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SINGULAR LIMIT OF DISSIPATIVE HYPERBOLIC EQUATIONS WITH MEMORY

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Abstract. We consider a class of weakly damped semilinear hyperbolic equations with memory, expressed by a convolution integral. We study the passage to the singular limit when the memory kernel collapses into the Dirac mass at zero, and we establish a convergence result for a proper family of exponential attractors.

1. Formulation of the Problem.

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. For $u = u(x,t) : \Omega \times \mathbb{R} \to \mathbb{R}$, we consider, on the time-interval $\mathbb{R}^+ = (0,\infty)$, the differential equation

$$u_{tt} + \alpha u_t - [\beta + k_\varepsilon(0)]\Delta u - \int_0^\infty k'_\varepsilon(s)\Delta u(t-s)ds + \phi(u) = f, \tag{1}$$

arising, for instance, in the theory of viscoelasticity (cf. [4, 9]). Here, $\alpha > 0$, $\beta > 0$, $\phi : \mathbb{R} \to \mathbb{R}$ is a nonlinear term of (at most) cubic growth, $f : \Omega \to \mathbb{R}$ is the external force, whereas the memory kernel

$$k_\varepsilon(s) = \frac{1}{\varepsilon} k\Big(\frac{s}{\varepsilon}\Big), \qquad \varepsilon \in (0,1],$$

is the rescaling of a smooth decreasing function $k : [0,\infty) \to [0,\infty)$ such that

$$\int_0^\infty k(s)ds = \gamma > 0.$$

Equation (1) is supplemented with the Dirichlet boundary condition

$$u(t)\,|_{\partial\Omega} = 0, \qquad \forall t \in \mathbb{R},$$

and the initial conditions

$$u(t)\,|_{t\le 0} = u_0(t), \qquad u_t(0) = u_1,$$

where $u_0(t)$ and u_1 are prescribed data. Indeed, the convolution integral requires the knowledge of $u(t)$ for all negative times. Since k vanishes at ∞, integrating by parts we can rewrite (1) as

$$u_{tt} + \alpha u_t - \beta\Delta u - \int_0^\infty k_\varepsilon(s)\Delta u_t(t-s)ds + \phi(u) = f. \tag{2}$$

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When $\varepsilon \to 0$, the function k_ε converges in the sense of the distributions to $\gamma\delta_0$, where δ_0 is the Dirac mass at zero. Hence, we formally obtain the limiting equation

$$u_{tt} + \alpha u_t - \beta\Delta u - \gamma\Delta u_t + \phi(u) = f, \tag{3}$$

that is, the (strongly) damped wave equation. In this paper we show that the convergence of (2) to (3) is in fact not just formal. A similar analysis for the reaction-diffusion equation with memory has been carried out in [1].

Following a well-established procedure first devised by Dafermos [2], we set the problem in the so-called *history space setting*. Namely (cf. [8]), putting

$$\mu(s) = -k'(s) \qquad \text{and} \qquad \mu_\varepsilon(s) = \frac{1}{\varepsilon^2}\mu\Big(\frac{s}{\varepsilon}\Big),$$

and introducing the *past history* $\eta^t(s) = u(t) - u(t-s)$, we translate (2) into the system

$$\begin{cases} u_{tt} + \alpha u_t - \beta\Delta u - \displaystyle\int_0^\infty \mu_\varepsilon(s)\Delta\eta(s)ds + \phi(u) = f, \\ \eta_t = -\eta_s + u_t, \end{cases}$$

for $t \in \mathbb{R}^+$, with the boundary conditions (now for $t \geq 0$)

$$u(t)\,|_{\partial\Omega} = 0, \qquad \eta^t\,|_{\partial\Omega} = 0, \qquad \eta^t(0) = 0,$$

and the initial conditions

$$u(0) = u_0, \qquad u_t(0) = u_1, \qquad \eta^0 = \eta_0,$$

having set $u_0 = u_0(0)$ and $\eta_0(s) = u_0(0) - u_0(-s)$. As pointed out in [5], upon making some reasonable assumptions on the kernel and on the nonlinearity, there is a complete equivalence between the two formulations.

