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On the Well-Posedness of a Class of Vector Schrödinger Equations

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Abstract

We investigate the local and global well-posedness for a class of nonlinear Schrödinger systems with an external time independent electromagnetic field and nonlocal nonlinearities. New conditions related to the growth of the nonlocal term are detected which allow the solvability of the problem.

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1 Introduction

In a recent paper [13], Ma and Zhao obtained an important classification result for the positive solitary solutions of the nonlinear Choquard equation

$$\mathrm{i}u_t + \Delta u + 2u|x|^{-1} * |u|^2 = 0, \qquad u \in H^1(\mathbb{R}^N).$$

This equation can be seen as a special case of the generalized nonlocal Schrödinger equation

$$\mathrm{i}u_t + \Delta u + \mu u|u|^{\mu-2}(|x|^{-\Gamma} * |u|^{\mu}) = 0, \qquad u \in H^1(\mathbb{R}^N),$$

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where $\mu \geq 2$ and $\Gamma \in (0, N)$. These PDEs emerge in various branches of mathematical physics such as quantum mechanics and physics of laser beams and were firstly rigorously investigated by Lieb, Simon and Lions (see e.g. [9, 10, 12] and the references therein). It is thus important to handle nonlocal nonlinearities in the existence results for nonlinear Schrödinger equations. In general, we can have the simultaneous presence of local nonlinearities, nonlocal ones, and external potentials. The aim of this paper is to study the well posedness for coupled nonlinear Schrödinger equations with electro-magnetic potentials, local nonlinearities and a class of nonlocal nonlinearities containing, as a particular case, that of the above Choquard equations. More precisely, on $\mathbb{R}^N$ with $N \geq 3$, we consider the following system

$$\begin{cases} \mathrm{i}\partial_t \Phi_j = L_A \Phi_j + V(x)\Phi_j - g_j(|x|, |\Phi_1|^2, \dots, |\Phi_m|^2)\Phi_j - \sum_{i=1}^m W_{ij} * h(|\Phi_i|)\frac{h'(|\Phi_j|)}{|\Phi_j|}\Phi_j, \\ \Phi_j(0,x) = \Phi_j^0(x), \\ 1 \leq j \leq m, \end{cases} \tag{1.1}$$

where, for all $1 \leq j \leq m$, $\Phi_j^0 : \mathbb{R}^N \to \mathbb{C}$ is the initial datum and $\Phi_j : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{C}$ is the unknown. Moreover, $V : \mathbb{R}^N \to \mathbb{R}$ and $A : \mathbb{R}^N \to \mathbb{R}^N$ represent electric and magnetic potentials satisfying suitable assumptions that will be stated in the following. The magnetic operator L_A is defined as

$$L_A \phi := \left(\frac{\nabla}{\mathrm{i}} - A(x)\right)^2 \phi = -\Delta\phi - \frac{2}{\mathrm{i}}A(x)\cdot\nabla\phi + |A(x)|^2\phi - \frac{1}{\mathrm{i}}\operatorname{div} A(x)\phi.$$

The magnetic field B is $B = \nabla \times A$ in $\mathbb{R}^3$ and can be thought (and identified) in general dimension as a 2-form $\mathcal{H}^B$ of coefficients $(\partial_i A_j - \partial_j A_i)$. Finally, $h : \mathbb{R}_+ \to \mathbb{R}_+$ and $g_j : \mathbb{R}_+ \times \mathbb{R}_+^m \to \mathbb{R}$ are suitable nonlinear functions. It is possible to write (1.1) in the form

$$\begin{cases} \mathrm{i}\partial_t \Phi = \mathcal{E}_A'(\Phi) \\ \Phi(0,x) = \Phi^0(x) \end{cases}$$

where $\Phi^0 = (\Phi_1^0, \dots, \Phi_m^0)$ and $\mathcal{E}_A$ is the energy functional

$$\begin{aligned} \mathcal{E}_A(\Phi) = {} & \frac{1}{2}\sum_{j=1}^m \int \left| \left(\frac{\nabla}{\mathrm{i}} - A(x)\right)\Phi_j \right|^2 + \frac{1}{2}\int V(x)|\Phi|^2 - \int G(|x|, |\Phi_1|^2, \dots, |\Phi_m|^2) \\ & - \frac{1}{2}\sum_{i,j=1}^m \iint W_{ij}(|x-y|)h(|\Phi_i(x)|)h(|\Phi_j(y)|)\, dxdy, \end{aligned}$$

where $W_{ij} = W_{ji}$ and $G : (0, \infty) \times \mathbb{R}_+^m \to \mathbb{R}$ satisfy the following conditions

$$\frac{\partial G}{\partial s_j} = \tfrac{1}{2} g_j(|x|, s_1, \dots, s_m),$$

for every $j = 1, \dots, m$. Both h and g_j satisfy requirements that will be stated later on. We point out that the general Schrödinger system (1.1) we aim to study contains, as particular cases, physically meaningful situations (see e.g. the discussion in [18, Section 1.1]).

The line of the proof follows a classical result due to Cazenave [1] (see also [4] and [2]), which relies essentially on Lipschitz control for the nonlinearities on suitable L^p spaces. Although the paper does not introduce new techniques, it detects new (also for the scalar case $m = 1$) exponents related to the nonlocal nonlinearity, which to the authors' knowledge are not present in the current literature. Roughly speaking, thinking about the important case where, say, for some $\mu, \Gamma > 0$,

$$W_{ij}(|x-y|)h(|\Phi_i(x)|)h(|\Phi_j(y)|) = \frac{|\Phi_i(x)|^\mu |\Phi_j(y)|^\mu}{|x-y|^\Gamma}, \quad \text{for all } x, y \in \mathbb{R}^N,$$

the solvability conditions (for what concerns the local term) are

$$2 \le \mu \le \frac{4N+4-\Gamma}{2N}, \qquad 2 \le \mu < \frac{2N+2-\Gamma}{N},$$

respectively for local (for $\Gamma < 4$) and global ($\Gamma < 2$) well posedness.

The core of the paper is contained in the estimates performed in Section 3.2 where suitable growth assumptions on W_{ij} and h yielding the desired local existence (cf. Theorem 2.1) are detected. It is particularly delicate to achieve these estimates for what concerns the integral controls on the nonlocal terms. Finally, by strengthening the assumptions, the global existence of the solutions (cf. Theorem 2.2) is obtained. To the authors knowledge, the results are new also in the scalar case ($m = 1$) and in absence of the external magnetic potential ($A = 0$).

