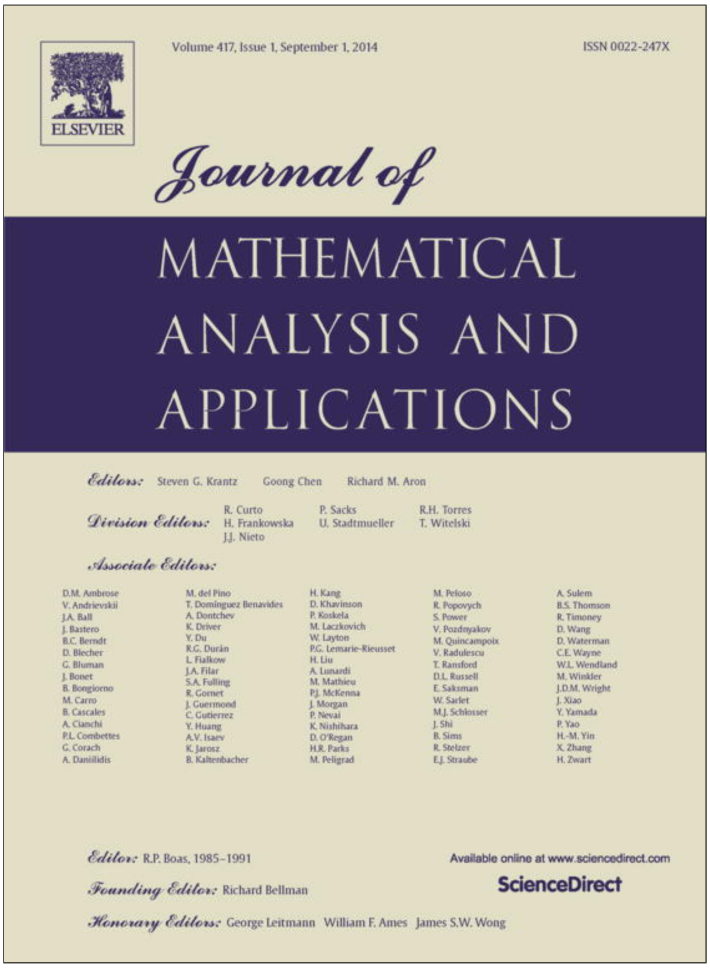
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Soliton dynamics for the generalized Choquard equation ☆

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ABSTRACT

We investigate the soliton dynamics for a class of nonlinear Schrödinger equations with a non-local nonlinear term. In particular, we consider what we call *generalized Choquard equation* where the nonlinear term is $(|x|^{\theta-N} * |u|^p)|u|^{p-2}u$. This problem is particularly interesting because the ground state solutions are not known to be unique or non-degenerate.

1. Introduction

The *soliton dynamics* of the nonlinear Schrödinger equation

$$i\varepsilon\frac{\partial\psi}{\partial t} = -\frac{\varepsilon^2}{2m}\Delta\psi + V(x)\psi - f\big(|\psi|\big)\psi \quad \text{in } (0,\infty)\times\mathbb{R}^N$$

in the last decade has been the object of many mathematical studies. In the case of pure power nonlinearities we just mention the fundamental papers [5,13,18]. Even if the results are accomplished by completely different methods, in all these papers the non-degeneracy of the ground states of the stationary equation plays a fundamental role in getting the modulational equation originally devised by Weinstein in [32,33]. Recently, a new approach was developed in [3,4] not requiring the non-degeneracy of the ground states.

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Another important class of nonlinearities are the non-local Hartree type nonlinearities, i.e.

$$f(|\psi|)\psi = \left(\frac{1}{|x|} * |\psi|^2\right)\psi.$$

Hartree nonlinearities arise in several examples of mathematical physics, as the mean field limit of weakly interacting molecules (see [23] and the references therein), in the Pekar theory of polarons (see [29,30] and [21] for further references), in Schrödinger–Newton systems [15] or, with a semi-relativistic differential operator, in boson stars modeling [14]. In the Hartree case, the non-degeneracy of ground states has been investigated only recently by Lenzmann in [20] and the solitonic dynamics in [11].

The goal of this paper is to obtain a soliton dynamics behavior for a general class of Hartree type nonlinearities for which, currently, neither *uniqueness* nor *non-degeneracy* of ground states are known, by exploiting the techniques of [3,4]. This further corroborates the usefulness and impact of the ideas developed in these papers on a problem which has recently attracted the attention of many researchers, especially in the stationary case.

We consider the following *generalized Choquard equation*

$$i\varepsilon\frac{\partial\psi}{\partial t} = -\frac{\varepsilon^2}{2m}\Delta\psi + V(x)\psi - \big(I_\theta * |\psi|^p\big)|\psi|^{p-2}\psi \quad \text{with } (t,x)\in(0,\infty)\times\mathbb{R}^N, \tag{$\mathcal{GC}$}$$

where $N \geqslant 3$, $\psi : [0,\infty)\times\mathbb{R}^N \to \mathbb{C}$, ε is the Planck constant, $m > 0$ and

$$I_\theta(x) := \frac{\Gamma(\frac{N-\theta}{2})}{\Gamma(\frac{\theta}{2})\pi^{N/2}2^\theta|x|^{N-\theta}},$$

with $\theta \in (0,N)$ a real parameter and

$$p \in \left(1+\frac{\theta}{N}, 1+\frac{2+\theta}{N}\right). \tag{1.1}$$

Moreover let $V : \mathbb{R}^N \to \mathbb{R}$ be a C^2-function satisfying

(V0) $V \geqslant 0$;
(V1) $|\nabla V(x)| \leqslant (V(x))^b$ for $|x| > R_1 > 1$ and $b \in (0,1)$;
(V2) $V(x) \geqslant |x|^a$ for $|x| > R_1 > 1$ and $a > 1$.

By the rescaling $\hat\psi(t,x) = m^{-\frac{\theta}{4(p-1)}}\varepsilon^{\frac{\alpha-\gamma(2p-1)}{2(p-1)}}\psi(t,x/\sqrt{m})$, Eq. $(\mathcal{GC})$ can be written as

$$i\varepsilon\frac{\partial\hat\psi}{\partial t} = -\frac{\varepsilon^2}{2}\Delta\hat\psi + \hat V(x)\hat\psi - \varepsilon^{\gamma(2p-1)-\alpha}\big(I_\theta * |\hat\psi|^p\big)|\hat\psi|^{p-2}\hat\psi, \tag{1.2}$$

where α and γ are real parameters and $\hat V(x) = V(x/\sqrt{m})$. We remark that in [11] it has been treated the physical case ($N = 3$, $\theta = 2$, $p = 2$), passing from $(\mathcal{GC})$ to (1.2) by using the same rescaling with $\gamma = 0$ and $\alpha = 2$. Hence in this paper we study the problem

$$\begin{cases} i\varepsilon\dfrac{\partial\psi}{\partial t} = -\dfrac{\varepsilon^2}{2}\Delta\psi + V(x)\psi - \varepsilon^{\gamma(2p-1)-\alpha}\big(I_\theta * |\psi|^p\big)|\psi|^{p-2}\psi \\ \psi(0,x) = U_\varepsilon(x)e^{\frac{i}{\varepsilon}x\cdot v}, \end{cases} \tag{$\mathcal{P}_\varepsilon$}$$

where $v \in \mathbb{R}^N$ and

$$U_\varepsilon(x) = \varepsilon^{-\gamma} U\big(\varepsilon^{-\beta} x\big), \tag{1.3}$$

U being a real solution of

$$\frac{1}{2}\Delta U + \big(I_\theta * |U|^p\big)|U|^{p-2}U = \omega U, \tag{1.4}$$

with $\omega > 0$, and $\beta \in \mathbb{R}$. Concerning local and global well-posedness of solutions in $H^1(\mathbb{R}^N)$ to $(\mathcal{P}_\varepsilon)$, as well as for conservation laws, we can follow the arguments in [17, Theorems 2.1 and 2.4]. However, we need more regularity for the solution, in particular we need to assume that the solution ψ to problem $(\mathcal{P}_\varepsilon)$ is in $C([0,\infty), H^2(\mathbb{R}^N)) \cap C^1((0,\infty), L^2(\mathbb{R}^N))$. In this direction a result of local existence can be found in [8, Theorem 4.8.1 and Corollary 4.8.6] for the case $p = 2$. The solutions to problem (1.4) have recently been an object of various deep investigations from the point of view of regularity, qualitative properties such as symmetry and asymptotic behavior and concentration properties of semiclassical states. We refer the reader to [9,19,26–28]. Concerning uniqueness of positive radial solutions to (1.4), to our knowledge, after the original contribution due to Lieb [22], a result can be found in [17] in the particular case $\theta = 2$. Finally, regarding the nondegeneracy of the ground states of (1.4), the only case where it is known is, to our knowledge, when $N = 3$, $\theta = 2$ and $p = 2$, see [20,31].

