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Calculus of Variations

Multiple solutions to logarithmic Schrödinger equations with periodic potential

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Abstract We study a class of logarithmic Schrödinger equations with periodic potential which come from physically relevant situations and obtain the existence of infinitely many geometrically distinct solutions.

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1 Introduction and results

We consider the equation

$$-\Delta u + V(x)u = Q(x)u \log u^2 \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where the external potential V and the term Q are 1-periodic functions of the variables $x_1, \ldots, x_N$, $Q \in C^1(\mathbb{R}^N)$, $\min_{\mathbb{R}^N} Q > 0$ and $\min_{\mathbb{R}^N}(V + Q) > 0$. The problem is formally associated with the energy functional $J : H^1(\mathbb{R}^N) \to \mathbb{R} \cup \{+\infty\}$ defined by

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$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + (V(x) + Q(x))u^2)dx - \frac{1}{2}\int_{\mathbb{R}^N} Q(x)u^2 \log u^2\, dx.$$

Problem (1.1) admits applications related to quantum mechanics, quantum optics, nuclear physics, transport and diffusion phenomena, open quantum systems, effective quantum gravity, theory of superfluidity and Bose–Einstein condensation (see e.g. [22] and the references therein). We stress that, specifically, periodic potentials V can play a significant rôle in crystals and in artificial crystals formed by light beams. Although the logarithmic Schrödinger equation has been ruled out as a fundamental quantum wave equation by very accurate experiments on neutron diffraction, it is currently under discussion if this equation can be adopted as a simplified model for some physical phenomena. We refer the reader to [7–9] for existence and uniqueness of solutions of the associated Cauchy problem in a suitable functional framework and to a study of orbital stability, with respect to radial perturbations, of the ground state solution (see [3–5]). In light of a simple modification (see formula (2.3) in Sect. 2) of the standard logarithmic Sobolev inequality [15]

$$\int_{\mathbb{R}^N} u^2 \log u^2 \le \frac{a^2}{\pi}\|\nabla u\|_2^2 + (\log \|u\|_2^2 - N(1+\log a))\|u\|_2^2, \quad \text{for } u \in H^1(\mathbb{R}^N) \text{ and } a > 0, \tag{1.2}$$

it is easy to see that $J(u) > -\infty$ for all $u \in H^1(\mathbb{R}^N)$, but there exist u in $H^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} u^2 \log u^2 dx = -\infty$. Thus, in general, J fails to be finite and C^1 smooth on $H^1(\mathbb{R}^N)$.

Due to this loss of smoothness, in order to study existence of solutions, to the best of our knowledge, at least three approaches were used so far in the literature. On the one hand, in [7], the idea is to work in a suitable Banach space W endowed with a Luxemburg type norm in order to make the functional $J : W \to \mathbb{R}$ well defined and C^1 smooth. On the other hand, in [14] the authors penalize the nonlinearity around the origin and try to obtain a priori estimates to get a nontrivial solution at the limit. However, the drawback of these indirect approaches is that the Palais–Smale condition cannot be obtained, due to a loss of coercivity of the functional J, and, in general, no multiplicity result can be obtained by the Lusternik–Schnirelmann category theory. Recently, in [11], in the case of constant potentials V and Q, the existence of infinitely many weak solutions was obtained by considering the functional J on $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ as merely lower semicontinuous and by applying the nonsmooth critical point theory of [12], originally formulated to tackle semilinear elliptic equations with one-sided growth conditions, and based upon the general theory developed in [6,10]. The restriction to the space of radially symmetric functions in $H^1(\mathbb{R}^N)$ is related to having the Palais–Smale condition (in a suitable sense) satisfied at an arbitrary energy level.

In this paper, we shall work in the unrestricted space $H^1(\mathbb{R}^N)$ and we exploit the fact that the functional J, although being nonsmooth, can be decomposed into the sum of a C^1 functional and a convex lower semicontinuous functional. If u is a solution to (1.1), so are the elements of its orbit under the action of $\mathbb{Z}^N$, $\mathcal{O}(u) := \{u(\cdot - k) : k \in \mathbb{Z}^N\}$, and two solutions are said to be *geometrically distinct* if $\mathcal{O}(u_1) \cap \mathcal{O}(u_2) = \emptyset$.

By adapting some arguments in [19] and using tools from [18], we prove the following result.

Theorem 1.1 *Equation* (1.1) *has infinitely many geometrically distinct solutions.*

Furthermore, setting:

$$\mathcal{N} := \left\{ u \in H^1(\mathbb{R}^N) \setminus \{0\} : \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx = \int_{\mathbb{R}^N} Q(x)u^2 \log u^2\, dx \right\}, \tag{1.3}$$

we have the following existence result for ground state solutions to (1.1).

Theorem 1.2 *The infimum* $\inf_{\mathcal{N}} J > 0$ *is attained at a solution* u *to Eq.* (1.1). *Moreover,* $u(x) > 0$ *for all* $x \in \mathbb{R}^N$ *or* $u(x) < 0$ *for all* $x \in \mathbb{R}^N$.

In the case of constant potentials, the ground state solution is known explicitly (it is called the *Gausson* in the physical literature [3–5]) and, as proved in [11], it is nondegenerate, that is to say the dimension of the nullspace of the linearized operator is N, i.e. smallest possible.

In what follows by a *solution* to (1.1) we shall always mean a function $u \in H^1(\mathbb{R}^N)$ such that $u^2 \log u^2 \in L^1(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv)\, dx = \int_{\mathbb{R}^N} Q(x)uv \log u^2\, dx, \quad \text{for all } v \in C_0^\infty(\mathbb{R}^N).$$

Note that since $|u \log u^2| \le C(1 + |u|^q)$, where $q > 1$, we may use local estimates and standard bootstrap to assert that u is, in fact, a classical solution (cf. [17, Appendix B]).