1.1 Notations and framework

We shall assume that $G(\cdot, s_1, \dots, s_m)$ is measurable on $[0, \infty)$, for all $s_1, \dots, s_m$. For all $1 \le n \le m$, every $(m-1)$ tuple of s_i and r in $[0, \infty)$, $s_n \mapsto G(r, \dots, s_n, \dots)$ is continuous on $\mathbb{R}_+$. We denote by $\mathcal{H}^1_A = \mathcal{H}^1_A(\mathbb{R}^N) = (H^1_A(\mathbb{R}^N))^m$, where $H^1_A = H^1_A(\mathbb{R}^N)$ is the Hilbert space defined as the closure of $C^\infty_c(\mathbb{R}^N; \mathbb{C})$ under the scalar product

$$\forall \phi, \psi \in H^1_A : \quad (\phi, \psi)_{H^1_A} = \Re\Big(\int D\phi \cdot \overline{D\psi} + \phi\overline{\psi} \Big),$$

where $D\phi$ is the matrix $(D_1\phi, \dots, D_N\phi)$ and $D_j = \mathrm{i}^{-1}\partial_j - A_j(x)$, with induced norm

$$\|\phi\|^2_{H^1_A} = \int \Big| \frac{1}{\mathrm{i}}\nabla\phi - A(x)\phi \Big|^2 + \int |\phi|^2 < \infty.$$

Recall that the Diamagnetic inequality

$$|\nabla|\phi|| \le \Big| \Big(\frac{\nabla}{\mathrm{i}} - A(x) \Big) \phi \Big| \tag{1.2}$$

holds for every $\phi \in H^1_A(\mathbb{R}^N)$ (see e.g. [11]). The inequality also holds for vectors $\Phi \in \mathcal{H}^1_A$. In turn, the space $\mathcal{H}^1_A$ is equipped with the norm

$$\|\Phi\|^2_{\mathcal{H}^1_A} = \Big\| \Big(\frac{\nabla}{\mathrm{i}} - A(x) \Big) \Phi \Big\|^2_{\mathcal{L}^2} + \|\Phi\|^2_{\mathcal{L}^2} = \sum_{j=1}^m \Big\| \Big(\frac{\nabla}{\mathrm{i}} - A(x) \Big) \Phi_j \Big\|^2_{L^2} + \|\Phi\|^2_{\mathcal{L}^2},$$

where $\mathcal{L}^p = \mathcal{L}^p(\mathbb{R}^N) = (L^p(\mathbb{R}^N))^m$ and $\|\Phi\|^2_{\mathcal{L}^p} = \sum_{j=1}^m \|\Phi_j\|^2_{L^p}$ for every $\Phi \in \mathcal{L}^p$. We denote by $L^q_w(\mathbb{R}^N)$ with $q > 1$ the weak L^q-space (see [11]) defined as the set of measurable functions f equipped with the norm

$$\|f\|_{q,w} = \sup_{D \subset \mathbb{R}^N,\ \mathcal{M}(D)<\infty} (\mathcal{M}(D))^{-1/q'} \int_D |f(x)| < +\infty,$$

where $\mathcal{M}$ denotes the Lebesque measure on $\mathbb{R}^N$ and q' is the conjugate exponent to q. The dual space of $\mathcal{H}^1_A$ is denoted by $\mathcal{H}'_A$. We denote $\mathcal{H}^1 = \mathcal{H}^1(\mathbb{R}^N) = (H^1(\mathbb{R}^N))^m$ equipped with the standard norm $\|\Phi\|^2_{\mathcal{H}^1} = \|\nabla\Phi\|^2_{\mathcal{L}^2} + \|\Phi\|^2_{\mathcal{L}^2}$ and $\mathcal{H}^{-1}(\mathbb{R}^N) = (H^{-1}(\mathbb{R}^N))^m$. By means of inequality (1.2), the space $\mathcal{H}^1_A$ is continuously embedded in $\mathcal{L}^p$ for all $p \in [2, 2^*]$ where $2^* = 2N/(N-2)$ for $N \geq 3$ and there exists $C > 0$ independent on A such that

$$\forall \Phi \in \mathcal{H}^1_A : \quad \|\Phi\|_{\mathcal{L}^p} \leq C\|\Phi\|_{\mathcal{H}^1_A}. \tag{1.3}$$

Furthermore, $(\mathcal{L}^p)' = \mathcal{L}^{p'} \subset \mathcal{H}'_A$ where p' denotes the conjugate of p.

2 The main results

We now formulate the assumptions on the Schrödinger system under investigation.

2.1 Assumptions on the magnetic potential

We suppose that $A \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$ and there exist $C_\alpha > 0$, $\alpha \in \mathbb{N}^n$ with:

(A) $\forall \alpha \in \mathbb{N}^n$, $|\alpha| \geq 1$, $\sup_{x\in\mathbb{R}^N} |\partial^\alpha A| \leq C_\alpha$

(B) $\exists \varepsilon > 0$, $\forall |\alpha| \geq 1$, $\sup_{x\in\mathbb{R}^N} |\partial^\alpha B| \leq C_\alpha \left(1 + |x|\right\rangle^{-1-\varepsilon}$.

2.2 Assumptions on the external potentials

We suppose that the external potentials V and W_{ij} satisfy:

(V) $V \in L^p(\mathbb{R}^N)$, for some $p > N/2$;

(W) $W_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$, $W_{ij}(|x|) \in L^q_w(\mathbb{R}^N)$ with $q > \max\{1, N/4\}$ and $W_{ij} = W_{ji}$.

2.3 Assumptions on the local nonlinearities

We suppose that the local nonlinearity satisfy:

(g) For every $j = 1, \dots, m$, the complex valued functions

$$f_j(x, \Phi) = g_j(|x|, |\Phi_1|^2, \dots, |\Phi_m|^2)\Phi_j$$

are measurable in $x \in \mathbb{R}^N$ and continuous in $\Phi \in \mathbb{C}^m$ almost everywhere on $\mathbb{R}^N$, with $f_j(x, \mathbf{0}) = \mathbf{0}$. There exist constant C and $\alpha \in [0, \frac{4}{N-2})$ such that

$$|f_j(x, \Psi) - f_j(x, \Phi)| \leq C\left(|\Phi|^\alpha + |\Psi|^\alpha\right)|\Psi - \Phi|, \quad \text{for a.e. } x \in \mathbb{R}^N \text{ and all } \Phi, \Psi \in \mathbb{C}^m.$$

(G) There exists $K > 0$ such that, for all $r \geq 0$ and $s_1, \ldots, s_m \geq 0$,

$$0 \leq G(r, s_1, \ldots, s_m) \leq K\Big(\sum_{j=1}^{m} s_j + \sum_{j=1}^{m} s_j^{\frac{l_j+2}{2}}\Big), \qquad 0 < l_j < \frac{4}{N-2}.$$

This last assumption will guarantee, via Gagliardo-Nirenberg and Diamagnetic inequalities, that the corresponding term in the energy functional is finite.

2.4 Assumptions on the nonlocal nonlinearities

We suppose that the nonlocal nonlinearity satisfy:

(h) $h : \mathbb{R}_+ \to \mathbb{R}_+$ is C^1 and non-decreasing, $h(0) = 0$ and there exist $C, D, E > 0$ such that

$$h(s) \leq Cs^\mu, \quad |h'(s)| \leq Ds^{\mu-1}, \quad |h''(s)| \leq Es^{\mu-2},$$

for all $s \in \mathbb{R}_+$, where

$$2 \leq \mu \leq 2 - \frac{1}{2q} + \frac{2}{N}. \tag{2.1}$$

Notice that this inequality is nonempty due to the condition $q > N/4$.