A problem similar to $(\mathcal{P}_\varepsilon)$ arises in the study of Eq. $(\mathcal{GC})$ with so-called semi-classical wave packets (or coherent states) as initial data, see for example [6], and also [7] where the same problem has been studied for the nonlinear Schrödinger equation with local nonlinear term. The main difference with our approach is that in the papers [6,7] the idea is to fix initial conditions with $\beta = \frac{1}{2}$ and $\gamma = \frac{N}{4}$ in (1.3), and then to study the behavior of the solution varying the power of ε in front of the nonlinear term. Instead, we choose the initial conditions according to the values of γ and α, see conditions (2.2), (2.8) and (2.9) below.

The following is the main result of the paper.

Theorem 1.1. *Assume that conditions* (V0)–(V2) *hold, that the solution ψ to problem $(\mathcal{P}_\varepsilon)$ is in $C([0,\infty), H^2(\mathbb{R}^N)) \cap C^1((0,\infty), L^2(\mathbb{R}^N))$, that*

$$\beta = \frac{\alpha + 2 - \gamma}{\theta + 2} > 1,$$

and p is as in (1.1)*. Then the barycenter*

$$q_\varepsilon(t) := \frac{1}{\|\psi(t)\|_2^2} \int_{\mathbb{R}^N} x\big|\psi(t,x)\big|^2\, dx, \tag{1.5}$$

of the solution ψ to problem $(\mathcal{P}_\varepsilon)$ satisfies the Cauchy problem

$$\begin{cases} \ddot{q}_\varepsilon(t) + \nabla V\big(q_\varepsilon(t)\big) = H_\varepsilon(t), \\ q_\varepsilon(0) = 0, \\ \dot{q}_\varepsilon(0) = v, \end{cases} \tag{1.6}$$

where $\|H_\varepsilon\|_{L^\infty(0,\infty)} \to 0$ as $\varepsilon \to 0^+$.

As explained above, we obtain as a corollary the same result for Eq. $(\mathcal{GC})$, namely

Corollary 1.2. *Under the same assumptions of Theorem 1.1, the solution $\psi(t,x)$ to Eq. $(\mathcal{GC})$ with initial condition*

$$\psi(0,x) = \varepsilon^{\frac{\gamma-\alpha}{2(p-1)}} U\left(\varepsilon^{-\beta}x\right) e^{\frac{i}{\varepsilon}x\cdot v}$$

has a barycenter $q_\varepsilon(t)$ *which satisfies Eq.* (1.6) *with* $\|H_\varepsilon\|_{L^\infty(0,\infty)} \to 0$ *as* $\varepsilon \to 0^+$.

We remark that, contrarily to the results obtained for example in [5], we have no information about the shape of the solution. This is due to the fact that we use no information about the uniqueness or non-degeneracy of the ground states, so we cannot conclude that the solution stays close to some specific function. However, by using the only information that the minimizing sequences for the constrained variational problem associated to Eq. (1.4) are relatively compact, we show that the solution is concentrated in a certain point (see Proposition 3.1). Then, we provide the dynamics of the barycenter and we estimate the distance between the concentration point and the barycenter. In this way, we are able to prove that the barycenter dynamics is approximatively that of a point particle moving under the effect of the external potential V.

The paper is organized as follows. In Section 2, we give some preliminary results on the relations between the parameters, on the first integrals of our equation and on the existence and properties of the ground states. In particular, the ground states of Eq. (1.4) are constrained minimizers for a functional J on the set of functions with fixed L^2-norm (see Lemmas 2.3, 2.4 and also [27]) and, as explained above, we show the pre-compactness, up to translations, of the minimizing sequences for J (see Lemma 2.5). Finally, in Section 3 we show the concentration behavior in the semi-classical limit and we conclude by Section 4 proving Theorem 1.1.

In the paper we denote by C a generic positive constant which can change from line to line.

2. Preliminary tools

2.1. Relations between the parameters

Let $\omega_\varepsilon \in \mathbb{R}$ and U_ε as in (1.3). We require that

$$\psi(t,x) = U_\varepsilon(x) e^{i\frac{\omega_\varepsilon}{\varepsilon}t}$$

solves $(\mathcal{P}_\varepsilon)$ with $V \equiv 0$, so that V can be interpreted as a perturbation term. Hence we ask that ψ solves

$$i\varepsilon\frac{\partial\psi}{\partial t} = -\frac{\varepsilon^2}{2}\Delta\psi - \varepsilon^{\gamma(2p-1)-\alpha}\left(I_\theta * |\psi|^p\right)|\psi|^{p-2}\psi, \tag{2.1}$$

i.e. that U_ε solves

$$\frac{\varepsilon^2}{2}\Delta U_\varepsilon + \varepsilon^{\gamma(2p-1)-\alpha}\left(I_\theta * |U_\varepsilon|^p\right)|U_\varepsilon|^{p-2}U_\varepsilon = \omega_\varepsilon U_\varepsilon.$$

So we establish a relation between β and the other parameters. Since

$$\left[\left(I_\theta * |U_\varepsilon|^p\right)|U_\varepsilon|^{p-2}U_\varepsilon\right](x) = e^{i\frac{\omega_\varepsilon}{\varepsilon}t}\varepsilon^{\beta\theta-\gamma(2p-1)}\left[\left(I_\theta * |U|^p\right)|U|^{p-2}U\right]\left(\varepsilon^{-\beta}x\right),$$

then

$$\frac{\varepsilon^{2-\gamma-2\beta}}{2}\Delta U + \varepsilon^{\beta\theta-\alpha}\left(I_\theta * |U|^p\right)|U|^{p-2}U = \omega_\varepsilon\varepsilon^{-\gamma}U.$$

Thus, $\psi(t,x) = U_\varepsilon(x)e^{i\frac{\omega_\varepsilon}{\varepsilon}t}$ is a solution of (2.1) if

$$\beta = \frac{\alpha + 2 - \gamma}{\theta + 2} \tag{2.2}$$

and

$$\omega_\varepsilon = \omega \varepsilon^{2-2\beta}.$$

In the following we always assume (2.2).

2.2. The first integrals of NSE

Noether's theorem states that any invariance for a one-parameter group of the Lagrangian implies the existence of an integral of motion (see e.g. [16]). Now we describe the first integrals for $(\mathcal{P}_\varepsilon)$ which will be relevant for this paper, namely the *hylenic charge* and the *energy*.