Notation. C, C_1, C_2 etc. will denote positive constants whose exact values are inessential. $\langle .\,, .\rangle$ is the duality pairing between E' and E, where E is a Hilbert (more generally, Banach) space and E' is its dual. $\|\cdot\|_p$ is the norm of the space $L^p(\mathbb{R}^N)$. $2^* := 2N/(N-2)$ if $N \ge 3$ and $2^* := \infty$ if $N = 1$ or 2. $B_R(x)$ denotes the open ball centered at x and having radius R and $S_R(x) := \partial B_R(x)$. For a functional J, we set $J^b := \{u \in E : J(u) \le b\}$, $J_a := \{u \in E : J(u) \ge a\}$ as well as $J_a^b := J_a \cap J^b$.

2 Proof of Theorem 1.1

In order to prove Theorem 1.1, we first need to state some preliminary results.

2.1 Preliminary results

We shall denote by E the Hilbert space of functions $u \in H^1(\mathbb{R}^N)$ and we endow it with the norm

$$\|u\| := \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + (V(x) + Q(x))u^2) dx \right)^{1/2}.$$

Furthermore, let us set:

$$F_1(s) := \begin{cases} -\frac{1}{2}s^2 \log s^2, & |s| < \delta, \\ -\frac{1}{2}s^2(\log \delta^2 + 3) + 2\delta|s| - \frac{1}{2}\delta^2, & |s| > \delta, \end{cases}$$

$$F_2(s) := \begin{cases} 0, & |s| < \delta, \\ \frac{1}{2}s^2 \log(s^2/\delta^2) + 2\delta|s| - \frac{3}{2}s^2 - \frac{1}{2}\delta^2, & |s| > \delta. \end{cases}$$

Then $F_2(s) - F_1(s) = \frac{1}{2}s^2 \log s^2$ and F_1 is convex, provided $\delta > 0$ is sufficiently small which we assume from now on. Moreover, $F_1, F_2 \in C^1(\mathbb{R})$. Re-write the functional J as $J(u) = \Phi(u) + \Psi(u)$, $u \in E$, where we have set:

$$\Phi(u) := \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} Q(x)F_2(u)\, dx,$$

$$\Psi(u) := \int_{\mathbb{R}^N} Q(x)F_1(u)\, dx.$$

Choosing $p \in (2, 2^*)$, we have $|F_2'(s)| \le C|s|^{p-1}$ for some $C > 0$ and all $s \in \mathbb{R}$, and hence it follows that $\Phi \in C^1(E, \mathbb{R})$ [21, Lemma 3.10]. Note that Ψ is convex, $\Psi \ge 0$ and

$\Psi(u) = +\infty$ for certain $u \in E$. Moreover, it is easy to see by Fatou's lemma that Ψ (and therefore also J) is lower semicontinuous (cf. [11, Proposition 2.9]).

Remark 2.1 Theorems 1.1 and 1.2 remain valid for any F_1, F_2 such that $F_1, F_2 \in C^1(\mathbb{R}^N)$, F_1 is convex and nonnegative, $F_1(0) = 0$, $|F_1'(s)|, |F_2'(s)| \leq C(1 + |s|^{p-1})$ for some $C > 0$ and $p \in (2, 2^*)$, and $F_2'(s)/s \to 0$ as $s \to 0$. Some additional conditions may also be needed in order to ensure that the corresponding functional J satisfies the conclusion of Lemma 2.9. The proof is the same except that small modifications are necessary at some places because $|F_2'(s)| \leq C|s|^{p-1}$ may not hold. Instead, for each $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $|F_2'(s)| \leq \varepsilon|s| + C_\varepsilon|s|^{p-1}$.

Lemma 2.2 *If $\Omega \subset \mathbb{R}^N$ is a bounded domain, then Ψ (and hence J) is of class C^1 in $H^1(\Omega)$.*

Proof Since $|F_1'(s)| \leq C(1 + |s|^{p-1})$, the conclusion follows from [21, Lemma 2.16]. In [21] the result is stated in $H_0^1(\Omega)$ but the argument remains valid in $H^1(\Omega)$. □

We shall need the following definitions, essentially taken from [18]:

Definition 2.3 (i) The set

$$\partial J(u) := \{w \in E' : \langle \Phi'(u), v - u\rangle + \Psi(v) - \Psi(u) \geq \langle w, v - u\rangle, \quad \text{for all } v \in E$$

is called the subdifferential of J at u.

(ii) $u \in E$ is a critical point of J if $J(u) < +\infty$ and $0 \in \partial J(u)$, i.e.

$$\langle \Phi'(u), v - u\rangle + \Psi(v) - \Psi(u) \geq 0, \quad \text{for all } v \in E.$$

(iii) (u_n) is a Palais–Smale sequence if $(J(u_n))$ is bounded and there exist $\varepsilon_n \to 0^+$ such that

$$\langle \Phi'(u_n), v - u_n\rangle + \Psi(v) - \Psi(u_n) \geq -\varepsilon_n\|v - u_n\|, \quad \text{for all } v \in E.$$

(iv) The set $D(J) := \{u \in E : J(u) < +\infty\}$ is called the effective domain of J.

Lemma 2.4 *If $u \in D(J)$, then $\partial J(u) \neq \emptyset$, i.e. there exists $w \in E'$ such that*

$$\langle \Phi'(u), v - u\rangle + \Psi(v) - \Psi(u) \geq \langle w, v - u\rangle, \quad \textit{for all } v \in E.$$

Moreover, this w is unique and satisfies

$$\langle \Phi'(u), z\rangle + \int_{\mathbb{R}^N} Q(x)F_1'(u)z\,dx = \langle w, z\rangle \quad \textit{for all } z \in E \textit{ such that } F_1'(u)z \in L^1(\mathbb{R}^N).$$

Proof Assume by contradiction that for each $w \in E'$ there exists $v \in E$ such that

$$\langle \Phi'(u), v - u\rangle + \Psi(v) - \Psi(u) - \langle w, v - u\rangle < 0.$$

Let $u_n \in C_0^\infty(\mathbb{R}^N)$, $u_n \to u$ in $\|\cdot\|$ as $n \to \infty$. Then, by the lower semicontinuity of Ψ,

$$\begin{aligned}\limsup_{n\to\infty} &\big(\langle \Phi'(u_n), v - u_n\rangle + \Psi(v) - \Psi(u_n) - \langle w, v - u_n\rangle\big)\\ &\leq \langle \Phi'(u), v - u\rangle + \Psi(v) - \Psi(u) - \langle w, v - u\rangle < 0.\end{aligned}$$