Remark 2.1 If we set $H(s) = \frac{h'(s)}{s}$ for all $s \in \mathbb{R}_+$, there is a constant $C > 0$ such that

$$|H(|z|)z - H(|w|)w| \leq C(|z| + |w|)^{\mu-2}|z - w|, \quad \text{for all } z, w \in \mathbb{C}, \tag{2.2}$$

$$|h(|z|) - h(|w|)| \leq C(|z|^{\mu-1} + |w|^{\mu-1})|z - w|, \quad \text{for all } z, w \in \mathbb{C}. \tag{2.3}$$

These follow by the growths of the maps $\{s \mapsto h'(s)\}$ and $\{s \mapsto h''(s)\}$. Concerning (2.2), see for instance (2-1) of Lemma 2.1 in Damascelli, AIHPC **15** (1998), applied with $A(\eta) := H(|\eta|)\eta : \mathbb{R}^2 \to \mathbb{R}^2$. The only required condition is (1-3) in the same paper, fulfilled by the growths of h', h''.

2.5 Statements of the results

We now state the main results of the paper.

First, we have the following local existence result for (1.1).

Theorem 2.1 *Assume (A), (B), (V), (W), (g) and (h). Then, for every $\Phi^0 \in \mathcal{H}^1_A(\mathbb{R}^N)$, there exist $T_* > 0$ and $T^* > 0$ and a unique, maximal solution*

$$\Phi \in C((-T_*, T^*), \mathcal{H}^1_A) \cap C^1((-T_*, T^*), \mathcal{H}'_A)$$

to system (1.1)*. Furthermore, charge and energy are conserved, that is*

$$\|\Phi(t)\|_{\mathcal{L}^2} = \|\Phi^0\|_{\mathcal{L}^2}, \qquad \mathcal{E}_A(\Phi(t)) = \mathcal{E}_A(\Phi^0)$$

for all $t \in (-T_*, T^*)$.

Then, by strengthening the assumptions, we have the following global existence result:

Theorem 2.2 *Assume (A), (B), (V), (W), (G) and (h) with $q > N/2$, V bounded from below and*

$$0 < l_j < \frac{4}{N}, \qquad 2 \le \mu < 2 - \frac{1}{q} + \frac{2}{N}.$$

Then, for all $\Phi^0 \in \mathcal{H}^1_A(\mathbb{R}^N)$, for the maximal solution to system (1.1), we have

$$\Phi \in C(\mathbb{R}, \mathcal{H}^1_A) \cap C^1(\mathbb{R}, \mathcal{H}'_A),$$

namely Φ is global.

Remark 2.2 The bound $\mu < 2 - 1/q + 2/N$ (global existence) for the growth of h gives, as natural, a more stringent condition than $\mu \le 2 - 1/2q + 2/N$ (local existence). Thus, they get close as q becomes large. For the important case $\mu = 2$ the local solvability condition is equivalent to $q \ge N/4$, while the global solvability condition is equivalent to $q > N/2$.

Remark 2.3 If W_{ij} is a convolution kernel of the form, say, $|x|^{-\Gamma}$, for some $\Gamma > 0$, it follows that W_{ij} belongs to the space $L^q_w(\mathbb{R}^N)$ where $q = \frac{N}{\Gamma}$ (cf. [11]). Then, the solvability conditions

$$2 \le \mu \le 2 - \frac{1}{2q} + \frac{2}{N}, \qquad 2 \le \mu < 2 - \frac{1}{q} + \frac{2}{N},$$

become, respectively,

$$2 \le \mu \le \mu_{\rm loc}, \quad \mu_{\rm loc} := \frac{4N + 4 - \Gamma}{2N}, \qquad 2 \le \mu < \mu_{\rm glo}, \quad \mu_{\rm glo} := \frac{2N + 2 - \Gamma}{N}.$$

If $N = 3$ and $\Gamma = 1$ (the physically relevant case of Coulomb kernels), these conditions read as

$$2 \le \mu < \frac{15}{6}, \qquad 2 \le \mu < \frac{7}{3},$$

for local and global solvability respectively.

Remark 2.4 A natural question is the following: does $\mu_{\rm glo}$ correspond to the critical threshold for the global well-posedness? does it correspond to the orbital stability threshold? In particular, in 3D with $A = 0$, the Coulomb kernel, and in presence only of the nonlocal nonlinearity, the candidate critical problem seems

$$\mathrm{i}u_t + \Delta u + \tfrac{7}{3} u|u|^{\frac{1}{3}}(|x|^{-1} * |u|^{\frac{7}{3}}) = 0, \qquad u \in H^1(\mathbb{R}^3).$$

3 Proofs of the results

In this section, we start with the local well posedness of the Cauchy problem (1.1). We denote by $\mathcal{L}(X, Y)$ the Banach space of linear, continuous operators from X to Y, with the norm topology. We consider, for all $j = 1, \dots, m$, problem (1.1) equivalently written as

$$\begin{cases} \mathrm{i}\partial_t \Phi - L_A \Phi + \tilde{g}(\Phi) = 0, \\ \Phi(0, x) = \Phi^0(x), \end{cases} \tag{3.1}$$

where $\tilde{g}_j(\Phi) = -\tilde{g}_{1,j}(\Phi) + \tilde{g}_{2,j}(\Phi) + \tilde{g}_{3,j}(\Phi)$, with

$$\begin{aligned}\tilde{g}_{1,j}(\Phi) &= V(x)\Phi_j,\\ \tilde{g}_{2,j}(\Phi) &= g_j(|x|, |\Phi_1|^2, \dots, |\Phi_m|^2)\Phi_j,\\ \tilde{g}_{3,j}(\Phi) &= \sum_{i=1}^{m} W_{ij} * h(|\Phi_i|)\frac{h'(|\Phi_j|)}{|\Phi_j|}\Phi_j,\end{aligned}$$

3.1 A property of $(I + \varepsilon L_A)^{-1}$

L_A is a self-adjoint, positive operator on $\mathcal{L}^2$ and iL_A generates a group of isometries $\{T(t)\}_{t\in\mathbb{R}}$ where $T(t) = e^{-itL_A}$ in $\mathcal{L}^2$. The proof of the following statement follows the line of Lemma 9.1.3 in [1] that covers the scalar case for the particular magnetic potential $A = (y, -x, 0)$.

For the sake of completeness we include the argument of the proof for the general case.