Following [1], the *hylenic charge* (or simply *charge*) is defined as the quantity which is preserved by the invariance of the Lagrangian with respect to the action

$$\psi \mapsto e^{i\theta}\psi.$$

For the equation in $(\mathcal{P}_\varepsilon)$ the charge is nothing else but the L^2-norm, namely

$$C(\psi) = \int |\psi|^2 = \int u^2.$$

The *energy*, by definition, is the quantity which is preserved by the time invariance of the Lagrangian. It has the form

$$E_\varepsilon(\psi) = \frac{\varepsilon^2}{2} \int |\nabla \psi|^2 + \int V(x)|\psi|^2 - \frac{\varepsilon^{\gamma(2p-1)-\alpha}}{p} \int \left(I_\theta * |\psi|^p\right)|\psi|^p.$$

Writing ψ in the polar form $ue^{\frac{i}{\varepsilon}S}$ we get

$$E_\varepsilon(\psi) = \frac{\varepsilon^2}{2} \int |\nabla u|^2 - \frac{\varepsilon^{\gamma(2p-1)-\alpha}}{p} \int \left(I_\theta * |u|^p\right)|u|^p + \int \left(\frac{1}{2}|\nabla S|^2 + V(x)\right)u^2. \tag{2.3}$$

Thus the energy has two components: the *internal energy* (which, sometimes, is also called *binding energy*)

$$J_\varepsilon(u) = \frac{\varepsilon^2}{2} \int |\nabla u|^2 - \frac{\varepsilon^{\gamma(2p-1)-\alpha}}{p} \int \left(I_\theta * |u|^p\right)|u|^p$$

and the *dynamical energy*

$$G(u, S) = \int \left(\frac{1}{2}|\nabla S|^2 + V(x)\right)u^2$$

which is composed by the *kinetic energy*

$$\frac{1}{2} \int |\nabla S|^2 u^2$$

and the *potential energy*

$$\int V(x)u^2.$$

Finally we define the *momentum*

$$p_\varepsilon(t,x) := \frac{1}{\varepsilon^{N-1}} \mathfrak{Im}\big(\bar{\psi}(t,x)\nabla\psi(t,x)\big), \quad x \in \mathbb{R}^N,\ t \in [0,\infty). \tag{2.4}$$

Arguing as in [11, Lemma 3.3], if $\psi \in C([0,\infty), H^2(\mathbb{R}^N)) \cap C^1((0,\infty), L^2(\mathbb{R}^N))$, the map

$$t \mapsto \int p_\varepsilon(t,x)\,dx$$

is C^1 and, on the solutions, the following identities hold:

$$\varepsilon^{-N}\partial_t\big|\psi(t,x)\big|^2 = -\operatorname{div}\big(p_\varepsilon(t,x)\big), \quad t \in [0,\infty),\ x \in \mathbb{R}^N, \tag{2.5}$$

$$\partial_t \int p_\varepsilon(t,x)\,dx = -\varepsilon^{-N} \int \nabla V(x)\big|\psi(t,x)\big|^2\,dx, \quad t \in [0,\infty). \tag{2.6}$$

2.3. Rescaling of internal energy and charge

If we consider again U_ε as in (1.3), since

$$\int \big(I_\theta * |U_\varepsilon|^p\big)|U_\varepsilon|^p = \varepsilon^{\beta(N+\theta)-2p\gamma} \int \big(I_\theta * |U|^p\big)|U|^p$$

by (2.2), we have

$$J_\varepsilon(U_\varepsilon) = \varepsilon^{2-2\gamma+\beta(N-2)} J(U) \tag{2.7}$$

where

$$J(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{1}{p}\int \big(I_\theta * |u|^p\big)|u|^p.$$

As pointed out in [27], J belongs to class C^1 on $H^1(\mathbb{R}^N)$ and for every $u, v \in H^1(\mathbb{R}^N)$

$$\big\langle J'(u), v\big\rangle = \int \nabla u \cdot \nabla v - 2\int \big(I_\theta * |u|^p\big)|u|^{p-2}uv.$$

Moreover, computing the charge of a rescaled function, we have

$$C(U_\varepsilon) = \varepsilon^{N\beta-2\gamma} C(U).$$

We can choose, without loss of generality, that

$$N\beta - 2\gamma = 0, \tag{2.8}$$

in order to have the same charge for any rescaling and to simplify the notations.

Thus, combining (2.7) and (2.8) we get

$$J_\varepsilon(U_\varepsilon) = \varepsilon^{2(1-\beta)} J(U),$$

so, when

$$\beta > 1 \tag{2.9}$$

we have that $J_\varepsilon(U_\varepsilon) \to +\infty$ for $\varepsilon \to 0^+$ which will be the key tool for the main result of this paper.

Remark 2.1. In the physical case ($N = 3$ and $\theta = 2$), in order to satisfy conditions (2.2), (2.8) and (2.9), we can have any couple (α, γ) on the line $3\alpha + 6 - 11\gamma = 0$ with $\gamma > 3/2$. This choice implies that for $p = 2$ the power $\gamma(2p - 1) - \alpha$ is, in the notation of [6], super-critical, indeed $\gamma(2p - 1) - \alpha - \frac{3}{2} < 0$.

2.4. Ground states

Let $\omega > 0$. A ground state for (1.4) is a solution that realizes the minimum of the energy

$$E_\omega(u) = \frac{1}{2}\int |\nabla u|^2 + \omega \int |u|^2 - \frac{1}{p}\int \big(I_\theta * |u|^p\big)|u|^p$$

on the set

$$\mathcal{N}_\omega = \left\{u \in H^1\big(\mathbb{R}^N\big) \setminus \{0\} \;\Big|\; \frac{1}{2}\int |\nabla u|^2 + \omega \int |u|^2 = \int \big(I_\theta * |u|^p\big)|u|^p\right\}.$$

A ground state can be found in several ways. In the recent paper [27], for instance, the authors minimize

$$S_{\theta,p}(u) = \frac{\|\nabla u\|_2^2 + \omega\|u\|_2^2}{(\int (I_\theta * |u|^p)|u|^p)^{1/p}} \quad \text{in } H^1\big(\mathbb{R}^N\big) \setminus \{0\}.$$

This way allows to obtain a sharp result on the existence with respect to the parameter p. In the following lemma we summarize some results obtained in [27].

Lemma 2.2. *Let $N \geqslant 3$, $\theta \in (0, N)$ and $p \in (1 + \theta/N, (N + \theta)/(N - 2))$. We have that (1.4) admits a ground state solution U in $H^1(\mathbb{R}^N)$. Moreover each ground state U of (1.4) is in $L^1(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$, it has fixed sign and there exist $x_0 \in \mathbb{R}^N$ and a monotone real function $v \in C^\infty(0, \infty)$ such that $U(x) = v(|x - x_0|)$ a.e. in $\mathbb{R}^N$.*

In the following, we consider only positive ground state.

Another way to look for ground states is to minimize J on

$$\Sigma_\nu = \big\{u \in H^1\big(\mathbb{R}^N\big) \;\big|\; \|u\|_2^2 = \nu\big\}$$

for $\nu > 0$ (cf. Lemma 2.4). In fact, under our assumptions on p, for every $u \in H^1(\mathbb{R}^N)$,

$$0 < Np - \theta - N < 2 \tag{2.10}$$

and

$$\frac{2Np}{N + \theta} \in \big(2, 2^*\big), \quad \text{with } 2^* = \frac{2N}{N - 2}, \tag{2.11}$$

so that $|u|^p \in L^{\frac{2N}{N+\theta}}(\mathbb{R}^N)$. Thus, by the Hardy–Littlewood–Sobolev and Gagliardo–Nirenberg inequalities, we have

$$\int \big(I_\theta * |u|^p\big)|u|^p \leqslant C\|u\|_{2Np/(N+\theta)}^{2p} \leqslant C\|\nabla u\|_2^{Np-\theta-N}\|u\|_2^{2p-Np+N+\theta}. \tag{2.12}$$

Hence, for all $u \in \Sigma_\nu$,

$$J(u) \geqslant \frac{1}{2}\|\nabla u\|_2^2 - C\nu^{\frac{2p-Np+N+\theta}{2}}\|\nabla u\|_2^{Np-\theta-N} \tag{2.13}$$

and so, by (2.10), we get that J is bounded from below on Σ_ν. Moreover we notice that if $p \in [1 + (2+\theta)/N, (N+\theta)/(N-2))$, J is unbounded from below on Σ_ν and if p satisfies (1.1), $\inf_{u\in H^1(\mathbb{R}^N)\setminus\{0\}} S_{\theta,p}(u)$ can be written in terms of $\inf_{u\in\Sigma_\nu} J(u)$ and any minimizer of $S_{\theta,p}$ in $H^1(\mathbb{R}^N)\setminus\{0\}$ is, up to suitable dilation and rescaling, a minimizer of J on Σ_ν. This last method seems to be the best for our arguments. So, for the sake of completeness we give some details. First of all we give the following preliminary result.