So $\partial J(u_n) = \emptyset$ for almost all $n \geq 1$ which is impossible because $u_n \in C_0^\infty(\mathbb{R}^N)$ and hence, using Lemma 2.2 and convexity of Ψ,

$$\langle \Phi'(u_n), v - u_n\rangle + \Psi(v) - \Psi(u_n) \geq \langle \Phi'(u_n) + \Psi'(u_n), v - u_n\rangle \quad \text{for all } v.$$

Let now $v = u + tz$, where $z \in C_0^\infty(\mathbb{R}^N)$ and let $w \in \partial J(u)$. Then

$$\langle \Phi'(u), z\rangle + \int_{\mathbb{R}^N} Q(x) \frac{F_1(u+tz) - F_1(u)}{t}\, dx \geq \langle w, z\rangle.$$

Since the integrand is 0 for $x \notin \operatorname{supp} z$, we can pass to the limit as $t \to 0$ (cf. Lemma 2.2) and we obtain

$$\langle \Phi'(u), z\rangle + \int_{\mathbb{R}^N} Q(x) F_1'(u) z\, dx \geq \langle w, z\rangle.$$

Since this also holds for $-z$,

$$\langle \Phi'(u), z\rangle + \int_{\mathbb{R}^N} Q(x) F_1'(u) z\, dx = \langle w, z\rangle, \quad \text{for all } z \in C_0^\infty(\mathbb{R}^N).$$

By density of $C_0^\infty(\mathbb{R}^N)$ in E, w is unique and the equality above holds for all $z \in E$ such that $F_1'(u)z \in L^1(\mathbb{R}^N)$. □

Definition 2.5 Let $u \in D(J)$. Then the unique element $w \in E'$ of $\partial J(u)$ introduced in Lemma 2.4 will be denoted by $J'(u)$.

An immediate consequence of Lemma 2.4 is the following

Corollary 2.6 *Suppose $(J(u_n))$ is bounded. Then $(u_n) \subset E$ is a Palais–Smale sequence if and only if $J'(u_n) \to 0$ in E' as $n \to \infty$, or equivalently,*

$$\lim_{n\to\infty} \sup\{\langle J'(u_n), v\rangle : v \in C_0^\infty(\mathbb{R}^N), \|v\| = 1\} = 0.$$

It follows from Lemma 2.4 that if $u \in D(J)$, then

$$\langle J'(u), v\rangle = \langle \Phi'(u), v\rangle + \int_{\mathbb{R}^N} Q(x) F_1'(u) v\, dx \quad \text{for all } v \in E \text{ such that } F_1'(u)v \in L^1(\mathbb{R}^N).$$

Next we construct a vector field of pseudo-gradient type. Denote the set of critical points of the functional J by K.

Lemma 2.7 *There exist a locally finite countable covering (W_j) of $D(J) \setminus K$, a set of points $(u_j) \subset D(J) \setminus K$ and a locally Lipschitz continuous vector field $H : D(J) \setminus K \to E$ with the following properties:*

(i) *The diameter of W_j and the distance from u_j to W_j tend to 0 as $j \to \infty$.*

(ii) *$\|H(u)\| \leq 1$ and $\langle J'(u), H(u)\rangle > z(u)$, where $z(u) := \min \frac{1}{2}\|J'(u_j)\|$ for all j such that $u \in W_j$.*

(iii) *H has locally compact support, i.e. for each $u_0 \in D(J)\setminus K$ there exists a neighbourhood U_0 of u_0 in $D(J) \setminus K$ and $R > 0$ such that $\operatorname{supp} H(u) \subset B_R(0)$ for all $u \in U_0$.*

(iv) *$J(u) > J(u_j) - \gamma_j$ for all j such that $u \in W_j$, where $\gamma_j > 0$ and $\gamma_j \to 0$ as $j \to \infty$.*

(v) *H is odd in u.*

Remark 2.8 The special properties of the covering (W_j) and the field H in Lemma 2.7 will be essential in the proofs of Lemmas 2.13 and 2.14.

Proof Since E is separable, there exists a countable dense set of points $(\widetilde{u}_k) \subset D(J) \setminus K$. For each k we can choose $\widetilde{v}_k \in C_0^\infty(\mathbb{R}^N)$, $\|\widetilde{v}_k\| = 1$, such that $\langle J'(\widetilde{u}_k), \widetilde{v}_k\rangle > \frac{1}{2}\|J'(\widetilde{u}_k)\|$. Since $\widetilde{v}_k$ has compact support, $u \mapsto \langle J'(u), \widetilde{v}_k\rangle$ is continuous according to Lemma 2.2 and hence

$$\langle J'(u), \widetilde{v}_k\rangle > \frac{1}{2}\|J'(\widetilde{u}_k)\|. \tag{2.1}$$

for all u in some neighbourhood $W(\widetilde{u}_k)$ of $\widetilde{u}_k$ in $D(J) \setminus K$. Moreover, we may assume that the diameter of $W(\widetilde{u}_k)$ tends to 0 as $k \to \infty$ and by the lower semicontinuity of J, $W(\widetilde{u}_k)$ may be chosen so that $J(u) > J(\widetilde{u}_k) - 1/k$ in $W(\widetilde{u}_k)$. Clearly, $(W(\widetilde{u}_k))$ is a covering of $D(J) \setminus K$. Since E is metric and hence paracompact, we can find a locally finite refinement (W_j) of $(W(\widetilde{u}_k))$ and for each W_j we choose $u_j := \widetilde{u}_{k_j}$ for some k_j such that $W_j \subset W(\widetilde{u}_{k_j})$ (note that u_j may not be in W_j). So (W_j) and (u_j) satisfy (i) and by (2.1), the inequality

$$\langle J'(u), v_j\rangle > \frac{1}{2}\|J'(u_j)\|$$

holds for $u_j = \widetilde{u}_{k_j}$, $v_j = \widetilde{v}_{k_j}$ and all $u \in W_j \subset W(\widetilde{u}_{k_j})$.

