DE GRUYTER Adv. Calc. Var. 2017; aop

Research Article

Andrea Pinamonti, Marco Squassina* and Eugenio Vecchi

Magnetic BV-functions and the Bourgain–Brezis–Mironescu formula

DOI: 10.1515/acv-2017-0019
Received March 27, 2017; accepted June 27, 2017

Abstract: We prove a general magnetic Bourgain–Brezis–Mironescu formula which extends the one obtained in [37] in the Hilbert case setting. In particular, after developing a rather complete theory of magnetic bounded variation functions, we prove the validity of the formula in this class.

Keywords: Fractional magnetic spaces, Bourgain–Brezis–Mironescu formula, BV-functions

MSC 2010: 49A50, 26A33, 82D99

Communicated by: Hitoshi Ishii

1 Introduction

The celebrated Bourgain–Brezis–Mironescu formula, (BBM) in short, appeared for the first time in [8, 9], and provided a new characterization for functions in the Sobolev space $W^{1,p}(\Omega)$, with $p \geq 1$ and for $\Omega \subset \mathbb{R}^N$ being a smooth bounded domain. To this end, the authors of [8, 9] perform a careful study of the limit properties of the Gagliardo semi-norm defined for the fractional Sobolev spaces $W^{s,p}(\Omega)$ with $0 < s < 1$. In particular, they considered the limit as $s \nearrow 1$. To be more precise, for any $W^{1,p}(\Omega)$ it holds

$$\lim_{s \nearrow 1}(1-s) \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dx\, dy = Q_{p,N} \int_\Omega |\nabla u|^p\, dx, \tag{BBM}$$

where $Q_{p,N}$ is defined by

$$Q_{p,N} = \frac{1}{p} \int_{\mathbb{S}^{N-1}} |\boldsymbol{\omega} \cdot h|^p\, d\mathcal{H}^{N-1}(h), \tag{1.1}$$

where $\mathbb{S}^{N-1} \subset \mathbb{R}^N$ denotes the unit sphere and $\boldsymbol{\omega}$ is an arbitrary unit vector of $\mathbb{R}^N$. This also allows to get the stability of (variational) eigenvalues for the fractional p-Laplacian operator as $s \nearrow 1$, see [10]. We recall that characterizations similar to (BBM) when $s \searrow 0$ were obtained in [30, 31].

In the following years, a huge effort in trying to extend the results proved in [8] has been made. One of the first extension was achieved by Nguyen in [32], where he provided a new characterization for functions in $W^{1,p}(\mathbb{R}^N)$. As we already mentioned, the (BBM)-formula proved in [8] covered the case of $\Omega \subset \mathbb{R}^N$ being a smooth and bounded domain, therefore it was quite natural to try to relax the assumptions on the open set $\Omega \subset \mathbb{R}^N$: this kind of problem was recently addressed in [25] and [26], where Leoni and Spector were able

Andrea Pinamonti: Dipartimento di Matematica, Università degli Studi di Trento, Via Sommarive 14, 38123 Povo (TN), Italy, e-mail: andrea.pinamonti@unitn.it
***Corresponding author: Marco Squassina:** Dipartimento di Matematica e Fisica, Università Cattolica del Sacro Cuore, Via dei Musei 41, I-25121 Brescia, Italy, e-mail: marco.squassina@dmf.unicatt.it
Eugenio Vecchi: Dipartimento di Matematica, Università di Bologna, Piazza di Porta S. Donato 5, 40126 Bologna, Italy, e-mail: eugenio.vecchi2@unibo.it

to provide a generalization of the (BBM)-formula to *any* open set $\Omega \subset \mathbb{R}^N$. The interest resulted from [8] led also to related new characterizations of Sobolev spaces in non-Euclidean contexts like the Heisenberg group (see [7, 18]).

One of the most challenging problems left open in [8] was to provide similar characterizations for functions of bounded variation. A positive answer to this question has been given by Davila in [20] and by Ponce in [34]. They completed the picture by showing that

$$\lim_{s\nearrow 1}(1-s)\int_\Omega\int_\Omega \frac{|u(x)-u(y)|}{|x-y|^{N+s}}\,dx\,dy = Q_{1,N}|Du|(\Omega)$$

for every bounded Lipschitz set $\Omega \subset \mathbb{R}^N$ and every $u \in \mathrm{BV}(\Omega)$. We also recall that the extension to any open set proved in [25, 26] concerns BV-functions as well, see also [35].