Lemma 2.3. *For every $\nu > 0$ we have that*

$$m_\nu := \inf_{u\in\Sigma_\nu} J(u) \in (-\infty, 0).$$

Proof. From the arguments above we know that J is bounded from below on Σ_ν. So it remains to prove that $m_\nu < 0$. To this end let $u \in \Sigma_\nu$ and define $u_\tau(x) := \tau^{N/2}u(\tau x)$ for $\tau > 0$ and $x \in \mathbb{R}^N$. Then $u_\tau \in \Sigma_\nu$ and

$$m_\nu \leqslant J(u_\tau) = \frac{\tau^2}{2}\int |\nabla u|^2 - \frac{\tau^{Np-\theta-N}}{p}\int \big(I_\theta * |u|^p\big)|u|^p.$$

By (2.10), taking $\tau > 0$ sufficiently small we get $m_\nu < 0$. □

Moreover, following step by step [10, proof of Lemma 2.6], we get

Lemma 2.4. *For every $\nu, \omega > 0$, the minimization problems*

$$\min_{u\in\Sigma_\nu} J(u) \quad \text{and} \quad \min_{u\in\mathcal{N}_\omega} E_\omega(u)$$

are equivalent. Moreover the L^2-norm of any ground state U of (1.4) is $\sqrt{\sigma}$ where

$$\sigma := \frac{N+\theta-(N-2)p}{2\omega(p-1)} \min_{u\in\mathcal{N}_\omega} E_\omega(u) \tag{2.14}$$

and

$$\min_{u\in\Sigma_\sigma} E_\omega(u) = \min_{u\in\mathcal{N}_\omega} E_\omega(u).$$

Proof. Let $\nu, \omega > 0$,

$$\mathcal{K}_{\Sigma_\nu} = \big\{m \in \mathbb{R}_- \;\big|\; \exists u \in \Sigma_\nu \text{ s.t. } J'\big|_{\Sigma_\nu}(u) = 0 \text{ and } J(u) = m\big\}$$

and

$$\mathcal{K}_{\mathcal{N}_\omega} = \big\{c \in \mathbb{R} \;\big|\; \exists u \in \mathcal{N}_\omega \text{ s.t. } E'_\omega(u) = 0 \text{ and } E_\omega(u) = c\big\}.$$

Let now $u \in \Sigma_\nu$ such that $J'|_{\Sigma_\nu}(u) = 0$ and $J(u) = m$ with $m < 0$. Then there exists $\gamma \in \mathbb{R}$ such that

$$\frac{1}{2}\Delta u + \big(I_\theta * |u|^p\big)|u|^{p-2}u = \gamma u \tag{2.15}$$

and so

$$\frac{1}{2}\|\nabla u\|_2^2 - \int \big(I_\theta * |u|^p\big)|u|^p = -\gamma\nu. \tag{2.16}$$

Thus, since $J(u) = m < 0$, by (2.16) we get

$$\frac{p-1}{2p}\|\nabla u\|_2^2 - m = \frac{\gamma\nu}{p}$$

and so $\gamma > 0$. Now let

$$w(x) := \tau^{\frac{\theta+2}{2(p-1)}} u(\tau x) \quad \text{with } \tau = \sqrt{\frac{\omega}{\gamma}}.$$

We have that w solves

$$-\frac{1}{2}\Delta w + \omega w - \big(I_\theta * |w|^p\big)|w|^{p-2}w = 0$$

and so $w \in \mathcal{N}_\omega$, $E'_\omega(w) = 0$ and $c = E_\omega(w) \in \mathcal{K}_{\mathcal{N}_\omega}$.

Vice versa, if $w \in \mathcal{N}_\omega$ such that $E'_\omega(w) = 0$ and $c = E_\omega(w)$, we consider

$$u(x) := \tau^{\frac{\theta+2}{2(p-1)}} w(\tau x) \quad \text{with } \tau = \left(\frac{\nu}{\|w\|_2^2}\right)^{\frac{p-1}{\theta+2-N(p-1)}}.$$

We have that $u \in \Sigma_\nu$, (2.15) holds for

$$\gamma = \omega\tau^2 = \omega\left(\frac{\nu}{\|w\|_2^2}\right)^{\frac{2(p-1)}{\theta+2-N(p-1)}}$$

and

$$m = \tau^{\frac{\theta+2p-N(p-1)}{p-1}}\big(c - \omega\|w\|_2^2\big) = \left(\frac{\nu}{\|w\|_2^2}\right)^{\frac{\theta+2p-N(p-1)}{\theta+2-N(p-1)}}\big(c - \omega\|w\|_2^2\big). \tag{2.17}$$

By [27, Proposition 3.1] (Pohožaev identity) and since we have $w \in \mathcal{N}_\omega$ and $E_\omega(w) = c$ we get the system

$$\begin{cases} \frac{N-2}{2}\|\nabla w\|_2^2 + \omega N\|w\|_2^2 - \frac{N+\theta}{p}\int \big(I_\theta * |w|^p\big)|w|^p = 0 \\ \frac{1}{2}\|\nabla w\|_2^2 + \omega\|w\|_2^2 - \int \big(I_\theta * |w|^p\big)|w|^p = 0 \\ \frac{1}{2}\|\nabla w\|_2^2 + \omega\|w\|_2^2 - \frac{1}{p}\int \big(I_\theta * |w|^p\big)|w|^p = c \end{cases}$$

from which

$$\|w\|_2^2 = \frac{N+\theta-(N-2)p}{2\omega(p-1)}c.$$

Thus (2.17) becomes

$$m = \frac{Np-2-N-\theta}{2(p-1)}\left(\frac{2\omega\nu(p-1)}{N+\theta-(N-2)p}\right)^{\frac{\theta+2p-N(p-1)}{\theta+2-N(p-1)}} c^{\frac{2(1-p)}{\theta+2-N(p-1)}}$$

and the first conclusion easily follows. The second part is a trivial consequence of the calculations of the first part. $\square$

By combining Lemma 2.2 and Lemma 2.4, we get that for every $\nu > 0$ the minimum of J in Σ_ν is attained. Furthermore, in order to obtain some uniform decay properties on the ground states, proceeding as in [2, Theorem 3.1], we prove the following result.

Lemma 2.5. *For every $\nu > 0$, every minimizing sequence for J in Σ_ν is relatively compact in $H^1(\mathbb{R}^N)$ up to a translation.*

Proof. Let $\{u_n\}$ be a minimizing sequence for J on Σ_ν. Without loss of generality, by Ekeland Variational Principle [12], we can assume that $\{u_n\}$ is a Palais–Smale sequence for J. By (2.13) we have that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and then there exists $u \in H^1(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$. Fixed $R > 0$, we have that there exist $c > 0$ and a subsequence $\{u_n\}$, such that

$$\sup_{n\in\mathbb{N}} \sup_{y\in\mathbb{R}^N} \int\limits_{B_R(y)} u_n^2 \geqslant c. \tag{2.18}$$

Indeed, if

$$\lim_n \sup_{y\in\mathbb{R}^N} \int\limits_{B_R(y)} u_n^2 = 0,$$

then, by [25, Lemma I.1], it follows that $u_n \to 0$ in $L^q(\mathbb{R}^N)$ for $q \in (2, 2^*)$. Thus, by (2.11) and (2.12), we have that

$$\int (I_\theta * |u_n|^p)|u_n|^p \to 0$$

and this is a contradiction since $m_\nu < 0$. Hence, by (2.18), for every $n \in \mathbb{N}$ there exists $y_n \in \mathbb{R}^N$ such that

$$\int\limits_{B_R(y_n)} u_n^2 \geqslant c.$$

So, if we take $v_n = u_n(\cdot + y_n)$, by using the compact embedding of $H^1_{\mathrm{loc}}(\mathbb{R}^N)$ into $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ we obtain a minimizing sequence whose weak limit is nontrivial. Moreover, the weak convergence implies immediately that $\|u\|_2^2 \leqslant \nu$,

$$\|u_n - u\|_2^2 + \|u\|_2^2 = \|u_n\|_2^2 + o_n(1), \tag{2.19}$$

$$\|\nabla u_n - \nabla u\|_2^2 + \|\nabla u\|_2^2 = \|\nabla u_n\|_2^2 + o_n(1) \tag{2.20}$$

and, by [27, Lemma 2.4],

$$\int (I_\theta * |u_n - u|^p)|u_n - u|^p + \int (I_\theta * |u|^p)|u|^p = \int (I_\theta * |u_n|^p)|u_n|^p + o_n(1). \tag{2.21}$$