Let $\widetilde{\rho}_j(u) := \operatorname{dist}(u, E \setminus W_j)$,

$$\rho_j(u) := \widetilde{\rho}_j(u) \Big/ \sum_{j=1}^{\infty} \widetilde{\rho}_j(u) \quad \text{and} \quad H(u) := \sum_{j=1}^{\infty} \rho_j(u) v_j.$$

It is easy to see that the properties (ii)-(iv) are satisfied (in (iv) we take $\gamma_j = 1/k_j$). Moreover, as J is even, H may be constructed so that $H(-u) = -H(u)$ (e.g. by taking $\pm W_j$, $\pm u_j$ etc.). Hence also (v) holds. $\square$

Lemma 2.9 *If (u_n) is a sequence such that $(J(u_n))$ is bounded above and $J'(u_n) \to 0$, then (u_n) is bounded. Moreover, since $\Psi \geq 0$, $(J(u_n))$ is also bounded below and hence it is a Palais–Smale sequence.*

Proof Let $(u_n) \subset E$ be a sequence with $(J(u_n))$ bounded above and $J'(u_n) \to 0$ as $n \to \infty$. Then, choosing $v = u_n$ as test function, we end up with

$$C_1\|u_n\|_2^2 \leq \int_{\mathbb{R}^N} Q(x)u_n^2\,dx = 2J(u_n) - \langle J'(u_n), u_n\rangle \leq C + o(1)\|u_n\|, \quad \text{as } n \to \infty. \tag{2.2}$$

Replacing u by $\sqrt{Q}u$ in (1.2) yields

$$\int_{\mathbb{R}^N} Qu^2 \log(Qu^2)\,dx \leq \frac{2a^2}{\pi}\left(\|\sqrt{Q}\nabla u\|_2^2 + \|u\nabla\sqrt{Q}\|_2^2\right) + \left(\log\|\sqrt{Q}u\|_2^2 - N(1+\log a)\right)\|\sqrt{Q}u\|_2^2.$$

This gives, taking $a > 0$ small enough,

$$\int_{\mathbb{R}^N} Q(x)u^2 \log u^2\,dx \leq \frac{1}{2}\|\nabla u\|_2^2 + C_2(\log\|u\|_2^2 + 1)\|u\|_2^2. \tag{2.3}$$

So using (2.2), we obtain

$$\begin{aligned} C \geq 2J(u_n) &= \|u_n\|^2 - \int_{\mathbb{R}^N} Q(x)u_n^2 \log u_n^2\,dx \geq \frac{1}{2}\|u_n\|^2 - C_2(\log\|u_n\|_2^2 + 1)\|u_n\|_2^2 \\ &\geq \frac{1}{2}\|u_n\|^2 - C_3(1 + \|u_n\|^{1+\delta}), \quad \text{as } n \to \infty, \end{aligned}$$

where we can take $\delta < 1$. Hence (u_n) is bounded, proving the first assertion. Then the second assertion immediately follows. □

Assume throughout the rest of this section that J has only *finitely many* critical orbits and choose a finite set $\mathcal{F} \subset K$ such that $\mathcal{F} = -\mathcal{F}$ and each critical orbit has a unique representative in K.

Lemma 2.10 $\kappa := \inf\{\|u - v\| : u, v \in K,\ u \neq v\} > 0.$

Proof This follows by a straightforward modification of the proof of [19, Lemma 2.13].

In the next three lemmas we adapt some arguments from [19]. □

Lemma 2.11 *Let* $(u_n^1), (u_n^2) \subset E$ *be two Palais–Smale sequences, Then either* $\|u_n^1 - u_n^2\| \to 0$ *as* $n \to \infty$ *or* $\limsup_{n\to\infty} \|u_n^1 - u_n^2\| \geq \kappa$.

Proof By Lemma 2.9, it follows that (u_n^1) and (u_n^2) are bounded in E. Choose $p \in (2, 2^*)$. Then $|F_2'(s)| \leq C|s|^{p-1}$ for some $C > 0$ and all $s \in \mathbb{R}$.

Suppose first $\|u_n^1 - u_n^2\|_p \to 0$ as $n \to \infty$. Then, by Definition 2.3(iii), we obtain

$$\langle \Phi'(u_n^1), u_n^2 - u_n^1\rangle + \Psi(u_n^2) - \Psi(u_n^1) \geq -\varepsilon_n \|u_n^2 - u_n^1\|,$$

and a similar inequality holds with the roles of u_n^1 and u_n^2 interchanged. Hence

$$\begin{aligned}
\|u_n^1 - u_n^2\|^2 &= \langle \Phi'(u_n^1), u_n^1 - u_n^2\rangle - \langle \Phi'(u_n^2), u_n^1 - u_n^2\rangle \\
&\quad + \int_{\mathbb{R}^N} Q(x)(F_2'(u_n^1) - F_2'(u_n^2))(u_n^1 - u_n^2)\, dx \\
&\leq 2\varepsilon_n \|u_n^1 - u_n^2\| + C\int_{\mathbb{R}^N} Q(x)(|u_n^1|^{p-1} + |u_n^2|^{p-1})|u_n^1 - u_n^2|\, dx \\
&\leq 2\varepsilon_n \|u_n^1 - u_n^2\| + D\|u_n^1 - u_n^2\|_p.
\end{aligned}$$

So $\|u_n^1 - u_n^2\| \to 0$.

