrt[4]{\varepsilon}\, \|A\bar{u}(t)\|.
\end{aligned}$$

Observe that

$$\int_0^t g_1(y)\, dy \le c(1+t)\sqrt{\varepsilon}, \quad \forall\, t \ge 0.$$

Besides, from (5.5),

$$\int_0^t g_2(y)\,dy \le c(1+t)^{3/2}\sqrt[4]{\varepsilon}, \qquad \forall\, t \ge 0.$$

Hence, recalling that $(\bar{u}(0), \bar{\eta}^0) = (0,0)$, the Gronwall Lemma and a subsequent integration on $(0,t)$ yield the thesis. □

Conclusion of the proofs of Theorem 5.1 and Theorem 5.2. As we saw, inequality (5.3) follows from Lemma 5.4, whereas by Lemma 5.5 we find the estimates (5.1)–(5.2). If in addition $u_0 \in B_{H^2}(R')$, for some $R' \ge 0$, then it is well known that $u(t)$ is uniformly bounded in H^2. By comparison, this furnishes a uniform bound of u_t in H^0 (with a bound depending on R'). Hence, when estimating $I_5(t)$, one has

$$\|u(t) - u(t-y)\| \le \int_{t-y}^t \|u_t(w)\|\,dw \le cy,$$

for some $c \ge 0$ depending on R'. Thus, completing the argument, one obtains $\sqrt[4]{\varepsilon}$ times a suitable constant depending on R' in place of $\sqrt[8]{\varepsilon}$. □

5.2. The Gurtin-Pipkin case. We now examine the case $\omega = 0$. Here, we have to take $f \in H^1$. As in the previous case, we will establish two results, depending on the regularity of initial data.

Theorem 5.6. *For every $R \ge 0$, $T > 0$, and $z = (u_0, \eta_0) \in B_{\mathcal{H}^1_\varepsilon}(R)$, there exists $K_{R,T} \ge 0$ such that, for every $t \in [0,T]$, there hold*

$$\|\mathbb{P}S_\varepsilon(t)z - S_0(t)\mathbb{P}z\|_{H^0} \le K_{R,T}\sqrt[16]{\varepsilon},$$

and

$$\|\mathbb{Q}_\varepsilon S_\varepsilon(t)z\|_{\mathcal{M}^0_\varepsilon} \le \|\eta_0\|_{\mathcal{M}^0_\varepsilon} e^{-\delta t/(4\varepsilon)} + K_{R,T}\sqrt{\varepsilon}.$$

With more regular initial data, we can improve the first estimate.

Theorem 5.7. *If in addition u_0 belongs to a bounded subset of H^2, then the term $\sqrt[16]{\varepsilon}$ above can be replaced by $\sqrt[8]{\varepsilon}$ times a constant depending on the H^2-bound of u_0.*

Proof. Both results are obtained repeating with minor modifications the arguments of the former case, except that now we make the restriction $t \in [0,T]$. In the sequel, the generic constant $c \ge 0$ may depend on R and T.

Arguing as in the proof of the subsequent Lemma 6.8 (now for $\omega = 0$), the reader will have no difficulties to see that, when $f \in H^1$,

$$\|S_\varepsilon(t)z\|_{\mathcal{H}^1_\varepsilon} \le c, \qquad \forall\, t \in [0,T].$$

As before, we introduce (u,η), $(\hat{u},\hat{\eta})$ and $(\bar{u},\bar{\eta})$. The analogue of Lemma 5.4 now reads

$$\max\{\|\hat{\eta}^t\|^2_{\mathcal{M}^0_\varepsilon}, \|\eta^t\|^2_{\mathcal{M}^0_\varepsilon}\} \le \|\eta_0\|^2_{\mathcal{M}^0_\varepsilon} e^{-\delta t/(2\varepsilon)} + c\varepsilon, \qquad \forall\, t \in [0,T]. \tag{5.8}$$

Concerning Lemma 5.5, the only difference here is the treatment of $I_5(t)$ (keeping in mind that now c depends also on T). For $y \in [0,t]$, with $t \in [0,T]$, we have

$$\begin{aligned}
&\|A^{1/2}u(t) - A^{1/2}u(t-y)\| \\
&\qquad \le \|u(t) - u(t-y)\|^{1/2}\, \|Au(t) - Au(t-y)\|^{1/2} \\
&\qquad \le c\sqrt[4]{y}\,[h(t) + h(t-y)],
\end{aligned}$$

where we put $h(t) = \|Au(t)\|^{1/2}\chi_{[0,\infty)}(t)$. Hence, for every $t \in [0,T]$,

$$\begin{aligned}
I_5(t) &\le 2\|A^{1/2}\bar{u}(t)\| \int_0^{\sqrt{\varepsilon}} \mu_\varepsilon(s)\Big[\int_0^{\min\{s,t\}} \|A^{1/2}u(t) - A^{1/2}u(t-y)\|\,dy\Big]\,ds \\
&\le \sqrt[8]{\varepsilon}\, g_3(t),
\end{aligned}$$

having set

$$g_3(t) = c\int_0^{\sqrt{\varepsilon}} \mu_\varepsilon(s)\Big[\int_0^s [h(t) + h(t-y)]\,dy\Big]\,ds.$$

By (5.5) (that still holds for $\varepsilon = 0$), we learn that

$$\int_0^T g_3(t)\,dt = c\int_0^{\sqrt{\varepsilon}} \mu_\varepsilon(s)\Big[\int_0^s\int_0^T [h(t) + h(t-y)]\,dt\,dy\Big]\,ds \le c.$$

So we end up with

$$\frac{d}{dt}(\|\bar{u}\|^2 + \|\bar{\eta}\|^2_{\mathcal{M}^0_\varepsilon}) \le 2\ell\|\bar{u}\|^2 + g_1 + \sqrt[8]{\varepsilon}\, g_3,$$

with g_1 as before. The Gronwall Lemma on $[0,T]$ then furnishes

$$\|\bar{u}(t)\|^2 \le c\sqrt[8]{\varepsilon}, \qquad \forall\, t \in [0,T]. \tag{5.9}$$

Collecting (5.8)–(5.9) we have proved Theorem 5.6. The proof of Theorem 5.7 follows as in the Coleman-Gurtin case. ◻

6. DISSIPATIVITY

Throughout the rest of the paper, we will focus on the asymptotic properties of the semigroup $S_\varepsilon(t)$ in the Coleman-Gurtin case $\omega > 0$. The case $\omega = 0$, much more critical in order to develop a global asymptotic analysis (due to the lack of regularizing effects), will possibly be the object of future investigations. There is no loss of generality to assume hereafter $\omega = \frac{1}{2}$.

In the sequel, we will state results valid for all $\varepsilon \in [0,1]$. However, we will limit ourselves to provide the proofs for $\varepsilon > 0$. The corresponding proofs for $\varepsilon = 0$ are actually easier, and can be immediately recovered just putting $\mu_0 \equiv 0$. Then, in the first equation the full term Au appears, whereas the second equation is vacuously true.

The dissipative character of the system is witnessed by the following result.

Theorem 6.1. *There exists $R_0 > 0$ such that the set $\mathcal{B}_\varepsilon^0 = B_{\mathcal{H}_\varepsilon^0}(R_0)$ is an absorbing set for $S_\varepsilon(t)$ on $\mathcal{H}_\varepsilon^0$, uniformly in ε. Namely, given any bounded set $\mathcal{B} \subset \mathcal{H}_\varepsilon^0$, and setting $R = \sup_{z\in\mathcal{B}} \|z\|_{\mathcal{H}_\varepsilon^0}$, there exists $t_0 = t_0(R) \geq 0$ such that*

$$S_\varepsilon(t)\mathcal{B} \subset \mathcal{B}_\varepsilon^0, \quad \forall\, t \geq t_0.$$

Remark 6.2. The uniformity with respect to ε means that neither the radius R_0 of the ball $\mathcal{B}_\varepsilon^0$ nor the entering time t_0 depend on ε.

The theorem is an immediate consequence of the following lemma.

Lemma 6.3. *There exist $\nu_0 > 0$ and $C_0 \geq 0$ such that*

$$\|S_\varepsilon(t)z\|_{\mathcal{H}_\varepsilon^0} \leq \|z\|_{\mathcal{H}_\varepsilon^0} e^{-\nu_0 t} + C_0, \quad \forall\, t \geq 0, \tag{6.1}$$

for any $z \in \mathcal{H}_\varepsilon^0$.

Proof. Multiply the first and the second equation of Problem $\mathbf{P}_\varepsilon$ by u and η, in the respective spaces. Taking (3.4) into account, and adding the results, yields

$$\frac{d}{dt}(\|u\|^2 + ||\eta||^2_{\mathcal{M}_\varepsilon^0}) + \|A^{1/2}u\|^2 + \frac{\delta}{\varepsilon}||\eta||^2_{\mathcal{M}_\varepsilon^0} + 2\langle\varphi(u), u\rangle \leq 2\langle f, u\rangle.$$

By (4.2), it is easily seen that

$$2\langle\varphi(u), u\rangle \geq -(1-3\nu)\|A^{1/2}u\|^2 - c,$$

for some $\nu > 0$ (possibly very small). Moreover, as $f \in H^0$ is constant in time,

$$2\langle f, u\rangle \leq \nu\|A^{1/2}u\|^2 + c.$$

Hence, for some $\nu_0 > 0$ small enough, we find the inequality

$$\frac{d}{dt}(\|u\|^2 + ||\eta||^2_{\mathcal{M}_\varepsilon^0}) + 2\nu_0(\|u\|^2 + ||\eta||^2_{\mathcal{M}_\varepsilon^0}) + \nu\|A^{1/2}u\|^2 \leq c, \tag{6.2}$$

and the Gronwall Lemma entails the desired conclusion. ◻

In light of the lemma above, we can choose R_0 to be any number strictly greater than C_0 to fulfill the thesis of Theorem 6.1. Accordingly,

$$t_0 = \frac{1}{\nu_0} \ln_+ \left(\frac{R}{R_0 - C_0} \right).$$

Remark 6.4. It is apparent from the proof that, if $f \equiv 0$ and $x\varphi(x) \geq 0$ for every $x \in \mathbb{R}$ (which occurs, for instance, if $\varphi' \geq 0$), then $S_\varepsilon(t)$ decays to zero exponentially fast, with a decay rate independent of ε. In this case, with reference to the next sections, the set $\{0\} \subset \mathcal{H}^0_\varepsilon$ is the (exponential) global attractor for $S_\varepsilon(t)$ on $\mathcal{H}^0_\varepsilon$.

