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Optimal solvability for a nonlocal problem at critical growth ☆

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Abstract

We provide optimal solvability conditions for a nonlocal minimization problem at critical growth involving an external potential function a. Furthermore, we get an existence and uniqueness result for a related nonlocal equation.

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Contents

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1. Introduction

1.1. Overview

Let Ω be a bounded domain of $\mathbb{R}^N$ with $N \geq 3$. In 1983, in the celebrated paper [5], Brezis and Nirenberg studied the solvability conditions for the semi-linear elliptic problem

$$\begin{cases} -\Delta u - \lambda\, u = u^{(N+2)/(N-2)}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

In particular, if $\lambda_1(\Omega)$ denotes the first eigenvalue of the Dirichlet–Laplacian in Ω, they proved that, if $N \geq 4$, then problem (1.1) admits a solution if $0 < \lambda < \lambda_1(\Omega)$ while for $N = 3$ there exists $\lambda^* \in (0, \lambda_1)$ such that (1.1) admits a solution if $\lambda^* < \lambda < \lambda_1(\Omega)$ and no solution for $0 < \lambda \leq \lambda^*$. Due to this phenomenon, $N = 3$ is often referred to in the literature as *critical dimension*.

In general λ^* is not given explicitly, except when Ω is a ball, in which case $\lambda^* = \lambda_1(\Omega)/4$. In addition, there is no solution to (1.1) when $\lambda \geq \lambda_1(\Omega)$ for any domain Ω (see [5, Remark 1.1]) and also for $\lambda \leq 0$ provided Ω is smooth and star-shaped (see [5, Remark 1.2]).

In the same paper the authors considered, for $N \geq 4$, the non-autonomous critical elliptic problem

$$\begin{cases} -\Delta u + a\, u = u^{(N+2)/(N-2)}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

and obtained the existence of a solution by assuming that $a \in L^\infty(\Omega)$ and that there exist $\delta > 0$ and an open subset $\Omega_0 \subset \Omega$ such that

$$a \leq -\delta, \quad \text{in } \Omega_0, \qquad \int_\Omega \left(|\nabla\varphi|^2 + a\,\varphi^2\right) dx \geq \delta \int_\Omega \varphi^2 dx, \qquad \text{for all } \varphi \in C_0^\infty(\Omega),$$

see [5, Section 4]. About the case of the critical dimension $N = 3$ for (1.2), no result is stated in [5].

After the striking achievements of [5], many works were devoted to the search of solvability conditions for (possibly sign-changing) solutions of the problem

$$\begin{cases} -\Delta u - \lambda u = |u|^{4/(N-2)} u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Solution are obtained for any $\lambda > 0$ if $N \geq 5$ and for all $\lambda > 0$ with $\lambda \notin \sigma(-\Delta)$ if $N \geq 4$. Here $\sigma(-\Delta)$ is the spectrum of the Dirichlet–Laplacian on Ω. We refer the reader e.g. to [7,13]. The main tool exploited is the Linking Theorem of Rabinowitz [19].

The previous results were then extended to the more general problem

$$\begin{cases} -\Delta_p u - \lambda |u|^{p-2} u = |u|^{p^*-2} u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $1 < p < N$, $-\Delta_p$ is the p-Laplacian operator and $p^* = N p/(N - p)$ is the critical Sobolev exponent. See for example [12,14] where *positive* solutions were found for $N \geq p^2$ and $0 < \lambda < \lambda_1(\Omega)$, via the Mountain Pass theorem of Ambrosetti and Rabinowitz [19]. Here $\lambda_1(\Omega)$ denotes the first eigenvalue of $-\Delta_p$ with Dirichlet boundary conditions.

For the general case of sign-changing solutions, a first result was obtained in [1] for λ below the second eigenvalue $\lambda_2(\Omega)$ of $-\Delta_p$ under some restrictions on p and N. Recently, Degiovanni and Lancelotti in [9] proved that, if the domain is of class $C^{1,\alpha}$ for some $\alpha \in (0, 1)$, then problem (1.3) admits a nontrivial solution for all $\lambda > 0$ provided that $(N^3 + p^3)/(N^2 + N) > p^2$.

Recently, in the framework of nonlocal problems, the following Brezis–Nirenberg type problem for the *fractional p-Laplacian of order s* was investigated

$$\begin{cases} (-\Delta_p)^s u - \lambda |u|^{p-2} u = |u|^{p_s^*-2} u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.4}$$

where $s \in (0, 1)$, $N > s\,p$, $\lambda > 0$ and $p_s^* = N p/(N - s\,p)$ is the fractional critical Sobolev exponent. This kind of problems has been first considered in [20,21], in the linear case $p = 2$. For a general p, in [17] the authors proved, among other results, that problem (1.4) has a nontrivial weak solution for all $\lambda > 0$ provided that $(N^3 + s^3 p^3)/N(N + s) > s\,p^2$ and Ω is of class $C^{1,1}$.

1.2. Main results

In this paper we return to the investigation of nonautonomous problems like (1.2), in the nonlinear nonlocal setting aiming to get optimal solvability conditions for the existence of ground state solutions. Let us set

$$[u]_{\mathcal{D}^{s,p}(\mathbb{R}^N)} := \left(\int\limits_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+s\,p}}\, dx\, dy \right)^{1/p},$$

and for $N > s\,p$ define the two spaces

$$\mathcal{D}^{s,p}(\mathbb{R}^N) := \{u \in L^{p_s^*}(\mathbb{R}^N) : [u]_{\mathcal{D}^{s,p}(\mathbb{R}^N)} < +\infty\},$$
$$\mathcal{D}_0^{s,p}(\Omega) := \{u \in \mathcal{D}^{s,p}(\mathbb{R}^N) : u = 0 \text{ on } \mathbb{R}^N \setminus \Omega\}.$$

The latter is endowed with the norm

$$\|u\|_{\mathcal{D}_0^{s,p}(\Omega)} := [u]_{\mathcal{D}^{s,p}(\mathbb{R}^N)}, \qquad u \in \mathcal{D}_0^{s,p}(\Omega).$$

Precisely, for $s\,p < N$, we aim to study the solvability conditions for the following minimization problem

$$\mathcal{S}_{p,s}(a) := \inf_{u\in\mathcal{D}_0^{s,p}(\Omega)} \left\{ [u]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \int_{\mathbb{R}^N} a\,|u|^p\,dx \,:\, \|u\|_{L^{p^*_s}(\mathbb{R}^N)} = 1 \right\},$$

where $a \in L^{N/sp}(\Omega)$ is given. By Lagrange Multipliers Rule, minimizers of the previous problem (provided they exist) are constant sign weak solutions of

$$\begin{cases} (-\Delta_p)^s\,u + a\,|u|^{p-2}\,u = \mu\,|u|^{p^*_s-2}\,u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.5}$$

with $\mu = \mathcal{S}_{p,s}(a)$. Namely, they satisfy

$$\int_{\mathbb{R}^{2N}} \frac{J_p(u(x)-u(y))\,\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+s\,p}}\,dx\,dy + \int_{\mathbb{R}^N} a\,|u|^{p-2}\,u\,\varphi\,dx = \mu \int_{\mathbb{R}^N} |u|^{p^*_s-2}\,u\,\varphi\,dx,$$

for every $\varphi \in \mathcal{D}_0^{s,p}(\Omega)$. Throughout the paper we will use the notation for $1 < p < +\infty$

$$J_p(t) = |t|^{p-2}\,t, \qquad t \in \mathbb{R}.$$

We also introduce the sharp Sobolev constant

$$\mathcal{S}_{p,s} := \inf_{u\in\mathcal{D}^{s,p}(\mathbb{R}^N)\setminus\{0\}} \frac{[u]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)}}{\|u\|^p_{L^{p^*_s}(\mathbb{R}^N)}}. \tag{1.6}$$

Finally, we use the standard notations

$$a_+ = \max\{a,\,0\}, \qquad a_- = \max\{-a, 0\}, \qquad B_R(x_0) = \{x \in \mathbb{R}^N \,:\, |x-x_0| < R\}.$$

The main result of the paper is the following

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. The following facts hold:*

1. *If $a \geq 0$, then $\mathcal{S}_{p,s}(a)$ does not admit a solution.*
2. *Let $N > s\,p^2$. Assume that there exist $\sigma > 0$, $R > 0$ and $x_0 \in \Omega$ such that*

$$a_- \geq \sigma, \qquad a.\,e.\ on\ B_R(x_0) \subset \Omega.$$

Then $\mathcal{S}_{p,s}(a)$ admits a solution.

3. *Let $s\,p < N \le s\,p^2$. For any $R > 0$ there exists $\sigma = \sigma(R, N, s, p) > 0$ such that if*

$$a_- \ge \sigma, \qquad \text{a. e. on } B_R(x_0) \subset \Omega,$$

then $\mathcal{S}_{p,s}(a)$ admits a solution.

The conditions on a for the existence of solutions in Theorem 1.1 are essentially optimal, see the discussion of Remark 4.1. In Proposition 3.4 we also prove that the solution is unique when $\mathcal{S}_{p,s}(a) \le 0$.

The precise form of the optimizers for the best constant in the fractional Sobolev embedding is still unknown (and proving it seems currently out of reach), although minimizers were conjectured to be of the form

$$c\,U\left(\frac{x - x_0}{\varepsilon}\right), \qquad \text{where } U(x) = (1 + |x|^{p'})^{-(N-sp)/p}, \qquad x \in \mathbb{R}^N,$$

$c \ne 0$, $x_0 \in \mathbb{R}^N$ and $\varepsilon > 0$. This would be consistent with the special case $p = 2$, where the form of optimizers is known, see [8]. In the proof of Theorem 1.1, a key tool is the use of suitable truncations of extremals U of the Sobolev inequality, introduced in [17] (see Section 2 below). In fact, the knowledge of their decay at infinity (recently proved in [3]) is enough to conclude.

Then, we also have the following result

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. Let $a \in L^{N/sp}(\Omega)$ be such that*

$$\mathcal{S}_{p,s}(a) < 0. \tag{1.7}$$

Then problem (1.5)

i) does not admit positive solutions if $\mu \ge 0$;
ii) admits a unique positive solution if $\mu < 0$.