Suppose now $\|u_n^1 - u_n^2\|_p \not\to 0$. By Lions' lemma [16, Lemma I.1], [21, Lemma 1.21], we can find $\varepsilon > 0$ and $(y_n) \subset \mathbb{R}^N$ with

$$\int_{B_1(y_n)} (u_n^1 - u_n^2)^2\, dx \geq \varepsilon,$$

after passing to a subsequence. Since J is invariant under translations $u \mapsto u(\cdot - k)$, $k \in \mathbb{Z}^N$, we may assume the sequence (y_n) is bounded. Hence, passing to a subsequence once more, $u_n^1 \rightharpoonup u^1$, $u_n^2 \rightharpoonup u^2$ and $u^1 \neq u^2$. The functional Ψ is lower semicontinuous and hence weakly lower semicontinuous (by convexity). So $\Psi(u^1) < \infty$ and therefore $u^1 \in D(J)$. Moreover, since $\langle J'(u_n^1), v\rangle \to 0$ for all $v \in C_0^\infty(\mathbb{R}^N)$, it is easy to see that $u^1 \in K$. Similarly, $u^2 \in K$. Hence

$$\limsup_{n\to\infty} \|u_n^1 - u_n^2\| \geq \|u^1 - u^2\| \geq \kappa,$$

concluding the proof. □

Remark 2.12 For the purpose of the next section we note that if there are finitely many critical orbits below a certain level $d > 0$, then the conclusions of Lemmas 2.10, 2.11 as well as of Lemmas 2.13 and 2.14 below remain valid on J^d. The proofs go through unchanged except that we need to show that $u^1, u^2 \in J^d$ in the proof of Lemma 2.11. Since (u_n^1) is bounded and $J'(u^1) = 0$, we have

$$d \geq J(u_n^1) = J(u_n^1) - \frac{1}{2}\langle J'(u_n^1), u_n^1\rangle + o(1) = \frac{1}{2}\int_{\mathbb{R}^N} Q(x)(u_n^1)^2 dx + o(1)$$
$$\geq \frac{1}{2}\int_{\mathbb{R}^N} Q(x)(u^1)^2 dx + o(1) = J(u^1) - \frac{1}{2}\langle J'(u^1), u^1\rangle + o(1) = J(u^1) + o(1).$$

So $u^1 \in J^d$ and similarly, $u^2 \in J^d$.

Consider now the flow η given by

$$\begin{cases} \frac{d}{dt}\eta(t,u) = -H(\eta(t,u)), \\ \eta(0,u) = u, \ u \in D(J) \setminus K, \end{cases} \tag{2.4}$$

and let $(T^-(u), T^+(u))$ be the maximal existence time for the trajectory $t \mapsto \eta(t,u)$.

Lemma 2.13 *Let $u \in D(J) \setminus K$. Then either $\lim_{t\to T^+(u)} \eta(t,u)$ exists and is a critical point of J or $\lim_{t\to T^+(u)} J(\eta(t,u)) = -\infty$. In the latter case $T^+(u) = +\infty$.*

Proof Since $\eta(s,u) = u - \int_0^s H(\eta(\tau,u))\,d\tau$ and H has locally compact support, $\tau \mapsto J(\eta(\tau,u))$ is continuously differentiable. To see this, consider

$$\frac{1}{h}(\Psi(\eta(t+h,u)) - \Psi(\eta(t,u))) = \frac{1}{h}\int_{\mathbb{R}^N} Q(x)(F_1(\eta(t+h,u)) - F_1(\eta(t,u)))\,dx.$$

Since $\eta(s,u(x)) = u(x)$ for all $s \in [0,t+h]$ and $|x|$ large enough, we can pass to the limit as $h \to 0$ using Lemma 2.2 and we obtain, by Lemma 2.7,

$$\frac{d}{dt}J(\eta(t,u)) = -\langle J'(\eta(t,u)), H(\eta(t,u))\rangle \leq -z(\eta(t,u)) < 0.$$

So $t \mapsto J(\eta(t,u))$ is decreasing.

Suppose $T^+(u) < \infty$ and let $0 \leq s < t < T^+(u)$. Then

$$\|\eta(t,u) - \eta(s,u)\| \leq \int_s^t \|H(\eta(\tau,u))\|\,d\tau \leq t - s.$$

Hence the limit exists and if it is not a critical point, then $\eta(\cdot,u)$ can be continued for $t > T^+(u)$.

Suppose $T^+(u) = +\infty$ and $J(\eta(t,u))$ is bounded below. It suffices to show that for each $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that $\|\eta(t_\varepsilon,u) - \eta(t,u)\| < \varepsilon$ whenever $t \geq t_\varepsilon$. Assuming the contrary, we can find $\varepsilon \in (0,\kappa/2)$ and $(t_n) \subset \mathbb{R}^+$ with $t_n \to +\infty$ and $\|\eta(t_n,u) - \eta(t_{n+1},u)\| = \varepsilon$ for all $n \geq 1$. Choose the smallest $t_n^1 \in (t_n, t_{n+1})$ such that $\|\eta(t_n,u) - \eta(t_n^1,u)\| = \varepsilon/3$ and let $\kappa_n := \min\{z(\eta(s,u)) : s \in [t_n, t_n^1]\}$. Then $\kappa_n > 0$ and

$$\frac{\varepsilon}{3} = \|\eta(t_n^1,u) - \eta(t_n,u)\| \leq \int_{t_n}^{t_n^1} \|H(\eta(s,u))\|\,ds \leq t_n^1 - t_n$$
$$\leq \frac{1}{\kappa_n}\int_{t_n}^{t_n^1} \langle J'(\eta(s,u)), H(\eta(s,u))\rangle\,ds = \frac{1}{\kappa_n}\Big(J(\eta(t_n,u)) - J(\eta(t_n^1,u))\Big).$$