Assume by contradiction that $\|u\|_2^2 = \tau < \nu$. Since, by (2.19),

$$a_n = \frac{\sqrt{\nu - \tau}}{\|u_n - u\|_2} \to 1$$

and, by (2.20) and (2.21),

$$J(u_n - u) + J(u) = m_\nu + o_n(1),$$

we have that

$$J\big(a_n(u_n - u)\big) + J(u) = J(u_n - u) + J(u) + o_n(1) = m_\nu + o_n(1).$$

Then, since $\|a_n(u_n - u)\|_2^2 = \nu - \tau$, we get

$$m_{\nu-\tau} + m_\tau \leqslant m_\nu + o_n(1). \tag{2.22}$$

But, if we consider, for $\mu > 0$, $\Sigma_\nu^\mu = \{u \in \Sigma_\nu \mid \int (I_\theta * |u|^p)|u|^p \geqslant \mu\}$, we can prove that there exists $\mu > 0$ such that

$$m_\nu = \inf_{u \in \Sigma_\nu^\mu} J(u). \tag{2.23}$$

Indeed, since $\Sigma_\nu^\mu \subset \Sigma_\nu$, we have $m_\nu \leqslant \inf_{u\in\Sigma_\nu^\mu} J(u)$. If we suppose by contradiction that, for every $\mu > 0$, $m_\nu < \inf_{u\in\Sigma_\nu^\mu} J(u)$, then we can construct a minimizing sequence $\{u_n\}$ such that

$$J(u_n) \to m_\nu \quad \text{and} \quad \int \big(I_\theta * |u_n|^p\big)|u_n|^p \to 0.$$

Thus

$$0 \leqslant \frac{1}{2}\|\nabla u_n\|_2^2 = J(u_n) + \frac{1}{p}\int \big(I_\theta * |u_n|^p\big)|u_n|^p \to m_\nu < 0.$$

Then, by using (2.23), it is easy to check that for every $\tau > 1$

$$m_{\tau\nu} < \tau m_\nu.$$

Thus, as proved in [24, Lemma II.1], we have that for all $\tau \in (0, \nu)$

$$m_\nu < m_\tau + m_{\nu-\tau}$$

which is in contradiction with (2.22). Hence $u \in \Sigma_\nu$, $\|u_n - u\|_2 = o_n(1)$ and, by applying the Gagliardo–Nirenberg inequality as in the second part of (2.12), we have that

$$\|u_n - u\|_{2Np/(N+\theta)} = o_n(1). \tag{2.24}$$

It remains to show that $\|\nabla u_n - \nabla u\|_2 = o_n(1)$. Since $\{u_n\}$ is a Palais–Smale sequence, there exists $\{\lambda_n\} \subset \mathbb{R}$ such that for every $v \in H^1(\mathbb{R}^N)$

$$\big\langle J'(u_n) - \lambda_n u_n, v\big\rangle = o_n(1)$$

and, since $\{u_n\}$ is bounded

$$\big\langle J'(u_n) - \lambda_n u_n, u_n\big\rangle = o_n(1).$$

Then we obtain that $\{\lambda_n\}$ is bounded and

$$\big\langle J'(u_n) - J'(u_m) - \lambda_n u_n + \lambda_m u_m, u_n - u_m\big\rangle \to 0 \quad \text{as } m, n \to +\infty.$$

Since, by the Hardy–Littlewood–Sobolev inequality and (2.24)

$$\left|\int (I_\theta * |u_n|^p)|u_n|^{p-2}u_n(u_n - u_m)\right| \leqslant C\|u_n\|_{2Np/(N+\theta)}^{p+2N(p-1)/(N+\theta)}\|u_n - u_m\|_{2Np/(N+\theta)} \to 0$$

and

$$\lambda_n\langle u_n, u_n - u_m\rangle \to 0$$

as $m, n \to +\infty$, we have that $\{u_n\}$ is a Cauchy sequence in $H^1(\mathbb{R}^N)$ and we conclude. □

We close this section by showing the following uniform estimate on the ground states.

Lemma 2.6. *For every $\lambda > 0$ there exists $R > 0$ such that for every ground state U there exists $q(U) \in \mathbb{R}^N$ such that*

$$\int\limits_{\mathbb{R}^N \setminus B_R(q(U))} U^2 < \lambda.$$

Proof. Assume by contradiction that there exists $\lambda > 0$ such that, for any $n \in \mathbb{N}$, there exists a ground state U_n such that for every $q \in \mathbb{R}^N$

$$\int\limits_{\mathbb{R}^N \setminus B_n(q)} U_n^2 \geqslant \lambda$$

and so

$$\inf_{q\in\mathbb{R}^N} \int\limits_{\mathbb{R}^N \setminus B_n(q)} U_n^2 \geqslant \lambda. \tag{2.25}$$

Then $\{U_n\}$ is a minimizing sequence and by virtue of Lemma 2.5 is relatively compact up to a translation $\{q_n\} \subset \mathbb{R}^N$. Thus there exists a ground state U with $U_n(\cdot - q_n) \to U$ in $H^1(\mathbb{R}^N)$ and

$$\inf_{q\in\mathbb{R}^N} \int\limits_{\mathbb{R}^N \setminus B_n(q)} U_n^2 \leqslant \int\limits_{\mathbb{R}^N \setminus B_n(-q_n)} U_n^2 = \int\limits_{\mathbb{R}^N \setminus B_n(0)} U_n^2(\cdot - q_n) = \int\limits_{\mathbb{R}^N \setminus B_n(0)} U^2 + o_n(1) = o_n(1),$$

which is in contradiction with (2.25). □

Remark 2.7. Of course, without loss of generality we can take $q(U) = 0$ in Lemma 2.6 for radially symmetric ground states U.

Throughout the rest of the paper, we will consider radially symmetric ground states U.

3. Concentration results

In this section we prove a concentration property of the solution of $(\mathcal{P}_\varepsilon)$ with suitable initial data; more exactly, we prove that, fixed $t \in (0, \infty)$, this solution is a function on $\mathbb{R}^N$ with one peak localized in a ball with center depending on t and radius not depending on t. In order to prove this result, it is sufficient to assume that problem $(\mathcal{P}_\varepsilon)$ admits global solutions ψ which satisfy the conservation of the energy and of the L^2-norm. Given K, $\varepsilon > 0$, let

$$B_\varepsilon^K = \left\{ \begin{array}{l} \psi(0,x) = u_\varepsilon(0,x) e^{\frac{i}{\varepsilon} S_\varepsilon(0,x)} \text{ with:} \\ u_\varepsilon(0,x) = \varepsilon^{-\gamma}[(U+w)(\varepsilon^{-\beta}(x-q))], \\ U \text{ radial ground state solution of (1.4)}, \\ q \in \mathbb{R}^N, \\ w \in H^1(\mathbb{R}^N) \text{ s.t. } \|U+w\|_2^2 = \|U\|_2^2 = \sigma \text{ and } \|w\| < K\varepsilon^{2(\beta-1)}, \\ \|\nabla S_\varepsilon(0,x)\|_\infty \leqslant K, \\ \int_{\mathbb{R}^N} V(x) u_\varepsilon^2(0,x)\, dx \leqslant K \end{array} \right\} \tag{3.1}$$

the set of *admissible* initial data, where $\|\cdot\|$ denotes the $H^1(\mathbb{R}^N)$-norm. Of course, here σ satisfies (2.14). In the following, if $m \in \mathbb{R}$ we denote with J^m the sublevels of J. The main result of this section is

Proposition 3.1. *Let $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$, $V \geqslant 0$, $\beta > 1$ and fix $K > 0$. For all $\lambda > 0$, there exist $\hat{R} > 0$ and $\varepsilon_0 > 0$ such that, for any $\varepsilon < \varepsilon_0$, ψ solution of $(\mathcal{P}_\varepsilon)$ with initial data $\psi(0,x) \in B_\varepsilon^K$ and $t \in (0,\infty)$, there exists $\hat{q}_\varepsilon(t) \in \mathbb{R}^N$ for which*

$$\frac{1}{\|\psi(t)\|_2^2} \int\limits_{\mathbb{R}^N \setminus B_{\hat{R}\varepsilon^\beta}(\hat{q}_\varepsilon(t))} \left|\psi(t,x)\right|^2 dx < \lambda.$$

Here $\hat{q}_\varepsilon(t)$ depends on ψ.