Remark 6.5. Actually, Theorem 6.1 also holds for the Gurtin-Pipkin model.

Corollary 6.6. *For any $R \geq 0$ there exists $Q_0 = Q_0(R)$ such that*

$$\sup_{\|z\|_{\mathcal{H}^0_\varepsilon} \leq R} \int_t^{t+1} \|A^{1/2} \mathbb{P} S_\varepsilon(y) z\|^2 \, dy \leq Q_0, \qquad \forall\, t \geq 0. \tag{6.3}$$

Proof. Integrate (6.2) on $(t, t+1)$ and use (6.1). ◻

The next step is to show the existence of an absorbing set in $\mathcal{Z}^1_\varepsilon$.

Theorem 6.7. *There exists $R_1 > 0$ such that the set $\mathcal{B}^1_\varepsilon = B_{\mathcal{Z}^1_\varepsilon}(R_1)$ is an absorbing set for $S_\varepsilon(t)$ on $\mathcal{Z}^1_\varepsilon$, uniformly in ε.*

The proof of this theorem is based on the subsequent lemma.

Lemma 6.8. *Given $R \geq 0$ and $\rho \geq 0$, there exist $C_1 = C_1(\rho) > 0$ and a positive function Ψ_1 vanishing at infinity such that, if $\|z\|_{\mathcal{Z}^1_\varepsilon} \leq R$ and $\|z\|_{\mathcal{H}^0_\varepsilon} \leq \rho$, there holds*

$$\|S_\varepsilon(t) z\|_{\mathcal{Z}^1_\varepsilon} \leq R\Psi_1(t) + C_1, \qquad \forall\, t \geq 0. \tag{6.4}$$

Moreover,

$$\sup_{\|z\|_{\mathcal{Z}^1_\varepsilon} \leq R} \int_t^{t+1} \|A \mathbb{P} S_\varepsilon(y) z\|^2 \, dy \leq Q_1, \qquad \forall\, t \geq 0, \tag{6.5}$$

for some $Q_1 = Q_1(R)$.

Proof. In this proof, the generic constant $c \geq 0$ may depend on ρ, but not on R. Arguing like in the proof of Lemma 6.3, except that now the multiplications are carried out in H^1 and $\mathcal{M}^1_\varepsilon$, respectively, we come to the differential inequality

$$\frac{d}{dt}(\|A^{1/2}u\|^2 + ||\eta||^2_{\mathcal{M}^1_\varepsilon}) + \|Au\|^2 + \frac{\delta}{\varepsilon}||\eta||^2_{\mathcal{M}^1_\varepsilon} + 2\langle A^{1/2}\varphi(u), A^{1/2}u\rangle \leq 2\langle f, Au\rangle.$$

Observe that $2\langle f, Au\rangle \le \frac{1}{4}\|Au\|^2 + c$. Moreover, from (4.3), there holds

$$\begin{aligned}2\langle A^{1/2}\varphi(u), A^{1/2}u\rangle &= 2\langle \varphi'(u)\nabla u, \nabla u\rangle \\ &\ge -2\ell\|A^{1/2}u\|^2 \ge -\frac{1}{4}\|Au\|^2 - c,\end{aligned}$$

where the last passage follows by interpolation and by Lemma 6.3. Therefore, we can choose $\nu_1 > 0$ small enough such that

$$\frac{d}{dt}(\|A^{1/2}u\|^2 + \|\eta\|^2_{\mathcal{M}^1_\varepsilon}) + 2\nu_1(\|A^{1/2}u\|^2 + \|\eta\|^2_{\mathcal{M}^1_\varepsilon}) + \frac{1}{4}\|Au\|^2 \le c. \tag{6.6}$$

Applying the Gronwall Lemma first, and then integrating (6.6) on $(t, t+1)$, we prove the estimate

$$\|S_\varepsilon(t)z\|_{\mathcal{H}^1_\varepsilon} \le Re^{-\nu_1 t} + c, \quad \forall\, t \ge 0, \tag{6.7}$$

along with (6.5). Here we used the fact that $\|z\|_{\mathcal{H}^1_\varepsilon} \le \|z\|_{\mathcal{Z}^1_\varepsilon} \le R$.

We are left to show the required control on the remaining part of the term $\|S_\varepsilon(t)z\|_{\mathcal{Z}^1_\varepsilon}$. From (6.7), there is $\tau_1 = \tau_1(R)$ such that $\|A^{1/2}u(t)\| \le c$ for all $t \ge \tau_1$. Thus, by Lemma 3.3 and Lemma 3.4 we get

$$\varepsilon\|T_\varepsilon\eta^t\|^2_{\mathcal{M}^0_\varepsilon} + \sup_{x\ge 1} x\mathbb{T}^\varepsilon_{\eta^t}(x) \le \begin{cases} R^2(e^{-\delta t} + \Psi(t)) + c, & \forall\, t \ge \tau_1, \\ cR^2, & \forall\, t \in [0, \tau_1]. \end{cases} \tag{6.8}$$

Using (6.7)–(6.8), and keeping in mind Corollary 3.5, we easily recover (6.4). □

Remark 6.9. It is clear from the proof above that Lemma 6.8 holds the same replacing $\mathcal{Z}^1_\varepsilon$ with $\mathcal{H}^1_\varepsilon$.

In light of Lemma 6.3 and Lemma 6.8, it is apparent that, selecting $\rho > R_0$, any number strictly greater than $C_1(\rho)$ is an admissible choice for R_1 in order to fulfill the thesis of Theorem 6.7.

When $\varepsilon = 0$, a stronger result holds. Namely, $\mathcal{B}^1_0$ absorbs bounded subsets of $\mathcal{H}^0_0$. To see that, it is enough to apply Lemma 3.8 to (6.6), on account of the integral estimate (6.3). Unfortunately, this cannot occur when $\varepsilon > 0$, due to the hyperbolic character of the equation for η, that prevents any kind of regularization of the solution. Nonetheless, it is still true that $\mathcal{B}^1_\varepsilon$ is exponentially attracting in $\mathcal{H}^0_\varepsilon$.

Theorem 6.10. *There exist $\kappa_1 > 0$ and a positive increasing function Γ_1 such that, up to (possibly) enlarging the radius R_1,*

$$\mathrm{dist}_{\mathcal{H}^0_\varepsilon}(S_\varepsilon(t)\mathcal{B}, \mathcal{B}^1_\varepsilon) \le \Gamma_1(R)e^{-\kappa_1 t}, \quad \forall\, t \ge 0,$$

for every set $\mathcal{B} \subset B_{\mathcal{H}^0_\varepsilon}(R)$.

In view of Theorem 6.1 and estimate (6.1), it is enough to show that

$$\mathrm{dist}_{\mathcal{H}^0_\varepsilon}(S_\varepsilon(t)\mathcal{B}^0_\varepsilon, \mathcal{B}^1_\varepsilon) \leq R_0 e^{-\kappa_1 t}, \qquad \forall\, t \geq 0.$$

The proof of this fact is based on a suitable decomposition of the solution $S_\varepsilon(t)z$. Recalling (4.3), we set

$$\varphi_0(x) = \varphi(x) + \ell x, \qquad x \in \mathbb{R}.$$

Obviously, $\varphi_0'(x) \geq 0$ for every $x \in \mathbb{R}$. Then, for $z = (u_0, \eta_0)$, we write $S_\varepsilon(t)z$ as the sum

$$S_\varepsilon(t)z = L_\varepsilon(t)z + K_\varepsilon(t)z,$$

where $L_\varepsilon(t)z = (v(t), \xi^t)$ and $K_\varepsilon(t)z = (w(t), \zeta^t)$ solve the problems

$$\begin{cases} v_t + \frac{1}{2}Av + \int_0^\infty \mu_\varepsilon(s)A\xi(s)\,ds + \varphi_0(u) - \varphi_0(w) = 0, \\ \xi_t = T_\varepsilon \xi + v, \\ (v(0), \xi^0) = (u_0, \eta_0), \end{cases}$$

and

$$\begin{cases} w_t + \frac{1}{2}Aw + \int_0^\infty \mu_\varepsilon(s)A\zeta(s)\,ds + \varphi_0(w) - \ell u = f, \\ \zeta_t = T_\varepsilon \zeta + w, \\ (w(0), \zeta^0) = (0, 0). \end{cases}$$

Notice that, in general, $L_\varepsilon(t)$ and $K_\varepsilon(t)$ are not semigroups. Let us establish some properties of these maps.

Lemma 6.11. *There exists $\kappa_1 > 0$ such that*

$$\sup_{z \in \mathcal{B}^0_\varepsilon} \|L_\varepsilon(t)z\|_{\mathcal{H}^0_\varepsilon} \leq R_0 e^{-\kappa_1 t}, \qquad \forall\, t \geq 0.$$

Proof. Repeat, with the obvious changes, the proof of Lemma 6.3, noting that

$$\langle \varphi_0(u) - \varphi_0(w), v \rangle = \langle \varphi_0(u) - \varphi_0(w), u - w \rangle \geq 0,$$

since $\varphi_0' \geq 0$. $\square$

Lemma 6.12. *There holds*

$$\sup_{t \geq 0} \sup_{z \in \mathcal{B}^0_\varepsilon} \|K_\varepsilon(t)z\|_{\mathcal{Z}^1_\varepsilon} \leq c,$$

for some $c = c(R_0)$.