In order to try to give a more complete overview of the subject, we have to mention that, parallel to the fractional theory of Sobolev spaces, there exists a quite developed theory of fractional s-perimeters (e.g. [16]), and also in this framework there have been several contributions concerning their analysis in the limits $s \nearrow 1$ and $s \searrow 0$ (see e.g. [2, 17, 21, 23, 28, 29]).

Very recently the results we have mentioned have been discovered to have interesting applications in image processing, see for instance [12–15]. One of the latest generalizations of (BBM) appeared very recently in [37] in the context of magnetic Sobolev spaces $W^{1,2}_A(\Omega)$. In fact, an important role in the study of particles which interact with a magnetic field $B = \nabla \times A$, $A : \mathbb{R}^3 \to \mathbb{R}^3$, is assumed by another *extension* of the Laplacian, namely the *magnetic Laplacian* $(\nabla - \mathrm{i}A)^2$ (see [6, 27, 36]), yielding to nonlinear Schrödinger equations like

$$-(\nabla - \mathrm{i}A)^2 u + u = f(u), \tag{1.2}$$

which have extensively been studied (see e.g. [5] and references therein), where $(\nabla - \mathrm{i}A)^2$ is defined in weak sense as the differential of the integral functional

$$W^{1,2}_A(\Omega) \ni u \mapsto \int_\Omega |\nabla u - \mathrm{i}A(x)u|^2\,dx. \tag{1.3}$$

If $A : \mathbb{R}^N \to \mathbb{R}^N$ is a smooth function and $s \in (0, 1)$, a nonlocal magnetic counterpart of (1.2), i.e.

$$(-\Delta)^s_A u(x) = c(N,s)\lim_{\varepsilon\searrow 0}\int_{B^c_\varepsilon(x)} \frac{u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)}{|x-y|^{N+2s}}\,dy, \qquad \lim_{s\nearrow 1}\frac{c(N,s)}{1-s} = \frac{4N\Gamma(\frac{N}{2})}{2\pi^{N/2}},$$

was introduced in [19, 24] for complex-valued functions. We point out that $(-\Delta)^s_A$ coincides with the usual fractional Laplacian for $A = 0$. The motivations for the introduction of this operator are carefully described in [19, 24] and fall into the framework of the general theory of Lévy processes. It is thus natural wondering about the consistency of the norms associated with the above fractional magnetic operator in the singular limit $s \nearrow 1$, with the energy functional (1.3).

The aim of this paper is to continue the study of the validity of a magnetic counterpart of (BBM), extending the results of [37] to arbitrary magnetic fractional Sobolev spaces and to magnetic BV-functions. We refer the reader to Sections 2 and 3 for the definitions. On the other hand, while for $p \geq 1$ the spaces $W^{1,p}_A(\Omega)$ have a wide background, to the best of our knowledge no notion of *magnetic bounded variations space* containing $W^{1,1}_A(\Omega)$ seems to be previously available in the literature.

As already recalled, this indeed holds for the Hilbert case $p = 2$, as stated in the following

Theorem (M. Squassina, B. Volzone [37]). *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set with Lipschitz boundary and let $A \in C^2(\bar{\Omega}, \mathbb{R}^N)$. Then, for every $u \in W^{1,2}_A(\Omega)$, we have*

$$\lim_{s\nearrow 1}(1-s)\int_\Omega\int_\Omega \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy = Q_{2,N}\int_\Omega |\nabla u - \mathrm{i}A(x)u|^2\,dx,$$

where $Q_{2,N}$ is the positive constant defined in (1.1) *with $p = 2$.*

The goal of this paper is twofold: first we aim to extend this formula to the case of general magnetic spaces $W^{1,p}_A$ for $p \geq 1$, and secondly we introduce a suitable notion of *magnetic bounded variation* $|Du|_A(\Omega)$ and we prove that a (BBM)-formula holds also in that case.