For the proof of this proposition we need some technical results.

Lemma 3.2. *For any $\lambda > 0$ there exist $\hat{R} = \hat{R}(\lambda) > 0$ and $\delta = \delta(\lambda) > 0$ such that, for any $u \in J^{m_\sigma+\delta} \cap \Sigma_\sigma$, there exists $\hat{q} \in \mathbb{R}^N$ such that*

$$\frac{1}{\sigma} \int\limits_{\mathbb{R}^N \setminus B_{\hat{R}}(\hat{q})} u^2 < \lambda. \tag{3.2}$$

Proof. First of all we prove that for any $\lambda > 0$, there exists $\delta > 0$ such that, for all $u \in J^{m_\sigma+\delta} \cap \Sigma_\sigma$, there exist $\hat{q} \in \mathbb{R}^N$ and a ground state U of (1.4) such that

$$u = U(\cdot - \hat{q}) + w \quad \text{and} \quad \|w\| \leqslant \lambda.$$

Indeed, let us assume by contradiction that there exist $\lambda > 0$ and a minimizing sequence $\{u_n\}$ such that for every $q_n \in \mathbb{R}^N$ and U ground state

$$\left\|u_n - U(\cdot - q_n)\right\| > \lambda. \tag{3.3}$$

Since, by Lemma 2.5, $\{u_n\}$ is relatively compact up to translations, there exists a ground state $U \in H^1(\mathbb{R}^N)$ such that $w_n = u_n - U(\cdot - q_n) \to 0$ in $H^1(\mathbb{R}^N)$ and this contradicts (3.3).

Now, let us fix $\lambda > 0$. We can suppose that $\lambda < 1$. Then, for $\sqrt{\sigma}\lambda$, there exists $\delta > 0$ such that, for all $u \in J^{m_\sigma+\delta} \cap \Sigma_\sigma$, there exists $\hat{q} \in \mathbb{R}^N$ and a ground state U such that $u = U(\cdot - \hat{q}) + w$ and $\|w\| \leqslant \sqrt{\sigma}\lambda$. Moreover, by Lemma 2.6, there exists $\hat{R} > 0$ such that, for every ground state U,

$$\int\limits_{\mathbb{R}^N \setminus B_{\hat{R}}(0)} U^2 < \sigma\lambda(1-\sqrt{\lambda})^2.$$

Thus, if $u \in J^{m_\sigma+\delta} \cap \Sigma_\sigma$, we have

$$\begin{aligned}\frac{1}{\sigma}\int_{\mathbb{R}^N\setminus B_{\hat{R}}(\hat{q})} u^2 &\leqslant \frac{1}{\sigma}\int_{\mathbb{R}^N\setminus B_{\hat{R}}(\hat{q})} U^2(\cdot-\hat{q}) + \frac{1}{\sigma}\|w\|_2^2 + \frac{2}{\sigma}\|w\|_2\Bigg(\int_{\mathbb{R}^N\setminus B_{\hat{R}}(\hat{q})} U^2(\cdot-\hat{q})\Bigg)^{1/2} \\ &= \frac{1}{\sigma}\int_{\mathbb{R}^N\setminus B_{\hat{R}}(0)} U^2 + \frac{1}{\sigma}\|w\|_2^2 + \frac{2}{\sigma}\|w\|_2\Bigg(\int_{\mathbb{R}^N\setminus B_{\hat{R}}(0)} U^2\Bigg)^{1/2} \\ &< \lambda(1-\sqrt{\lambda})^2 + \lambda^2 + 2\lambda\sqrt{\lambda}(1-\sqrt{\lambda}) = \lambda\end{aligned}$$

which concludes the proof. □

As a consequence of the previous lemma, we can describe the concentration properties of the solutions of $(\mathcal{P}_\varepsilon)$.

Lemma 3.3. *For any $\lambda > 0$, there exist $\delta = \delta(\lambda) > 0$ and $\hat{R} = \hat{R}(\lambda) > 0$ such that for any ψ solution of $(\mathcal{P}_\varepsilon)$ with $\varepsilon^\gamma|\psi(t, \varepsilon^\beta x)| \in J^{m_\sigma+\delta} \cap \Sigma_\sigma$ for all $t \in (0, \infty)$, there exists $\hat{q}_\varepsilon(t) \in \mathbb{R}^N$, which depends on λ, ε, t and ψ, for which*

$$\frac{1}{\sigma}\int_{\mathbb{R}^N\setminus B_{\varepsilon^\beta\hat{R}}(\hat{q}_\varepsilon(t))} \big|\psi(t,x)\big|^2\,dx < \lambda.$$

Proof. Let $\lambda > 0$ be fixed. By Lemma 3.2 we have that there exist $\delta = \delta(\lambda) > 0$ and $\hat{R} = \hat{R}(\lambda) > 0$ such that for any $u \in J^{m_\sigma+\delta} \cap \Sigma_\sigma$, there exists $\hat{q} \in \mathbb{R}^N$ such that (3.2) holds. So we fix ε, t and ψ solution of $(\mathcal{P}_\varepsilon)$, such that $v(x) = \varepsilon^\gamma|\psi(t, \varepsilon^\beta x)| \in J^{m_\sigma+\delta} \cap \Sigma_\sigma$. We have that there exists $\bar{q} = \bar{q}(v) \in \mathbb{R}^N$ such that, using (2.8),

$$\frac{1}{\sigma}\int_{\mathbb{R}^N\setminus B_{\hat{R}}(\bar{q})} |v|^2 = \frac{1}{\sigma}\int_{\mathbb{R}^N\setminus B_{\varepsilon^\beta\hat{R}}(\varepsilon^\beta\bar{q})} \big|\psi(t,x)\big|^2\,dx < \lambda.$$

Then we conclude taking $\hat{q}_\varepsilon(t) = \varepsilon^\beta\bar{q}$, which depends on λ, ε, t and ψ, while $\hat{R}$ depends only upon the value of λ. □

Now we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. If ψ is a solution of $(\mathcal{P}_\varepsilon)$ with *admissible* initial datum, then, by the conservation of the energy E_ε and by (3.1), (2.3) and (2.2), we have

$$E_\varepsilon(\psi) \leqslant \varepsilon^{2(1-\beta)}J(U+w) + \frac{K^2\sigma}{2} + K. \tag{3.4}$$

Moreover, since J is C^1 in $H^1(\mathbb{R}^N)$, we have

$$J(U+w) \leqslant m_\sigma + C\|w\| \leqslant m_\sigma + C\varepsilon^{2(\beta-1)}. \tag{3.5}$$

So, combining (3.4) and (3.5), we obtain

$$E_\varepsilon(\psi) \leqslant \varepsilon^{2(1-\beta)}m_\sigma + C. \tag{3.6}$$

Thus, in light of (3.6) and because $V(x) \geqslant 0$, if $u_\varepsilon(t,x) = |\psi(t,x)|$, we get

$$J_\varepsilon(u_\varepsilon) = E_\varepsilon(\psi) - G(u_\varepsilon, S_\varepsilon) \leqslant \varepsilon^{2(1-\beta)} m_\sigma + C. \tag{3.7}$$

Then, by (2.2), (2.8) and (3.7) we get

$$J\big(\varepsilon^\gamma u_\varepsilon\big(t, \varepsilon^\beta x\big)\big) = \varepsilon^{2(\beta-1)} J_\varepsilon(u_\varepsilon) \leqslant m_\sigma + \varepsilon^{2(\beta-1)} C.$$

So, since, by the conservation of the hylenic charge,

$$\big\|\varepsilon^\gamma u_\varepsilon\big(t, \varepsilon^\beta x\big)\big\|_2^2 = \big\|\varepsilon^\gamma u_\varepsilon\big(0, \varepsilon^\beta x\big)\big\|_2^2 = \|U + w\|_2^2 = \sigma,$$

if $\beta > 1$ and for ε small we can apply Lemma 3.3 and we conclude. $\square$

4. Proof of the main result

4.1. Barycenter and concentration point

In this subsection, we provide the dynamics of the barycenter and we estimate the distance between the concentration point and the barycenter of a solution ψ for a potential satisfying (V0) and (V2).