Proof. Mimicking the proof of Lemma 6.8, we get

$$\frac{d}{dt}(\|A^{1/2}w\|^2 + \|\zeta\|^2_{\mathcal{M}^1_\varepsilon}) + 2\nu_1(\|A^{1/2}w\|^2 + \|\zeta\|^2_{\mathcal{M}^1_\varepsilon}) + \frac{1}{4}\|Aw\|^2 \leq c + 2\ell\langle u, Aw\rangle.$$

Since $u(t)$ is uniformly bounded in H^0,

$$2\ell\langle u, Aw\rangle \leq \frac{1}{8}\|Aw\|^2 + c.$$

Thus, the Gronwall Lemma together with the condition $(w(0), \zeta^0) = (0, 0)$ bear the estimate

$$\sup_{t\geq 0}\sup_{z\in\mathcal{B}^0_\varepsilon}\|K_\varepsilon(t)z\|_{\mathcal{H}^1_\varepsilon} \leq c.$$

The thesis then follows applying Lemma 3.3 and Lemma 3.4 (with null initial data), and by Corollary 3.5. ◻

On account of Lemma 6.11 and Lemma 6.12, and redefining R_1 to be greater than the constant $c(R_0)$ above, we get at once the desired inequality

$$\mathrm{dist}_{\mathcal{H}^0_\varepsilon}(S_\varepsilon(t)\mathcal{B}^0_\varepsilon, \mathcal{B}^1_\varepsilon) \leq R_0 e^{-\kappa_1 t}, \qquad \forall\, t \geq 0.$$

In the sequel, we agree to redefine the radius R_1 so that Theorem 6.10 holds true.

Remark 6.13. Integrating the differential inequality in the proof of Lemma 6.12 on $(0, t)$, we also find the estimate

$$\sup_{t\geq 0}\int_t^{t+1}\|Aw(y)\|^2\,dy < \infty.$$

7. Global Attractors

After Theorem 6.10, we learn that $S_\varepsilon(t)$ is *asymptotically compact* on the phase-space $\mathcal{H}^0_\varepsilon$. In other words, there exists a compact attracting set (namely, $\mathcal{B}^1_\varepsilon$) for $S_\varepsilon(t)$. Thus, by means of well-known results of the theory of dynamical systems (see, e.g., [24]), there holds the following result.

Theorem 7.1. *For every $\varepsilon \in [0, 1]$, the strongly continuous semigroup $S_\varepsilon(t)$ acting on the phase-space $\mathcal{H}^0_\varepsilon$ possesses a connected global attractor $\mathcal{A}_\varepsilon$ which is bounded in $\mathcal{H}^1_\varepsilon$, uniformly with respect to ε.*

Recall that the global attractor is the (unique) compact set which is at the same time attracting (with respect to the Hausdorff semidistance) and fully invariant for the semigroup. Also, the attractor can be explicitly described as the section at time $t = 0$ of the set of all complete bounded trajectories of the system.

If the nonlinearity φ and the source term f are more regular, so is the attractor. This issue will be discussed in detail in Section 9. Besides, $S_\varepsilon(t)$ is injective on $\mathcal{A}_\varepsilon$.

Proposition 7.2. *The semigroup $S_\varepsilon(t)$ uniquely extends to a strongly continuous group of operators $\{\tilde{S}_\varepsilon(t)\}_{t\in\mathbb{R}}$ on $\mathcal{A}_\varepsilon$.*

Proof. The result follows from the invariance of $\mathcal{A}_\varepsilon$ and the backwards uniqueness property of $S_\varepsilon(t)$ on $\mathcal{A}_\varepsilon$. We prove this last property for the case $\varepsilon > 0$ (cf. [9], where a similar situation is encountered), whereas the case $\varepsilon = 0$ is classical (see [24]). Denoting by $(\bar{u}_0, \bar{\eta}_0)$ the difference of two initial data, let $(\bar{u}, \bar{\eta})$ be the difference of the corresponding solutions. Assume that $(\bar{u}(\tau), \bar{\eta}^\tau) = (0, 0)$ at a certain time $\tau > 0$. We suppose that τ belongs to the support of μ_ε (the other case, which might be empty, is easier). Then, from the representation formula (3.5), we see that $\bar{\eta}_0 = 0$, and so $\bar{\eta}^t = 0$ for $t \in [0, \tau]$. But then the equation for $\bar{\eta}$ implies that $\bar{u} = 0$ in $[0, \tau]$. We conclude that $(\bar{u}_0, \bar{\eta}_0) = (0, 0)$. $\square$

Finally, the family of global attractors $\{\mathcal{A}_\varepsilon\}$ is upper semicontinuous at $\varepsilon = 0$, with respect to the Hausdorff semidistance in $\mathcal{H}^0_\varepsilon$.

Theorem 7.3. *There holds*

$$\lim_{\varepsilon\to 0}[\mathrm{dist}_{\mathcal{H}^0_\varepsilon}(\mathcal{A}_\varepsilon, \mathbb{L}_\varepsilon \mathcal{A}_0)] = 0.$$

Equivalently,

$$\lim_{\varepsilon\to 0}[\mathrm{dist}_{H^0}(\mathbb{P}\mathcal{A}_\varepsilon, \mathcal{A}_0) + \sup_{z\in\mathcal{A}_\varepsilon} \|\mathbb{Q}_\varepsilon z\|_{\mathcal{M}^0_\varepsilon}] = 0.$$

We need first a preparatory lemma.

Lemma 7.4. *The family of maps*

$$\{u \in C(\mathbb{R}, H^0) \mid u(t) = \mathbb{P}\tilde{S}_\varepsilon(t)z, \text{ with } z \in \mathcal{A}_\varepsilon \text{ for some } \varepsilon \in (0, 1]\}$$

is equicontinuous and equibounded in H^1.

Proof. Notice first that $\tilde{S}_\varepsilon(t)z$ is well defined, in light of Proposition 7.2. The equiboundedness in H^1 follows at once from the fact that $u(t) \in \mathbb{P}\mathcal{A}_\varepsilon$ for every $t \in \mathbb{R}$. Exploiting the invariance of the attractor, it is then enough to prove the equicontinuity on the time-interval $[1, 2]$. Directly from the equation, we see that

$$\|u_t\|^2 \le c\left(\|Au\|^2 + \|\varphi(u)\|^2 + \frac{1}{\varepsilon}\|\eta\|^2_{\mathcal{M}^1_\varepsilon} + \|f\|^2\right).$$

Using the Agmon inequality,

$$\|\varphi(u)\|^2 \le c(1 + \|Au\|^2).$$

On the other hand, repeating the proof of Lemma 5.4, taking now the products in $\mathcal{M}^1_\varepsilon$, we obtain

$$\|\eta^t\|^2_{\mathcal{M}^1_\varepsilon} \le ce^{-\delta/(2\varepsilon)} + c\varepsilon\|Au(t)\|^2, \quad \forall\, t \in [1,2].$$

In conclusion,

$$\|u_t\|^2 \le c\left(1 + \|Au\|^2 + \frac{1}{\varepsilon}e^{-\delta/(2\varepsilon)}\right),$$

so that the integral estimate (6.5) bears

$$\int_1^2 \|u_t(y)\|^2\,dy \le c.$$

This yields the desired ($\frac{1}{2}$-Hölder) equicontinuity. □

Proof of Theorem 7.3. The idea of the proof is borrowed from [12], along the lines of [15]. Assume by contradiction that there exists $\varepsilon_n \in (0,1]$, with $\varepsilon_n \to 0$, and a corresponding sequence $z_n \in \mathcal{A}_{\varepsilon_n}$ such that

$$\inf_{z_0 \in \mathcal{A}_0} \|z_n - \mathbb{L}_{\varepsilon_n} z_0\|_{\mathcal{H}^0_{\varepsilon_n}} \ge c > 0.$$

In view of Proposition 7.2, denote

$$u_n(t) = \mathbb{P}\tilde{S}_{\varepsilon_n}(t)z_n \quad \text{and} \quad \eta^t_n = \mathbb{Q}_{\varepsilon_n}\tilde{S}_{\varepsilon_n}(t)z_n,$$

for every $t \in \mathbb{R}$. Thanks to Lemma 7.4, we are in a position to apply Ascoli's theorem. Therefore, there exists $u_* \in C(\mathbb{R}, H^0)$ such that, up to a subsequence,

$$\lim_{n\to\infty} \|u_n - u_*\|_{C([-T,T],H^0)} = 0,$$

for every $T > 0$. In addition,

$$\sup_{t\in\mathbb{R}} \|u_*(t)\| < \infty.$$

Furthermore, from Lemma 5.4 and the invariance of the attractor,

$$\lim_{n\to\infty} \|\eta^0_n\|_{\mathcal{M}^0_{\varepsilon_n}} = 0.$$

Therefore, setting $z_* = (u_*(0), 0)$, we get

$$\lim_{n\to\infty} \|z_n - \mathbb{L}_{\varepsilon_n} z_*\|_{\mathcal{H}^0_{\varepsilon_n}} = 0.$$

We reach the contradiction if we show that $u_*(0) \in \mathcal{A}_0$, which occurs if and only if u_* is a complete bounded trajectory of $S_0(t)$. The boundedness has already been proved, and the remaining assertion is a consequence of Theorem 5.1. □

8. FURTHER DISSIPATIVITY

In this section we show that, if we require sufficient regularity to the nonlinearity φ and to the source term f, then the absorbing property holds true in higher-order spaces as well, provided that the nonlinearity satisfies a further growth restriction. Throughout this section, let $m \geq 2$ be any integer. We make the following additional assumptions (besides setting $\omega = \frac{1}{2}$):

$$f \in H^{m-1}, \ \varphi \in C^{m-1}(\mathbb{R}), \quad \text{with } \gamma < 4, \tag{8.1}$$

where, with reference to (4.4), γ is the growth rate of φ'. Moreover, we require that

$$\text{if } m \geq 4, \quad \text{then } \varphi^{(\iota)}(0) = 0 \text{ for } \iota = 2, \dots, 2\left[\frac{m}{2}\right] - 2. \tag{8.2}$$

Observe however that, to include the physically significant nonlinearity $\varphi(x) = x^3 - x$, we should restrict to $m \leq 5$.