Remark 1.3. We can rephrase condition (1.7) also in terms of the following Poincaré-type constant

$$\lambda(\Omega, a) := \inf_{u \in \mathcal{D}_0^{s,p}(\Omega)} \left\{ [u]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \int_{\mathbb{R}^N} a_+\,|u|^p\,dx \,:\, \int_{\mathbb{R}^N} a_-\,|u|^p\,dx = 1 \right\}.$$

Indeed, we can show that

$$\lambda(\Omega; a) < 1 \qquad \Longleftrightarrow \qquad \mathcal{S}_{p,s}(a) < 0,$$

and also

$$\lambda(\Omega; a) = 1 \qquad \Longleftrightarrow \qquad \mathcal{S}_{p,s}(a) = 0,$$

see Remark 3.5. For a discussion on sufficient conditions ensuring $\lambda(\Omega; a) < 1$, and hence (1.7), we refer the reader to Remark 5.1.

Remark 1.4 *(The case of the p-Laplacian).* Although we can only formally choose the limiting value $s = 1$ in the previous statements, the same proofs in the paper would provide existence and non-existence results for the following quasilinear local problem

$$\mathcal{S}_p(a) = \inf_{u \in \mathcal{D}_0^{1,p}(\Omega)} \left\{ \int_{\mathbb{R}^N} |\nabla u|^p \, dx + \int_{\mathbb{R}^N} a \, |u|^p \, dx : \|u\|_{L^{Np/(N-p)}(\mathbb{R}^N)} = 1 \right\}.$$

Finally, we stress that already in the semi-linear case $p = 2$, Theorem 1.1 and Theorem 1.2 cover situations which were previously open, such as the critical case of dimension $N = 3$ and also the case of weaker integrability assumptions on the external potential a.

2. Preliminaries

2.1. Some known results

We start with an elementary inequality, whose proof is based on Calculus and we omit it.

Lemma 2.1. *If $\gamma < 0$, then we have*

$$(1-t)^\gamma \le 1 + 2\,t\,(2^{-\gamma} - 1), \qquad \text{for every } 0 \le t \le \frac{1}{2}. \tag{2.1}$$

If $0 \le \gamma \le 1$, then we have

$$(1-t)^\gamma \ge 1 + 2\,t\,(2^{-\gamma} - 1), \qquad \text{for every } 0 \le t \le \frac{1}{2}. \tag{2.2}$$

The following is a discrete version of the celebrated *Picone's inequality*. See [2, Proposition 4.2] and [11, Lemma 2.6] for a proof.

Proposition 2.2 *(Discrete Picone inequality). Let $1 < p < \infty$ and let $a, b, c, d \in [0, +\infty)$, with $a, b > 0$. Then*

$$J_p(a-b) \left[\frac{c^p}{a^{p-1}} - \frac{d^p}{b^{p-1}} \right] \le |c-d|^p. \tag{2.3}$$

Moreover, if equality holds in (2.3), then

$$\frac{a}{b} = \frac{c}{d}.$$

We recall that in [3, Theorem 1.1] the following result was proved.

Theorem 2.3 *(Decay of extremals). Let $U \in \mathcal{D}^{s,p}(\mathbb{R}^N)$ be any minimizer for (1.6). Then $U \in L^\infty(\mathbb{R}^N)$ is a constant sign, radially symmetric and monotone function with*

$$\lim_{|x|\to\infty} |x|^{\frac{N-s\,p}{p-1}}\, U(x) = U_\infty,$$

for some constant $U_\infty \in \mathbb{R}\setminus\{0\}$.

Let us now fix $U \in \mathcal{D}^{s,p}(\mathbb{R}^N)$ a positive minimizer of (1.6), such that

$$[U]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} = \|U\|^{p_s^*}_{L^{p_s^*}(\mathbb{R}^N)} = \mathcal{S}_{p,s}^{\frac{N}{s\,p}}, \qquad \text{and} \qquad U(0)=1.$$

Since this is a radially symmetric function, with a slight abuse of notation we write $U(r)$ in place of $U(x)$, with $|x|=r$. Based upon the above decay estimate, we can infer the following result (see [17, Lemma 2.2]).

Lemma 2.4. *With the notation above, there exists $\theta > 1$ such that for all $r \geq 1$,*

$$\frac{U(\theta\, r)}{U(r)} \leq \frac{1}{2}.$$

In Section 3 we will need to use some truncations of the Sobolev extremal U. To this aim, for $\varepsilon > 0$ we first set

$$U_\varepsilon(r) := \varepsilon^{\frac{s\,p-N}{p}}\, U\left(\frac{r}{\varepsilon}\right), \qquad r \geq 0,$$

which still solves (1.6). Then by following [17], for $\delta > 0$ we introduce

$$u_{\varepsilon,\delta}(r) := \begin{cases} U_\varepsilon(r), & r \leq \delta \\ U_\varepsilon(\delta)\, \dfrac{U_\varepsilon(r) - U_\varepsilon(\theta\delta)}{U_\varepsilon(\delta) - U_\varepsilon(\theta\,\delta)}, & \delta < r \leq \theta\,\delta \\ 0, & r > \theta\,\delta, \end{cases} \tag{2.4}$$

where θ is the constant appearing in Lemma 2.4. We then recall from [17, Lemma 2.7] the following crucial energy estimates for $u_{\varepsilon,\delta}$.

Lemma 2.5 *(Energy estimates). There exists $\mathcal{C} = \mathcal{C}(N,p,s) > 0$ such that for any $\varepsilon \leq \delta/2$,*

$$[u_{\varepsilon,\delta}]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} \leq \left(\mathcal{S}_{p,s}\right)^{\frac{N}{sp}} + \mathcal{C}\left(\frac{\varepsilon}{\delta}\right)^{\frac{N-s\,p}{p-1}},$$

$$\|u_{\varepsilon,\delta}\|^p_{L^{p_s^*}(\mathbb{R}^N)} \geq \left[\left(\mathcal{S}_{p,s}\right)^{\frac{N}{sp}} - \mathcal{C}\left(\frac{\varepsilon}{\delta}\right)^{N/(p-1)}\right]^{\frac{N-s\,p}{N}}.$$

2.2. *Levels of compactness*

The next result tells us that, in order to have loss of compactness, a certain amount of energy is necessary.

Lemma 2.6. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let us suppose that $s\,p < N$. Let $\mu \in \mathbb{R} \setminus \{0\}$, the functional*

$$\mathcal{K}(u) := \frac{1}{p}\,[u]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \frac{1}{p}\int_\Omega a\,|u|^p\,dx - \frac{\mu}{p^*_s}\int_\Omega |u|^{p^*_s}\,dx, \qquad u \in \mathcal{D}^{s,p}_0(\Omega), \tag{2.5}$$

satisfies the Palais–Smale condition:

- *at every energy level $c \in \mathbb{R}$, if $\mu < 0$;*
- *at every energy level c such that*

$$c < \frac{s}{N}\,\mu\left(\frac{\mathcal{S}_{p,s}}{\mu}\right)^{\frac{N}{s\,p}},$$

if $\mu > 0$.

Proof. We discuss the two cases separately.

<u>*Case* $\mu < 0$.</u> Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{D}^{s,p}_0(\mathbb{R}^N)$ be a Palais–Smale sequence at level c, i.e.

$$\frac{1}{p}\,[u_n]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \frac{1}{p}\int_\Omega a\,|u_n|^p\,dx - \frac{\mu}{p^*_s}\int_\Omega |u_n|^{p^*_s}\,dx = c + o_n(1)$$

and

$$\begin{aligned}\sup_{\|\varphi\|_{\mathcal{D}^{s,p}_0(\Omega)}=1}\Bigg|\int_{\mathbb{R}^{2N}} &\frac{J_p(u_n(x)-u_n(y))\,\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+s\,p}}\,dx\,dy \\ &+ \int_\Omega a\,|u_n|^{p-2}\,u_n\,\varphi\,dx - \mu\int_\Omega |u_n|^{p^*_s-2}\,u_n\,\varphi\,dx\Bigg| = o_n(1).\end{aligned} \tag{2.6}$$

The first condition implies that for n sufficiently large we have

$$\begin{aligned} c+1 &\geq \frac{1}{p}\,[u_n]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} - \frac{1}{p}\,\|a_-\|_{L^{N/sp}(\Omega)}\left(\int_\Omega |u_n|^{p^*_s}\,dx\right)^{\frac{p}{p^*_s}} - \frac{\mu}{p^*_s}\int_\Omega |u_n|^{p^*_s}\,dx \\ &\geq \frac{1}{p}\,[u_n]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} - \frac{p^*_s-p}{p\,p^*_s}\,\varepsilon^{-\frac{p}{p^*_s-p}}\,\|a_-\|^{\frac{p^*_s}{p^*_s-p}}_{L^{N/sp}(\Omega)} - \frac{\mu+\varepsilon}{p^*_s}\int_\Omega |u_n|^{p^*_s}\,dx, \end{aligned}$$

where we used Hölder's and Young's inequalities. By choosing $\varepsilon = -\mu > 0$, we get that the sequence is bounded in $\mathcal{D}^{s,p}_0(\Omega)$. Thus we can infer weak convergence in $\mathcal{D}^{s,p}_0(\Omega)$ and $L^{p^*_s}(\Omega)$ (up to a subsequence) to a function $u \in \mathcal{D}^{s,p}_0(\Omega)$. By weak convergence, we obtain

$$
\int_{\mathbb{R}^{2N}} \frac{J_p(u(x)-u(y))\big((u_n-u)(x)-(u_n-u)(y)\big)}{|x-y|^{N+s\,p}}\,dx\,dy + \int_{\mathbb{R}^N} a\,|u|^{p-2}\,u\,(u_n-u)\,dx
$$
$$
-\mu \int_{\mathbb{R}^N} |u|^{p_s^*-2}\,u\,(u_n-u)\,dx = o_n(1).
$$