In order to state the main result we need to introduce some notation: let $p \geq 1$ be fixed and let us consider the normed space $(\mathbb{C}^N, |\cdot|_p)$, with

$$|z|_p := (|(\Re z_1, \dots, \Re z_N)|^p + |(\Im z_1, \dots, \Im z_N)|^p)^{\frac{1}{p}}, \tag{1.4}$$

where $|\cdot|$ is the Euclidean norm of $\mathbb{R}^N$ and $\Re a, \Im a$ denote the real and imaginary parts of $a \in \mathbb{C}$, respectively. Notice that $|z|_p = |z|$ whenever $z \in \mathbb{R}^N$, which makes our next statements consistent with the case $A = 0$ and u being a real-valued function [8, 11, 20, 34].

Theorem 1.1 (General magnetic Bourgain–Brezis–Mironescu limit). *Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be of class C^2. Then, for any bounded extension domain $\Omega \subset \mathbb{R}^N$,*

$$\lim_{s \nearrow 1}(1-s) \int_\Omega \int_\Omega \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|_1}{|x-y|^{N+s}}\,dx\,dy = Q_{1,N}|Du|_A(\Omega)$$

for all $u \in \mathrm{BV}_A(\Omega)$, where $Q_{p,N}$ is defined in (1.1). *Furthermore, for any $p \geq 1$ and any Lipschitz bounded domain $\Omega \subset \mathbb{R}^N$,*

$$\lim_{s \nearrow 1}(1-s) \int_\Omega \int_\Omega \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|_p^p}{|x-y|^{N+ps}}\,dx\,dy = Q_{p,N} \int_\Omega |\nabla u - \mathrm{i}A(x)u|_p^p\,dx$$

for all $u \in W^{1,p}_A(\Omega)$.

We refer to Definition 3.11 for a precise explanation of *extension domain*. We stress that the definitions of both the magnetic Sobolev spaces $W^{1,p}_A(\Omega)$ and of the magnetic BV-spaces $\mathrm{BV}_A(\Omega)$ made in Sections 2 and 3 are consistent, in the case of zero magnetic potential A, with the classical spaces $W^{1,p}(\Omega)$ and $\mathrm{BV}(\Omega)$, respectively. Moreover, it holds $|Du|_A(\Omega) = |Du|(\Omega)$, so that Theorem 1.1 is consistent with the classical formulas of [8, 20, 34].

In particular, in the spirit of [11], as a byproduct of Theorem 1.1, if $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $A : \mathbb{R}^N \to \mathbb{R}^N$ is of class C^2 and we have

$$\lim_{s \nearrow 1}(1-s) \int_\Omega \int_\Omega \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|_p^p}{|x-y|^{N+ps}}\,dx\,dy = 0, \quad u \in W^{1,p}_A(\Omega),$$

then we get

$$\begin{cases} \nabla \Re u = -\Im u A, \\ \nabla \Im u = \Re u A, \end{cases}$$

namely the direction of $\nabla \Re u$, $\nabla \Im u$ is that of the magnetic potential A. In the particular case $A = 0$, consistently with the results of [11], this implies that u is a constant function.

We finally notice that for a Borel set $E \subset \Omega$, denoting $E^c = \Omega \setminus E$, the quantity

$$P_s(E; A) := \frac{1}{2} \int_E \int_E \frac{|1 - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}|_1}{|x-y|^{N+s}}\,dx\,dy + \frac{1}{2} \int_E \int_{E^c} \frac{1}{|x-y|^{N+s}}\,dx\,dy + \frac{1}{2} \int_{E^c} \int_E \frac{|e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}|_1}{|x-y|^{N+s}}\,dx\,dy$$

plays the rôle of a nonlocal s-perimeter of E depending on A, which reduces for $A = 0$ to the classical notion of fractional s-perimeter of E in Ω