Then we have the following result.

Theorem 8.1. *There exists $R_m > 0$ such that the set $\mathcal{B}_\varepsilon^m = B_{\mathcal{Z}_\varepsilon^m}(R_m)$ is an absorbing set for $S_\varepsilon(t)$ on $\mathcal{Z}_\varepsilon^m$, uniformly in ε.*

In light of Lemma 6.8, the thesis immediately follows exploiting an inductive argument on $m \geq 2$, on account of the following lemma.

Lemma 8.2. *Given $R \geq 0$ and $\rho \geq 0$, there exist $C_m = C_m(\rho) > 0$, and a positive function Ψ_m vanishing at infinity such that, if $\|z\|_{\mathcal{Z}_\varepsilon^m} \leq R$ and $\|z\|_{\mathcal{Z}_\varepsilon^{m-1}} \leq \rho$, there holds*

$$\|S_\varepsilon(t)z\|_{\mathcal{Z}_\varepsilon^m} \leq R\Psi_m(t) + C_m, \quad t \geq 0. \tag{8.3}$$

Moreover,

$$\sup_{\|z\|_{\mathcal{Z}_\varepsilon^m} \leq R} \int_t^{t+1} \|A^{(m+1)/2}\mathbb{P}S_\varepsilon(y)z\|^2\,dy \leq Q_m, \quad \forall\, t \geq 0, \tag{8.4}$$

for some $Q_m = Q_m(R)$.

Proof. Let $m \geq 2$ be a fixed integer. In this proof, the generic constant $c \geq 0$ may depend on ρ, but not on R. Notice that the term

$$\varepsilon||T_\varepsilon\eta^t||^2_{\mathcal{M}_\varepsilon^0} + \sup_{x\geq 1} x\mathbb{T}^\varepsilon_{\eta^t}(x),$$

can be estimated as in (6.8). Hence, in order to prove (8.3), it will suffice to show that

$$\|S_\varepsilon(t)z\|_{\mathcal{H}_\varepsilon^m} \leq cRe^{-\nu_m t} + c, \quad \forall\, t \geq 0, \tag{8.5}$$

for some $\nu_m > 0$. Arguing as in the proof of Lemma 6.8, taking this time the products in H^m and $\mathcal{M}_\varepsilon^m$, we find the inequality

$$\frac{d}{dt}(\|A^{m/2}u\|^2 + \|\eta\|^2_{\mathcal{M}_\varepsilon^m}) + \|A^{(m+1)/2}u\|^2 + \frac{\delta}{\varepsilon}\|\eta\|^2_{\mathcal{M}_\varepsilon^m} \\ \leq -2\langle\varphi(u), A^m u\rangle + 2\langle f, A^m u\rangle.$$

Since $f \in H^{m-1}$,

$$2\langle f, A^m u\rangle \leq \frac{1}{4}\|A^{(m+1)/2}u\|^2 + c.$$

Now we have to proceed differently, according to the value of m.

CASE $m = 2$. By virtue of the Agmon inequality and Theorem 6.7,

$$\|\varphi'(u)\|_{L^\infty} \leq c(1 + \|u\|^\gamma_{L^\infty}) \leq c(1 + \|Au\|^{\gamma/2}),$$

so that we deduce the estimate

$$\begin{aligned} -2\langle\varphi(u), A^2 u\rangle &= -2\langle\varphi'(u)\nabla u, \nabla Au\rangle \\ &\leq c(1 + \|Au\|^{\gamma/2})\|A^{1/2}u\|\,\|A^{3/2}u\| \\ &\leq \frac{1}{4}\|A^{3/2}u\|^2 + c\|Au\|^\gamma + c. \end{aligned}$$

Hence we obtain

$$\frac{d}{dt}(\|Au\|^2 + \|\eta\|^2_{\mathcal{M}_\varepsilon^2}) + 2\nu_2(\|Au\|^2 + \|\eta\|^2_{\mathcal{M}_\varepsilon^2}) + \frac{1}{4}\|A^{3/2}u\|^2 \leq c + c\|Au\|^\gamma.$$

for some $\nu_2 > 0$ sufficiently small. Setting $\sigma = \max\{(\gamma - 2)/2, 0\} \in [0, 1)$, the right-hand side of the inequality above is less than or equal to

$$c + c\|Au\|^2(\|Au\|^2 + \|\eta\|^2_{\mathcal{M}_\varepsilon^2})^\sigma.$$

Thanks to (6.5), we can apply Lemma 3.7, which entails (8.5). With a subsequent integration on $(t, t+1)$, we recover the integral estimate (8.4).

CASE $m > 2$. We exploit an inductive argument on m. Assuming that the result holds for $m - 1$, we have

$$\sup_{t\geq 0}\ \sup_{\|z\|_{\mathcal{H}_\varepsilon^{m-1}}\leq\rho} \|A^{(m-1)/2}u(t)\| \leq c.$$

Moreover, since now $u(t)$ is uniformly bounded in $L^\infty(\Omega)$,

$$\sup_{t\geq 0}\sum_{\iota=0}^{m-1}\|\varphi^{(\iota)}(u(t))\|_{L^\infty} \leq c.$$

Therefore, appealing to (8.2), we get that $\varphi(u) \in H^{m-1}$, and it is a standard matter to check that $\|A^{(m-1)/2}\varphi(u)\| \le c$. Accordingly,

$$\begin{aligned} -2\langle \varphi(u), A^m u\rangle &= -2\langle A^{(m-1)/2}\varphi(u), A^{(m+1)/2}u\rangle \\ &\le \frac{1}{4}\|A^{(m+1)/2}u\|^2 + c. \end{aligned}$$

In conclusion, we get

$$\frac{d}{dt}(\|A^{m/2}u\|^2 + \|\eta\|^2_{\mathcal{M}^m_\varepsilon}) + 2\nu_m(\|A^{m/2}u\|^2 + \|\eta\|^2_{\mathcal{M}^m_\varepsilon}) + \frac{1}{4}\|A^{(m+1)/2}u\|^2 \le c,$$

for a suitable $\nu_m > 0$. Arguing as in the previous case, we reach the thesis. □

Remark 8.3. The restriction $\gamma < 4$ (which is required only to treat the case $m = 2$) is actually unnecessary when $\varepsilon = 0$. Indeed, in that situation, the final differential inequality is

$$\frac{d}{dt}\|Au\|^2 + 2\nu_2\|Au\|^2 + \frac{1}{4}\|A^{3/2}u\|^2 \le c + c\|Au\|^\gamma.$$

Hence, if $\gamma = 4$, we reach the thesis exploiting Lemma 3.8. By the same token, $\mathcal{B}^m_0$ is an absorbing set in the phase-space $\mathcal{H}^0_0$.

Of course, if $\varepsilon > 0$ we cannot expect that $\mathcal{B}^m_\varepsilon$ is an absorbing set in $\mathcal{H}^0_\varepsilon$. Nonetheless, it is exponentially attracting.

Theorem 8.4. *There exist $\kappa_m > 0$ and an increasing positive function Γ_m such that, up to (possibly) enlarging the radius R_m,*

$$\mathrm{dist}_{\mathcal{H}^0_\varepsilon}(S_\varepsilon(t)\mathcal{B}, \mathcal{B}^m_\varepsilon) \le \Gamma_m(R)e^{-\kappa_m t}, \quad \forall\, t \ge 0,$$

for every set $\mathcal{B} \subset B_{\mathcal{H}^0_\varepsilon}(R)$.

The result is a direct consequence of Lemma 3.9, Theorem 4.5, Theorem 6.10, and Lemma 8.5 below.

Lemma 8.5. *There exists $\Gamma_m > 0$ such that, up to (possibly) enlarging the radius R_m,*

$$\mathrm{dist}_{\mathcal{H}^0_\varepsilon}(S_\varepsilon(t)\mathcal{B}^{m-1}_\varepsilon, \mathcal{B}^m_\varepsilon) \le \Gamma_m e^{-\kappa_1 t}, \quad \forall\, t \ge 0.$$

Proof. The proof follows step by step the one of Theorem 6.10, with the only difference that here we need the estimate $\sup_{t\ge 0}\sup_{z\in\mathcal{B}^{m-1}_\varepsilon}\|K_\varepsilon(t)z\|_{\mathcal{Z}^m_\varepsilon} \le c$, for some $c = c(R_{m-1})$. Since by Theorem 8.1 $u(t)$ is uniformly bounded in H^{m-1}, we have $2\ell\langle u, A^m w\rangle \leq \frac{1}{4}\|A^{(m+1)/2}w\|^2 + c$. Hence, by mimicking the proof of Lemma 8.2, there holds

$$\frac{d}{dt}(\|A^{m/2}w\|^2 + ||\zeta||^2_{\mathcal{M}^m_\varepsilon}) + 2\nu_m(\|A^{m/2}w\|^2 + ||\zeta||^2_{\mathcal{M}^m_\varepsilon}) + \frac{1}{2}\|A^{(m+1)/2}w\|^2 \\ \leq -2\langle\varphi_0(w), A^m w\rangle + c.$$

In order to derive the suitable differential inequality that, via a Gronwall Lemma, will provide the desired conclusion, we just follow the proof of Lemma 8.2, with the obvious changes. For the case $m = 2$, note that the final step is based on the inequality

$$\sup_{t\geq 0}\int_t^{t+1}\|Aw(y)\|^2\,dy < \infty,$$

formulated in Remark 6.13. □

Till the end of this work, we agree to redefine inductively the radii R_m so that Theorem 8.4 holds true.