Proposition 4.1. *Let ψ be a global solution of $(\mathcal{P}_\varepsilon)$ with initial data $\psi(0,x)$ such that*

$$\int |x|\big|\psi(0,x)\big|^2\,dx < +\infty.$$

Then the map $q_\varepsilon : \mathbb{R} \to \mathbb{R}^N$, where $q_\varepsilon(t)$ is given by (1.5), is well defined, is C^1 and

$$\dot{q}_\varepsilon(t) = \frac{\varepsilon^N}{\|\psi(t)\|_2^2} \int p_\varepsilon(t,x)\,dx \tag{4.1}$$

$$\ddot{q}_\varepsilon(t) = -\frac{1}{\|\psi(t)\|_2^2} \int \nabla V(x)\big|\psi(t,x)\big|^2\,dx \tag{4.2}$$

Proof. We prove that q_ε is well defined by a regularization argument. Let $\lambda > 0$ and

$$k_\lambda(t) = \int e^{-2\lambda|x|}|x|\big|\psi(t,x)\big|^2\,dx.$$

By (2.5) we have

$$k'_\lambda(t) = -\varepsilon^N \int e^{-2\lambda|x|}|x|\,\mathrm{div}\big(p_\varepsilon(t,x)\big)\,dx = \varepsilon^N \int e^{-2\lambda|x|}\big(1 - 2\lambda|x|\big)\frac{x}{|x|}\cdot p_\varepsilon(t,x)\,dx.$$

Thus, on account of (2.4),

$$\big|k'_\lambda(t)\big| \leqslant \varepsilon\big\|\psi(t)\big\|_2\big\|\nabla\psi(t)\big\|_2$$

and then

$$k_\lambda(t) = k_\lambda(0) + \int_0^t k'_\lambda(s)\,ds \leqslant \int |x|\big|\psi(0,x)\big|^2\,dx + \varepsilon\int_0^t \big\|\psi(s)\big\|_2\big\|\nabla\psi(s)\big\|_2\,ds.$$

Hence, using Fatou's Lemma, we get that for all $t \in (0, \infty)$

$$\int |x| \left|\psi(t,x)\right|^2 dx < +\infty$$

and so q_ε is well defined for all t. With the same regularization technique, we can also prove that q_ε is C^1 and that (4.1) holds by (2.5). Finally, Eq. (4.2) is a straightforward consequence of (4.1) and (2.6). □

Now, for $K > 0$ fixed, let ψ be a global solution of $(\mathcal{P}_\varepsilon)$ such that $\psi \in C([0,\infty), H^2(\mathbb{R}^N)) \cap C^1((0,\infty), L^2(\mathbb{R}^N))$ and the initial data $\psi(0,x) \in B_\varepsilon^K$. Moreover let $u_\varepsilon(t,x) = |\psi(t,x)|$.

Lemma 4.2. *There exists a constant $C > 0$ such that, for all $t \in \mathbb{R}$,*

$$\int V(x) u_\varepsilon^2(t,x)\, dx \leqslant C.$$

Proof. Since $\varepsilon^\gamma u_\varepsilon(t, \varepsilon^\beta x) \in \Sigma_\sigma$, then, by (2.2) and (2.8),

$$J_\varepsilon\big(u_\varepsilon(t,x)\big) = \varepsilon^{2(1-\beta)} J\big(\varepsilon^\gamma u_\varepsilon\big(t, \varepsilon^\beta x\big)\big) \geqslant \varepsilon^{2(1-\beta)} m_\sigma. \tag{4.3}$$

Moreover, as in the proof of Proposition 3.1, inequality (3.6) holds and so, using (4.3), we get

$$\int V(x) u_\varepsilon^2(t,x)\, dx = E_\varepsilon(\psi) - J_\varepsilon(u_\varepsilon) - \frac{1}{2}\int |\nabla S|^2 u_\varepsilon^2(t,x)\, dx \leqslant C. \qquad \square$$

The following lemma shows the boundedness for the barycenter $q_h(t)$ defined in (1.5).

Lemma 4.3. *There exists $K_1 > 0$ such that for all $t \in [0,\infty)$, $|q_\varepsilon(t)| \leqslant K_1$.*

Proof. By Lemma 4.2 and assumption (V2) we get that for any $R_2 \geqslant R_1$ and for any $t \in [0, \infty)$,

$$C \geqslant \int_{\mathbb{R}^N \setminus B_{R_2}(0)} V(x) u_\varepsilon^2(t,x)\, dx \geqslant R_2^{a-1} \int_{\mathbb{R}^N \setminus B_{R_2}(0)} |x| u_\varepsilon^2(t,x)\, dx. \tag{4.4}$$

Hence

$$\left| \int x u_\varepsilon^2(t,x)\, dx \right| \leqslant \int_{\mathbb{R}^N \setminus B_{R_1}(0)} |x| u_\varepsilon^2(t,x)\, dx + \int_{B_{R_1}(0)} |x| u_\varepsilon^2(t,x)\, dx \leqslant \frac{C}{R_1^{a-1}} + R_1 \left\|u_\varepsilon(t)\right\|_2^2,$$

so that $|q_\varepsilon(t)| \leqslant R_1 + C/(R_1^{a-1}\sigma)$. □

Remark 4.4. By the inequality (4.4) we have also that, if R_2 is large enough, for all $t \in [0, \infty)$

$$\frac{1}{\|u_\varepsilon(t)\|_2^2} \int_{\mathbb{R}^N \setminus B_{R_2}(0)} u_\varepsilon^2(t,x)\, dx \leqslant \frac{C}{\sigma R_2^a} < \frac{1}{2}.$$

Now we show the boundedness of the concentration point $\hat{q}_\varepsilon(t)$ defined in Lemma 3.3.

Lemma 4.5. *If $0 < \lambda < 1/2$ and R_2 large enough we get that*

(1) *for ε small enough*

$$\sup_{t\in[0,\infty)} \left|\hat{q}_\varepsilon(t)\right| < R_2 + \hat{R}(\lambda)\varepsilon^\beta < R_2 + 1;$$

(2) *for all $R_3 \geqslant R_2$ and ε small enough*

$$\sup_{t\in[0,\infty)} \left|q_\varepsilon(t) - \hat{q}_\varepsilon(t)\right| < \frac{3C}{\sigma R_3^{a-1}} + 3R_3\lambda + \hat{R}(\lambda)\varepsilon^\beta.$$

Proof. By Proposition 3.1, with $\lambda < 1/2$, and by Remark 4.4, it is obvious that the ball $B_{\hat{R}(\lambda)\varepsilon^\beta}(\hat{q}_\varepsilon(t)) \not\subset \mathbb{R}^N \setminus B_{R_2}(0)$ and

$$B_{\hat{R}(\lambda)\varepsilon^\beta}\big(\hat{q}_\varepsilon(t)\big) \subset B_{R_2+2\hat{R}(\lambda)\varepsilon^\beta}(0).$$

Because $\hat{R}(\lambda)$ does not depend on ε, we can assume ε so small that $2\hat{R}(\lambda)\varepsilon^\beta < 1$. Then

$$\left|\hat{q}_\varepsilon(t)\right| < R_2 + 2\hat{R}(\lambda)\varepsilon^\beta < R_2 + 1, \tag{4.5}$$

$$B_{\hat{R}(\lambda)\varepsilon^\beta}\big(\hat{q}_\varepsilon(t)\big) \subset B_{R_2+1}(0), \tag{4.6}$$

and (4.5) implies (1).