$$P_s(E) = \int_E \int_{E^c} \frac{1}{|x-y|^{N+s}}\,dx\,dy.$$

Then the main result, Theorem 1.1, reads as

$$\lim_{s \nearrow 1}(1-s)P_s(E, A) = Q_{1,N}|D\mathbf{1}_E|_A(\Omega).$$

The structure of the paper is as follows. In Section 2 we introduce magnetic Sobolev spaces $W^{1,p}_A(\Omega)$. In Section 3 we define the magnetic BV space $\mathrm{BV}_A(\Omega)$ and we prove that several classical results for BV-functions hold also for functions belonging to $\mathrm{BV}_A(\Omega)$. In particular, we prove a structure result (Lemma 3.6), a result about the extension to $\mathbb{R}^N$ for Lipschitz domains (Lemma 3.12), the semi-continuity of the variation (Lemma 3.7), a magnetic counterpart of the classical Anzellotti–Giaquinta Approximation Theorem (Lemma 3.10) and, finally, a compactness result (Lemma 3.14). In Sections 4, 5 and 6 we finally prove Theorem 1.1.

2 Magnetic Sobolev spaces

In order to avoid confusion with the different uses of the symbol $v \cdot w$, we define

$$v \cdot w := \begin{cases} \sum_{i=1}^{N} v_i w_i & \text{if } v, w \in \mathbb{R}^N, \\ \sum_{i=1}^{N} (\Re v_i + \mathrm{i}\Im v_i)(\Re w_i + \mathrm{i}\Im w_i) & \text{if } v, w \in \mathbb{C}^N. \end{cases}$$

Let Ω be an open set of $\mathbb{R}^N$. For any $p \geq 1$ we denote by $L^p(\Omega, \mathbb{C})$ the Lebesgue space of complex-valued functions $u : \Omega \to \mathbb{C}$ such that

$$\|u\|_{L^p(\Omega)} = \Big(\int_\Omega |u(x)|_p^p \, dx \Big)^{\frac{1}{p}} < \infty,$$

where $|\cdot|_p$ is as in (1.4). For a locally bounded function $A : \mathbb{R}^N \to \mathbb{R}^N$, we consider the semi-norm

$$[u]_{W^{1,p}_A(\Omega)} := \Big(\int_\Omega |\nabla u - \mathrm{i}A(x)u|_p^p \, dx \Big)^{\frac{1}{p}},$$

and define $W^{1,p}_A(\Omega)$ as the space of functions $u \in L^p(\Omega, \mathbb{C})$ such that $[u]_{W^{1,p}_A(\Omega)} < \infty$ with norm

$$\|u\|_{W^{1,p}_A(\Omega)} := (\|u\|^p_{L^p(\Omega)} + [u]^p_{W^{1,p}_A(\Omega)})^{\frac{1}{p}}.$$

The space $W^{1,p}_{0,A}(\Omega)$ will denote the closure of the space $C_c^\infty(\Omega)$ in $W^{1,p}_A(\Omega)$. For any $s \in (0,1)$ and $p \geq 1$, the magnetic Gagliardo semi-norm is defined as

$$[u]_{W^{s,p}_A(\Omega)} := \Big(\int_\Omega \int_\Omega \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y)|_p^p}{|x-y|^{N+ps}} \, dx \, dy \Big)^{\frac{1}{p}}.$$

We denote by $W^{s,p}_A(\Omega)$ the space of functions $u \in L^p(\Omega, \mathbb{C})$ such that $[u]_{W^{s,p}_A(\Omega)} < \infty$ normed with

$$\|u\|_{W^{s,p}_A(\Omega)} := (\|u\|^p_{L^p(\Omega)} + [u]^p_{W^{s,p}_A(\Omega)})^{\frac{1}{p}}.$$

For $A = 0$ this is consistent with the usual space $W^{s,p}(\Omega)$ with norm $\|\cdot\|_{W^{s,p}(\Omega)}$.