9. Robust Exponential Attractors

In this last section, we state and prove the main result about the asymptotic behavior of solutions for the Coleman-Gurtin model (as usual, we put $\omega = \frac{1}{2}$). Again, we have to make the requirement that (4.4) holds with $\gamma < 4$.

Theorem 9.1. *Assume that $\gamma < 4$. Then for every $\varepsilon \in [0,1]$ there exists a set $\mathcal{E}_\varepsilon$, compact in $\mathcal{H}^0_\varepsilon$ and bounded in $\mathcal{Z}^1_\varepsilon$, which satisfies the following conditions.*

(i) *$\mathcal{E}_\varepsilon$ is positively invariant for $S_\varepsilon(t)$, that is,*

$$S_\varepsilon(t)\mathcal{E}_\varepsilon \subset \mathcal{E}_\varepsilon, \quad \forall\, t \geq 0.$$

(ii) *There exist $\kappa > 0$ and a positive increasing function M (both independent of ε) such that, for every bounded set $\mathcal{B} \subset B^0_{\mathcal{H}_\varepsilon}(R)$, there holds*

$$\operatorname{dist}_{\mathcal{H}^0_\varepsilon}(S_\varepsilon(t)\mathcal{B}, \mathcal{E}_\varepsilon) \leq M(R)e^{-\kappa t}, \quad \forall\, t \geq 0.$$

(iii) *The fractal dimension of $\mathcal{E}_\varepsilon$ in $\mathcal{H}^0_\varepsilon$ is uniformly bounded with respect to ε.*

(iv) *There exist $\Theta \geq 0$ and $\tau \in (0, \frac{1}{8}]$ such that*

$$\operatorname{dist}^{\mathrm{sym}}_{\mathcal{H}^0_\varepsilon}(\mathcal{E}_\varepsilon, \mathbb{L}_\varepsilon\mathcal{E}_0) \leq \Theta\varepsilon^\tau.$$

The last property (iv) witnesses the robustness of the family $\{\mathcal{E}_\varepsilon\}$ with respect to the singular limit $\varepsilon \to 0$, and it is equivalent to

$$\operatorname{dist}^{\mathrm{sym}}_{H^0}(\mathbb{P}\mathcal{E}_\varepsilon, \mathcal{E}_0) + \sup_{z\in\mathcal{E}_\varepsilon}\|\mathbb{Q}_\varepsilon z\|_{\mathcal{M}^0_\varepsilon} \leq \Theta\varepsilon^\tau.$$

Since $\mathcal{E}_\varepsilon$ is a compact attracting set, it follows that $\mathcal{A}_\varepsilon \subset \mathcal{E}_\varepsilon$. As a byproduct we have the following result.

Corollary 9.2. *If $\gamma < 4$, the global attractor $\mathcal{A}_\varepsilon$ has finite fractal dimension, uniformly with respect to ε.*

With further hypotheses on φ and f, we can strengthen the thesis.

Theorem 9.3. *Assume in addition that* (8.1)–(8.2) *hold for $m \geq 2$. Then $\mathcal{E}_\varepsilon$ is bounded in $\mathcal{Z}_\varepsilon^m$ and the constant τ appearing in* (iv) *is replaced by 2τ.*

Before going to the proofs of the theorems, let us briefly discuss the further regularity of $\mathcal{E}_\varepsilon$ (and, consequently, of $\mathcal{A}_\varepsilon$). From Theorem 9.3 we see that $\mathcal{E}_\varepsilon$ is bounded in $\mathcal{H}_\varepsilon^m$. Indeed, $\mathcal{E}_\varepsilon$ is contained by construction in $\mathcal{B}_\varepsilon^m$. Hence, $\mathcal{E}_\varepsilon$ is as regular as φ and f permit.

Proposition 9.4. *Let $\gamma < 4$, $f \in C^\infty(\bar{\Omega})$, and $\varphi \in C^\infty(\mathbb{R})$ with $\varphi^{(\iota)}(0) = 0$ for all $\iota \neq 1$. Then $\mathcal{E}_\varepsilon$ (and hence $\mathcal{A}_\varepsilon$) belongs to $\mathcal{B}_\varepsilon^m$, for every $m \geq 0$. In particular,*

$$\mathbb{P}\mathcal{A}_\varepsilon \subset \mathbb{P}\mathcal{E}_\varepsilon \subset C^\infty(\bar{\Omega}).$$

Remark 9.5. The global attractor $\mathcal{A}_0$ fulfills the regularity properties above *without* the constraint $\gamma < 4$, for it belongs to $\mathcal{B}_0^m$ (cf. Remark 8.3).

Actually, one could easily recast all the calculations made so far by replacing, for $m \geq 1$, the space $\mathcal{L}_\varepsilon^m$ with $\mathcal{L}_\varepsilon^m \cap H_\mu^1(\mathbb{R}^+, H^m)$. This yields the following result.

Corollary 9.6. *Let $\varepsilon > 0$, $\gamma < 4$, $f \in C^\infty(\bar{\Omega})$, and $\varphi \in C^\infty(\mathbb{R})$ with $\varphi^{(\iota)}(0) = 0$ for all $\iota \neq 1$. Then*

$$\mathbb{Q}_\varepsilon\mathcal{A}_\varepsilon \subset \mathbb{Q}_\varepsilon\mathcal{E}_\varepsilon \subset C^\infty([0, s_\infty) \times \bar{\Omega}),$$

where $s_\infty = \sup\{s \mid \mu_\varepsilon(s) > 0\}$ (possibly, $s_\infty = \infty$).

Proof of Theorem 9.1. We want to exploit the abstract Theorem A.2 given in Appendix A. Preliminarily notice that, due to the exponential attraction property provided by Theorem 6.10, and using the transitivity of exponential attraction (i.e., Lemma 3.9) together with estimate (4.9), it suffices to show that

$$\operatorname{dist}_{\mathcal{H}_\varepsilon^0}(S_\varepsilon(t)\mathcal{B}_\varepsilon^1, \mathcal{E}_\varepsilon) \leq M_1 e^{-\kappa t}, \quad \forall\, t \geq 0,$$

for some $M_1 \geq 0$, in place of the stronger condition (ii). According to the notations of Appendix A, we set

$$X = Y^\dagger = H^0, \quad X' = H^\beta, \quad Y = H^1, \quad Y' = H^{1+\beta},$$

where $\beta = (4 - \gamma)/2$. Without loss of generality, we may assume $\gamma > 2$, so that $\beta \in (0, 1)$. Besides, we set $\mathcal{B}_\varepsilon = \mathcal{B}_\varepsilon^1$, and we denote by t_1 the entering time of $\mathcal{B}_\varepsilon$

into itself. So, in particular, $S_\varepsilon(t)\mathcal{B}_\varepsilon \subset \mathcal{B}_\varepsilon$ for every $t \geq t_1$. It is readily seen that $\mathcal{B}_\varepsilon$ is closed in $\mathcal{H}_\varepsilon$ (which, in our concrete situation, is the space $\mathcal{H}^0_\varepsilon$). We are now left to verify hypotheses (H1)–(H5) of Theorem A.2, that will guarantee the existence of a set $\mathcal{E}_\varepsilon \subset \mathcal{B}_\varepsilon$ with the desired properties.

Hence, let $t^\star \geq t_1$ to be determined later, and set $S_\varepsilon = S_\varepsilon(t^\star)$. Conditions (H2)–(H3), with $\Sigma(\varepsilon) = \sqrt[8]{\varepsilon}$, immediately follow from Theorem 5.1 (cf. (5.4)), whereas (H4) is a consequence of (4.9). Besides, due to (4.9), to prove (H5) (for $\alpha = \frac{1}{2}$) it is enough to show that there exists a positive constant $c(\varepsilon)$ such that

$$\|S_\varepsilon(t_1)z - S_\varepsilon(t_2)z\|_{\mathcal{H}_\varepsilon} \leq c(\varepsilon)\sqrt{|t_1 - t_2|},$$

for every $t_1, t_2 \in [t^\star, 2t^\star]$ and every $z \in \mathcal{B}_\varepsilon$. But this follows from the bound

$$\|u_t\|_{L^2(t^\star,2t^\star;H^0)} + \|\eta_t\|_{L^2(t^\star,2t^\star;\mathcal{M}^0_\varepsilon)} \leq c(\varepsilon),$$

which is in turn a consequence of (6.4)–(6.5) (cf. the proof of Lemma 7.4) and Lemma 3.3.