Proposition 3.1 *Let $\varepsilon > 0$ and $1 \le p < \infty$. Then the operator*

$$(I + \varepsilon L_A)^{-1} : \mathcal{L}^p \to \mathcal{L}^p,$$

is continuous and

$$\|(I + \varepsilon L_A)^{-1}\|_{\mathscr{L}(\mathcal{L}^p,\mathcal{L}^p)} \le 1.$$

Proof. Let $\varepsilon > 0$, $\Phi_\varepsilon \in \mathcal{H}^1_A$ and $g \in \mathcal{H}'_A$ verify

$$(I + \varepsilon L_A)\Phi_\varepsilon = g. \tag{3.2}$$

For $g \in \mathcal{H}'_A$ with $g \in \mathcal{L}^p$, $\Phi_\varepsilon = J^A_\varepsilon g \in \mathcal{H}^1_A$ with $J^A_\varepsilon = (I + \varepsilon L_A)^{-1}$ denotes the unique solution of equation (3.2). We aim to prove that $\Phi_\varepsilon \in \mathcal{L}^p$ and $\|\Phi_\varepsilon\|_{\mathcal{L}^p} \le \|g\|_{\mathcal{L}^p}$, yielding the desired assertions. Without loss of generality, we can consider the proof for the case $p \ge 2$. In fact, the case $1 \le p \le 2$ can be easily recovered by virtue of a duality argument. Let $\theta \in C^1(\mathbb{R}_+)$ be positive and bounded, with θ' positive, the map $\{s \mapsto \theta'(s)s\}$ bounded and $\theta(0) = 0$. By multiplying $(L_A + \frac{I}{\varepsilon})\Phi_\varepsilon = \frac{1}{\varepsilon} g$ in $\mathcal{L}^2$ by the test function $\phi(\Phi_\varepsilon) = \Phi_\varepsilon\theta(|\Phi_\varepsilon|^2)$, we reach

$$\langle L_A\Phi_\varepsilon, \phi(\Phi_\varepsilon)\rangle_{\mathcal{L}^2} + \frac{1}{\varepsilon}\langle \Phi_\varepsilon, \phi(\Phi_\varepsilon)\rangle_{\mathcal{L}^2} = \frac{1}{\varepsilon}\langle g, \phi(\Phi_\varepsilon)\rangle_{\mathcal{L}^2}.$$

Observe that

$$\langle L_A\Phi_\varepsilon, \phi(\Phi_\varepsilon)\rangle_{\mathcal{L}^2} \ge 0. \tag{3.3}$$

Indeed (following the proof of [1, Lemma 9.1.3]) we can write

$$\langle L_A\Phi_\varepsilon, \phi(\Phi_\varepsilon)\rangle_{\mathcal{L}^2} = \lim_{m\to\infty}\langle L_A\Phi_\varepsilon, \rho_m\phi(\Phi_\varepsilon)\rangle_{\mathcal{L}^2}$$

where, for $m \ge 1$, $\rho_m(x) = \rho(\frac{x}{m})$ with $\rho \in C^\infty_c(\mathbb{R}^N)$, $0 \le \rho \le 1$, $\rho(x) = 1$ for $|x| \le 1$ and $\rho(x) = 0$ for $|x| \ge 2$. Furthermore, we have

$$\langle L_A\Phi_\varepsilon, \rho_m\phi(\Phi_\varepsilon)\rangle_{\mathcal{L}^2} = I_m(\Phi_\varepsilon) - \Re\int_{\mathbb{R}^N}\frac{1}{i}\operatorname{div} A|\Phi_\varepsilon|^2\rho_m\theta(|\Phi_\varepsilon|^2)$$

where the second term is zero and

$$I_m(\Phi_\varepsilon)=\sum_{j=1}^m \Re\Big(\int \nabla\Phi_{\varepsilon,j}\cdot\nabla(\rho_m\theta(|\Phi_\varepsilon|^2)\overline{\Phi_{\varepsilon,j}})\Big)-2\sum_{j=1}^m \Im\Big(\int(\rho_m\theta(|\Phi_\varepsilon|^2)\overline{\Phi_{\varepsilon,j}}A\cdot\nabla\Phi_{\varepsilon,j}\Big)$$
$$+\int \rho_m|A|^2\theta(|\Phi_\varepsilon|^2)|\Phi_\varepsilon|^2.$$

Since by Cauchy Schwarz inequality

$$2\Im\sum_{j=1}^m\Big(\int(\rho_m\theta(|\Phi_\varepsilon|^2)\overline{\Phi_{\varepsilon,j}})A\cdot\nabla\Phi_{\varepsilon,j}\Big)\le\int\rho_m|A|^2\theta(|\Phi_\varepsilon|^2)|\Phi_\varepsilon|^2+\int\rho_m\theta(|\Phi_\varepsilon|^2)|\nabla\Phi_\varepsilon|^2,$$

we have

$$\begin{aligned}\langle L_A\Phi_\varepsilon,\rho_m\phi(\Phi_\varepsilon)\rangle_{\mathcal{L}^2}&\ge\sum_{j=1}^m\Re\Big(\int\nabla\Phi_{\varepsilon,j}\cdot\nabla(\rho_m\theta(|\Phi_\varepsilon|^2)\overline{\Phi_{\varepsilon,j}})\Big)-\int\rho_m\theta(|\Phi_\varepsilon|^2)|\nabla\Phi_\varepsilon|^2\\&\ge-\frac12\int\Theta(|\Phi_\varepsilon|^2)\Delta\rho_m,\end{aligned}$$

where $\Theta(s)=\int_0^s\theta(\sigma)\,d\sigma$. In the last inequality we used the fact that ρ_m,θ' are positive and

$$\sum_{j=1}^m\Re\big(\nabla\Phi_{\varepsilon,j}\cdot\nabla(\rho_m\theta(|\Phi_\varepsilon|^2)\overline{\Phi_{\varepsilon,j}})\big)$$
$$-\rho_m\theta(|\Phi_\varepsilon|^2)|\nabla\Phi_\varepsilon|^2=\frac12\rho_m\theta'(|\Phi_\varepsilon|^2)\big|\nabla|\Phi_\varepsilon|^2\big|^2+\frac12\nabla\rho_m\cdot\nabla(\Theta(|\Phi_\varepsilon|^2)).$$

Taking into account the boundedness of θ, the remaining integral vanishes, as $m\to\infty$, yielding (3.3). Hence,

$$\int|\Phi_\varepsilon|^2\theta(|\Phi_\varepsilon|^2)\le\int|g||\Phi_\varepsilon|\theta(|\Phi_\varepsilon|^2).$$

For $\delta>0$ we can choose, for instance, the function $\theta(s)=\big(\frac{s}{1+\delta s}\big)^{\frac{p}{2}-1}$, which satisfies the requirements mentioned at the beginning of the proof. In turn, we obtain

$$\int\frac{|\Phi_\varepsilon|^p}{(1+\delta|\Phi_\varepsilon|^2)^{\frac{p}{2}-1}}\le\int|g|^p.$$

Letting δ go to zero, by Fatou's Lemma, we obtain the desired assertion.

3.2 Integral Lipschitz estimates

In this section we prove the following result concerning the L^p Lipschitz estimates fulfilled by the nonlinearities involved in the Schrödinger system, local and nonlocal.