3 Magnetic BV-spaces

In this section we introduce a suitable notion of magnetic bounded variation functions. Let Ω be an open set of $\mathbb{R}^N$. We recall that a real-valued function $u \in L^1(\Omega)$ is of bounded variation, and we shall write $u \in \mathrm{BV}(\Omega)$, if

$$|Du|(\Omega) = \sup\Big\{ \int_\Omega u(x) \operatorname{div} \varphi(x) \, dx : \varphi \in C_c^\infty(\Omega, \mathbb{R}^N),\ \|\varphi\|_{L^\infty(\Omega)} \leq 1 \Big\} < \infty.$$

The space $\mathrm{BV}(\Omega)$ is endowed with the norm

$$\|u\|_{\mathrm{BV}(\Omega)} := \|u\|_{L^1(\Omega)} + |Du|(\Omega).$$

The space of complex-valued bounded variation functions $\mathrm{BV}(\Omega, \mathbb{C})$ is defined as the class of Borel functions $u : \Omega \to \mathbb{C}$ such that $\Re u, \Im u \in \mathrm{BV}(\Omega)$. The $\mathbb{C}$-total variation of u is defined by

$$|Du|(\Omega) := |D\Re u|(\Omega) + |D\Im u|(\Omega).$$

More generally, it is possible to define a notion of variation for functions $u : \Omega \to E$, where $\Omega \subset \mathbb{R}^N$ is an open set and (E, d) is a locally compact metric space. We refer the interested reader to [1].

We are now ready to define the magnetic BV-functions.

Definition 3.1 (A-bounded variation functions). Let $\Omega \subset \mathbb{R}^N$ be an open set and let $A : \mathbb{R}^N \to \mathbb{R}^N$ be a locally bounded function. A function $u \in L^1(\Omega, \mathbb{C})$ is said to be of A-bounded variation and we write $u \in \mathrm{BV}_A(\Omega)$ if

$$|Du|_A(\Omega) := C_{1,A,u}(\Omega) + C_{2,A,u}(\Omega) < \infty,$$

where we have set

$$C_{1,A,u}(\Omega) := \sup\Big\{\int_\Omega \Re u(x) \operatorname{div} \varphi(x) - A(x) \cdot \varphi(x) \Im u(x)\, dx : \varphi \in C_c^\infty(\Omega, \mathbb{R}^N),\ \|\varphi\|_{L^\infty(\Omega)} \le 1\Big\},$$

$$C_{2,A,u}(\Omega) := \sup\Big\{\int_\Omega \Im u(x) \operatorname{div} \varphi(x) + A(x) \cdot \varphi(x) \Re u(x)\, dx : \varphi \in C_c^\infty(\Omega, \mathbb{R}^N),\ \|\varphi\|_{L^\infty(\Omega)} \le 1\Big\}.$$

A function $u \in L^1_{\mathrm{loc}}(\Omega, \mathbb{C})$ is said to be of locally A-bounded variation and we write $u \in \mathrm{BV}_{A,\mathrm{loc}}(\Omega)$, provided that it holds

$$|Du|_A(U) < \infty \quad \text{for every open set } U \Subset \Omega.$$

We stress that for $A \equiv 0$, the previous definition is consistent with the one of $\mathrm{BV}(\Omega)$. In order to justify our definition, we will collect in the following some properties of the space $\mathrm{BV}_A(\Omega)$. These properties are the natural generalization to the magnetic setting of the classical theory [3, 22, 38].

Lemma 3.2 (Extension of $|Du|_A|$). *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set, $A : \mathbb{R}^N \to \mathbb{R}^N$ locally bounded and $u \in \mathrm{BV}_A(\Omega)$. Let $E \subset \Omega$ be a Borel set. Then*

$$|Du|_A(E) := \inf\{C_{1,A,u}(U) : E \subset U,\ U \subset \Omega \text{ open}\} + \inf\{C_{2,A,u}(U) : E \subset U,\ U \subset \Omega \text{ open}\}$$

extends $|Du|_A(\cdot)$ to a Radon measure in Ω. For any open set $U \subset \Omega$, $C_{1,A,u}(U)$ and $C_{2,A,u}(U)$ are defined requiring the test functions to be supported in U and $|Du|_A(\emptyset) := 0$.