Since $J(\eta(t_n,u)) - J(\eta(t_n^1,u)) \to 0$, it follows that $\kappa_n \to 0$. Hence we can find $s_n^1 \in [t_n, t_n^1]$ such that $z(\eta(s_n^1,u)) \to 0$ as $n \to \infty$. So by Lemma 2.7 there exist u_n^1 (where $u_n^1 = u_{j_n}$ for some j_n) such that $J'(u_n^1) \to 0$, $J(u_n^1) \leq J(\eta(s_n^1,u)) + \gamma_n^1$ and $\|u_n^1 - \eta(s_n^1,u)\| \to 0$. Here it is important that the diameter of W_{j_n} and the distance from u_{j_n} to W_{j_n} in Lemma 2.7 tend to 0 and that (iv) of this lemma gives a uniform bound from above for $J(u_n^1)$. Similarly we can first find the largest $t_n^2 \in [t_n^1, t_{n+1}]$ for which $\|\eta(t_{n+1}, u) - \eta(t_n^2, u)\| = \varepsilon/3$ and then $s_n^2 \in [t_n^1, t_{n+1}]$ and u_n^2 such that $J'(u_n^2) \to 0$, $J(u_n^2) \le J(\eta(s_n^2, u)) + \gamma_n^2$ and $\|u_n^2 - \eta(s_n^2, u)\| \to 0$. Since $(J(u_n^1))$, $(J(u_n^2))$ are bounded above, (u_n^1) and (u_n^2) are Palais–Smale sequences according to Lemma 2.9. Hence

$$\frac{\varepsilon}{3} \le \limsup_{n\to\infty} \|u_n^1 - u_n^2\| \le 2\varepsilon < \kappa,$$

a contradiction to Lemma 2.11. This completes the proof. □

Let $d > 0$ and choose $\varepsilon_0 > 0$ such that $J_{d-2\varepsilon_0}^{d+2\varepsilon_0} \cap K = K_d := \{u \in K : J(u) = d\}$. Denote

$$U_\delta(K_d) := \{u \in E : \operatorname{dist}(u, K_d) < \delta\}.$$

Lemma 2.14 *For each $\delta > 0$ there exists $\varepsilon \in (0, \varepsilon_0)$ such that*

$$\lim_{t\to T^+(u)} J(\eta(t, u)) < d - \varepsilon, \quad \text{whenever } u \in J^{d+\varepsilon} \setminus U_\delta(K_d).$$

Moreover, $\eta(t, u) \notin U_{\delta/2}(K_d) \cap J_{d-\epsilon}$ for any $t \in [0, T^+(u))$.

Proof Assume without loss of generality that $\delta < \kappa$. Let

$$\tau := \inf\{z(u) : u \in J_{d-2\varepsilon_0}^{d+2\varepsilon_0} \cap U_\delta(K_d) \setminus U_{\delta/2}(K_d)\}.$$

We show that $\tau > 0$. Arguing by contradiction, we find a sequence $w_n^1 \in J_{d-2\varepsilon_0}^{d+2\varepsilon_0} \cap U_\delta(K_d) \setminus U_{\delta/2}(K_d)$ with $z(w_n^1) \to 0$ and then, using Lemma 2.7, u_n^1 (where $u_n^1 = u_{j_n}$ for some j_n) such that $J'(u_n^1) \to 0$, $\|u_n^1 - w_n^1\| \to 0$ and $J(u_n^1) \le J(w_n^1) + \gamma_n^1$ ($\gamma_n^1 \to 0$). Hence $J(u_n^1)$ is bounded above, so (u_n^1) is a Palais–Smale sequence by Lemma 2.9. Using Lemma 2.10 and $\mathbb{Z}^N$-invariance of J we may assume $w_n^1 \in U_\delta(u_0) \setminus U_{\delta/2}(u_0)$ for some $u_0 \in K_d$. Set $u_n^2 := u_0$ for all n. This is obviously a Palais–Smale sequence and we have

$$\frac{\delta}{2} \le \limsup_{n\to\infty} \|u_n^1 - u_n^2\| \le \delta < \kappa,$$

a contradiction to Lemma 2.11. So $\tau > 0$. If the conclusion of the lemma is false, there exists $w \in K_d$ such that $\eta(t, u)$ will enter $U_{\delta/2}(w)$. Set

$$\begin{aligned} t_1 &:= \sup\{t \in [0, T^+(u)) : \eta(t, u) \notin U_\delta(w)\}, \\ t_2 &:= \inf\{t \in (t_1, T^+(u)) : \eta(t, u) \in U_{\delta/2}(w)\}. \end{aligned}$$

Then

$$\frac{\delta}{2} \le \|\eta(t_2, u) - \eta(t_1, u)\| \le \int_{t_1}^{t_2} \|H(\eta(s, u))\|\, ds \le t_2 - t_1$$

and therefore

$$J(\eta(t_2, u)) - J(\eta(t_1, u)) = -\int_{t_1}^{t_2} \langle J'(\eta(s, u)), H(\eta(s, u))\rangle\, ds \le -\tau(t_2 - t_1) \le -\frac{\tau\delta}{2}.$$

So $J(\eta(t_2, u)) \le d + \varepsilon - \tau\delta/2 < d - \varepsilon$ for ε small. Hence $\eta(t, u)$ cannot enter $U_{\delta/2}(w) \cap J_{d-\varepsilon}$. □

Lemma 2.15 *There exist $\rho, b > 0$ such that $J(u) \geq 0$ for all $u \in B_\rho(0)$ and $J(u) \geq b$ for all $u \in S_\rho(0)$.*

Proof Recalling that $\Psi \geq 0$ and $|F_2'(s)| \leq C|s|^{p-1}$, we obtain $J(u) \geq \Phi(u) = \frac{1}{2}\|u\|^2 + o(\|u\|^2)$. Hence the conclusion. □

In the proof of Theorem 1.1 we shall need a variant of Benci's pseudoindex [1,2] which we now introduce. Let $\Sigma := \{A \subset D(J) : A = -A \text{ and } A \text{ is compact}\}$ and

$$\mathcal{H} := \{h : D(J) \to E,\ h \text{ odd homeomorphism onto } h(D(J)) \\ \text{and } J(h(u)) \leq J(u) \text{ for all } u \in D(J)\}.$$

Denote Krasnoselskii's genus of $A \in \Sigma$ by $\gamma(A)$ [17] and set

$$i^*(A) := \min_{h \in \mathcal{H}} \gamma(h(A) \cap S_\rho(0)),$$

where ρ is as in Lemma 2.15.