From (2.6) and using that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{D}_0^{s,p}(\Omega)$, we obtain

$$
\int_{\mathbb{R}^{2N}} \frac{J_p(u_n(x)-u_n(y))\big((u_n-u)(x)-(u_n-u)(y)\big)}{|x-y|^{N+s\,p}}\,dx\,dy
$$
$$
+\int_{\mathbb{R}^N} a\,|u_n|^{p-2}\,u_n\,(u_n-u)\,dx - \mu \int_{\mathbb{R}^N} |u_n|^{p_s^*-2}\,u_n\,(u_n-u)\,dx = o_n(1).
$$

By subtracting the last two displays, we obtain

$$
\int_{\mathbb{R}^{2N}} \frac{\Big(J_p(u_n(x)-u_n(y))-J_p(u(x)-u(y))\Big)\big((u_n-u)(x)-(u_n-u)(y)\big)}{|x-y|^{N+s\,p}}\,dx\,dy
$$
$$
+\int_{\mathbb{R}^N} a\left(|u_n|^{p-2}\,u_n-|u|^{p-2}\,u\right)(u_n-u)\,dx
$$
$$
-\mu \int_{\mathbb{R}^N} \left(|u_n|^{p_s^*-2}\,u_n-|u|^{p_s^*-2}\,u\right)(u_n-u)\,dx = o_n(1).
$$

By using that $-\mu>0$ and the monotonicity of $t\mapsto |t|^{p_s^*-2}\,t$, we get

$$
\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{\Big(J_p(u_n(x)-u_n(y))-J_p(u(x)-u(y))\Big)\big((u_n-u)(x)-(u_n-u)(y)\big)}{|x-y|^{N+s\,p}}\,dx\,dy\\
&+\int_{\mathbb{R}^N} a\left(|u_n|^{p-2}\,u_n-|u|^{p-2}\,u\right)(u_n-u)\,dx \le o_n(1).
\end{aligned}
\tag{2.7}
$$

We now observe that by Lemma 2.8 below

$$
\int_{\mathbb{R}^N} a\left(|u_n|^{p-2}\,u_n-|u|^{p-2}\,u\right)(u_n-u)\,dx = o_n(1),
$$

and that

$$
\Big(J_p(u_n(x)-u_n(y))-J_p(u(x)-u(y))\Big)\big((u_n-u)(x)-(u_n-u)(y)\big)\ge 0,
$$

by monotonicity of $t \mapsto J_p(t)$. Thus from (2.7) we can finally infer

$$\int\limits_{\mathbb{R}^{2N}} \frac{\Big(J_p(u_n(x)-u_n(y)) - J_p(u(x)-u(y))\Big)\Big((u_n-u)(x)-(u_n-u)(y)\Big)}{|x-y|^{N+s\,p}}\,dx\,dy = o_n(1).$$

By standard monotonicity inequalities, we eventually infer the strong convergence to u.

Case $\mu > 0$. It is an easy variant of the proof of [18, Proposition 3.1] where the compactness is obtained for a constant potential $a \equiv -\mu$, for $\mu > 0$. □

Remark 2.7. When $\mu = 0$, in general the functional

$$\mathcal{K}(u) := \frac{1}{p}\,[u]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \frac{1}{p}\int\limits_{\mathbb{R}^N} a\,|u|^p\,dx, \qquad u \in \mathcal{D}^{s,p}_0(\Omega),$$

does not satisfy the Palais–Smale condition, unless some conditions on a are imposed. For example, let us define the first eigenvalue of the fractional p-Laplacian of order s, i.e.

$$\lambda_1(\Omega) := \inf_{u\in\mathcal{D}^{s,p}_0(\Omega)} \left\{ [u]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} : \int\limits_{\mathbb{R}^N} |u|^p\,dx = 1 \right\}. \tag{2.8}$$

Then by taking

$$a = -\lambda_1(\Omega),$$

and ϕ_1 a minimizer of (2.8), the sequence

$$u_n = n\,\phi_1, \qquad n \in \mathbb{N},$$

is a Palais–Smale sequence for $\mathcal{K}$ (at level 0), but the sequence is not even bounded.

It is easily seen that in this case $\mathcal{K}$ satisfies the Palais–Smale condition at every level, when

$$\|a_-\|_{L^{N/sp}(\Omega)} < \mathcal{S}_{p,s}.$$

Lemma 2.8. *Let $1 < p < \infty$ and $s \in (0,1)$ be such that $s\,p < N$. Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. Let $\{u_n\} \subset \mathcal{D}^{s,p}_0(\Omega)$ be such that*

$$[u_n]_{\mathcal{D}^{s,p}(\mathbb{R}^N)} \le C, \qquad \textit{for every } n \in \mathbb{N}.$$

Then there exists $u \in \mathcal{D}^{s,p}_0(\Omega)$ such that (up to a subsequence)

$$\Big\{ \big(|u_n|^{p-2}\,u_n - |u|^{p-2}\,u\big)\,(u_n - u) \Big\}_{n\in\mathbb{N}},$$

converges weakly to 0 in $L^{p^*_s/p}(\Omega)$.

Proof. The hypothesis implies there exists $u \in \mathcal{D}_0^{s,p}(\Omega)$ such that (up to a subsequence) the sequence converges weakly in $\mathcal{D}_0^{s,p}(\Omega)$ and strongly in $L^p(\Omega)$ to u, by compactness of the embedding $\mathcal{D}_0^{s,p}(\Omega) \hookrightarrow L^p(\Omega)$. In particular, up to extracting a further subsequence, we can suppose that u_n converges almost everywhere to u.

We now observe that

$$\Big|\Big(|u_n|^{p-2}\,u_n - |u|^{p-2}\,u\Big)\,(u_n - u)\Big| \le C \left\{ \begin{array}{ll} |u_n - u|^p, & \text{if } 1 < p \le 2, \\ \Big(|u_n|^{p-2} + |u|^{p-2}\Big)\,|u_n - u|^2, & \text{if } p > 2, \end{array} \right.$$

for some $C = C(p) > 0$. By using Sobolev and Hölder inequalities, this implies that the sequence

$$v_n := \Big(|u_n|^{p-2}\,u_n - |u|^{p-2}\,u\Big)\,(u_n - u),$$

is bounded in $L^{p_s^*/p}(\Omega)$. Moreover, v_n converges almost everywhere to 0 by the first part of the proof. Then the conclusion follows from [15, Lemme 4.8]. □

3. Analysis of the ground state level

Throughout this section, Ω will be an open bounded set and we will assume $s\,p < N$. For a given potential $a \in L^{N/sp}(\Omega)$, we want to study some properties of the ground state level

$$\mathcal{S}_{p,s}(a) := \inf_{u \in \mathcal{D}_0^{s,p}(\Omega)} \left\{ [u]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \int_{\mathbb{R}^N} a\,|u|^p\,dx \,:\, \|u\|_{L^{p_s^*}(\mathbb{R}^N)} = 1 \right\}.$$

We first observe that $\mathcal{S}_{p,s}(a) > -\infty$, indeed for every admissible function u we have

$$[u]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \int_{\mathbb{R}^N} a\,|u|^p\,dx \ge [u]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} - \|a\|_{L^{N/sp}(\Omega)} \ge \mathcal{S}_{p,s} - \|a\|_{L^{N/sp}(\Omega)},$$

by Hölder's and Sobolev inequalities. We observe that when $a \equiv 0$, the problem above coincides with the determination of the best constant in the inequality

$$[u]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} \ge c\,\|u\|^p_{L^{p_s^*}(\Omega)}, \qquad \text{for } u \in \mathcal{D}_0^{s,p}(\Omega).$$

As in the local case, such a constant does not depend on Ω and is never attained on proper subsets of $\mathbb{R}^N$. This is the content of the next result.

Lemma 3.1. *For every open set $E \subset \mathbb{R}^N$ we have*

$$\mathcal{S}_{p,s}(0) = \mathcal{S}_{p,s},$$

and $\mathcal{S}_{p,s}(0)$ is not attained in $\mathcal{D}_0^{s,p}(E)$, whenever $|\mathbb{R}^N \setminus E| > 0$.

Proof. Let $\varepsilon > 0$, then there exists $\varphi_\varepsilon \in C_0^\infty(\mathbb{R}^N)$ such that

$$\|\varphi_\varepsilon\|_{L^{p_s^*}(\mathbb{R}^N)} = 1 \qquad \text{and} \qquad [\varphi_\varepsilon]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} \le \mathcal{S}_{p,s} + \varepsilon.$$

Let $\lambda > 1$, by taking

$$\varphi_{\lambda,\varepsilon}(x) = \lambda^{\frac{N-s\,p}{p}}\, \varphi_\varepsilon(\lambda\, x),$$

we still have that

$$\|\varphi_{\lambda,\varepsilon}\|_{L^{p_s^*}(\mathbb{R}^N)} = 1 \qquad \text{and} \qquad [\varphi_{\lambda,\varepsilon}]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} \le \mathcal{S}_{p,s} + \varepsilon.$$

Moreover, by taking λ large enough and up to translations, we have $\varphi_{\lambda,\varepsilon} \in C_0^\infty(E)$. Thus we can use it as a test function in the problem defining $\mathcal{S}_{p,s}(0)$ and obtain

$$\mathcal{S}_{p,s}(0) \le [\varphi_{\lambda,\varepsilon}]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} \le \mathcal{S}_{p,s} + \varepsilon.$$

By arbitrariness of ε, this gives $\mathcal{S}_{p,s}(0) \le \mathcal{S}_{p,s}$. The reverse inequality is straightforward, thanks to the continuous embedding $\mathcal{D}_0^{s,p}(E) \hookrightarrow \mathcal{D}^{s,p}(\mathbb{R}^N)$.