For the sake of simplicity, from now on we set $\alpha = \beta = \gamma = 1$. Also, we assume that $k(0) = 1$, which can always be obtained by rescaling k. Observe that we have

$$\int_0^\infty \mu_\varepsilon(s)ds = \frac{1}{\varepsilon} \qquad \text{and} \qquad \int_0^\infty s\mu_\varepsilon(s)ds = 1.$$

Notation. Given a Banach space $\mathcal{H}$, we denote by $B_\mathcal{H}(R)$ the closed ball in $\mathcal{H}$ of radius $R \geq 0$ centered at zero. Let $A = -\Delta$ be the Laplace operator on $\big(L^2(\Omega), \langle\cdot,\cdot\rangle, \|\cdot\|\big)$ with domain $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Then, for $r \in \mathbb{R}$, we have the hierarchy of compactly nested Hilbert spaces $H^r = \mathcal{D}(A^{r/2})$ endowed with the inner products $\langle u_1, u_2\rangle_{H^r} = \langle A^{r/2}u_1, A^{r/2}u_2\rangle$. Concerning the memory kernel, we assume $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, with $\mu \geq 0$, satisfying the condition

$$\mu'(s) + \delta\mu(s) \leq 0, \qquad \forall s \in \mathbb{R}^+, \text{ for some } \delta > 0. \tag{4}$$

Then, we introduce the weighted Hilbert spaces $\mathcal{M}_\varepsilon^r = L^2_{\mu_\varepsilon}(\mathbb{R}^+, H^{r+1})$ endowed with the usual inner products. Let $T_\varepsilon = -\partial_s$ be the linear operator on $\mathcal{M}_\varepsilon^0$ with domain $\mathcal{D}(T_\varepsilon) = \big\{\eta \in \mathcal{M}_\varepsilon^0 : \eta_s \in \mathcal{M}_\varepsilon^0,\, \eta(0) = 0\big\}$. On account of (4), there holds (cf. [5])

$$\langle T_\varepsilon\eta, \eta\rangle_{\mathcal{M}_\varepsilon^0} \leq -\frac{\delta}{2\varepsilon}\|\eta\|^2_{\mathcal{M}_\varepsilon^0}, \qquad \forall \eta \in \mathcal{D}(T_\varepsilon). \tag{5}$$

Next, we introduce the Banach spaces (see [1]) $\mathcal{L}_\varepsilon^r = \big\{\eta : \|\eta\|_{\mathcal{L}_\varepsilon^r} < \infty\big\}$, where

$$\|\eta\|^2_{\mathcal{L}_\varepsilon^r} = \|\eta\|^2_{\mathcal{M}_\varepsilon^r} + |||\eta|||^2_\varepsilon,$$

having set

$$|||\eta|||_{\varepsilon}^{2} = \varepsilon \|T_{\varepsilon}\eta\|_{\mathcal{M}_{\varepsilon}^{0}}^{2} + \varepsilon \sup_{x\geq 1} x \int_{(0,\frac{1}{x})\cup(x,\infty)} \mu_{\varepsilon}(s)\|A^{1/2}\eta(s)\|^{2} ds.$$

Due to [8, Lemma 5.5], the embedding $\mathcal{L}_{\varepsilon}^{r} \subset \mathcal{M}_{\varepsilon}^{0}$ is compact for every $r > 0$ (contrary to the embedding $\mathcal{M}_{\varepsilon}^{r} \subset \mathcal{M}_{\varepsilon}^{0}$ which is only continuous). Finally, for $\varepsilon \in [0,1]$, we define the product Banach spaces

$$\mathcal{H}_{\varepsilon}^{r} = \begin{cases} H^{r+1} \times H^{r} \times \mathcal{M}_{\varepsilon}^{r}, & \text{if } \varepsilon > 0, \\ H^{r+1} \times H^{r}, & \text{if } \varepsilon = 0, \end{cases} \qquad \mathcal{Z}_{\varepsilon}^{r} = \begin{cases} H^{r+1} \times H^{r} \times \mathcal{L}_{\varepsilon}^{r}, & \text{if } \varepsilon > 0, \\ H^{r+1} \times H^{r}, & \text{if } \varepsilon = 0. \end{cases}$$

When $\varepsilon = 0$, we agree to interpret (u, v, η) just as (u, v). We will also make use of the *lifting map* $\mathbb{L}_{\varepsilon} : \mathcal{H}_{0}^{0} \to \mathcal{H}_{\varepsilon}^{0}$, and of the *projection maps* $\mathbb{P}$ and $\mathbb{Q}_{\varepsilon}$ on $\mathcal{H}_{\varepsilon}^{0}$, given by

$$\mathbb{L}_{\varepsilon}(u, v) = (u, v, 0), \qquad \mathbb{P}(u, v, \eta) = (u, v), \qquad \mathbb{Q}_{\varepsilon}(u, v, \eta) = \eta.$$

2. **The Dissipative Dynamical System.** Let $f \in H^{0}$ be independent of time, and let $\phi \in C^{2}(\mathbb{R})$ satisfy the dissipation and the growth conditions

$$|\phi''(r)| \leq c(1 + |r|), \qquad \forall r \in \mathbb{R}, \tag{6}$$

$$\liminf_{|r|\to\infty} \frac{\phi(r)}{r} > -\lambda_{1}, \tag{7}$$

where λ_{1} is the first eigenvalue of A. Then, we consider the following family of problems, depending on $\varepsilon \in [0,1]$.