To prove (2), first we estimate the difference between the barycenter and the concentration point. We have

$$\left|q_\varepsilon(t) - \hat{q}_\varepsilon(t)\right| = \frac{1}{\|u_\varepsilon(t)\|_2^2}\left|\int \big(x - \hat{q}_\varepsilon(t)\big)u_\varepsilon^2(t,x)\,dx\right| \leqslant I_1 + I_2 + I_3$$

where

$$I_1 = \frac{1}{\|u_\varepsilon(t)\|_2^2}\left|\int\limits_{\mathbb{R}^N\setminus B_{R_3}(0)} \big(x - \hat{q}_\varepsilon(t)\big)u_\varepsilon^2(t,x)\,dx\right|,$$

$$I_2 = \frac{1}{\|u_\varepsilon(t)\|_2^2}\left|\int\limits_{A_2} \big(x - \hat{q}_\varepsilon(t)\big)u_\varepsilon^2(t,x)\,dx\right|,$$

$$I_3 = \frac{1}{\|u_\varepsilon(t)\|_2^2}\left|\int\limits_{A_3} \big(x - \hat{q}_\varepsilon(t)\big)u_\varepsilon^2(t,x)\,dx\right|,$$

$A_2 = B_{R_3}(0) \setminus B_{\hat{R}(\lambda)\varepsilon^\beta}(\hat{q}_\varepsilon(t))$, $A_3 = B_{R_3}(0) \cap B_{\hat{R}(\lambda)\varepsilon^\beta}(\hat{q}_\varepsilon(t))$ and $R_3 \geqslant R_2$. Obviously

$$I_3 \leqslant \hat{R}(\lambda)\varepsilon^\beta.$$

Moreover, by (1) and Proposition 3.1 we have

$$I_2 \leqslant [2R_3 + 1]\lambda < 3R_3\lambda.$$

Finally, by (4.4), (1) and Remark 4.4 we have

$$\begin{aligned} I_1 &\leqslant \frac{1}{\|u_\varepsilon(t)\|_2^2}\left|\int\limits_{\mathbb{R}^N\setminus B_{R_3}(0)} |x|u_\varepsilon^2(t,x)\,dx\right| + \frac{|\hat{q}_\varepsilon(t)|}{\|u_\varepsilon(t)\|_2^2}\left|\int\limits_{\mathbb{R}^N\setminus B_{R_3}(0)} u_\varepsilon^2(t,x)\,dx\right| \\ &< \frac{C}{\sigma R_3^{a-1}} + \frac{(R_3+1)C}{\sigma R_3^a} < \frac{3C}{\sigma R_3^{a-1}} \end{aligned}$$

and we conclude using the independence of $t \in [0,\infty)$. □

We notice that R_1, R_2 and R_3 defined in this section do not depend on λ.

4.2. Equation of the traveling soliton

We prove that the barycenter dynamics is approximatively that of a point particle moving under the effect of an external potential V satisfying our assumptions.

Theorem 4.6. *Assume that V satisfies* (V0), (V1), (V2). *Given $K > 0$, let ψ be a global solution of Eq.* $(\mathcal{P}_\varepsilon)$, *with initial data in B_ε^K. If ε is small enough, then we have*

$$\ddot{q}_\varepsilon(t) + \nabla V\big(q_\varepsilon(t)\big) = H_\varepsilon(t)$$

where $\|H_\varepsilon(t)\|_{L^\infty(0,\infty)} \to 0$ as $\varepsilon \to 0^+$.

Proof. By (4.2) it is sufficient to estimate

$$H_\varepsilon(t) = \big[\nabla V\big(q_\varepsilon(t)\big) - \nabla V\big(\hat{q}_\varepsilon(t)\big)\big] + \frac{1}{\|u_\varepsilon(t)\|_2^2}\int \big[\nabla V\big(\hat{q}_\varepsilon(t)\big) - \nabla V(x)\big]u_\varepsilon^2(t,x)\,dx.$$

We set

$$M = \max\big\{\big|\partial^\alpha V(\tau)\big| \;\big|\; \alpha = 1,2 \text{ and } |\tau| \leqslant K_1 + R_2 + 1\big\}$$

where K_1 is defined in Lemma 4.3 and R_2 is defined in Remark 4.4. By Lemma 4.3 and Lemma 4.5, for any $R_3 \geqslant R_2$, we get

$$\big|\nabla V\big(q_\varepsilon(t)\big) - \nabla V\big(\hat{q}_\varepsilon(t)\big)\big| \leqslant M\left(\frac{3C}{\sigma R_3^{a-1}} + 3R_3\lambda + \hat{R}(\lambda)\varepsilon^\beta\right). \tag{4.7}$$

Moreover, we consider

$$\frac{1}{\|u_\varepsilon(t)\|_2^2}\left|\int\limits_{\mathbb{R}^N} \big[\nabla V\big(\hat{q}_\varepsilon(t)\big) - \nabla V(x)\big]u_\varepsilon^2(t,x)\,dx\right| \leqslant L_1 + L_2 + L_3$$

with

$$\begin{aligned} L_1 &= \frac{1}{\|u_\varepsilon(t)\|_2^2}\int\limits_{B_{\hat{R}(\lambda)\varepsilon^\beta}(\hat{q}_\varepsilon(t))} \big|\nabla V\big(\hat{q}_\varepsilon(t)\big) - \nabla V(x)\big|u_\varepsilon^2(t,x)\,dx, \\ L_2 &= \frac{1}{\|u_\varepsilon(t)\|_2^2}\int\limits_{\mathbb{R}^N\setminus B_{\hat{R}(\lambda)\varepsilon^\beta}(\hat{q}_\varepsilon(t))} \big|\nabla V(x)\big|u_\varepsilon^2(t,x)\,dx, \\ L_3 &= \frac{1}{\|u_\varepsilon(t)\|_2^2}\int\limits_{\mathbb{R}^N\setminus B_{\hat{R}(\lambda)\varepsilon^\beta}(\hat{q}_\varepsilon(t))} \big|\nabla V\big(\hat{q}_\varepsilon(t)\big)\big|u_\varepsilon^2(t,x)\,dx. \end{aligned}$$

By Proposition 3.1 and Lemma 4.5 we have

$$L_3 < M\lambda \tag{4.8}$$

and

$$L_1 \leqslant M\hat{R}(\lambda)\varepsilon^\beta. \tag{4.9}$$

Finally,

$$L_2 \leqslant M\lambda + \left(\frac{C}{\sigma}\right)^b \lambda^{1-b}, \tag{4.10}$$

since, by (V1), (4.6), Proposition 3.1 and (4.4), for $R_2 \geqslant R_1$,

$$\begin{aligned}
&\frac{1}{\|u_\varepsilon(t)\|_2^2} \int\limits_{\mathbb{R}^N \setminus B_{R_2+1}(0)} \left|\nabla V(x)\right| u_\varepsilon^2(t,x)\,dx \\
&\leqslant \frac{1}{\|u_\varepsilon(t)\|_2^2} \left(\int\limits_{\mathbb{R}^N \setminus B_{R_2+1}(0)} \left|\nabla V(x)\right|^{1/b} u_\varepsilon^2(t,x)\,dx \right)^b \left(\int\limits_{\mathbb{R}^N \setminus B_{R_2+1}(0)} u_\varepsilon^2(t,x)\,dx \right)^{1-b} \\
&\leqslant \left(\frac{1}{\|u_\varepsilon(t)\|_2^2} \int\limits_{\mathbb{R}^N \setminus B_{R_2+1}(0)} V(x) u_\varepsilon^2(t,x)\,dx \right)^b \lambda^{1-b} \leqslant \left(\frac{C}{(R_2+1)^{a-1}\sigma} \right)^b \lambda^{1-b} \\
&\leqslant \left(\frac{C}{\sigma}\right)^b \lambda^{1-b}
\end{aligned}$$

and, again by Proposition 3.1, we have

$$\frac{1}{\|u_\varepsilon(t)\|_2^2} \int\limits_{B_{R_2+1}(0) \setminus B_{\hat{R}(\lambda)\varepsilon^\beta}(\hat{q}_\varepsilon(t))} \left|\nabla V(x)\right| u_\varepsilon^2(t,x)\,dx \leqslant M\lambda.$$

So, by (4.7), (4.8), (4.9) and (4.10), we have

$$\left|H_\varepsilon(t)\right| \leqslant \frac{3CM}{\sigma R_3^{a-1}} + \left(\frac{C}{\sigma}\right)^b \lambda^{1-b} + M(2+3R_3)\lambda + 2M\hat{R}(\lambda)\varepsilon^\beta.$$

At this point we can have $\|H_\varepsilon(t)\|_{L^\infty(0,\infty)}$ arbitrarily small choosing firstly R_3 sufficiently large, secondly λ sufficiently small and, finally, ε small enough. □

Proof of Theorem 1.1. By Theorem 4.6 we immediately conclude the proof of Theorem 1.1. □

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