We now suppose that $\mathcal{S}_{p,s}(0)$ is attained by $u \in \mathcal{D}_0^{s,p}(E) \setminus \{0\}$. By extending it to 0 outside E, we have $u \in \mathcal{D}^{s,p}(\mathbb{R}^N)$ and thus u is an extremal for the Sobolev inequality on the whole $\mathbb{R}^N$, thanks to the first part of the proof. Observe that u can be taken to be non-negative, due to Proposition 3.2 below. By optimality, we get that u is a constant sign supersolution of $(-\Delta_p)^s$, i.e. $(-\Delta_p)^s u \ge 0$, in $\mathbb{R}^N$. Thus by the minimum principle of [2, Proposition A.1], we should have $u > 0$ almost everywhere. But this contradicts the fact that $u \equiv 0$ in $\mathbb{R}^N \setminus E$. $\square$

Proposition 3.2. *Let us assume that the variational problem defining $\mathcal{S}_{p,s}(a)$ admits a solution $u \in \mathcal{D}_0^{s,p}(\Omega)$. Then u has constant sign and $u \neq 0$ almost everywhere in Ω.*

Proof. We observe that the function $|u|$ is still admissible for the variational problem and

$$\big[|u|\big]_{\mathcal{D}^{s,p}(\mathbb{R}^N)} \le [u]_{\mathcal{D}^{s,p}(\mathbb{R}^N)},$$

with inequality being strict if both u_+ and u_- are nontrivial. By minimality of u, this implies that u must have constant sign. Let us assume for example that $u \ge 0$ almost everywhere in Ω, then by a simple application of the Lagrange Multipliers Rule we get that u is a non negative solution of (1.5), where $\mu = \mathcal{S}_{p,s}(a)$. By appealing to the minimum principle of [4, Proposition B.3] we get that $u > 0$ almost everywhere. $\square$

Proposition 3.3. *Let $\mu \le 0$, then the problem*

$$\begin{cases} (-\Delta_p)^s\, u + a\, u^{p-1} = \mu\, u^{p_s^*-1}, & \textit{in } \Omega, \\ u > 0, & \textit{in } \Omega, \\ u = 0, & \textit{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

admits at most

- *one solution, if* $\mu < 0$*;*
- *one solution with unit* $L^{p_s^*}$ *norm, if* $\mu = 0$.

Proof. We use an idea due to Brezis and Oswald, based on Picone's inequality (see [6]). We assume that the problem above admits two solutions $u_1, u_2 \in \mathcal{D}_0^{s,p}(\Omega)$. We fix $\varepsilon > 0$ and define $u_{i,\varepsilon} = \min\{u_i,\ 1/\varepsilon\}$ for $i = 1, 2$. We have

$$\int\limits_{\mathbb{R}^{2N}} \frac{J_p(u_1(x) - u_1(y))\big(\varphi(x) - \varphi(y)\big)}{|x-y|^{N+s\,p}}\,dx\,dy + \int\limits_{\mathbb{R}^N} a\,u_1^{p-1}\,\varphi\,dx = \mu \int\limits_{\mathbb{R}^N} u_1^{p_s^*-1}\,\varphi\,dx,$$

$$\int\limits_{\mathbb{R}^{2N}} \frac{J_p(u_2(x) - u_2(y))\big(\varphi(x) - \varphi(y)\big)}{|x-y|^{N+s\,p}}\,dx\,dy + \int\limits_{\mathbb{R}^N} a\,u_2^{p-1}\,\varphi\,dx = \mu \int\limits_{\mathbb{R}^N} u_2^{p_s^*-1}\,\varphi\,dx.$$

We test these equations respectively with

$$\varphi_1 := \frac{u_{2,\varepsilon}^p}{(u_1+\varepsilon)^{p-1}} - u_{1,\varepsilon}, \qquad \varphi_2 := \frac{u_{1,\varepsilon}^p}{(u_2+\varepsilon)^{p-1}} - u_{2,\varepsilon}.$$

By adding the two resulting identities, we obtain

$$\begin{aligned}
&\int\limits_{\mathbb{R}^{2N}} \frac{J_p(u_1(x) - u_1(y))\left(\dfrac{u_{2,\varepsilon}^p}{(u_1+\varepsilon)^{p-1}}(x) - \dfrac{u_{2,\varepsilon}^p}{(u_1+\varepsilon)^{p-1}}(y)\right)}{|x-y|^{N+s\,p}}\,dx\,dy \\
&\quad - \int\limits_{\mathbb{R}^{2N}} \frac{J_p(u_1(x) - u_1(y))\big(u_{1,\varepsilon}(x) - u_{1,\varepsilon}(y)\big)}{|x-y|^{N+s\,p}}\,dx\,dy \\
&\quad + \int\limits_{\mathbb{R}^{2N}} \frac{J_p(u_2(x) - u_2(y))\left(\dfrac{u_{1,\varepsilon}^p}{(u_2+\varepsilon)^{p-1}}(x) - \dfrac{u_{1,\varepsilon}^p}{(u_2+\varepsilon)^{p-1}}(y)\right)}{|x-y|^{N+s\,p}}\,dx\,dy \\
&\quad - \int\limits_{\mathbb{R}^{2N}} \frac{J_p(u_2(x) - u_2(y))\big(u_{2,\varepsilon}(x) - u_{2,\varepsilon}(y)\big)}{|x-y|^{N+s\,p}}\,dx\,dy \\
&\quad + \int\limits_{\mathbb{R}^N} a\,u_1^{p-1}\left(\frac{u_{2,\varepsilon}^p}{(u_1+\varepsilon)^{p-1}} - u_{1,\varepsilon}\right)dx + \int\limits_{\mathbb{R}^N} a\,u_2^{p-1}\left(\frac{u_{1,\varepsilon}^p}{(u_2+\varepsilon)^{p-1}} - u_{2,\varepsilon}\right)dx \\
&= \mu \int\limits_{\mathbb{R}^N} u_1^{p_s^*-1}\left(\frac{u_{2,\varepsilon}^p}{(u_1+\varepsilon)^{p-1}} - u_{1,\varepsilon}\right)dx + \mu \int\limits_{\mathbb{R}^N} u_2^{p_s^*-1}\left(\frac{u_{1,\varepsilon}^p}{(u_2+\varepsilon)^{p-1}} - u_{2,\varepsilon}\right)dx.
\end{aligned} \tag{3.1}$$

We now observe that

$$J_p(u_i(x) - u_i(y)) = J_p((u_i+\varepsilon)(x) - (u_i+\varepsilon)(y)), \qquad \text{for } i = 1, 2,$$

thus by Picone's inequality Proposition 2.2 we have

$$J_p(u_1(x)-u_1(y))\left(\frac{u_{2,\varepsilon}^p}{(u_1+\varepsilon)^{p-1}}(x)-\frac{u_{2,\varepsilon}^p}{(u_1+\varepsilon)^{p-1}}(y)\right)\le |u_2(x)-u_2(y)|^p,$$

where we also used that $t\mapsto \min\{|t|,1/\varepsilon\}$ is 1-Lipschitz. Similarly, we get

$$J_p(u_2(x)-u_2(y))\left(\frac{u_{1,\varepsilon}^p}{(u_2+\varepsilon)^{p-1}}(x)-\frac{u_{1,\varepsilon}^p}{(u_2+\varepsilon)^{p-1}}(y)\right)\le |u_1(x)-u_1(y)|^p.$$

We then pass to the limit in (3.1), by using Fatou's Lemma in the first and third terms and the Dominated Convergence Theorem in all the others. This yields

$$\begin{aligned}
&\int\limits_{\mathbb{R}^{2N}}\frac{J_p(u_1(x)-u_1(y))\left(\dfrac{u_2^p}{u_1^{p-1}}(x)-\dfrac{u_2^p}{u_1^{p-1}}(y)\right)}{|x-y|^{N+s\,p}}\,dx\,dy-\int\limits_{\mathbb{R}^{2N}}\frac{|u_1(x)-u_1(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy\\
&+\int\limits_{\mathbb{R}^{2N}}\frac{J_p(u_2(x)-u_2(y))\left(\dfrac{u_1^p}{u_2^{p-1}}(x)-\dfrac{u_1^p}{u_2^{p-1}}(y)\right)}{|x-y|^{N+s\,p}}\,dx\,dy-\int\limits_{\mathbb{R}^{2N}}\frac{|u_2(x)-u_2(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy\\
&\ge \mu\int\limits_{\mathbb{R}^N}u_1^{p_s^*-p}\,u_2^p\,dx-\mu\int\limits_{\mathbb{R}^N}u_1^{p_s^*}\,dx+\mu\int\limits_{\mathbb{R}^N}u_2^{p_s^*-p}\,u_1^p\,dx-\mu\int\limits_{\mathbb{R}^N}u_2^{p_s^*}\,dx.
\end{aligned}\tag{3.2}$$

We now use Picone's inequality in the left-hand side of (3.2). This gives

$$0\ge -\mu\int\limits_{\mathbb{R}^N}(u_1^p-u_2^p)\,(u_1^{p_s^*-p}-u_2^{p_s^*-p})\,dx.$$

If $\mu<0$, from the previous inequality we directly get that $u_1=u_2$.

If $\mu=0$, we go back to (3.2) and observe that this and Picone's inequality imply

$$\int\limits_{\mathbb{R}^{2N}}\frac{J_p(u_1(x)-u_1(y))\left(\dfrac{u_2^p}{u_1^{p-1}}(x)-\dfrac{u_2^p}{u_1^{p-1}}(y)\right)}{|x-y|^{N+s\,p}}\,dx\,dy=\int\limits_{\mathbb{R}^{2N}}\frac{|u_2(x)-u_2(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy.$$

By appealing to equality cases in Picone's inequality, we get the conclusion

$$\frac{u_1(x)}{u_1(y)}=\frac{u_2(x)}{u_2(y)},\qquad \text{for a. e. } x,y\in\Omega.$$

This implies that $u_1=c\,u_2$ for some positive constant c. □

In the local case, formally corresponding to $s = 1$, the assertion below was proved in [16].

Proposition 3.4. *Let us suppose that $\mathcal{S}_{p,s}(a) < \mathcal{S}_{p,s}$. Then the problem defining $\mathcal{S}_{p,s}(a)$ has a solution. Moreover, if $\mathcal{S}_{p,s}(a) \le 0$, then such a solution is unique, up to the choice of the sign.*

Proof. We first observe that if $\mathcal{S}_{p,s}(a) \le 0$, then the uniqueness follows by combining Propositions 3.2 and 3.3, since every minimizer is a constant sign solution of (1.5), with $\mu = \mathcal{S}_{p,s}(a)$.