Problem. *Given $z = (u_{0}, u_{1}, \eta_{0}) \in \mathcal{H}_{\varepsilon}^{0}$, find $(u, u_{t}, \eta) \in C([0,\infty), \mathcal{H}_{\varepsilon}^{0})$ solution to*

$$\begin{cases} u_{tt} + u_{t} + Au + \displaystyle\int_{0}^{\infty} \mu_{\varepsilon}(s) A\eta(s) ds + \phi(u) = f, \\ \eta_{t} = T_{\varepsilon}\eta + u_{t}, \end{cases}$$

for $t > 0$, satisfying the initial condition $(u(0), u_{t}(0), \eta^{0}) = z$.

With the adopted notation, $\boldsymbol{P_{\varepsilon}}$ includes the limiting case $\varepsilon = 0$ associated with equation (3), provided that we interpret the integral term as Au_{t} (whereas the second equation is vacuously true). Throughout the paper, we will denote by $c \geq 0$ a generic constant. All the quantities appearing in the sequel are understood to be independent of $\varepsilon \in [0,1]$.

Theorem 1. *For every $\varepsilon \geq 0$, $\boldsymbol{P_{\varepsilon}}$ defines a strongly continuous semigroup (or dynamical system) $S_{\varepsilon}(t)$ on the phase-space $\mathcal{H}_{\varepsilon}^{0}$. In particular, the third component η of the solution has the explicit representation formula*

$$\eta^{t}(s) = \begin{cases} u(t) - u(t-s), & 0 < s \leq t, \\ \eta_{0}(s-t) + u(t) - u(0), & s > t. \end{cases}$$

Moreover, the continuous dependence estimate

$$\|S_{\varepsilon}(t)z_{1} - S_{\varepsilon}(t)z_{2}\|_{\mathcal{H}_{\varepsilon}^{0}} \leq ce^{ct}\|z_{1} - z_{2}\|_{\mathcal{H}_{\varepsilon}^{0}}$$

holds for some $c \geq 0$ and every $t \geq 0$ and $z_{1}, z_{2} \in \mathcal{H}_{\varepsilon}^{0}$.

Theorem 2. *There exists $R_0 \geq 0$ such that, given any bounded set $\mathcal{B} \subset \mathcal{H}^0_\varepsilon$ there is $t_0 = t_0(\mathcal{B}) \geq 0$ such that*

$$\sup_{z\in\mathcal{B}} \|S_\varepsilon(t)z\|_{\mathcal{H}^0_\varepsilon} \leq R_0, \qquad \forall t \geq t_0.$$

Moreover, denoting $R = \sup_{z\in\mathcal{B}} \|z\|_{\mathcal{H}^0_\varepsilon}$, there exist constants $C_0 \geq 0$ and $\Lambda_0 \geq 0$ (both depending on R) such that

$$\sup_{z\in\mathcal{B}} \sup_{t\geq 0} \|S_\varepsilon(t)z\|_{\mathcal{H}^0_\varepsilon} \leq C_0 \qquad \textit{and} \qquad \sup_{z\in\mathcal{B}} \int_0^\infty \|u_t(y)\|^2 dy \leq \Lambda_0.$$

In particular, the result says that the set $\mathcal{B}^0_\varepsilon = B_{\mathcal{H}^0_\varepsilon}(R_0)$ is a *bounded absorbing set* for $S_\varepsilon(t)$ on $\mathcal{H}^0_\varepsilon$, uniformly with respect to ε.

The proofs of the above theorems recast arguments used in the papers [6, 8], and are therefore omitted. Following [1], with the obvious formal changes, we also have

Lemma 1. *Let $\eta_0 \in \mathcal{D}(T_\varepsilon)$, and assume that $\|A^{1/2}u_t(t)\| \leq K$ for some $K > 0$ and every $t \geq 0$. Then $\eta^t \in \mathcal{D}(T_\varepsilon)$ and*

$$|\!|\!|\eta^t|\!|\!|^2_\varepsilon \leq 2(t+2)e^{-\delta t}|\!|\!|\eta_0|\!|\!|^2_\varepsilon + cK^2,$$

for all $t \geq 0$.