Lemma 2.16 *Let $A, B \in \Sigma$.*

(i) *If $A \subset B$, then $i^*(A) \leq i^*(B)$.*
(ii) *$i^*(A \cup B) \leq i^*(A) + \gamma(B)$.*
(iii) *If $g \in \mathcal{H}$, then $i^*(A) \leq i^*(g(A))$.*
(iv) *Let E_k be a k-dimensional subspace of $D(J)$. Then $i^*(E_k \cap \overline{B}_R(0)) \geq k$ whenever R is large enough.*

Proof (i) follows immediately from the properties of genus [17].
(ii) For each $h \in \mathcal{H}$,

$$i^*(A \cup B) \leq \gamma(h(A \cup B) \cap S_\rho(0)) = \gamma((h(A) \cup h(B)) \cap S_\rho(0)) \\ \leq \gamma(h(A) \cap S_\rho(0)) + \gamma(B).$$

Taking the minimum over all $h \in \mathcal{H}$ on the right-hand side we obtain the conclusion.
(iii) Since $J(g(u)) \leq J(u)$ for all $u \in D(J)$, $h \circ g \in \mathcal{H}$ if $h \in \mathcal{H}$. Hence

$$\min_{h \in \mathcal{H}} \gamma(h(A) \cap S_\rho(0)) \leq \min_{h \in \mathcal{H}} \gamma((h \circ g)(A) \cap S_\rho(0)).$$

(iv) It is easy to see that $J(u) \to -\infty$ uniformly for $u \in E_k$, $\|u\| \to \infty$. So $J(u) < 0$ on $E_k \setminus B_R(0)$ if R is large enough. Let $D := E_k \cap \overline{B}_R(0)$. Suppose $i^*(D) < k$, choose $h \in \mathcal{H}$ such that $\gamma(h(D) \cap S_\rho(0)) < k$ and an odd mapping $f : h(D) \cap S_\rho(0) \to \mathbb{R}^{k-1} \setminus \{0\}$. Let $U := h^{-1}(B_\rho(0)) \cap E_k$. Since $J(h(u)) \leq J(u) < 0$ when $u \in E_k \setminus B_R(0)$, $U \subset D$ according to Lemma 2.15 and hence U is an open and bounded neighbourhood of 0 in E_k. If $u \in \partial U$, then $h(u) \in S_\rho(0)$ and therefore $f \circ h : \partial U \to \mathbb{R}^{k-1} \setminus \{0\}$, contradicting the Borsuk–Ulam theorem [17, Proposition II.5.2], [21, Theorem D.17]. So $i^*(D) \geq k$. □

2.2 Proof of Theorem 1.1 completed

Let

$$d_k := \inf_{i^*(A) \geq k} \sup_{u \in A} J(u).$$

Since there exist sets of arbitrarily large pseudoindex i^*, d_k is well defined for all $k \geq 1$ and it follows from Lemma 2.15 that $d_k \geq b$.

We shall show that $K_{d_k} \neq \emptyset$ and $d_k < d_{k+1}$ for all k, and this contradicts our assumption that there are only finitely many critical orbits. Put $d \equiv d_k$. By Lemma 2.10, $\gamma(K_d) = 0$ (if $K_d = \emptyset$) or 1. Let $U := U_\delta(K_d)$ where δ is so small that $\gamma(\overline{U}) = \gamma(K_d)$ and choose $\varepsilon > 0$ as in Lemma 2.14. Choose $A \in \Sigma$ such that $i^*(A) \geq k$ and $\sup_{u \in A} J(u) \leq d + \varepsilon$. We need to modify the flow η. Let $\chi_1 : E \to [0, 1]$ be locally Lipschitz continuous and such that $\chi_1 = 0$ on $J^{d-2\varepsilon_0}$, $\chi_1 > 0$ otherwise. Since $\{u \in E : J(u) > d + 2\varepsilon_0\}$ is an open set, there exists a locally Lipschitz continuous function $\chi_2 : E \to [0, 1]$ such that $\chi_2 = 1$ on $J^{d+2\varepsilon_0} \setminus U_{\delta/2}(K_d)$ and $\chi_2 = 0$ in a neighbourhood of $K \cap J_{d-2\varepsilon_0}$. Put $\chi(u) = \chi_1(u)\chi_2(u)$. The flow $\widetilde{\eta}$ given by

$$\begin{cases} \frac{d}{dt}\widetilde{\eta}(t, u) = -\chi(\widetilde{\eta}(t, u))H(\widetilde{\eta}(t, u)), \\ \widetilde{\eta}(0, u) = u, \; u \in D(J) \end{cases}$$

is defined for all $t > 0$ an has the same flow lines on $J_{d-\varepsilon}^{d+\varepsilon} \setminus U_{\delta/2}(K_d)$ as η. Choose T so that $J(\widetilde{\eta}(T, u)) < d - \varepsilon$ for all $u \in A \setminus U$. Such T exists because A is compact and $\lim_{t \to T^+(u)} J(\eta(t, u)) \leq d - 2\varepsilon_0$. The properties of pseudoindex give

$$k \leq i^*(A) \leq i^*(A \setminus U) + \gamma(\overline{U}) \leq i^*(\widetilde{\eta}(T, A \setminus U)) + \gamma(\overline{U}) \leq k - 1 + \gamma(K_d).$$

So $\gamma(K_d) \neq 0$ and hence $K_d \neq \emptyset$. If $d \equiv d_k = d_{k+1}$, then $i^*(A) \geq k + 1$ and therefore $\gamma(K_d) \geq 2$ which is impossible. So $d_k < d_{k+1}$ for all k. □

3 Proof of Theorem 1.2

Let $u \in D(J) \setminus \{0\}$. Then the map $s \mapsto J(su)$ admits a unique maximum point on $(0, \infty)$. In fact, if $\varphi : (0, \infty) \to \mathbb{R}$ is the map defined by

$$\varphi(s) := J(su) = J(u)s^2 - s^2 \log s \int_{\mathbb{R}^N} Q(x)u^2 dx, \quad s > 0,$$

we have $\varphi(s) > 0$ for $s > 0$ sufficiently small and $\varphi(s) < 0$ for all $s > 0$ large enough. Moreover, $\varphi'(s) = 0$ with $s > 0$ if and only if

$$J(u) = \frac{2 \log s + 1}{2} \int_{\mathbb{R}^N} Q(x)u^2 dx,$$

which proves the claim. Since $\varphi'(s) = \langle J'(su), u\rangle$, the ray $\{su : s > 0\}$ intersects the Nehari manifold $\mathcal{N}$ (see definition (1.3)) at exactly one point. Moreover, there exists $s_0 > 0$ such that for all $u \in D(J) \cap S_1(0)$, $s \mapsto \Phi(su)$ is increasing when $0 < s < s_0$. Since $s \mapsto \Psi(su)$ is increasing for all $s > 0$ (by convexity), $\mathcal{N}$ is bounded away from 0. Alternatively one can observe that, if $u \in \mathcal{N}$, then inequality (2.3) yields

$$\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) dx \leq \frac{1}{2}\|\nabla u\|_2^2 + C_2(\log \|u\|_2^2 + 1)\|u\|_2^2.$$