Theorem 3.1 *Assume that conditions (V), (W), (g) and (h) hold. Then, for all $k=1,2,3$, we have*

1. $\tilde{g}_k \in C(\mathcal{H}^1_A, \mathcal{H}'_A)$ *and* $\tilde{G}_k \in C^1(\mathcal{H}^1_A, \mathbb{R})$ *with* $\tilde{g}_k = \tilde{G}'_k$;
2. *there exists* $\rho_k \in [2, \frac{2N}{N-2})$ *such that*
$$\tilde{g}_k : \mathcal{H}^1_A \to \mathcal{L}^{\rho'_k} \hookrightarrow \mathcal{H}'_A; \tag{3.4}$$
3. *there exists* $r_k \in [2, \frac{2N}{N-2})$ *such that for every* $M > 0$, *there is* $C(M) > 0$ *with*
$$\|\tilde{g}_k(\Psi) - \tilde{g}_k(\Phi)\|_{\mathcal{L}^{\rho'_k}} \leq C(M)\|\Psi - \Phi\|_{\mathcal{L}^{r_k}}, \tag{3.5}$$
for every $\Psi,\ \Phi \in \mathcal{H}^1_A$ *such that* $\|\Psi\|_{\mathcal{H}^1_A} + \|\Phi\|_{\mathcal{H}^1_A} \leq M$;
4. *for every* $\Phi \in \mathcal{H}^1_A$ *and any* $j = 1, \dots, m$,
$$\Im(\tilde{g}_{k,j}(\Phi)\overline{\Phi_j}) = 0 \quad a.e.\ on\ \mathbb{R}^N. \tag{3.6}$$

Proof. For any $k = 1, 2, 3$ the first assertion is satisfied by the definition of $\tilde{g}_k$ and with
$$\tilde{G}_1(\Phi) = \frac{1}{2}\int V(x)|\Phi|^2, \quad \tilde{G}_2(\Phi) = \int G(|x|, |\Phi_1|^2, \dots, |\Phi_m|^2),$$
and
$$\tilde{G}_3(\Phi) = \frac{1}{2}\sum_{i,j=1}^m \iint W_{ij}(|x-y|)h(|\Phi_i(x)|)h(|\Phi_j(y)|).$$
The continuity of $\tilde{g}_k$ from $\mathcal{H}^1_A$ to $\mathcal{H}'_A$ is a consequence of inequality (3.5). Let now $k = 1$. Concerning conditions (3.4) and (3.5), for all $j = 1, \dots, m$, we obtain
$$\begin{aligned}\|\tilde{g}_{1,j}(\Psi) - \tilde{g}_{1,j}(\Phi)\|^{\rho'_1}_{L^{\rho'_1}} &\leq \int |V(x)|^{\rho'_1}|\Psi_j - \Phi_j|^{\rho'_1} \\ &\leq C\Big(\int |V(x)|^p\Big)^{\frac{\rho'_1}{p}}\Big(\int |\Psi_j - \Phi_j|^{\frac{p\rho'_1}{p-\rho'_1}}\Big)^{\frac{p-\rho'_1}{p}} \\ &\leq C\|V\|^{\rho'_1}_{L^p}\|\Psi_j - \Phi_j\|^{\rho'_1}_{L^{r_1}}\end{aligned}$$
for $\rho_1 \in [2, 2^*)$ and for r_1 which satisfies
$$2 \leq r_1 = \frac{p\rho'_1}{p - \rho'_1} < 2^*.$$
The choice of $\rho_1 = \frac{2p}{p-1}$ then leads to $r_1 = \rho_1 \in [2, 2^*)$ since $p > \frac{N}{2}$. Finally, condition (3.6) follows since V is real-valued.

Let now $k = 2$. Concerning (3.4) and (3.5), they follow from assumption (g) and from the Diamagnetic and Sobolev inequality, with the choice $r_2 = \rho_2 = \alpha + 2$. Indeed, by assumption (g),
$$\begin{aligned}\|\tilde{g}_{2,j}(\Psi) - \tilde{g}_{2,j}(\Phi)\|^{\rho'_2}_{L^{\rho'_2}} &= \int \big|g_j(|x|, |\Psi_1|^2, \dots, |\Psi_m|^2)\Psi_j - g_j(|x|, |\Phi_1|^2, \dots, |\Phi_m|^2)\Phi_j\big|^{\rho'_2} \\ &\leq C\big(\|\Psi\|^{\alpha\rho'_2}_{\mathcal{L}^\vartheta} + \|\Phi\|^{\alpha\rho'_2}_{\mathcal{L}^\vartheta}\big)\,\|\Psi - \Phi\|^{\rho'_2}_{\mathcal{L}^{r_2}} \\ &\leq C(M)\,\|\Psi - \Phi\|^{\rho'_2}_{\mathcal{L}^{r_2}}\end{aligned}$$

where $r_2 = \rho_2 = \alpha + 2$ (recall that $2 \le \alpha + 2 < 2^*$), so that

$$2 \le \vartheta = \frac{\alpha \rho_2' r_2}{r_2 - \rho_2'} = \alpha + 2 < 2^*.$$

Condition (3.6) is obvious by the definition of $g_{2,j}$ is real valued.

Finally let $k = 3$. Recall the Hardy-Littlewood inequality ([11, formula (9), pp.107])

$$\|g * f\|_{L^r(\mathbb{R}^N)} \le C\|g\|_{L^q_w(\mathbb{R}^N)}\|f\|_{L^s(\mathbb{R}^N)}, \quad \forall f \in L^s(\mathbb{R}^N), \quad \frac{1}{s} + \frac{1}{q} = 1 + \frac{1}{r}. \tag{3.7}$$

Concerning condition (3.4), by assumption (h), (3.7) and Hölder inequality, for all j, we have

$$\begin{aligned}
\|\tilde g_{3,j}(\Phi)\|^{\rho_3'}_{L^{\rho_3'}} &\le C\sum_{i=1}^m \int |W_{ij} * h(|\Phi_i|)|^{\rho_3'} \left|\frac{h'(|\Phi_j|)}{|\Phi_j|}\Phi_j\right|^{\rho_3'} \\
&\le C\sum_{i=1}^m \int |W_{ij} * h(|\Phi_i|)|^{\rho_3'}\, |h'(|\Phi_j|)|^{\rho_3'} \\
&\le C\sum_{i=1}^m \Big(\int |W_{ij} * h(|\Phi_i|)|^{\frac{\rho_3'\rho_3}{\rho_3-\rho_3'}}\Big)^{\frac{\rho_3-\rho_3'}{\rho_3}} \Big(\int |h'(|\Phi_j|)|^{\rho_3}\Big)^{\frac{\rho_3'}{\rho_3}} \\
&\le C\sum_{i=1}^m \Big(\int |W_{ij} * h(|\Phi_i|)|^{\frac{\rho_3}{\rho_3-2}}\Big)^{\frac{\rho_3-\rho_3'}{\rho_3}} \Big(\int |\Phi_j|^{\rho_3(\mu-1)}\Big)^{\frac{\rho_3'}{\rho_3}} \\
&\le C\sum_{i=1}^m \|W_{ij} * h(|\Phi_i|)\|^{\frac{\rho_3}{\rho_3-1}}_{L^{\frac{\rho_3}{\rho_3-2}}} \|\Phi_j\|^{\rho_3'(\mu-1)}_{L^{\rho_3(\mu-1)}} \\
&\le C\sum_{i=1}^m \|W_{i,j}\|^{\rho_3'}_{L^q_w} \|\Phi_i\|^{\rho_3'\mu}_{L^{\mu p_0}} \|\Phi_j\|^{\rho_3'(\mu-1)}_{L^{\rho_3(\mu-1)}}
\end{aligned}$$

where $p_0 = \frac{\rho_3 q}{2\rho_3 q - 2q - \rho_3}$ for $\rho_3 \in [2, 2^*)$ which satisfies

$$\begin{cases} 2 \le \rho_3(\mu - 1) \le 2^* \\ \frac{2}{\mu} \le p_0 = \frac{\rho_3 q}{2\rho_3 q - 2q - \rho_3} \le \frac{2^*}{\mu}. \end{cases}$$