3. **Exponentially Attracting Sets.** Our first step is to show that $\boldsymbol{P_\varepsilon}$ enjoys a strong dissipativity property. Namely, there exists a regular (exponentially) attracting set which, in addition, absorbs itself under the action of $S_\varepsilon(t)$.

Theorem 3. *There exist $R_1 \geq 0$, $\kappa > 0$ and $M \geq 0$ such, denoting $\mathcal{B}_\varepsilon = B_{\mathcal{Z}^1_\varepsilon}(R_1)$, there holds*

$$\mathrm{dist}_{\mathcal{H}^0_\varepsilon}\big(S_\varepsilon(t)\mathcal{B}^0_\varepsilon, \mathcal{B}_\varepsilon\big) \leq Me^{-\kappa t}, \qquad \forall t \geq 0. \tag{8}$$

Moreover, there is $t_1 \geq 0$ such that

$$S_\varepsilon(t)\mathcal{B}_\varepsilon \subset \mathcal{B}_\varepsilon, \qquad \forall t \geq t_1. \tag{9}$$

As a consequence, by means of standard arguments of the theory of dynamical systems (cf. [10]), we obtain

Theorem 4. *The semigroup $S_\varepsilon(t)$ possesses a connected global attractor $\mathcal{A}_\varepsilon$ on $\mathcal{H}^0_\varepsilon$ which is bounded in $\mathcal{Z}^1_\varepsilon$.*

The rest of the section is devoted to the proof of Theorem 3, which requires several passages. We restrict to the case $\varepsilon > 0$; the corresponding arguments for $\varepsilon = 0$ can be easily deduced. Besides, we will diffusely make use without explicit mention of the Young, the Hölder and the Poincaré inequalities, as well as of the Sobolev embeddings.

Lemma 2. *Let $\sigma \in [0,1]$ be given, and assume that $\|z\|_{\mathcal{H}^\sigma_\varepsilon} \leq R$, for some $R \geq 0$. Then there exist constants $C_\sigma \geq 0$ and $\Lambda_\sigma \geq 0$ (both depending on R) such that*

$$\|S_\varepsilon(t)z\|_{\mathcal{H}^\sigma_\varepsilon} \leq C_\sigma, \qquad \forall t \geq 0,$$

and

$$\int_\tau^t \|A^{\sigma/2}u_t(y)\|^2 dy \leq \Lambda_\sigma\big(1 + \sqrt{t-\tau}\,\big), \qquad \forall t \geq \tau \geq 0.$$

Proof. The result for $\sigma = 0$ is contained in Theorem 2. Therefore, let $\sigma \in (0,1]$. In the following, the generic constant c will depend *only* on the $\mathcal{H}^0_\varepsilon$-norm of the initial data z. For $\nu \in [0,\nu_0]$, with $\nu_0 > 0$ to be fixed later, set $\psi = u_t + \nu u$ and define the functional

$$\Phi(t) = \|A^{(1+\sigma)/2}u(t)\|^2 + \|A^{\sigma/2}\psi(t)\|^2 + \|\eta^t\|^2_{\mathcal{M}^\sigma_\varepsilon} + \mathcal{G}(t) + c,$$

where

$$\mathcal{G}(t) = 2\langle \phi(u(t)), A^\sigma u(t)\rangle - 2\langle f, A^\sigma u(t)\rangle.$$

Choosing the above constant c large enough and ν_0 small enough, it is readily seen that

$$\frac{1}{2}\|S_\varepsilon(t)z\|^2_{\mathcal{H}^\sigma_\varepsilon} \le \Phi(t) \le 2\|S_\varepsilon(t)z\|^2_{\mathcal{H}^\sigma_\varepsilon} + c,$$

for all $\nu \in [0,\nu_0]$. Multiplying the first equation of Problem $\boldsymbol{P_\varepsilon}$ by $A^\sigma\psi$ in H^0 and the second by η in $\mathcal{M}^\sigma_\varepsilon$ bears

$$\begin{aligned}
&\frac{d}{dt}\Phi + 2\nu\|A^{(1+\sigma)/2}u\|^2 + 2(1-\nu)\|A^{\sigma/2}\psi\|^2 + \frac{\delta}{\varepsilon}\|\eta\|^2_{\mathcal{M}^\sigma_\varepsilon} + \nu\mathcal{G}(t)\\
&\le 2\nu(1-\nu)\langle A^{\sigma/2}u, A^{\sigma/2}\psi\rangle - 2\nu\langle \eta, u\rangle_{\mathcal{M}^\sigma_\varepsilon} + 2\langle \phi'(u)u_t, A^\sigma u\rangle.
\end{aligned}$$