Proof. We note that

$$\nu_1(E) := \inf\{C_{1,A,u}(U) : E \subset U,\ U \subset \Omega \text{ open}\}$$

is the variation measure associated with

$$\varphi \mapsto \int_\Omega \Re u(x) \operatorname{div} \varphi(x) - A(x) \cdot \varphi(x) \Im u(x)\, dx,$$

and by [22, Theorem 1.38] it is a Radon measure. The same argument applies to

$$\nu_2(E) := \inf\{C_{2,A,u}(U) : E \subset U,\ U \subset \Omega \text{ open}\}$$

and the thesis follows. □

Lemma 3.3 (Local inclusion of Sobolev functions). *Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be locally bounded. Then*

$$W^{1,1}_{\mathrm{loc}}(\Omega) \subset \mathrm{BV}_{A,\mathrm{loc}}(\Omega).$$

Proof. Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, $U \Subset \Omega$ open and consider $\varphi \in C_c^\infty(U, \mathbb{R}^N)$ with $\|\varphi\|_{L^\infty(U)} \le 1$. Then

$$\begin{aligned}
\int_U \Re u(x) \operatorname{div} \varphi(x) - A(x)\cdot\varphi(x)\Im u(x)\,dx + \int_U \Im u(x) \operatorname{div} \varphi(x) + A(x)\cdot\varphi(x)\Re u(x)\,dx \\
= -\int_U (\nabla \Re u(x) + A(x)\Im u(x))\cdot\varphi(x)\,dx - \int_U (\nabla \Im u(x) - A(x)\Re u(x))\cdot\varphi(x)\,dx \\
\le \int_{\bar U} |\nabla \Re u(x) + A(x)\Im u(x)|\,dx + \int_{\bar U} |\nabla \Im u(x) - A(x)\Re u(x)|\,dx \\
\le \int_{\bar U} |\nabla \Re u(x)|\,dx + \int_{\bar U} |\nabla \Im u(x)|\,dx + \|A\|_{L^\infty(\bar U)}\Big(\int_{\bar U} (|\Re u(x)| + |\Im u(x)|)\,dx\Big) < \infty,
\end{aligned}$$

which, taking the supremum over φ, concludes the proof. □

Next we prove that for $W^{1,1}_A(\Omega)$-functions the magnetic bounded variation semi-norm $|Du|_A(\Omega)$ boils down to the usual local magnetic semi-norm.

Lemma 3.4 (BV_A-norm on $W^{1,1}_A$). *Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be locally bounded. Assume that $u \in W^{1,1}_A(\Omega)$. Then $u \in \mathrm{BV}_A(\Omega)$ and it holds*

$$|Du|_A(\Omega) = \int_\Omega |\nabla u - \mathrm{i}A(x)u|_1\,dx.$$

Furthermore, if $u \in \mathrm{BV}_A(\Omega) \cap C^\infty(\Omega)$, then $u \in W^{1,1}_A(\Omega)$.

Proof. If $u \in W^{1,1}_A(\Omega)$, then we have

$$\nabla \Re u + A \Im u \in L^1(\Omega), \quad \nabla \Im u - A \Re u \in L^1(\Omega).$$

For every $\varphi \in C_c^\infty(\Omega, \mathbb{R}^N)$ with $\|\varphi\|_{L^\infty(\Omega)} \le 1$, we have

$$\Big|\int_\Omega \Re u(x) \operatorname{div}\varphi(x) - A(x)\cdot\varphi(x)\Im u(x)\,dx\Big| = \Big|\int_\Omega \nabla \Re u(x)\cdot\varphi(x) + A(x)\cdot\varphi(x)\Im u(x)\,dx\Big| \le \int_\Omega |\nabla \Re u + A\Im u|\,dx,$$

as well as

$$\Big|\int_\Omega \Im u(x) \operatorname{div}\varphi(x) + A(x)\cdot\varphi(x)\Re u(x)\,dx\Big| = \Big|\int_\Omega \nabla \Im u(x)\cdot\varphi(x) - A(x)\cdot\varphi(x)\Re u(x)\,dx\Big| \le \int_\Omega |\nabla \Im u - A\Re u|\,dx,$$

which, taking the supremum over φ, proves $u \in \mathrm{BV}_A(\Omega, \mathbb{C})$ and