Thus, we are left to prove (H1). To this aim, we decompose the map $S_\varepsilon(t)$ into the sum

$$S_\varepsilon(t) = L_\varepsilon(t) + K_\varepsilon(t),$$

where, for $z \in \mathcal{B}_\varepsilon$, $L_\varepsilon(t)z = (v(t), \xi^t)$ and $K_\varepsilon(t)z = (w(t), \zeta^t)$ solve the problems

$$\begin{cases} v_t + \frac{1}{2}Av + \int_0^\infty \mu_\varepsilon(s)A\xi(s)\,ds = 0, \\ \xi_t = T_\varepsilon\xi + v, \\ L_\varepsilon(0)z = z, \end{cases}$$

and

$$\begin{cases} w_t + \frac{1}{2}Aw + \int_0^\infty \mu_\varepsilon(s)A\zeta(s)\,ds + \varphi(u) = f, \\ \zeta_t = T_\varepsilon\zeta + w, \\ K_\varepsilon(0)z = 0. \end{cases}$$

It is readily seen that $L_\varepsilon(t)$ is a strongly continuous semigroup of linear operators on $\mathcal{H}_\varepsilon$. Moreover, it is exponentially stable, with a decay rate independent of ε (cf. Remark 6.4). Hence, choosing any $\lambda < \frac{1}{2}$, we can fix $t^\star \geq t_1$ large enough such that

$$\|L_\varepsilon z_1 - L_\varepsilon z_2\|_{\mathcal{H}_\varepsilon} \leq \lambda\|z_1 - z_2\|_{\mathcal{H}_\varepsilon}, \quad \forall\, z_1, z_2 \in \mathcal{B}_\varepsilon,$$

where we set $L_\varepsilon = L_\varepsilon(t^\star)$. Let now $z_1, z_2 \in \mathcal{B}_\varepsilon$. Denoting the difference $(\bar{w}(t), \bar{\zeta}) = K_\varepsilon(t)z_1 - K_\varepsilon(t)z_2$, we end up with the system

$$\begin{cases} \bar{w}_t + \frac{1}{2}A\bar{w} + \displaystyle\int_0^\infty \mu_\varepsilon(s)A\bar{\zeta}(s)\,ds + \varphi(u^1) - \varphi(u^2) = 0, \\ \bar{\zeta}_t = T_\varepsilon\bar{\zeta} + \bar{w}, \\ \bar{z}(0) = 0, \end{cases}$$

where $u^i(t) = \mathbb{P}S_\varepsilon(t)z_i$. Multiplying the first equation by $\bar{w}$ in H^β, and the second by $\bar{\zeta}$ in $\mathcal{M}_\varepsilon^\beta$, exploiting (3.4) and adding the resulting equations, we get

$$\frac{d}{dt}(\|A^{\beta/2}\bar{w}\|^2 + \|\bar{\zeta}\|^2_{\mathcal{M}_\varepsilon^\beta}) + \|A^{(1+\beta)/2}\bar{w}\|^2 \leq 2\langle\varphi(u^2) - \varphi(u^1), A^\beta\bar{w}\rangle.$$

Using (4.4) and the generalized Hölder inequality with exponents

$$\left\{2, \frac{3}{1-\beta}, \frac{6}{1+2\beta}, \infty\right\},$$

we get

$$2\langle\varphi(u_2)-\varphi(u_1), A^\beta\bar{w}\rangle \leq g\|u^1 - u^2\|\,\|A^\beta\bar{w}\|_{L^{6/(1+2\beta)}},$$

with

$$g = c(1 + \|u^1\|^2_{L^\infty}\|u^1\|^{\gamma-2}_{L^6} + \|u^2\|^2_{L^\infty}\|u^2\|^{\gamma-2}_{L^6}).$$

Due to the continuous embedding $H^{1-\beta} \subset L^{6/(1+2\beta)}(\Omega)$,

$$\|A^\beta\bar{w}\|_{L^{6/(1+2\beta)}} \leq c\|A^{(1+\beta)/2}\bar{w}\|.$$

On the other hand, the uniform bound on $u^i(t)$ in H^1 given by (6.4) and the Agmon inequality entail $g \leq c(1+\|Au^1\|+\|Au^2\|)$. Finally, from the continuous dependence estimate (4.9),

$$\|u^1(t) - u^2(t)\| \leq c\|z_1 - z_2\|_{\mathcal{H}_\varepsilon}, \quad \forall\, t \in [0, t^\star].$$

Collecting all the information obtained so far, we deduce the differential inequality

$$\frac{d}{dt}(\|A^{\beta/2}\bar{w}\|^2+\|\bar{\zeta}\|^2_{\mathcal{M}_\varepsilon^\beta})+\frac{1}{2}\|A^{(1+\beta)/2}\bar{w}\|^2 \leq c(1+\|Au^1\|^2+\|Au^2\|^2)\|z_1-z_2\|^2_{\mathcal{H}_\varepsilon},$$

for every $t \in [0, t^\star]$. Notice that, from (6.5),

$$\int_0^{t^\star}(1 + \|Au^1(y)\|^2 + \|Au^2(y)\|^2)\,dy \leq c.$$

Therefore, integration on $(0, t^\star)$ leads to

$$\|K_\varepsilon(t)z_1 - K_\varepsilon(t)z_2\|_{\mathcal{H}^\beta_\varepsilon} \leq c\|z_1 - z_2\|_{\mathcal{H}_\varepsilon}, \quad \forall\, t \in [0, t^\star]. \tag{9.1}$$

Also, we find

$$\int_0^{t^\star} \|A^{(1+\beta)/2}\bar{w}(y)\|^2\, dy \leq c\|z_1 - z_2\|^2_{\mathcal{H}_\varepsilon}. \tag{9.2}$$

Hence, setting $K_\varepsilon = K_\varepsilon(t^\star)$, we have in particular

$$\|K_\varepsilon z_1 - K_\varepsilon z_2\|_{\mathcal{H}^\beta_\varepsilon} \leq c\|z_1 - z_2\|_{\mathcal{H}_\varepsilon},$$

To complete the proof of (H1) we need to prove the inequality above with $\mathcal{H}'_\varepsilon$ in place of $\mathcal{H}^\beta_\varepsilon$. This amounts to showing that

$$\sup_{x\geq 1} x\mathbb{T}^\varepsilon_\eta(x) \leq c\|z_1 - z_2\|^2_{\mathcal{H}^0_\varepsilon},$$

$$\varepsilon\|T_\varepsilon\eta\|^2_{\mathcal{M}^{-1}_\varepsilon} \leq c\|z_1 - z_2\|^2_{\mathcal{H}^0_\varepsilon},$$

where we put for simplicity $\eta = \bar{\zeta}^{t^\star}$. The first inequality is an immediate consequence of (9.2) and Lemma 3.6 (where $\bar{w}$ plays the role of u). The second one is proved recasting Lemma 3.3 in $\mathcal{M}^{-1}_\varepsilon$ on $[0, t^\star]$, using the estimate (9.1) to control $\bar{w}$ in H^0. In both cases it is crucial to use the fact that $\bar{\zeta}^0 = 0$. This finishes the proof of Theorem 9.1. ◻

Proof of Theorem 9.3. The argument here is exactly the same, except that now we set $\mathcal{B}_\varepsilon = \mathcal{B}^m_\varepsilon$. Due to the exponential attraction property given by Theorem 8.4, it will suffice to prove

$$\mathrm{dist}_{\mathcal{H}^0_\varepsilon}(S_\varepsilon(t)\mathcal{B}^m_\varepsilon, \mathcal{E}_\varepsilon) \leq M_1 e^{-\kappa t}, \quad \forall\, t \geq 0,$$

for some $M_1 \geq 0$, in place of (ii). On account of Theorem 5.2, here $\Sigma(\varepsilon) = \sqrt[4]{\varepsilon}$. The exponential attractor $\mathcal{E}_\varepsilon$ found in this way will belong to $\mathcal{B}^m_\varepsilon$. ◻

Appendix A. The Abstract Result

1.1. The setting. Let X, X', $Y^\dagger$, Y, Y' be reflexive Banach spaces with embeddings $X' \Subset X$ and $Y' \Subset Y \subset Y^\dagger$. Let $\mu \in C(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ be a nonnegative decreasing function. For $\varepsilon \in (0, 1]$, we set

$$\mu_\varepsilon(s) = \frac{1}{\varepsilon^2}\mu\left(\frac{s}{\varepsilon}\right), \quad s \in \mathbb{R}^+,$$

and we consider the Banach spaces $\mathcal{M}'_\varepsilon = L^2_{\mu_\varepsilon}(\mathbb{R}^+, Y')$, $\mathcal{M}_\varepsilon = L^2_{\mu_\varepsilon}(\mathbb{R}^+, Y)$, $\mathcal{M}^\dagger_\varepsilon = L^2_{\mu_\varepsilon}(\mathbb{R}^+, Y^\dagger)$, endowed with the usual norms, together with the space

$$\mathcal{W}_\varepsilon = \{\eta \in \mathcal{M}'_\varepsilon \mid \eta_s \in \mathcal{M}^\dagger_\varepsilon,\ \sup_{x\geq 1} x\mathbb{T}^\varepsilon_\eta(x) < \infty\},$$

where

$$\mathbb{T}^\varepsilon_\eta(x) = \varepsilon \int_{(0,1/x)\cup(x,\infty)} \mu_\varepsilon(s)\|\eta(s)\|^2_Y\, ds, \quad x \geq 1,$$

and η_s is the distributional derivative of η with respect to the internal variable s. Then, $\mathcal{W}_\varepsilon$ is a Banach space with the norm

$$\|\eta\|^2_{\mathcal{W}_\varepsilon} = \|\eta\|^2_{\mathcal{M}'_\varepsilon} + \varepsilon\|\eta_s\|^2_{\mathcal{M}^\dagger_\varepsilon} + \sup_{x\geq 1} x\mathbb{T}^\varepsilon_\eta(x).$$

Finally, for $\varepsilon \in [0,1]$, we define the product Banach spaces

$$\mathcal{H}_\varepsilon = \begin{cases} X \times \mathcal{M}_\varepsilon, & \text{if } \varepsilon > 0,\\ X, & \text{if } \varepsilon = 0,\end{cases}$$

$$\mathcal{H}'_\varepsilon = \begin{cases} X' \times \mathcal{W}_\varepsilon, & \text{if } \varepsilon > 0,\\ X', & \text{if } \varepsilon = 0,\end{cases}$$

normed by

$$\|(u,\eta)\|^2_{\mathcal{H}_\varepsilon} = \|u\|^2_X + \|\eta\|^2_{\mathcal{M}_\varepsilon},$$
$$\|(u,\eta)\|^2_{\mathcal{H}'_\varepsilon} = \|u\|^2_{X'} + \|\eta\|^2_{\mathcal{W}_\varepsilon}.$$

It is understood that, when $\varepsilon = 0$, the pair (u,η) reads just u. Using an abstract form of Lemma 3.1, we have the compact embedding $\mathcal{H}'_\varepsilon \Subset \mathcal{H}_\varepsilon$. The *lifting map* $\mathbb{L}_\varepsilon : \mathcal{H}_0 \to \mathcal{H}_\varepsilon$, and the *projection map* $\mathbb{P} : \mathcal{H}_\varepsilon \to \mathcal{H}_0$ are given by

$$\mathbb{L}_\varepsilon u = \begin{cases} (u,0), & \text{if } \varepsilon > 0,\\ u, & \text{if } \varepsilon = 0,\end{cases}$$

$$\mathbb{P}(u,\eta) = u.$$

Before proceeding, we need the following technical lemma.