Due to (5), it is quite immediate to check that, provided that ν_0 is small enough,

$$\begin{aligned}
&2\nu(1-\nu)\langle A^{\sigma/2}u, A^{\sigma/2}\psi\rangle - 2\nu\langle \eta, u\rangle_{\mathcal{M}^\sigma_\varepsilon}\\
&\le \nu\|A^{(1+\sigma)/2}u\|^2 + c\nu\|A^{\sigma/2}\psi\|^2 + \frac{\delta}{2\varepsilon}\|\eta\|^2_{\mathcal{M}^\sigma_\varepsilon}.
\end{aligned}$$

Concerning the last term, in light of (6) and the inequality (for $\sigma = 1$ this is the Agmon inequality, interpreting $\frac{6}{1-\sigma}$ as ∞)

$$\|u\|^2_{L^{6/(1-\sigma)}} \le c\|A^{1/2}u\|\|A^{(1+\sigma)/2}u\|,$$

there holds

$$\begin{aligned}
2\langle \phi'(u)u_t, A^\sigma u\rangle &\le c\big(1 + \|u\|^2_{L^{6/(1-\sigma)}}\big)\|u_t\|\|A^\sigma u\|_{L^{6/(1+2\sigma)}}\\
&\le c\big(1 + \|A^{(1+\sigma)/2}u\|\big)\|u_t\|\|A^{(1+\sigma)/2}u\|\\
&\le c\|u_t\| + c\|u_t\|\Phi.
\end{aligned}$$

Hence, up to further reducing ν_0, we end up with the differential inequality

$$\frac{d}{dt}\Phi + \nu\Phi + \|A^{\sigma/2}\psi\|^2 \le c\|u_t\|\Phi + c\|u_t\| + c\nu, \tag{10}$$

that holds for all $\nu \in [0,\nu_0]$. From Theorem 2,

$$\int_\tau^t \|u_t(y)\|dy \le c\sqrt{t-\tau}.$$

Then, exploiting a generalized Gronwall lemma (cf. [6, Lemma 2.2]), we obtain

$$\|S_\varepsilon(t)z\|^2_{\mathcal{H}^\sigma_\varepsilon} \le c\|z\|^2_{\mathcal{H}^\sigma_\varepsilon} e^{-\frac{\nu}{2}t} + c, \qquad \forall t \ge 0. \tag{11}$$

In particular, this yields the first assertion of the lemma. The second one follows by setting $\nu = 0$ and integrating (10) on (τ, t). $\square$

The next step is to show that $S_\varepsilon(t)$ has an asymptotic regularizing property. In order to pursue this aim, we need to find a suitable decomposition of the semigroup.

It is quite immediate to check that a function ϕ satisfying (6) can be written as the sum $\phi = \phi_0 + \phi_1$, where $\phi_0, \phi_1 \in C^2(\mathbb{R})$ fulfill

$$|\phi_0''(r)| \le c(1+|r|), \qquad \forall r \in \mathbb{R}, \tag{12}$$

$$\phi_0'(0) = 0, \tag{13}$$

$$\phi_0(r)r \ge 0, \qquad \forall r \in \mathbb{R}, \tag{14}$$

$$|\phi_1'(r)| \le c, \qquad \forall r \in \mathbb{R}. \tag{15}$$

Then, for $z = (u_0, u_1, \eta_0) \in \mathcal{B}_0^\varepsilon$, we write

$$S_\varepsilon(t)z = L_\varepsilon(t)z + K_\varepsilon(t)z,$$

where $L_\varepsilon(t)z = (v(t), v_t(t), \xi^t)$ and $K_\varepsilon(t)z = (w(t), w_t(t), \zeta^t)$ solve the problems

$$\begin{cases} v_{tt} + v_t + Av + \displaystyle\int_0^\infty \mu_\varepsilon(s)A\psi(s)ds + \phi_0(v) = 0, \\ \psi_t = T_\varepsilon \psi + v_t, \\ (v(0), v_t(0), \xi^0) = z, \end{cases}$$

and

$$\begin{cases} w_{tt} + w_t + Aw + \displaystyle\int_0^\infty \mu_\varepsilon(s)A\zeta(s)ds + \phi(u) - \phi_0(v) = f, \\ \zeta_t = T_\varepsilon \zeta + w_t, \\ (w(0), w_t(0), \zeta^0) = 0. \end{cases}$$