Then, if $\|u\|$ is so small that $C_2(\log \|u\|_2^2 + 1) \leq \inf V$ one gets the contradiction $u = 0$. Let

$$\Gamma := \{\alpha \in C([0, 1], E) : \alpha(0) = 0, \; J(\alpha(1)) < 0\}$$

and

$$c := \inf_{\alpha \in \Gamma} \sup_{s \in [0,1]} J(\alpha(s)).$$

By Lemma 2.15, $c \geq b > 0$ (and c is the Mountain Pass level). Clearly, $c \leq c_{\mathcal{N}} := \inf_{\mathcal{N}} J$. Assume that, for some $\varepsilon_0 > 0$, there exists no nontrivial solution below the energy level $c+\varepsilon_0$. According to Remark 2.12, we can use Lemma 2.14 with $U_\delta(K_c) = \emptyset$ and a sufficiently small $\varepsilon < \varepsilon_0$. Let $\chi : E \to [0,1]$ be a locally Lipschitz continuous function such that $\chi = 0$ on $J^{c/2}$, $\chi > 0$ otherwise, and consider the flow

$$\begin{cases} \frac{d}{dt}\widehat{\eta}(t,u) = -\chi(\widehat{\eta}(t,u))H(\widehat{\eta}(t,u)), \\ \widehat{\eta}(0,u) = u, \ \ u \in J^{c+\varepsilon}. \end{cases}$$

Choosing $\alpha \in \Gamma$ such that $\sup_{s\in[0,1]} J(\alpha(s)) \leq c+\varepsilon$ and setting $\beta(s) := \widehat{\eta}(T,\alpha(s))$, where T is large enough, we obtain $\sup_{s\in[0,1]} J(\beta(s)) < c$ which is a contradiction because $\beta \in \Gamma$. Hence there exists a sequence of nontrivial solutions (u_n) with $\limsup_{n\to\infty} J(u_n) \leq c$ (we do not exclude the possibility that $u_n = u$ for all n and some u). Since $u_n \in \mathcal{N}$ and $c \leq c_{\mathcal{N}}$, it follows that $c = c_{\mathcal{N}}$ and thus $J(u_n) \to c$. Obviously, (u_n) is a Palais–Smale sequence, hence it is bounded according to Lemma 2.9 and we may assume that $u_n \rightharpoonup u$ in E as $n \to \infty$. As we have seen earlier, u is a solution for (1.1). If $\|u_n\|_p \to 0$ for some $p \in (2, 2^*)$, then

$$0 = \langle J'(u_n), u_n \rangle \geq \|u_n\|^2 - C\int_{\{u_n^2 \geq 1/e\}} |u_n|^p \, dx,$$

yielding $\|u_n\| \to 0$ as $n \to \infty$, contrary to the fact that $(u_n) \subset \mathcal{N}$. Hence, by means of Lions' lemma [16, Lemma I.1], [21, Lemma 1.21], we have

$$\int_{B_1(y_n)} u_n^2 \, dx \geq \varepsilon,$$

for some sequence $(y_n) \subset \mathbb{R}^N$ and some $\varepsilon > 0$. As in the proof of Lemma 2.11 we may assume, making translations if necessary, that (y_n) is bounded. So for the (translated) sequence (u_n) we have $u_n \rightharpoonup u \neq 0$ as $n \to \infty$. Notice that $J(u) = \inf_{\mathcal{N}} J$. In fact, $J(u) \leq c$ by the same argument as in Remark 2.12 and obviously, $J(u) \geq c$. Hence u is a ground state solution.

Finally, the solution u has constant sign. In fact, let us write $u = u^+ - u^-$. Then $J(u) = J(u^+) + J(u^-)$ and $u^+, u^- \in D(J)$. Moreover, $0 = \langle J'(u), u^+ \rangle = \langle J'(u^+), u^+ \rangle$, so either $u^+ \in \mathcal{N}$ or $u^+ = 0$. A similar conclusion holds for u^-. Hence, either one of the functions u^+, u^- is equal to 0 or $J(u) \geq 2c$, which yields a contradiction. Suppose $u = u^+$. Then, by a slight variant of the argument in [11, Section 3.1] it follows from the maximum principle (see [20, Theorem 1]) that $u(x) > 0$, for a.e. $x \in \mathbb{R}^N$. □

4 A note on the p-Laplacian

Our arguments also allow to prove Theorems 1.1 and 1.2 for the equation

$$-\Delta_p u + V(x)|u|^{p-2}u = Q(x)|u|^{p-2}u \log|u|^p, \quad u \in W^{1,p}(\mathbb{R}^N), \tag{4.1}$$

where $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian $(1 < p < N)$ and V, Q satisfy the conditions stated at the beginning of Sect. 1. Here one needs to make use of a p-logarithmic Sobolev inequality, see e.g. [13] and the references therein. The functional corresponding to (4.1) is

$$J(u) = \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u|^p + (V(x) + Q(x))|u|^p)\,dx \\ - \frac{1}{p}\int_{\mathbb{R}^N} Q(x)|u|^p \log|u|^p\,dx, \quad u \in W^{1,p}(\mathbb{R}^N).$$

In order to show boundedness of the sequence (u_n) with $J(u_n)$ bounded above and $J'(u_n) \to 0$, one needs to use [13, formula (3)] with u replaced by $Q^{1/p}u$ and modify the proof of Lemma 2.9 in a suitable way. We omit the easy but somewhat tedious details.

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