Observe that the choice of $\rho_3 = \frac{4q}{2q-1} \in [2, 2^*)$ in the above inequalities leads to $p_0 = \frac{2q}{2q-1}$ and to the restriction

$$2 \le \mu < 1 + \frac{2^*(2q-1)}{4q},$$

which is compatible with the range of μ in condition (h). Condition (3.5) is also satisfied since, for each $j = 1, \dots, m$, we have

$$
\begin{aligned}
&\|\tilde{g}_{3,j}(\Psi) - \tilde{g}_{3,j}(\Phi)\|_{L^{\rho_3'}}^{\rho_3'} \\
&\le C \sum_{i=1}^{m} \int \Big| W_{ij} * h(|\Psi_i|) \frac{h'(|\Psi_j|)}{|\Psi_j|} \Psi_j - W_{ij} * h(|\Phi_i|) \frac{h'(|\Phi_j|)}{|\Phi_j|} \Phi_j \Big|^{\rho_3'} \\
&\le C \sum_{i=1}^{m} (\mathbb{I}_i + \mathbb{J}_i),
\end{aligned}
$$

where, for $i = 1, \dots, m$ we have set

$$
\begin{aligned}
\mathbb{I}_i &:= \int \Big| W_{ij} * h(|\Psi_i|) \Big|^{\rho_3'} \Big| \frac{h'(|\Psi_j|)}{|\Psi_j|} \Psi_j - \frac{h'(|\Phi_j|)}{|\Phi_j|} \Phi_j \Big|^{\rho_3'} \\
\mathbb{J}_i &:= \int \Big| W_{ij} * (h(|\Psi_i|) - h(|\Phi_i|)) \Big|^{\rho_3'} \Big| h'(|\Phi_j|) \Big|^{\rho_3'}.
\end{aligned}
$$

In light of inequality (2.2), by Hölder inequality, the Hardy-Littlewood inequality (3.7) and Sobolev and Diamagnetic inequalities, we have that

$$
\begin{aligned}
\mathbb{I}_i &= \int \Big| W_{ij} * h(|\Psi_i|) \Big|^{\rho_3'} \Big| H(|\Psi_j|)\Psi_j - H(|\Phi_j|)\Phi_j \Big|^{\rho_3'} \\
&\le \Big(\int \Big| W_{ij} * h(|\Psi_i|) \Big|^{\frac{\rho_3}{\rho_3 - 2}} \Big)^{\frac{\rho_3 - 2}{\rho_3 - 1}} \Big(\int \Big| H(|\Psi_j|)\Psi_j - H(|\Phi_j|)\Phi_j \Big|^{\rho_3} \Big)^{\frac{\rho_3'}{\rho_3}} \\
&\le C \|W_{i,j}\|_{L^q_w}^{\rho_3'} \|\Psi_i\|_{L^{\mu p_0}}^{\rho_3' \mu} \Big(\int (|\Psi_j| + |\Phi_j|)^{\rho_3(\mu - 2)} |\Psi_j - \Phi_j|^{\rho_3} \Big)^{\frac{\rho_3'}{\rho_3}}.
\end{aligned}
$$

Then, if $\mu = 2$, we have

$$
\mathbb{I}_i \le C \|W_{i,j}\|_{L^q_w}^{\rho_3'} \|\Psi_i\|_{L^{2p_0}}^{2\rho_3'} \|\Psi_j - \Phi_j\|_{L^{r_3}}^{\rho_3'},
$$

with $\rho_3 = r_3 = \frac{4q}{2q-1} = 2p_0$. If instead $\mu > 2$, then we get

$$
\begin{aligned}
\mathbb{I}_i &\le C \|W_{i,j}\|_{L^q_w}^{\rho_3'} \|\Psi_i\|_{L^{\mu p_0}}^{\rho_3' \mu} \| |\Psi_j| + |\Phi_j| \|_{L^{\frac{(\mu-2) r_3 \rho_3}{r_3 - \rho_3}}}^{\rho_3'(\mu - 2)} \|\Psi_j - \Phi_j\|_{L^{r_3}}^{\rho_3'} \\
&\le C(M) \|\Psi_j - \Phi_j\|_{L^{r_3}}^{\rho_3'},
\end{aligned}
$$

where $\rho_3 \in [2, 2^*)$ and $r_3 \in [2, 2^*)$ must satisfy the following conditions

$$
\begin{cases}
2 \le \mu p_0 \le 2^* \\
2 \le \frac{(\mu-2) r_3 \rho_3}{r_3 - \rho_3} \le 2^*.
\end{cases}
\tag{3.8}
$$

Observe that, by the choice of $\rho_3 = \frac{4q}{2q-1} \in [2, 2^*)$, the above inequalities are satisfied by

$$
\frac{4q 2^*}{2^*(2q-1) - 4q(\mu - 2)} \le r_3 \le \frac{4q}{6q - 1 - 2q\mu}.
\tag{3.9}
$$

The range of μ in assumption (h) ensures that $r_3 \in [2, 2^*)$. Dealing with the second term in the above sum, by means of inequality (2.3), we have

$$\begin{aligned}
\mathbb{J}_i &= \int \Big| W_{ij} * (h(|\Psi_i|) - h(|\Phi_i|)) \Big|^{\rho_3'} \Big| h'(|\Phi_j|) \Big|^{\rho_3'} \\
&\le \Big(\int \Big| W_{ij} * (h(|\Psi_i|) - h(|\Phi_i|)) \Big|^{\frac{\rho_3}{\rho_3-2}} \Big)^{\frac{\rho_3-2}{\rho_3-1}} \Big(\int \Big| h'(|\Phi_j|) \Big|^{\rho_3} \Big)^{\frac{\rho_3'}{\rho_3}} \\
&\le \|W_{ij} * (h(|\Psi_i|) - h(|\Phi_i|))\|^{\rho_3'}_{L^{\frac{\rho_3}{\rho_3-2}}} \|h'(|\Phi_j|)\|^{\rho_3'}_{L^{\rho_3}} \\
&\le C\|W_{ij}\|^{\rho_3'}_{L^q_w} \|(h(|\Psi_i|) - h(|\Phi_i|))\|^{\rho_3'}_{L^{p_0}} \|\Phi_j\|^{\rho_3'(\mu-1)}_{L^{\rho_3(\mu-1)}} \\
&\le C\|W_{ij}\|^{\rho_3'}_{L^q_w} \Big(\|\Phi_i\|^{\rho_3'(\mu-1)}_{L^{\frac{(\mu-1)p_0 r_3}{r_3-p_0}}} + \|\Psi_i\|^{\rho_3'(\mu-1)}_{L^{\frac{(\mu-1)p_0 r_3}{r_3-p_0}}} \Big) \times \\
&\times \|\Psi_i - \Phi_i\|^{\rho_3'}_{L^{r_3}} \|\Phi_j\|^{\rho_3'(\mu-1)}_{L^{\rho_3(\mu-1)}} \le C(M)\|\Psi_i - \Phi_i\|^{\rho_3'}_{L^{r_3}},
\end{aligned}$$