Adapting a quite standard argument used in [7, Lemma 5.5] to the present case with memory, by force of (14) we can prove the existence of $\Gamma \ge 0$ and $\kappa_0 > 0$ such that

$$\|L_\varepsilon(t)z\|_{\mathcal{H}_\varepsilon^0} \le \Gamma e^{-\kappa_0 t}, \qquad \forall t \in \mathbb{R}^+. \tag{16}$$

Lemma 3. *Let $\sigma \in [0, \frac{3}{4}]$ be fixed, and assume that $\|z\|_{\mathcal{H}_\varepsilon^\sigma} \le R$, for some $R \ge 0$. Then there exists $\rho_\sigma = \rho_\sigma(R)$ such that*

$$\|K_\varepsilon(t)z\|_{\mathcal{H}_\varepsilon^{(1+4\sigma)/4}} \le \rho_\sigma, \qquad \forall t \in \mathbb{R}^+.$$

Proof. In this proof, $c > 0$ will depend only on R. For $\nu > 0$ to be fixed, consider the energy functional

$$\Phi_c(t) = \|A^{(5+4\sigma)/8}w(t)\|^2 + \|A^{(1+4\sigma)/8}\psi_c(t)\|^2 + \|\zeta^t\|^2_{\mathcal{M}_\varepsilon^{(1+4\sigma)/4}} + \mathcal{G}_c(t) + c,$$

where $\psi_c = w_t + \nu w$ and

$$\mathcal{G}_c(t) = 2\langle \phi(u(t)) - \phi_0(v(t)), A^{(1+4\sigma)/4}w(t)\rangle - 2\langle f, A^{(1+4\sigma)/4}w(t)\rangle.$$

Fixing the above constant c large enough and ν small enough, we have

$$\frac{1}{2}\|K_\varepsilon(t)z_0\|^2_{\mathcal{H}_\varepsilon^{(1+4\sigma)/4}} \le \Phi_c(t) \le 2\|K_\varepsilon(t)z_0\|^2_{\mathcal{H}_\varepsilon^{(1+4\sigma)/4}} + c.$$

This, in light of (6), (15) and Lemma 2, follows at once from the inequalities

$$\begin{aligned} &2\big|\langle \phi_0(u) - \phi_0(v), A^{(1+4\sigma)/4}w\rangle\big| \\ &\le c\big(1 + \|u\|^2_{L^6} + \|v\|^2_{L^6}\big)\|w\|_{L^{12/(5-4\sigma)}}\|A^{(1+4\sigma)/4}w\|_{L^{12/(3+4\sigma)}} \\ &\le c\|A^{(1+4\sigma)/8}w\|\|A^{(5+4\sigma)/8}w\| \\ &\le \frac{1}{4}\|A^{(5+4\sigma)/8}w\|^2 + c, \end{aligned}$$

and

$$\begin{aligned}2\big|\langle\phi_1(u), A^{(1+4\sigma)/4}w\rangle\big| &\le c\big(1+\|u\|_{L^{12/(9-4\sigma)}}\big)\|A^{(1+4\sigma)/4}w\|_{L^{12/(3+4\sigma)}}\\ &\le c\|A^{(5+4\sigma)/8}w\|\\ &\le \frac{1}{4}\|A^{(5+4\sigma)/8}w\|^2+c.\end{aligned}$$

Multiplying the first equation of the system by $A^{(1+4\sigma)/4}\psi_c$ and the second by ζ in $\mathcal{M}_\varepsilon^{(1+4\sigma)/4}$, arguing as in the proof of Lemma 2 we get the differential inequality

$$\begin{aligned}&\frac{d}{dt}\Phi_c+\nu\Phi_c+\|A^{(1+4\sigma)/8}\psi_c\|^2\\ &\le 2\langle(\phi_0'(u)-\phi_0'(v))u_t, A^{(1+4\sigma)/4}w\rangle+2\langle\phi_0'(v)w_t, A^{(1+4\sigma)/4}w\rangle\\ &\quad+2\langle\phi_1'(u)u_t, A^{(1+4\sigma)/4}w\rangle+c,\end{aligned}$$

provided that $\nu>0$ is small enough. Due to (12),

$$\begin{aligned}&2\langle(\phi_0'(u)-\phi_0'(v))u_t, A^{(1+4\sigma)/4}w\rangle\\ &\le c\|w\|_{L^{12}}(1+\|u\|_{L^6}+\|v\|_{L^6})\|u_t\|_{L^{6/(3-2\sigma)}}\|A^{(1+\sigma)/4}w\|_{L^{12/(3+4\sigma)}}\\ &\le c\|A^{\sigma/2}u_t\|\|A^{5/8}w\|\|A^{(5+4\sigma)/8}w\|\\ &\le c\|A^{\sigma/2}u_t\|\|A^{(5+4\sigma)/8}w\|^2\\ &\le c\|A^{\sigma/2}u_t\|\Phi_c.\end{aligned}$$