We now come to the existence part and divide the proof in two cases.

Case $\mathcal{S}_{p,s}(a) \neq 0$. Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{D}_0^{s,p}(\Omega)$ be a sequence such that $\|u_n\|_{L^{p_s^*}(\mathbb{R}^N)} = 1$ and

$$\lim_{n\to\infty} \left([u_n]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \int_{\mathbb{R}^N} a\,|u_n|^p\,dx \right) = \mathcal{S}_{p,s}(a).$$

By the constrained version of Ekeland's variational principle (see [10, Theorem 3.1]) applied to the following functionals on $\mathcal{D}_0^{s,p}(\Omega)$

$$J(u) := \frac{1}{p}\,[u]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \frac{1}{p}\int_{\mathbb{R}^N} a\,|u|^p\,dx \qquad \text{and} \qquad G(u) := \frac{1}{p_s^*}\int_{\mathbb{R}^N} |u|^{p_s^*}\,dx,$$

there exist $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ and $\{\widetilde{u}_n\}_{n\in\mathbb{N}} \subset \mathcal{D}_0^{s,p}(\Omega)$ such that

$$J(\widetilde{u}_n) \le J(u_n) \qquad \text{and} \qquad G(\widetilde{u}_n) = G(u_n) = \frac{1}{p_s^*},$$

and

$$\lim_{n\to\infty} \left(\sup_{\|\varphi\|_{\mathcal{D}_0^{s,p}(\Omega)}=1} \left|\langle J'(\widetilde{u}_n) - \lambda_n\, G'(\widetilde{u}_n), \varphi\rangle\right| \right) = 0.$$

Since the sequence $\{\widetilde{u}_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{D}_0^{s,p}(\Omega)$, we have

$$\langle J'(\widetilde{u}_n), \widetilde{u}_n\rangle - \lambda_n\,\langle G'(\widetilde{u}_n), \widetilde{u}_n\rangle = o_n(1),$$

which yields $\lambda_n = \mathcal{S}_{p,s}(a) + o_n(1)$. Therefore, a direct computation shows that $\{\widetilde{u}_n\}_{n\in\mathbb{N}} \subset \mathcal{D}_0^{s,p}(\Omega)$ is a Palais–Smale sequence for the functional (2.5) with $\mu = \mathcal{S}_{p,s}(a) \neq 0$, at the energy level

$$c := \frac{s}{N}\,\mathcal{S}_{p,s}(a) < \frac{s}{N}\,\mathcal{S}_{p,s}(a) \left(\frac{\mathcal{S}_{s,p}}{\mathcal{S}_{p,s}(a)} \right)^{\frac{N}{s\,p}}.$$

By Lemma 2.6, we can infer strong convergence of the minimizing sequence $\{\widetilde{u}_n\}_{n\in\mathbb{N}}$ in $\mathcal{D}_0^{s,p}(\Omega)$ and thus existence of a solution u to the problem defining $\mathcal{S}_{p,s}(a)$.

Case $\mathcal{S}_{p,s}(a) = 0$. We first observe that if $a \geq 0$ on Ω, then

$$\mathcal{S}_{p,s}(a) \geq \mathcal{S}_{p,s} > 0.$$

Thus $\mathcal{S}_{p,s}(a) = 0$ implies that $a_- \not\equiv 0$. By definition of $\mathcal{S}_{p,s}(a)$ and the fact that $\mathcal{S}_{p,s}(a) = 0$, this implies that we have the Poincaré-type inequality

$$[u]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \int_{\mathbb{R}^N} a_+\,|u|^p\,dx \geq \int_{\mathbb{R}^N} a_-\,|u|^p\,dx, \tag{3.3}$$

for every $u \in \mathcal{D}^{s,p}_0(\Omega)$. Let us consider the sharp constant in the previous Poincaré-type inequality, i.e.

$$\lambda(\Omega, a) := \inf\left\{[u]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \int_{\mathbb{R}^N} a_+\,|u|^p\,dx \,:\, \int_{\mathbb{R}^N} a_-\,|u|^p\,dx = 1\right\}. \tag{3.4}$$

It is standard routine to see that the infimum above is achieved, by a constant-sign function. Let us call ϕ_1 the positive solution. Observe that by (3.3), we have $\lambda(\Omega, a) \geq 1$. On the other hand, since $\mathcal{S}_{p,s}(a) = 0$ there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{D}^{s,p}_0(\Omega)$ such that

$$\|u_n\|_{L^{p^*_s}(\mathbb{R}^N)} = 1 \qquad \text{and} \qquad [u_n]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \int_{\mathbb{R}^N} a_+\,|u_n|^p\,dx - \int_{\mathbb{R}^N} a_-\,|u_n|^p\,dx = o_n(1). \tag{3.5}$$

We now observe that by Sobolev inequality we have

$$\int_{\mathbb{R}^N} a_-\,|u_n|^p\,dx = [u_n]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \int_{\mathbb{R}^N} a_+\,|u_n|^p\,dx + o_n(1) \geq \mathcal{S}_{p,s} + o_n(1),$$

where we used that every u_n has unit norm in $L^{p^*_s}$. We can thus divide the second equation in (3.5) by $\int_{\mathbb{R}^N} a_-\,|u_n|^p\,dx$ and obtain

$$\lambda(\Omega, a) \leq \lim_{n\to\infty} \frac{[u_n]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \int_{\mathbb{R}^N} a_+\,|u_n|^p\,dx}{\int_{\mathbb{R}^N} a_-\,|u_n|^p\,dx} = 1.$$

This finally implies that $\lambda(\Omega, a) = 1$ and thus the function

$$v_0 := \frac{\phi_1}{\|\phi_1\|_{L^{p^*_s}(\mathbb{R}^N)}},$$

is such that

$$[v_0]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \int\limits_{\mathbb{R}^N} a\,|v_0|^{p-2}\,v_0\,dx = 0 \qquad \text{and} \qquad \|v_0\|_{L^{p_s^*}(\mathbb{R}^N)} = 1,$$

i.e. v_0 is a solution of the problem defining $\mathcal{S}_{p,s}(a)$. □

Remark 3.5. The quantity $\lambda(\Omega, a)$ defined by (3.4) is the first eigenvalue of the following eigenvalue problem

$$\begin{cases} (-\Delta_p)^s\,u + a_+\,|u|^{p-2}\,u = \lambda\,a_-\,|u|^{p-2}\,u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

It is not difficult to see that

$$\mathcal{S}_{p,s}(a) = 0 \qquad \Longleftrightarrow \qquad \lambda(\Omega, a) = 1.$$

Indeed, the implication $\Longrightarrow$ has been proven above. The converse implication goes as follows: we suppose $\lambda(\Omega, a) = 1$ and take ϕ_1 a solution of problem (3.4). From inequality (3.3), we have

$$[u]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \int\limits_{\mathbb{R}^N} a\,|u|^p\,dx \geq 0,$$

for every $u \in \mathcal{D}^{s,p}_0(\Omega)$ with unit $L^{p_s^*}$ norm. This implies that $\mathcal{S}_{p,s}(a) \geq 0$. On the other hand, the function ϕ_1 gives equality in the previous inequality, thus

$$\mathcal{S}_{p,s}(a) \leq \frac{[\phi_1]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \int\limits_{\mathbb{R}^N} a\,|\phi_1|^p\,dx}{\left(\int\limits_{\mathbb{R}^N} |\phi_1|^{p_s^*}\,dx\right)^{\frac{p}{p_s^*}}} = 0,$$

as well.

Actually, we can also show that

$$\mathcal{S}_{p,s}(a) < 0 \qquad \Longleftrightarrow \qquad \lambda(\Omega; a) < 1. \tag{3.6}$$

If $\mathcal{S}_{p,s}(a) < 0$, by the previous result there exists a minimizer ϕ for the problem defining $\mathcal{S}_{p,s}(a)$. Thus in particular we get

$$[\phi]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \int\limits_{\mathbb{R}^N} a_+\,|\phi|^p\,dx - \int\limits_{\mathbb{R}^N} a_-\,|\phi|^p\,dx = \mathcal{S}_{p,s}(a) < 0,$$

which immediately implies $\lambda(\Omega; a) < 1$. On the other hand, if we assume $\lambda(\Omega; a) < 1$, the minimizer ϕ_1 of problem (3.4) now verifies

$$[\phi_1]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \int_{\mathbb{R}^N} a_+ |\phi_1|^p \, dx = \lambda(\Omega; a) \int_{\mathbb{R}^N} a_- |\phi_1|^p \, dx < \int_{\mathbb{R}^N} a_- |\phi_1|^p \, dx.$$

By using the function $\phi_1/\|\phi_1\|_{L^{p^*_s}}$ as a competitor for the problem defining $\mathcal{S}_{p,s}(a)$ and appealing to the previous estimate, we then get $\mathcal{S}_{p,s}(a) < 0$.

Remark 3.6. It is not difficult to see that

$$\|a_-\|_{L^{N/sp}(\Omega)} < \mathcal{S}_{p,s} \quad \Longrightarrow \quad \mathcal{S}_{p,s}(a) > 0.$$

Indeed, by Hölder's and Sobolev inequalities

$$\begin{aligned}
[u]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \int_{\mathbb{R}^N} a\,|u|^p \, dx &\geq [u]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} a_- |u|^p \, dx \\
&\geq \mathcal{S}_{p,s}\, \|u\|^p_{L^{p^*_s}} - \|a_-\|_{L^{N/sp}(\Omega)}\, \|u\|^p_{L^{p^*_s}} \\
&= \mathcal{S}_{p,s} - \|a_-\|_{L^{N/sp}(\Omega)} > 0,
\end{aligned}$$

for every function with unit $L^{p^*_s}$ norm.

4. Proof of Theorem 1.1

We proceed to prove each point separately.