where again $p_0 = \frac{\rho_3 q}{2\rho_3 q - 2q - \rho_3}$. Taking once again $\rho_3 = \frac{4q}{2q-1} \in [2, 2^*)$, we have $p_0 = \frac{2q}{2q-1}$ and $r_3 \in [2, 2^*)$ must satisfy this time the system

$$\begin{cases} 2 \le \rho_3(\mu - 1) \le 2^* \\ 2 \le \frac{(\mu-1)p_0 r_3}{(r_3 - p_0)} \le 2^*. \end{cases}$$

Observe that, these inequalities can be fulfilled by the same r_3 found for inequalities of the set (3.8) (just compare with the extrema of the interval in formula (3.9)). Therefore, we conclude that

$$\|\tilde{g}_3(\Psi) - \tilde{g}_3(\Phi)\|_{\mathcal{L}^{\rho_3'}} \le C(M)\|\Psi - \Phi\|_{\mathcal{L}^{r_3}}.$$

Finally, condition (3.6) is obvious since W_{ij}, h and h' are real-valued functions.

3.3 A priori uniqueness property

We say that we have (local) uniqueness for problem (3.1) if, for every interval J containing 0 of sufficiently small size, and for every $\Phi^0 \in \mathcal{H}^1_A$, any two solutions of (3.1)

$$\Phi, \Psi \in L^\infty(J, \mathcal{H}^1_A) \cap W^{1,\infty}(J, \mathcal{H}'_A),$$

coincide. For all $j = 1, \dots, m$, writing the j-th equation of (3.1)

$$\mathrm{i}\partial_t \Phi_j - L_A \Phi_j + \tilde{g}_j(\Phi) = 0. \tag{3.10}$$

It follows that

$$\Psi_j(t) - \Phi_j(t) = \mathrm{i} \int_0^t T(t-s)(\tilde{g}(\Psi(s)) - \tilde{g}(\Phi(s))_j \, ds,$$

for all $t \in I$, where $T(t)$ is the propagator e^{-itL_A}. For the Strichartz type estimates for the magnetic operator L_A, we refer the reader to the paper by Yajima [20] (see also [5]). By assumptions (A) and (B) on the potentials, adapting the result in Yajima [20] proved for such $T(t)$ in the scalar case, we have the following $\mathcal{L}^p$-$\mathcal{L}^q$ estimates that we take from a work of Michael [14]:

Lemma 3.1 *Let $I = [-T, T]$, (q, r) and (γ_k, ρ_k) $(k = 1, 2, 3)$ pairs such that*

$$r, \rho_k \in [2, 2^*), \quad q, \gamma_k \in (2, \infty], \quad \frac{2}{q} = N\big(\frac{1}{2} - \frac{1}{r}\big), \quad \frac{2}{\gamma_k} = N\big(\frac{1}{2} - \frac{1}{\rho_k}\big).$$

Let $\tilde{g}_{k,j} \in L^{\gamma_k'}(I, L^{\rho_k'})$, where γ_k' and ρ_k' denote, respectively, the conjugate exponents of γ_k and ρ_k. Then, the solution of (3.10) with $\tilde{g}_j = -\tilde{g}_{1,j} + \tilde{g}_{2,j} + \tilde{g}_{3,j}$ with zero initial datum, satisfies

$$\|\Phi_j\|_{L^q(I,L^r)} \leq C \sum_{k=1}^{3} \|\tilde{g}_{k,j}\|_{L^{\gamma_k'}(I,L^{\rho_k'})}, \tag{3.11}$$

where the constant C depends only on the length of I and on A, B.

This, finally, allows to show the following

Theorem 3.2 *Problem* (3.1) *enjoys a priori uniqueness over time intervals of sufficiently small length.*

Proof. Let I be an interval containing 0. Let $\Psi, \Phi \in L^\infty(I, \mathcal{H}_A^1) \cap W^{1,\infty}(I, \mathcal{H}_A')$ be two solutions of (3.1). Let r_k and ρ_k the exponents for which the nonlinearity $\tilde{g}_k$ verifies Theorem 3.1. Therefore, if $\frac{2}{q_k} = N(\frac{1}{2} - \frac{1}{r_k})$ and $\frac{2}{\gamma_k} = N(\frac{1}{2} - \frac{1}{\rho_k})$, in light of Lemma 3.1, one can find $\delta > 0$ such that, for $j = 1, \ldots, m$ and $\ell = 1, 2, 3$, it holds

$$\begin{aligned}
\|(\Psi - \Phi)_j\|_{L^{q_\ell}(I,L^{r_\ell})} &\leq C \sum_{k=1}^{3} \|(\tilde{g}(\Psi) - \tilde{g}(\Phi))_{k,j}\|_{L^{\gamma_k'}(I,L^{\rho_k'})} \\
&\leq C(|I| + |I|^\delta) \sum_{k=1}^{3} \|\Psi - \Phi\|_{L^{q_k}(I,\mathcal{L}^{r_k})}.
\end{aligned}$$

Therefore, switching to the vector norm on the left hand side, adding the resulting inequalities over ℓ and, finally, choosing the size of I such that $C(|I| + |I|^\delta) < 1$, we get the inequality

$$(1 - C(|I| + |I|^\delta)) \sum_{k=1}^{3} \|\Psi - \Phi\|_{L^{q_k}(I,\mathcal{L}^{r_k})} \leq 0,$$

which yields the desired conclusion.

3.4 Proof of Theorem 2.1 concluded

We recall that L_A is a self-adjoint and positive operator on $\mathcal{L}^2$. Taking into account Proposition 3.1 of Section 3.1, Theorem 3.1 of Section 3.2, Theorem 3.2 of Section 3.3 the assertion follows by [1, Theorem 4.6.1] (see also [1, Remarks 4.6.3 and 4.3.4 on systems]).

3.5 Global well-posedness

In the previous section, we have established the local solvability of the Cauchy problem (3.1) in $\mathcal{H}^1_A$. In order to show Theorem 2.2 namely that the solution Φ is global, one needs to establish a priori estimates on $\|\Phi(t)\|_{\mathcal{H}^1_A}$ by using the conservation laws under some appropriate assumptions on the nonlinearities.