On account of (12)-(13) and (16),

$$\begin{aligned}2\langle\phi_0'(v)w_t, A^{(1+4\sigma)/4}w\rangle &\le (\|v\|_{L^3}+\|v\|_{L^6}^2)\|w_t\|_{L^{12/(5-4\sigma)}}\|A^{(1+4\sigma)/4}w\|_{L^{12/(3+4\sigma)}}\\ &\le c\|A^{1/2}v\|\|A^{(1+4\sigma)/8}w_t\|\|A^{(5+4\sigma)/8}w\|\\ &\le \|A^{(1+4\sigma)/8}\psi_c\|^2+c\|A^{1/2}v\|\Phi_c.\end{aligned}$$

Finally, from (15),

$$\begin{aligned}2\langle\phi_1'(u)u_t, A^{(1+4\sigma)/4}w\rangle &\le c\|u_t\|_{L^{6/(3-2\sigma)}}\|A^{(1+4\sigma)/4}w\|_{L^{12/(3+4\sigma)}}\\ &\le c\|A^{\sigma/2}u_t\|\|A^{(5+4\sigma)/8}w\|\\ &\le c\|A^{\sigma/2}u_t\|+c\|A^{\sigma/2}u_t\|\Phi_c.\end{aligned}$$

Collecting the above inequalities, and setting

$$g(t)=c\|A^{1/2}v(t)\|+c\|A^{\sigma/2}u_t(t)\|,$$

we end up with

$$\frac{d}{dt}\Phi_c+\nu\Phi_c\le g\Phi_c+g+c.$$

Since from Lemma 2 and (16)

$$\int_\tau^t g(y)dy\le c\big(1+\sqrt{t-\tau}\,\big),$$

the thesis follows from an application of the generalized Gronwall lemma [6, Lemma 2.2], recalling that $\Phi_c(0)=c$. $\square$

With R_0 as in Theorem 2, we now choose

$$\rho_0 = \rho_0(R_0), \qquad \rho_{1/4} = \rho_{1/4}(\rho_0), \qquad \rho_{1/2} = \rho_{1/2}(\rho_{1/4}), \qquad \rho_{3/4} = \rho_{3/4}(\rho_{1/2}),$$

and we define, for $j = 0, 1, 2, 3$,

$$\mathcal{B}_\varepsilon^{(j+1)/4} = \big\{ z \in \mathcal{H}_\varepsilon^{(j+1)/4} : \|z\|_{\mathcal{H}_\varepsilon^{(j+1)/4}} \le \rho_{j/4} \big\}.$$

Hence, applying recursively Lemma 3 along with (16), there exist constants $M_j \ge 0$, for $j = 0, 1, 2, 3$, such that

$$\mathrm{dist}_{\mathcal{H}_\varepsilon^0}(S_\varepsilon(t)\mathcal{B}_\varepsilon^{j/4}, \mathcal{B}_\varepsilon^{(j+1)/4}) \le M_j e^{-\nu_0 t}.$$

Finally, setting

$$\mathcal{B}_\varepsilon = \big\{ z \in \mathcal{Z}_\varepsilon^1 : \|z\|_{\mathcal{Z}_\varepsilon^1} \le R_1 \big\},$$

for some $R_1 \ge \rho_{3/4}$ large enough, using Lemma 1, we find the further exponential attraction property

$$\mathrm{dist}_{\mathcal{H}_\varepsilon^0}(S_\varepsilon(t)\mathcal{B}_\varepsilon^1, \mathcal{B}_\varepsilon) \le M_\star e^{-\nu_\star t},$$

for some $M_\star \ge 0$ and $\nu_\star > 0$. Thanks to the continuous dependence estimate of Theorem 1, the above chain of attractions can be connected, exploiting the so-called transitivity of the exponential attraction property (see [3, Theorem 5.1]), so to obtain the desired formula (8).