Lemma A.1. *The inequality*

$$\mathcal{N}_r(B_{\mathcal{H}'_\varepsilon}(1), \mathcal{H}_\varepsilon) \leq \mathcal{N}_r(B_{\mathcal{H}'_1}(1), \mathcal{H}_1)$$

holds for every $r > 0$ *and every* $\varepsilon \in [0,1]$.

Proof. The inequality is straightforward if $\varepsilon = 0$. If $\varepsilon > 0$, notice that $z = (u, \eta) \in \mathcal{H}_\varepsilon$ if and only if $\hat{z} = (u, \hat{\eta}) \in \mathcal{H}_1$, having set $\hat{\eta}(s) = (1/\sqrt{\varepsilon})\, \eta(\varepsilon s)$. In particular, $\|z\|_{\mathcal{H}_\varepsilon} = \|\hat{z}\|_{\mathcal{H}_1}$. The thesis follows from the simple observation that if $z \in B_{\mathcal{H}'_\varepsilon}(1)$, then $\hat{z} \in B_{\mathcal{H}'_1}(1)$. Indeed,

$$||\hat{z}||^2_{\mathcal{H}'_1} \leq ||u||^2_{X'} + ||\eta||^2_{\mathcal{M}'_\varepsilon} + \varepsilon^2 ||\eta_s||^2_{\mathcal{M}^\dagger_\varepsilon} + \varepsilon \sup_{x \geq 1} x \mathbb{T}^\varepsilon_\eta(x) \leq ||z||^2_{\mathcal{H}'_\varepsilon} \leq 1,$$

since $\varepsilon \leq 1$. ◻

1.2. The theorem. For every $\varepsilon \in [0, 1]$, let $S_\varepsilon(t) : \mathcal{H}_\varepsilon \to \mathcal{H}_\varepsilon$ be a strongly continuous semigroup of operators. Assume that there exist $R > 0$ and $t^\star > 0$, both independent of ε, and a family of closed sets $\mathcal{B}_\varepsilon \subset B_{\mathcal{H}_\varepsilon}(R)$ such that

$$S_\varepsilon(t)\mathcal{B}_\varepsilon \subset \mathcal{B}_\varepsilon, \quad \forall\, t \geq t^\star.$$

Theorem A.2. *Assume that there exist $\Lambda_j \geq 0$, $\lambda \in [0, \frac{1}{2})$, $\alpha \in (0, 1]$, and a continuous increasing function $\Sigma : [0, 1] \to [0, \infty)$ with $\Sigma(0) = 0$ (all independent of ε) such that the following conditions hold.*

(H1) *The map $S_\varepsilon = S_\varepsilon(t^\star)$ satisfies, for every $z_1, z_2 \in \mathcal{B}_\varepsilon$,*

$$S_\varepsilon z_1 - S_\varepsilon z_2 = L_\varepsilon(z_1, z_2) + N_\varepsilon(z_1, z_2),$$

where

$$\|L_\varepsilon(z_1, z_2)\|_{\mathcal{H}_\varepsilon} \leq \lambda \|z_1 - z_2\|_{\mathcal{H}_\varepsilon},$$
$$\|N_\varepsilon(z_1, z_2)\|_{\mathcal{H}'_\varepsilon} \leq \Lambda_1 \|z_1 - z_2\|_{\mathcal{H}_\varepsilon}.$$

(H2) *There holds*

$$\|S^n_\varepsilon z - \mathbb{L}_\varepsilon S^n_0 \mathbb{P} z\|_{\mathcal{H}_\varepsilon} \leq \Lambda^n_2 \Sigma(\varepsilon), \quad \forall\, z \in \mathcal{B}_\varepsilon,\ \forall\, n \in \mathbb{N}.$$

(H3) *There holds*

$$\|S_\varepsilon(t) z - \mathbb{L}_\varepsilon S_0(t) \mathbb{P} z\|_{\mathcal{H}_\varepsilon} \leq \Lambda_3 \Sigma(\varepsilon), \quad \forall\, z \in \mathcal{B}_\varepsilon,\ \forall\, t \in [t^\star, 2t^\star].$$

(H4) *The map*

$$z \mapsto S_\varepsilon(t) z : \mathcal{B}_\varepsilon \to \mathcal{B}_\varepsilon$$

is Lipschitz continuous, with a Lipschitz constant independent of ε and of $t \in [t^\star, 2t^\star]$. Here, $\mathcal{B}_\varepsilon$ is endowed with the metric topology of $\mathcal{H}_\varepsilon$.

(H5) *The map*

$$(t, z) \mapsto S_\varepsilon(t) z : [t^\star, 2t^\star] \times \mathcal{B}_\varepsilon \to \mathcal{B}_\varepsilon$$

is Hölder continuous of exponent α (with a constant that may depend on ε). Again, $\mathcal{B}_\varepsilon$ is endowed with the metric topology of $\mathcal{H}_\varepsilon$.

Then there exists a family of compact sets $\mathcal{E}_\varepsilon \subset \mathcal{B}_\varepsilon$, such that

$$S_\varepsilon(t)\mathcal{E}_\varepsilon \subset \mathcal{E}_\varepsilon, \quad \forall\, t \geq 0,$$

with the following additional properties.

(T1) *$\mathcal{E}_\varepsilon$ attracts $\mathcal{B}_\varepsilon$ with an exponential rate which is uniform with respect to ε, that is,*

$$\mathrm{dist}_{\mathcal{H}_\varepsilon}(S_\varepsilon(t)\mathcal{B}_\varepsilon, \mathcal{E}_\varepsilon) \leq M_1 e^{-\kappa t}, \quad \forall\, t \geq 0,$$

for some $\kappa > 0$.

(T2) *The fractal dimension of $\mathcal{E}_\varepsilon$ is uniformly bounded with respect to ε, that is,*

$$\dim_{\mathcal{H}_\varepsilon}[\mathcal{E}_\varepsilon] \leq M_2.$$

(T3) *There holds*

$$\mathrm{dist}^{\mathrm{sym}}_{\mathcal{H}_\varepsilon}(\mathcal{E}_\varepsilon, \mathbb{L}_\varepsilon\mathcal{E}_0) \leq M_3[\Sigma(\varepsilon)]^\tau,$$

for some $\tau \in (0,1]$.

The positive constants κ, τ and M_j are independent of ε, and they can be explicitly calculated.

The sets $\mathcal{E}_\varepsilon$ are called *exponential attractors* (with respect to the topology of $\mathcal{H}_\varepsilon$) for $S_\varepsilon(t)$ on $\mathcal{B}_\varepsilon$.

Remark A.3. Notice that condition (H1) is verified if the map S_ε admits the decomposition $S_\varepsilon = L_\varepsilon + K_\varepsilon$ such that, for every $z_1, z_2 \in \mathcal{B}_\varepsilon$,

$$\|L_\varepsilon z_1 - L_\varepsilon z_2\|_{\mathcal{H}_\varepsilon} \leq \lambda \|z_1 - z_2\|_{\mathcal{H}_\varepsilon},$$
$$\|K_\varepsilon z_1 - K_\varepsilon z_2\|_{\mathcal{H}'_\varepsilon} \leq \Lambda_1 \|z_1 - z_2\|_{\mathcal{H}_\varepsilon}.$$

Remark A.4. Observe that (T3) is equivalent to

$$\mathrm{dist}^{\mathrm{sym}}_X(\mathbb{P}\mathcal{E}_\varepsilon, \mathcal{E}_0) + \sup_{(u,\eta)\in\mathcal{E}_\varepsilon} \|\eta\|_{\mathcal{M}_\varepsilon} \leq M_4[\Sigma(\varepsilon)]^\tau,$$

for some positive constant M_4 independent of ε.

1.3. Sketch of the proof. The construction parallels the one of the proof of the main theorem in [7] (but see also [4]). Thus, we will limit ourselves to treat in detail only those passages where appreciable differences appear, and we address the reader to [7] for the rest.