1. *Case $a \geq 0$.* Let us prove that for $a \geq 0$ the problem does not admit any solution. By Lemma 3.1, we already know that this is true if $a \equiv 0$, thus let us assume that a is non-negative and

$$\|a\|_{L^{N/sp}(\Omega)} > 0.$$

It is sufficient to show that in this case $\mathcal{S}_{p,s}(a) \leq \mathcal{S}_{p,s}$. Indeed, let us assume the latter to be true. If $u \in \mathcal{D}^{s,p}_0(\Omega)$ is a solution for the problem defining $\mathcal{S}_{p,s}(a)$, we would get

$$\mathcal{S}_{p,s} \geq \mathcal{S}_{p,s}(a) = [u]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \int_{\mathbb{R}^N} a\,|u|^p \, dx \geq [u]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} \geq \mathcal{S}_{p,s}.$$

Then we have equalities everywhere and in particular

$$\int_{\mathbb{R}^N} a\,|u|^p \, dx = 0,$$

which implies that $u = 0$ almost everywhere on the support of a. This contradicts the properties of u contained in Proposition 3.2.

In order to prove $\mathcal{S}_{p,s}(a) \le \mathcal{S}_{p,s}$, we consider the functions $u_{\varepsilon,\delta} \in \mathcal{D}_0^{s,p}(\mathbb{R}^N)$ as defined in (2.4). We fix $\delta > 0$ small enough so that, up to translations, $u_{\varepsilon,\delta}$ has support contained in Ω. Then accordingly we take $\varepsilon \le \delta/2$. We have

$$\mathcal{S}_{p,s}(a) \le \frac{[u_{\varepsilon,\delta}]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \int\limits_{\mathbb{R}^N} a\,|u_{\varepsilon,\delta}|^p\,dx}{\|u_{\varepsilon,\delta}\|^p_{L^{p^*_s}(\mathbb{R}^N)}}. \tag{4.1}$$

By the estimates of Lemma 2.5 and[1]

$$\lim_{\varepsilon\to 0} \int\limits_{\mathbb{R}^N} a\,|u_{\varepsilon,\delta}|^p\,dx = 0,$$

when we let ε go to 0 in (4.1), we get $\mathcal{S}_{p,s}(a) \le \mathcal{S}_{p,s}$.

2. *Case* $N > s\,p^2$. We want to prove that in this case $\mathcal{S}_{p,s}(a) < \mathcal{S}_{p,s}$ and then apply Proposition 3.4. By assumption, there exist $\sigma > 0$, $R > 0$ and $x_0 \in \Omega$ such that

$$a(x) \le -\sigma \qquad \text{for a. e. } x \in B_R(x_0) \subset \Omega.$$

Without loss of generality, we can assume that $x_0 = 0$. Let $\theta > 1$ be the same constant appearing in Lemma 2.3, we choose $\delta = R/\theta$. Then we consider again the functions $u_{\varepsilon,\delta} \in \mathcal{D}_0^{s,p}(\Omega)$ as defined in (2.4), with ε

$$\varepsilon \le \varepsilon_0 := \delta\, \min\left\{\frac{1}{2}, \left(\frac{1}{2}\,\frac{\left(\mathcal{S}_{p,s}\right)^{N/sp}}{\mathcal{C}}\right)^{\frac{p-1}{N}}\right\}, \tag{4.2}$$

that will be chosen in a while. The constant $\mathcal{C}$ is the same as in Lemma 2.5. We fix α such that

$$s\,p < \alpha < \frac{N - s\,p}{p-1},$$

a choice which is feasible thanks to the assumption $N > s\,p^2$. We now set

$$a_\varepsilon(x) := -\varepsilon^\alpha\, u_{\varepsilon,\delta}^{p^*_s - p}(x) \in L^{N/sp}(\Omega),$$

and we enforce condition (4.2), by requiring that $\varepsilon > 0$ satisfies

$$\varepsilon \le \varepsilon_1 := \min\left\{\varepsilon_0,\, \sigma^{\frac{1}{\alpha - s\,p}}\right\}.$$

[1] The family $\{|u_{\varepsilon,\delta}|^p\}_{\varepsilon>0}$ is bounded in $L^{p^*_s/p}(\Omega)$ and converges to 0 almost everywhere in Ω, by construction. Then we can apply [15, Lemme 4.8] again and infer vanishing of the term $\int_{\mathbb{R}^N} a\,|u_{\varepsilon,\delta}|^p\,dx$.

Observe that with such a choice, we have (recall that we are taking $U(0)=1$)

$$a_\varepsilon(x) = -\varepsilon^\alpha\, u_{\varepsilon,\delta}^{p_s^*-p}(x) \geq -\varepsilon^\alpha\, u_{\varepsilon,\delta}^{p_s^*-p}(0) = -\varepsilon^\alpha\, U_\varepsilon^{p_s^*-p}(0) \geq -\sigma, \qquad \text{in } B_R(0).$$

Thanks to the hypothesis on a, this yields

$$a(x) \leq a_\varepsilon(x), \qquad \text{for a. e. } x \in B_R(0).$$

Then by using the test function $u_{\varepsilon,\delta}$ (which is supported in the ball $B_R(0) \subset \Omega$) we get

$$\begin{aligned}
\mathcal{S}_{p,s}(a) &\leq \frac{[u_{\varepsilon,\delta}]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \displaystyle\int_{\mathbb{R}^N} a\,|u_{\varepsilon,\delta}|^p\,dx}{\|u_{\varepsilon,\delta}\|^p_{L^{p_s^*}(\mathbb{R}^N)}} \\
&= \frac{[u_{\varepsilon,\delta}]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \displaystyle\int_{B_R(0)} a\,|u_{\varepsilon,\delta}|^p\,dx}{\|u_{\varepsilon,\delta}\|^p_{L^{p_s^*}(\mathbb{R}^N)}} \\
&\leq \frac{[u_{\varepsilon,\delta}]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \displaystyle\int_{B_R(0)} a_\varepsilon\,|u_{\varepsilon,\delta}|^p\,dx}{\|u_{\varepsilon,\delta}\|^p_{L^{p_s^*}(\mathbb{R}^N)}} =: \mathcal{R}(\varepsilon,\delta).
\end{aligned}$$

We need to estimate the last Rayleight-type quotient. Thanks to the definition of a_ε and Lemma 2.5 we can infer

$$\begin{aligned}
\mathcal{R}(\varepsilon,\delta) &= \frac{[u_{\varepsilon,\delta}]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)}}{\|u_{\varepsilon,\delta}\|^p_{L^{p_s^*}(\mathbb{R}^N)}} - \varepsilon^\alpha\,\|u_{\varepsilon,\delta}\|^{p_s^*-p}_{L^{p_s^*}(\mathbb{R}^N)} \\
&\leq \frac{\left(\mathcal{S}_{p,s}\right)^{\frac{N}{sp}} + \mathcal{C}\left(\dfrac{\varepsilon}{\delta}\right)^{\frac{N-s\,p}{p-1}}}{\left[\left(\mathcal{S}_{p,s}\right)^{\frac{N}{sp}} - \mathcal{C}\left(\dfrac{\varepsilon}{\delta}\right)^{\frac{N}{p-1}}\right]^{\frac{N-s\,p}{N}}} - \varepsilon^\alpha\left[\left(\mathcal{S}_{p,s}\right)^{\frac{N}{sp}} - \mathcal{C}\left(\frac{\varepsilon}{\delta}\right)^{\frac{N}{p-1}}\right]^{\frac{s\,p}{N}}.
\end{aligned}$$

We now apply (2.1) with

$$\gamma = -\frac{N-s\,p}{N}, \qquad t = \frac{\mathcal{C}}{\left(\mathcal{S}_{p,s}\right)^{\frac{N}{sp}}}\left(\frac{\varepsilon}{\delta}\right)^{\frac{N}{p-1}}$$

and (2.2) with

$$\gamma = \frac{s\,p}{N}, \qquad t = \frac{\mathcal{C}}{\left(\mathcal{S}_{p,s}\right)^{\frac{N}{sp}}}\left(\frac{\varepsilon}{\delta}\right)^{\frac{N}{p-1}}.$$

This gives

$$
\begin{aligned}
\mathcal{R}(\varepsilon,\delta) &\le \mathcal{S}_{p,s}\left[1+\frac{\mathcal{C}}{(\mathcal{S}_{p,s})^{\frac{N}{sp}}}\left(\frac{\varepsilon}{\delta}\right)^{\frac{N-s\,p}{p-1}}\right]\left[1+\frac{2\,\mathcal{C}}{(\mathcal{S}_{p,s})^{\frac{N}{sp}}}\left(\frac{\varepsilon}{\delta}\right)^{\frac{N}{p-1}}(2^{\frac{N-s\,p}{N}}-1)\right]\\
&\quad-\varepsilon^{\alpha}\,\mathcal{S}_{p,s}\left[1+\frac{2\,\mathcal{C}}{(\mathcal{S}_{p,s})^{\frac{N}{sp}}}\left(\frac{\varepsilon}{\delta}\right)^{\frac{N}{p-1}}(2^{-\frac{s\,p}{N}}-1)\right]\\
&=\mathcal{S}_{p,s}+\varepsilon^{\alpha}\left[-\mathcal{S}_{p,s}+\frac{\mathcal{C}}{(\mathcal{S}_{p,s})^{\frac{N}{sp}-1}}\left(\frac{\varepsilon}{\delta}\right)^{\frac{N-s\,p}{p-1}-\alpha}+\frac{2\,\mathcal{C}}{(\mathcal{S}_{p,s})^{\frac{N}{sp}-1}}(2^{\frac{N-s\,p}{N}}-1)\left(\frac{\varepsilon}{\delta}\right)^{\frac{N}{p-1}-\alpha}\right.\\
&\quad\left.-\frac{2\,\mathcal{C}}{(\mathcal{S}_{p,s})^{\frac{N}{sp}-1}}\left(\frac{\varepsilon}{\delta}\right)^{\frac{N}{p-1}}(2^{-\frac{s\,p}{N}}-1)+2\,\mathcal{S}_{p,s}\left(\frac{\mathcal{C}}{(\mathcal{S}_{p,s})^{\frac{N}{sp}}}\right)^{2}\left(\frac{\varepsilon}{\delta}\right)^{\frac{2\,N-s\,p}{p-1}-\alpha}\right].
\end{aligned}
$$

By recalling the choice of α, we have that there exists $\varepsilon_2=\varepsilon_2(N,s,p,\delta)>0$ such that for every $0<\varepsilon\le\varepsilon_2$ the term above into square brackets is negative.[2] Thus in particular

$$
\mathcal{S}_{p,s}(a)\le\mathcal{R}(\varepsilon,\delta)<\mathcal{S}_{p,s},\qquad \text{for every } 0<\varepsilon\le\min\{\varepsilon_1,\varepsilon_2\}.
$$

Then the existence of a solution follows from Proposition 3.4.