Thus, we are left to prove that $\mathcal{B}_\varepsilon$ fulfills the absorbing property (9). But this follows directly from (11) for $\sigma = 1$, and Lemma 1, upon (possibly) enlarging the radius $R_1 > 0$. This of course is not a problem, since the redefined $\mathcal{B}_\varepsilon$ (being larger) satisfies *a fortiori* relation (8). The proof of Theorem 3 is then concluded.

4. **Robust Exponential Attractors.** The exponential attraction property furnished by Theorem 3 allows us to prove a sharp result on the asymptotic behavior of solutions to problem $\boldsymbol{P_\varepsilon}$. Namely, the existence of a family of exponential attractors for the semigroups $S_\varepsilon(t)$ on the phase-space $\mathcal{H}_\varepsilon^0$, which is robust with respect to the singular limit $\varepsilon \to 0$.

Theorem 5. *For every $\varepsilon \in [0,1]$ there exists a set $\mathcal{E}_\varepsilon$, compact in $\mathcal{H}_\varepsilon^0$ and bounded in $\mathcal{Z}_\varepsilon^1$, which satisfies the following conditions.*

- (i) *$S_\varepsilon(t)\mathcal{E}_\varepsilon \subset \mathcal{E}_\varepsilon$, for every $t \ge 0$.*
- (ii) *There exist $\omega > 0$ and a positive increasing function J such that, for every bounded set $\mathcal{B} \subset B_{\mathcal{H}_\varepsilon^0}(R)$, there holds $\mathrm{dist}_{\mathcal{H}_\varepsilon^0}(S_\varepsilon(t)\mathcal{B}, \mathcal{E}_\varepsilon) \le J(R)e^{-\omega t}$.*
- (iii) *The fractal dimension of $\mathcal{E}_\varepsilon$ in $\mathcal{H}_\varepsilon^0$ is uniformly bounded with respect to ε.*
- (iv) *There exist $\Theta \ge 0$ and $\tau \in (0, \frac{1}{16}]$, such that $\mathrm{dist}^{\mathrm{sym}}_{\mathcal{H}_\varepsilon^0}(\mathcal{E}_\varepsilon, \mathbb{L}_\varepsilon\mathcal{E}_0) \le \Theta\varepsilon^\tau$.*

With usual notation, dist and $\mathrm{dist}^{\mathrm{sym}}$ denote the Hausdorff semidistance and symmetric distance (in $\mathcal{H}_\varepsilon^0$), respectively. Observe that (iv) can be equivalently rewritten as

$$\mathrm{dist}^{\mathrm{sym}}_{H^1 \times H^0}(\mathbb{P}\mathcal{E}_\varepsilon, \mathcal{E}_0) + \sup_{z \in \mathcal{E}_\varepsilon} \|\mathbb{Q}z\|_{\mathcal{M}_\varepsilon^0} \le \Theta\varepsilon^\tau.$$

Since $\mathcal{A}_\varepsilon \subset \mathcal{E}_\varepsilon$, as a byproduct we have

Corollary 1. *The fractal dimension of the global attractor $\mathcal{A}_\varepsilon$ in $\mathcal{H}_\varepsilon^0$ is uniformly bounded with respect to ε.*

The proof of Theorem 5 is based on an abstract result formulated by the same authors in [1, Theorem A.2]. We will not report here the argument, which basically parallels the corresponding one for the reaction-diffusion equation with memory [1]. We limit ourselves to say that the key step is Theorem 3 (which really constitutes the novelty respect to the parabolic case treated in [1]), together with the transitivity of the exponential attraction property. Incidentally, when checking the hypotheses of [1, Theorem A.2], one also obtains a result that has an interest in itself, that we report as

Proposition 1. *For every given $T > 0$ and $R \geq 0$ there exists $K_{T,R} \geq 0$ such that*

$$\|S_\varepsilon(t)z - \mathbb{L}_\varepsilon S_0(t)\mathbb{P}z\|_{\mathcal{H}^0_\varepsilon} \leq \|\mathbb{Q}_\varepsilon z\|_{\mathcal{M}^0_\varepsilon} e^{-\frac{\delta t}{4\varepsilon}} + K_{T,R} \sqrt[16]{\varepsilon},$$

for every $t \in [0,T]$ and every $z \in B_{\mathcal{H}^1_\varepsilon}(R)$.

Proposition 1 establishes the convergence of the solutions of $\boldsymbol{P_\varepsilon}$ to the solutions of $\boldsymbol{P_0}$ as $\varepsilon \to 0$ on finite-time intervals.

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