In the following, c will denote a generic positive constant independent of ε. We define

$$R_n = R\left(\frac{1}{2} + \lambda\right)^n, \quad n \in \mathbb{N},$$

and we put

$$N_0 = \max\{3, \mathcal{N}_{(1-2\lambda)/(4\Lambda_1)}(B_{\mathcal{H}_1'}(1), \mathcal{H}_1)\}.$$

First, we build (discrete) exponential attractors $\mathcal{E}_\varepsilon^d$ for the maps S_ε for $\varepsilon = 0$ or $\varepsilon > \varepsilon_0$, where $\varepsilon_0 > 0$ will be determined later (it might occur that $\varepsilon_0 \geq 1$, in which case the construction for $\varepsilon > \varepsilon_0$ is empty). This is done exactly as in [7]. Namely, we find for $n \in \mathbb{N}$ a family of finite sets $E_n(\varepsilon)$ such that

$$\begin{cases} E_n(\varepsilon) \subset S_\varepsilon^n \mathcal{B}_\varepsilon, \quad S_\varepsilon E_n(\varepsilon) \subset E_{n+1}(\varepsilon), \\ E_n(\varepsilon) \text{ is a } R_n\text{-net of } S_\varepsilon^n \mathcal{B}_\varepsilon, \\ \operatorname{card}[E_n(\varepsilon)] \leq N_0^{n+1}. \end{cases}$$

In the last inequality we can choose N_0 for all the involved ε by force of Lemma A.1. Then, we define

$$\mathcal{E}_\varepsilon^d = \overline{\bigcup_{n\in\mathbb{N}} E_n(\varepsilon)}^{\mathcal{H}_\varepsilon}.$$

Note that the analogous union (made for $\varepsilon = 0$) in [7] is taken over $\mathbb{N} \cup \{0\}$. The (compact) sets $\mathcal{E}_\varepsilon^d$ are exponential attractors for S_ε on $\mathcal{B}_\varepsilon$ that satisfy

$$\begin{aligned} &S_\varepsilon \mathcal{E}_\varepsilon^d \subset \mathcal{E}_\varepsilon^d \subset \mathcal{B}_\varepsilon, \\ &\operatorname{dist}_{\mathcal{H}_\varepsilon}(S_\varepsilon^n \mathcal{B}_\varepsilon, \mathcal{E}_\varepsilon^d) \leq R_n, \\ &\dim_{\mathcal{H}_\varepsilon}[\mathcal{E}_\varepsilon^d] \leq \frac{\ln N_0}{\ln(2/(2\lambda+1))}. \end{aligned}$$

Besides, it is apparent that

$$\operatorname{dist}_{\mathcal{H}_\varepsilon}^{\mathrm{sym}}(\mathcal{E}_\varepsilon^d, \mathbb{L}_\varepsilon \mathcal{E}_0^d) \leq c. \tag{A.1}$$

The next step is to determine ε_0, in order to construct $\mathcal{E}_\varepsilon^d$ for $\varepsilon \in (0, \varepsilon_0]$. Without loss of generality, we assume $\Lambda_2 \geq 1$ and we choose $\varepsilon_0 \in (0,1]$ such that, for all $\varepsilon \in (0, \varepsilon_0]$,

$$\nu(\varepsilon) = \frac{\ln \Sigma(\varepsilon)}{\ln\left(\frac{1}{2}+\lambda\right) - \ln \Lambda_2} \geq 1. \tag{A.2}$$

Hence, let $\varepsilon \in (0, \varepsilon_0]$. For every $n \in \mathbb{N}$, we find a set $\hat{E}_n \subset \mathcal{B}_0$ such that

$$S_0^n \hat{E}_n = E_n(0) \quad \text{and} \quad \operatorname{card}[\hat{E}_n] \leq N_0^{n+1},$$

and we put

$$\hat{E}_n(\varepsilon) = \hat{E}_n \times \{0\} \subset \mathcal{B}_\varepsilon.$$

Clearly, $\mathbb{P}\hat{E}_n(\varepsilon) = \hat{E}_n$, $\mathbb{L}_\varepsilon\hat{E}_n = \hat{E}_n(\varepsilon)$, $\mathrm{card}[\hat{E}_n(\varepsilon)] \le N_0^{n+1}$. Finally, we define $\tilde{E}_n(\varepsilon) = S_\varepsilon^n\hat{E}_n(\varepsilon) \subset \mathcal{B}_\varepsilon$. We shall prove the estimate

$$\mathrm{dist}_{\mathcal{H}_\varepsilon}(S_\varepsilon^n\mathcal{B}_\varepsilon, \tilde{E}_n(\varepsilon)) \le 2\Lambda_2^n\Sigma(\varepsilon) + R_n, \qquad \forall\, n \in \mathbb{N}. \tag{A.3}$$

Fix $z \in \mathcal{B}_\varepsilon$. Recalling (H2),

$$\|S_\varepsilon^n z - \mathbb{L}_\varepsilon S_0^n\mathbb{P}z\|_{\mathcal{H}_\varepsilon} \le \Lambda_2^n\Sigma(\varepsilon).$$

Observe that $\mathbb{L}_\varepsilon E_n(0)$ is a R_n-net of $\mathbb{L}_\varepsilon S_0^n\mathcal{B}_0$ in $\mathcal{H}_\varepsilon$. Thus there exists $\bar{z} \in E_n(0)$ such that

$$\|\mathbb{L}_\varepsilon S_0^n\mathbb{P}z - \mathbb{L}_\varepsilon\bar{z}\|_{\mathcal{H}_\varepsilon} \le R_n.$$

By the definition of $\hat{E}_n(\varepsilon)$, we can find $\hat{z} \in \hat{E}_n(\varepsilon)$ such that $\bar{z} = S_0^n\mathbb{P}\hat{z}$. Hence, applying (H2) once more,

$$\|S_\varepsilon^n\hat{z} - \mathbb{L}_\varepsilon\bar{z}\|_{\mathcal{H}_\varepsilon} = \|S_\varepsilon^n\hat{z} - \mathbb{L}_\varepsilon S_0^n\mathbb{P}\hat{z}\|_{\mathcal{H}_\varepsilon} \le \Lambda_2^n\Sigma(\varepsilon).$$

Collecting the three inequalities above, we deduce (A.3), as claimed. At this point, we set

$$\tau = \frac{\ln(\frac{1}{2}+\lambda)}{\ln(\frac{1}{2}+\lambda) - \ln\Lambda_2} \in (0,1],$$

so that, in light of (A.2), we find the equality

$$\Lambda_2^\nu\Sigma(\varepsilon) = \Big(\frac{1}{2}+\lambda\Big)^\nu = [\Sigma(\varepsilon)]^\tau,$$

for $\nu = \nu(\varepsilon)$. We distinguish two cases. When $1 \le n \le \nu$, we set $E_n(\varepsilon) = \tilde{E}_n(\varepsilon)$. Thanks to (H2) and (A.3) there hold

$$\mathrm{dist}_{\mathcal{H}_\varepsilon}^{\mathrm{sym}}(E_n(\varepsilon), \mathbb{L}_\varepsilon E_n(0)) \le c[\Sigma(\varepsilon)]^\tau, \tag{A.4}$$

$$\mathrm{dist}_{\mathcal{H}_\varepsilon}(S_\varepsilon^n\mathcal{B}_\varepsilon, E_n(\varepsilon)) \le cR_n. \tag{A.5}$$

For the case $n > \nu$, the sets $E_n(\varepsilon)$ are constructed by induction, paralleling the construction of $E_n(0)$ (cf. [7]; again, it is crucial to have Lemma A.1 at disposal), starting from the initial step $E_{[\nu]}(\varepsilon)$, and asking (A.5) to hold for all $n \in \mathbb{N}$. Then, for every $n \in \mathbb{N}$, the family $E_n(\varepsilon)$ fulfills (A.5) and

$$\begin{cases} E_n(\varepsilon) \subset S_\varepsilon^n\mathcal{B}_\varepsilon, \\ S_\varepsilon E_n(\varepsilon) \subset E_{n+1}(\varepsilon), \\ \mathrm{card}[E_n(\varepsilon)] \le N_0^{n+2}. \end{cases} \tag{A.6}$$

Finally, we define

$$\mathcal{E}_\varepsilon^d = \overline{\bigcup_{n\in\mathbb{N}} E_n(\varepsilon)}^{\mathcal{H}_\varepsilon}.$$

Clearly, $S_\varepsilon \mathcal{E}_\varepsilon^d \subset \mathcal{E}_\varepsilon^d \subset \mathcal{B}_\varepsilon$. Besides, by (A.1) and (A.4)–(A.6) (cf. [7]), we learn that

$$\begin{aligned}
&\mathrm{dist}_{\mathcal{H}_\varepsilon}(S_\varepsilon^n \mathcal{B}_\varepsilon, \mathcal{E}_\varepsilon^d) \le c R_n,\\
&\dim_{\mathcal{H}_\varepsilon}[\mathcal{E}_\varepsilon^d] \le c,\\
&\mathrm{dist}_{\mathcal{H}_\varepsilon}^{\mathrm{sym}}(\mathcal{E}_\varepsilon^d, \mathbb{L}_\varepsilon \mathcal{E}_0^d) \le c[\Sigma(\varepsilon)]^\tau.
\end{aligned}$$

The passage from the discrete to the continuous case is obtained by setting

$$\mathcal{E}_\varepsilon = \bigcup_{t\in[t^\star, 2t^\star]} S_\varepsilon(t)\mathcal{E}_\varepsilon^d.$$

The verification of (T1)–(T3) goes exactly like in [7].

Acknowledgments. This research was partially supported by the Italian MIUR Research Projects *Metodi Variazionali ed Equazioni Differenziali Nonlineari*, *Modellizzazione Matematica ed Analisi dei Problemi a Frontiera Libera*, and *Metodi Variazionali e Topologici nello Studio dei Fenomeni Nonlineari*

We are indebted to the Referee for a very careful reading and many valuable suggestions and comments.

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KEY WORDS AND PHRASES: equations with memory, singular limit, strongly continuous semigroups, global attractors, robust exponential attractors

2000 MATHEMATICS SUBJECT CLASSIFICATION: 35B25, 35B40, 35K57, 37L30, 45K05

Received: *June 7th, 2004; revised: October 16th, 2004.*