3. <u>*Case* $s\,p<N\le s\,p^2$.</u> Let us prove the second assertion. We fix $x_0\in\Omega$ and $R>0$ such that $B_R(x_0)\subset\Omega$. As before, we assume for simplicity that $x_0=0$. Then we choose

$$
0\le\alpha<\frac{N-s\,p}{p-1}.
$$

Observe that now we automatically get $\alpha<s\,p$ as well. We set again $\delta=R/\theta$ and

$$
a_\varepsilon(x):=-\varepsilon^{\alpha}\,u_{\varepsilon,\delta}^{p_s^*-p}(x).
$$

By arguing as in the estimate above for the term $\mathcal{R}(\varepsilon,\delta)$, we gain the existence of an explicit $\varepsilon_2>0$ such that

[2] The constant ε_2 is easily seen to have the following form

$$
\varepsilon_2=\left(\frac{\mathcal{S}_{p,s}}{C}\right)^{\frac{1}{\frac{N-s\,p}{p-1}-\alpha}}\frac{R}{\theta},
$$

with $C=C(N,p,s)>0$ that can be computed explicitly.

$$\frac{[u_{\varepsilon_2,\delta}]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \displaystyle\int_{\mathbb{R}^N} a_{\varepsilon_2}\,|u_{\varepsilon_2,\delta}|^p\,dx}{\|u_{\varepsilon_2,\delta}\|^p_{L^{p_s^*}(\mathbb{R}^N)}} < \mathcal{S}_{p,s}.$$

Observe that ε_2 can be taken so that $\varepsilon_2 \le \delta/2$. By construction, we have

$$\|a_{\varepsilon_2}\|_{L^\infty(B_\delta(0))} = U(0)^{p_s^*-p}\,\varepsilon_2^{\alpha-s\,p} = \varepsilon_2^{\alpha-s\,p},$$

and observe that $\alpha - s\,p < 0$. If we assume that

$$a_- \ge \sigma := \varepsilon_2^{\alpha-s\,p}, \qquad \text{a.e. in } B_R(0),$$

we have

$$a \le -\sigma \le a_{\varepsilon_2} \qquad \text{a.e. in } B_R(0).$$

We use $u_{\varepsilon_2,\delta}$ as a test function. By using the previous estimate and the fact that $u_{\varepsilon_2,\delta}$ is supported on $B_R(0)$, we get as before $\mathcal{S}_{p,s}(a) < \mathcal{S}_{p,s}$. By appealing again to Proposition 3.4, we can infer that $\mathcal{S}_{p,s}(a)$ has a solution. □

Remark 4.1 *(About the condition on a).* In order to understand optimality of the conditions on a, let us consider the case $s = 1$, $p = 2$ and $a = -\lambda$ for some constant $\lambda > 0$. In this case, solutions ϕ of

$$\mathcal{S}_{2,1}(-\lambda) := \inf_{u\in\mathcal{D}^{1,2}_0(\Omega)} \left\{ \int_\Omega |\nabla u|^2\,dx - \lambda \int_\Omega |u|^2\,dx \,:\, \|u\|_{L^{\frac{2N}{N-2}}(\Omega)} = 1 \right\}, \tag{4.3}$$

verify in weak sense

$$-\Delta\phi - \lambda\,\phi = \mu\,\phi^{\frac{N+2}{N-2}} \ \text{ in } \Omega, \qquad \phi > 0 \ \text{ in } \Omega, \qquad \phi = 0 \ \text{ on } \partial\Omega,$$

for $\mu = \mathcal{S}_{2,1}(-\lambda)$. Observe that the sign of the Lagrange multiplier μ depends on whether $\lambda < \lambda_1(\Omega)$ (in this case $\mu > 0$) or $\lambda > \lambda_1(\Omega)$ (in this case $\mu < 0$). Here $\lambda_1(\Omega)$ is the first eigenvalue of the Dirichlet–Laplacian on Ω. In the first case, by setting

$$\psi = \mu^{\frac{N-2}{4}}\,\phi,$$

we would get a nontrivial solution of

$$-\Delta\psi - \lambda\,\psi = \psi^{\frac{N+2}{N-2}} \ \text{ in } \Omega, \qquad \psi \ge 0 \ \text{ in } \Omega, \qquad \psi = 0 \ \text{ on } \partial\Omega.$$

We already recalled in the Introduction that a necessary condition for this to be possible is that

$$\lambda > \lambda^* > 0 \ \text{ if } N = 3, \qquad \lambda > 0 \ \text{ if } N \ge 4,$$

for a suitable $\lambda^* < \lambda_1(\Omega)$. In other words, by observing that in this case we (formally) have $s\,p = 2$ and $s\,p^2 = 4$, problem (4.3) can not have a solution for $s\,p < N < s\,p^2$ if $\lambda > 0$ is arbitrarily small, while for $N \geq s\,p^2$ the parameter $\lambda > 0$ can be taken as small as desired. This exactly fits into the statement of Theorem 1.1 when a is a negative constant.

For completeness, we also record the following results.

Lemma 4.2. *The map $\mathcal{T} : L^{N/sp}(\Omega) \to \mathbb{R}$ defined by*

$$\mathcal{T}(a) = \mathcal{S}_{p,s}(a), \qquad \text{for } a \in L^{N/sp}(\Omega),$$

is 1-Lipschitz. In other words, for every $a, a' \in L^{N/sp}(\Omega)$ we have

$$\left|\mathcal{S}_{p,s}(a) - \mathcal{S}_{p,s}(a')\right| \leq \|a - a'\|_{L^{N/sp}(\Omega)}. \tag{4.4}$$

Proof. We pick $u \in \mathcal{D}_0^{s,p}(\Omega)$ with $\|u\|_{L^{p^*_s}(\Omega)} = 1$. Then, Hölder's inequality yields

$$\mathcal{S}_{p,s}(a) \leq [u]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \int\limits_{\mathbb{R}^N} a\,|u|^p\,dx \leq [u]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \int\limits_{\mathbb{R}^N} a'\,|u|^p\,dx + \|a - a'\|_{L^{N/sp}(\Omega)}.$$

Thus $\mathcal{S}_{p,s}(a) \leq \mathcal{S}_{p,s}(a') + \|a' - a\|_{L^{N/sp}(\Omega)}$. Switching the role of a and a', we get (4.4). $\square$

Proposition 4.3. *Consider $\{a_k\}_{k\in\mathbb{N}} \subset L^{N/sp}(\Omega)$ converging to a in $L^{N/sp}(\Omega)$. Let us assume that $\mathcal{S}_{p,s}(a) < \mathcal{S}_{p,s}$. If u_k is a solution to $\mathcal{S}_{p,s}(a_k)$, then there exists a solution u to $\mathcal{S}_{p,s}(a)$ such that (up to a subsequence)*

$$\lim_{k\to\infty} [u_k - u]_{\mathcal{D}^{s,p}(\mathbb{R}^N)} = 0.$$

Proof. Let $a \in L^{N/sp}(\Omega)$ and consider a sequence $\{a_k\}_{k\in\mathbb{N}}$ converging to a in $L^{N/sp}(\Omega)$. From (4.4) we already know that $\mathcal{S}_{p,s}(a_k)$ converges to $\mathcal{S}_{p,s}(a)$. For k large enough, we thus have $\mathcal{S}_{p,s}(a_k) < \mathcal{S}_{p,s}$ as well. Then by Proposition 3.4, the problem defining $\mathcal{S}_{p,s}(a_k)$ does admit a solution u_k. Still by (4.4), we know that

$$\lim_{k\to\infty} \left([u_k]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \int\limits_{\mathbb{R}^N} a_k\,|u_k|^p\,dx \right) = \mathcal{S}_{p,s}(a), \tag{4.5}$$

thus, without loss of generality, we can assume

$$[u_k]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \int\limits_{\mathbb{R}^N} a_k\,|u_k|^p\,dx \leq \mathcal{S}_{p,s}(a) + 1.$$

Moreover, by hypothesis we have

$$\|a_k\|_{L^{N/sp}(\Omega)} \leq M,$$

thus by Hölder's inequality we get

$$\mathcal{S}_{p,s}(a)+1\geq [u_k]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)}+\int_{\mathbb{R}^N} a_k\,|u_k|^p\,dx\geq [u_k]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)}-M.$$

This implies that the sequence $\{u_k\}_{k\in\mathbb{N}}$ is equi-bounded in $\mathcal{D}^{s,p}_0(\Omega)$ and thus we can infer strong convergence (up to a subsequence) in $L^q(\Omega)$, for $q<p^*_s$, to some limit function $u\in\mathcal{D}^{s,p}_0(\Omega)$. We need to show that

$$\|u\|_{L^{p^*_s}(\mathbb{R}^N)}=1 \qquad \text{and} \qquad \mathcal{S}_{p,s}(a)\geq [u]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)}+\int_{\mathbb{R}^N} a\,|u|^p\,dx. \tag{4.6}$$

The second fact easily follows from (4.5), the lower semicontinuity of the Gagliardo seminorm and the weak convergence of $\{|u_k|^p\}_{k\in\mathbb{N}}$ in $L^{p^*_s/p}(\Omega)$ to $|u|^p$. Observe that the conditions (4.6) automatically give that u is a minimizer for $\mathcal{S}_{p,s}(a)$.