Proof. Let $I_0 = (-T_*, T^*)$. In light of Theorem 2.1, we have the conservation of energy and charge, that is $\|\Phi(t)\|_{\mathcal{L}^2} = \|\Phi^0\|_{\mathcal{L}^2} = M_0$ and $\mathcal{E}_A(\Phi(t)) = \mathcal{E}_A(\Phi^0)$, for all $t \in I_0$. In the following C will denote a generic positive constant, that may change from line to line and which depends only on the problem and on the initial datum. We have

$$
\begin{aligned}
\mathcal{E}_A(\Phi(t)) = \frac{1}{2}\sum_{j=1}^{m}\int \left|\left(\frac{\nabla}{\mathrm{i}} - A(x)\right)\Phi_j(t)\right|^2 + \frac{1}{2}\int V(x)|\Phi(t)|^2 \\
- \int G(|x|, |\Phi_1(t)|^2, \dots, |\Phi_m(t)|^2) \\
- \frac{1}{2}\sum_{i,j=1}^{m}\iint W_{ij}(|x-y|)h(|\Phi_i(t)|)h(|\Phi_j(t)|) = \mathcal{E}_A(\Phi^0) = C.
\end{aligned}
$$

Since V is bounded from below, and view of the conservation of the charge, we have that

$$
\begin{aligned}
\|\Phi(t)\|^2_{\mathcal{H}^1_A} &\le \left\|\left(\frac{\nabla}{\mathrm{i}} - A(x)\right)\Phi(t)\right\|^2_{\mathcal{L}^2} + \int V(x)|\Phi(t)|^2 + C \\
&\le C + C\int G(|x|, |\Phi_1(t)|^2, \dots, |\Phi_m(t)|^2) \\
&\quad + C\sum_{i,j=1}^{m}\iint W_{ij}(|x-y|)h(|\Phi_i(t)|)h(|\Phi_j(t)|).
\end{aligned}
$$

By applying the Gagliardo-Nirenberg (σ_j is equal to $Nl_j/(2(l_j+2))$ in the following), the Young and Diamagnetic inequalities, we obtain

$$
\begin{aligned}
\|\Phi_j(t)\|^{l_j+2}_{l_j+2} &\le C\|\Phi_j(t)\|^{(1-\sigma_j)(l_j+2)}_{L^2}\|\nabla|\Phi_j(t)|\|^{\sigma_j(l_j+2)}_{L^2} \le C_\varepsilon + \varepsilon C\|\nabla|\Phi_j(t)|\|^2_{L^2} \\
&\le C_\varepsilon + \varepsilon C\left\|\left(\frac{\nabla}{\mathrm{i}} - A(x)\right)\Phi_j(t)\right\|^2_{L^2} \le C_\varepsilon + \varepsilon C\|\Phi(t)\|^2_{\mathcal{H}^1_A}.
\end{aligned}
$$

Consequently, by the growth assumptions on G, we have

$$
\int G(|x|, |\Phi_1(t)|^2, \dots, |\Phi_m(t)|^2) \le C_\varepsilon + \varepsilon C\|\Phi(t)\|^2_{\mathcal{H}^1_A}
$$

Now, since assumptions (h) and (W) hold true, by the Hardy-Littlewood-Sobolev inequality for weak L^q kernels (cf. [11, formula (7), p.107]), by the Gagliardo-Nirenberg and the

Diamagnetic inequality (1.2), we have for all $i, j = 1, \dots, m$,

$$\begin{aligned}
&\iint W_{ij}(|x-y|)h(|\Phi_i(x)|)h(|\Phi_j(y)|)\,dxdy \le C\|W_{ij}\|_{L^q_w}\|\Phi_i\|^{\mu}_{L^{\frac{2q\mu}{2q-1}}}\|\Phi_j\|^{\mu}_{L^{\frac{2q\mu}{2q-1}}} \\
&\le C\|\nabla|\Phi_i|\|_{L^2}^{N\mu\left(\frac{1}{2}-\frac{2q-1}{2q\mu}\right)}\|\Phi_i\|_{L^2}^{\mu\left[1-N\left(\frac{1}{2}-\frac{(2q-1)}{2q\mu}\right)\right]}\|\nabla|\Phi_j|\|_{L^2}^{N\mu\left(\frac{1}{2}-\frac{2q-1}{2q\mu}\right)}\|\Phi_j\|_{L^2}^{\mu\left[1-N\left(\frac{1}{2}-\frac{2q-1}{2q\mu}\right)\right]} \\
&\le C\left\|\left(\frac{\nabla}{\mathrm{i}}-A(x)\right)\Phi_i\right\|_{L^2}^{N\mu\left(\frac{1}{2}-\frac{2q-1}{2q\mu}\right)}\|\Phi_i\|_{L^2}^{\mu\left[1-N\left(\frac{1}{2}-\frac{2q-1}{2q\mu}\right)\right]} \\
&\times\left\|\left(\frac{\nabla}{\mathrm{i}}-A(x)\right)\Phi_j\right\|_{L^2}^{N\mu\left(\frac{1}{2}-\frac{2q-1}{2q\mu}\right)}\|\Phi_j\|_{L^2}^{\mu\left[1-N\left(\frac{1}{2}-\frac{2q-1}{2q\mu}\right)\right]} \\
&\le C\left\|\left(\frac{\nabla}{\mathrm{i}}-A(x)\right)\Phi\right\|_{\mathcal{L}^2}^{2N\mu\left(\frac{1}{2}-\frac{2q-1}{2q\mu}\right)},
\end{aligned}$$

where we exploited $\|\Phi_i(t)\|_{L^2} \le \|\Phi^0\|_{\mathcal{L}^2} = C$ for all $i = 1, \dots, m$. Observe that, in order to perform the above inequality, we need to make sure that

$$2 \le \frac{2q\mu}{2q-1} \le 2^*.$$

This is true provided that the number μ belongs to the range $2 \le \mu \le 2^*(2q-1)/2q$, which is, in fact, compatible with the one assumed in (h) (and it is a fortiori compatible with the range of values assumed for this theorem), which is smaller. Now, taking into account that, by assumption, it holds

$$2N\mu\left(\frac{1}{2}-\frac{2q-1}{2q\mu}\right) < 2,$$

it follows that

$$\begin{aligned}
\sum_{i,j=1}^{m}\iint W_{ij}(|x-y|)h(|\Phi_i(t)|)h(|\Phi_j(t)|)\,dxdy &\le C\,\|\Phi(t)\|_{\mathcal{H}^1_A}^{2N\mu\left(\frac{1}{2}-\frac{2q-1}{2q\mu}\right)} \\
&\le C_\varepsilon + \varepsilon C\,\|\Phi(t)\|^2_{\mathcal{H}^1_A}.
\end{aligned}$$

Fixing now ε_0 sufficiently small, for all $t > 0$, we have

$$(1 - C\varepsilon_0)\|\Phi(t)\|^2_{\mathcal{H}^1_A} \le C_{\varepsilon_0}.$$

Hence, in light of this apriori estimate, the global existence follows by standard arguments.

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