In order to conclude, we need to prove that $\{u_k\}_{k\in\mathbb{N}}$ converges strongly in $L^{p^*_s}(\Omega)$. Observe that by minimality of u_k, we get

$$\begin{aligned}\int_{\mathbb{R}^{2N}} &\frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+s\,p}}\,dx\,dy+\int_{\mathbb{R}^N} a_k\,|u_k|^{p-2}\,u_k\,\varphi\,dx\\ &=\mathcal{S}_{p,s}(a_k)\int_{\mathbb{R}^N}|u_k|^{p^*_s-2}\,u_k\,\varphi\,dx,\end{aligned} \tag{4.7}$$

for every $\varphi\in\mathcal{D}^{s,p}_0(\Omega)$. We now distinguish two cases.

<u>*Case* $\mathcal{S}_{p,s}(a)\neq 0$.</u> If we consider the functional $\mathcal{K}$ introduced in (2.5) with $\mu=\mathcal{S}_{p,s}(a)$, recalling that u_k has unit norm in $L^{p^*_s}(\Omega)$, by (4.7) we obtain

$$\begin{aligned}|\langle\mathcal{K}'(u_k),\varphi\rangle| &=\Bigg|\int_{\mathbb{R}^{2N}}\frac{|u_k(x)-u_k(y)|^{p-2}\,(u_k(x)-u_k(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+s\,p}}\,dx\,dy\\ &\quad+\int_{\mathbb{R}^N} a\,|u_k|^{p-2}\,u_k\,\varphi\,dx-\mathcal{S}_{p,s}(a)\int_{\mathbb{R}^N}|u_k|^{p^*_s-2}\,u_k\,\varphi\,dx\Bigg|\\ &\leq\int_{\mathbb{R}^N}|a_k-a|\,|u_k|^{p-1}\,|\varphi|\,dx+|\mathcal{S}_{p,s}(a)-\mathcal{S}_{p,s}(a_k)|\int_{\mathbb{R}^N}|u_k|^{p^*_s-1}\,|\varphi|\,dx\\ &\leq\Big(\|a_k-a\|_{L^{N/sp}(\Omega)}+|\mathcal{S}_{p,s}(a)-\mathcal{S}_{p,s}(a_k)|\Big)\,\|\varphi\|_{L^{p^*_s}(\mathbb{R}^N)}.\end{aligned}$$

This shows that $\{u_k\}_{k\in\mathbb{N}}$ is a Palais–Smale sequence for $\mathcal{K}$ at the level $s/N\,\mathcal{S}_{p,s}(a)$. By recalling that $\mathcal{S}_{p,s}(a)<\mathcal{S}_{p,s}$, we obtain strong convergence in $\mathcal{D}^{s,p}_0(\Omega)$ by Lemma 2.6.

Case $\mathcal{S}_{p,s}(a)=0$. From (4.5), we get

$$[u_k]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \int_{\mathbb{R}^N} a_k\,|u_k|^p\,dx = o_k(1),$$

thus by using the weak convergence of $\{|u_k|^p\}_{k\in\mathbb{N}}$ and the strong convergence of $\{a_k\}_{k\in\mathbb{N}}$, we obtain

$$[u_k]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} = -\int_{\mathbb{R}^N} a\,|u|^p\,dx + o_k(1).$$

On the other hand, by testing (4.7) with u and then taking the limit as k goes to ∞, we obtain

$$[u]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} = -\int_{\mathbb{R}^N} a\,|u|^p\,dx.$$

The last two displays imply that

$$\lim_{k\to\infty}[u_k]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} = [u]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)}.$$

By uniform convexity of the space $\mathcal{D}^{s,p}_0(\Omega)$, we obtain the strong convergence in this case as well. □

5. Proof of Theorem 1.2

We consider the two cases separately.

Case $\mu \geq 0$. We proceed by contradiction. Let us assume that for a $\mu\geq 0$ there exists a positive solution $u_0\in\mathcal{D}^{s,p}_0(\Omega)\setminus\{0\}$. Thus u_0 satisfies

$$\int_{\mathbb{R}^{2N}} \frac{J_p(u_0(x)-u_0(y))\,\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+s\,p}}\,dx\,dy + \int_{\mathbb{R}^N} a\,u_0^{p-1}\,\varphi\,dx = \mu\int_{\mathbb{R}^N} u_0^{p_s^*-1}\,\varphi\,dx,$$

for every $\varphi\in\mathcal{D}^{s,p}_0(\Omega)$. We use the test function[3] $\varphi=\phi_1^p/u_0^{p-1}$, where ϕ_1 is the positive solution of (3.4). This gives

$$\mu\int_{\mathbb{R}^N} u_0^{p_s^*-p}\,\phi_1^p\,dx = \int_{\mathbb{R}^{2N}} \frac{J_p(u_0(x)-u_0(y))\left(\dfrac{\phi_1^p}{u_0^{p-1}}(x)-\dfrac{\phi_1^p}{u_0^{p-1}}(y)\right)}{|x-y|^{N+s\,p}}\,dx\,dy + \int_{\mathbb{R}^N} a\,\phi_1^p\,dx$$

[3] This test function is not admissible in principle, but it is sufficient to proceed as in the proof of Proposition 3.3. We prefer to avoid these technicalities here.

We can then apply Picone's inequality (2.3) and obtain

$$\mu \int_{\mathbb{R}^N} u_0^{p_s^*-p}\, \phi_1^p\, dx \leq [\phi_1]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} + \int_{\mathbb{R}^N} a_+\, \phi_1^p\, dx - \int_{\mathbb{R}^N} a_-\, \phi_1^p\, dx = \lambda(\Omega; a) - 1.$$

The right-hand side is strictly negative by (1.7) and (3.6), thus we get a contradiction.

Case $\mu < 0$. Since we are assuming $\mathcal{S}_{p,s}(a) < 0$, by Proposition 3.4, we can infer that the variational problem defining $\mathcal{S}_{p,s}(a)$ has a positive solution $u \in \mathcal{D}_0^{s,p}(\Omega) \setminus \{0\}$. As already observed, we have that u solves

$$\begin{cases} (-\Delta_p)^s\, u + a\, u^{p-1} = \mathcal{S}_{p,s}(a)\, u^{p_s^*-1}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

We fix $\mu < 0$, it is now sufficient to notice that

$$v = t\, u, \qquad \text{with } t = \left(\frac{\mu}{\mathcal{S}_{p,s}(a)}\right)^{\frac{1}{p-p_s^*}} > 0,$$

is the desired solution. Finally, we use Proposition 3.3 to infer its uniqueness.

Remark 5.1 *(Negative potentials).* For $1 \leq q < p$, let us define the sharp Poincaré constant for the embedding $\mathcal{D}_0^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$, i.e.

$$\lambda_{p,q}(\Omega) = \inf_{u \in \mathcal{D}_0^{s,p}(\Omega)} \left\{ [u]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)} \, : \, \int_{\mathbb{R}^N} |u|^q\, dx = 1 \right\}.$$

When the potential a is negative, i.e. when $a_+ \equiv 0$, then the condition $\mathcal{S}_{p,s}(a) < 0$ (and thus $\lambda(\Omega; a) < 1$ by (3.6)) is verified if

$$\left(\int_{\Omega} a_-^{-\frac{q}{p-q}}\, dx\right)^{\frac{p-q}{q}} < \frac{1}{\lambda_{p,q}(\Omega)}. \tag{5.1}$$

Indeed, observe that by Hölder's inequality

$$\int_{\mathbb{R}^N} |u|^q\, dx \leq \left(\int_{\mathbb{R}^N} a_-\, |u|^p\, dx\right)^{\frac{q}{p}} \left(\int_{\Omega} a_-^{-\frac{q}{p-q}}\, dx\right)^{\frac{p-q}{p}},$$

where we used that $p/q > 1$. Thus by using this we get

$$\lambda(\Omega;a)=\inf_{u\in\mathcal{D}_0^{s,p}(\Omega)\setminus\{0\}}\frac{[u]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)}}{\displaystyle\int_{\mathbb{R}^N} a_-\,|u|^p\,dx}\le\left(\int_\Omega a_-^{-\frac{q}{p-q}}\,dx\right)^{\frac{p-q}{q}}\inf_{u\in\mathcal{D}_0^{s,p}(\Omega)\setminus\{0\}}\frac{[u]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)}}{\left(\displaystyle\int_{\mathbb{R}^N}|u|^q\,dx\right)^{\frac{p}{q}}}$$

$$=\left(\int_\Omega a_-^{-\frac{q}{p-q}}\,dx\right)^{\frac{p-q}{q}}\lambda_{p,q}(\Omega)<1,$$

where the last estimate follows from (5.1). In the limit case $q=p$, we recall the definition of the first eigenvalue of the fractional p-Laplacian of order s

$$\lambda_1(\Omega)=\inf_{u\in\mathcal{D}_0^{s,p}(\Omega)}\left\{[u]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)}\,:\,\int_{\mathbb{R}^N}|u|^p\,dx=1\right\}.$$

Then a sufficient condition for $\lambda(\Omega;a)<1$ to hold is

$$\inf_\Omega a_->\lambda_1(\Omega).$$

The proof is as above.

Finally, if the potential a is a negative constant, i.e. $a\equiv-\lambda$ with $\lambda>0$, we observe that

$$\lambda(\Omega,a):=\inf_{u\in\mathcal{D}_0^{s,p}(\Omega)}\left\{[u]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)}\,:\,\int_{\mathbb{R}^N}|u|^p\,dx=\frac{1}{\lambda}\right\}$$

$$=\inf_{u\in\mathcal{D}_0^{s,p}(\Omega)}\left\{\frac{1}{\lambda}[u]^p_{\mathcal{D}^{s,p}(\mathbb{R}^N)}\,:\,\int_{\mathbb{R}^N}|u|^p\,dx=1\right\}=\frac{\lambda_1(\Omega)}{\lambda},$$

and condition (1.7) is *equivalent* to $\lambda>\lambda_1(\Omega)$.

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