$$|Du|_A(\Omega) \le \int_\Omega |\nabla u - \mathrm{i}A(x)u|_1\,dx. \tag{3.1}$$

Defining now $f, g \in L^\infty(\Omega, \mathbb{R}^N)$ by setting

$$f(x) := \begin{cases} -\dfrac{\nabla \Re u(x) + A(x)\Im u(x)}{|\nabla \Re u(x) + A(x)\Im u(x)|} & \text{if } x \in \Omega \text{ and } \nabla \Re u(x) + A(x)\Im u(x) \ne 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g(x) := \begin{cases} -\dfrac{\nabla \Im u(x) - A(x)\Re u(x)}{|\nabla \Im u(x) - A(x)\Re u(x)|} & \text{if } x \in \Omega \text{ and } \nabla \Im u(x) - A(x)\Re u(x) \ne 0, \\ 0 & \text{otherwise,} \end{cases}$$

we have $\|f\|_\infty, \|g\|_\infty \le 1$. By a standard approximation result, there exist two sequences $\{\varphi_n\}_{n\in\mathbb{N}}, \{\psi_n\}_{n\in\mathbb{N}}$ in $C_c^\infty(\Omega, \mathbb{R}^N)$ such that $\varphi_n \to f$ and $\psi_n \to g$ pointwise as $n \to \infty$, with $\|\varphi_n\|_{L^\infty(\Omega)}, \|\psi_n\|_{L^\infty(\Omega)} \le 1$ for all $n \in \mathbb{N}$. By the definition of $C_{1,A,u}(\Omega)$, after integration by parts, it follows that, for every $n \ge 1$,

$$C_{1,A,u}(\Omega) \ge -\sum_{i=1}^N \int_\Omega (\partial_{x_i} \Re u(x) + A^{(i)}(x)\Im u(x))\varphi_n^{(i)}(x)\,dx.$$

By the Dominated Convergence Theorem and the definition of f, letting $n \to \infty$, we obtain

$$C_{1,A,u}(\Omega) \geq \int_\Omega |\nabla \Re u(x) + A(x)\Im u(x)|\,dx.$$

Similarly, using the sequence $\{\psi_n\}_{n\in\mathbb{N}}$ and arguing in a similar fashion yields

$$C_{2,A,u}(\Omega) \geq \int_\Omega |\nabla \Im u(x) - A(x)\Re u(x)|\,dx,$$

which, on account of (1.4), proves the opposite of inequality (3.1), concluding the proof of the first statement. If $u \in \mathrm{BV}_A(\Omega) \cap C^\infty(\Omega)$, fix a compact set $K \subset \Omega$ with nonempty interior and consider

$$\tilde{f} := f\chi_{\mathrm{int}(K)}, \quad \tilde{g} := g\chi_{\mathrm{int}(K)}.$$

Then, as above, one can find two sequences $\{\varphi_n\}_{n\in\mathbb{N}}, \{\psi_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\mathrm{int}(K), \mathbb{R}^N)$ such that $\varphi_n \to f$ and $\psi_n \to g$ pointwise and $\|\varphi_n\|_{L^\infty(\mathrm{int}(K))}, \|\psi_n\|_{L^\infty(\mathrm{int}(K))} \leq 1$, for all $n \in \mathbb{N}$. Then we have

$$\begin{aligned} C_{1,A,u}(\Omega) &\geq \int_\Omega \Re u(x)\operatorname{div}\varphi_n(x) - A(x)\cdot\varphi_n(x)\Im u(x)\,dx \\ &= \int_K \Re u(x)\operatorname{div}\varphi_n(x) - A(x)\cdot\varphi_n(x)\Im u(x)\,dx \\ &= -\sum_{i=1}^N \int_K (\partial_{x_i}\Re u(x) + A^{(i)}(x)\Im u(x))\varphi_n^{(i)}(x)\,dx. \end{aligned}$$

Since $u \in C^\infty(\Omega)$, we have $\nabla\Re u + A\Im u \in L^1(K)$. Thus, by the Dominated Convergence Theorem,

$$C_{1,A,u}(\Omega) \geq \int_K |\nabla \Re u(x) + A(x)\Im u(x)|\,dx.$$

The conclusion follows using an exhaustive sequence of compacts via monotone convergence. □

We endow the space $\mathrm{BV}_A(\Omega, \mathbb{C})$ with the following norm:

$$\|u\|_{\mathrm{BV}_A(\Omega)} := \|u\|_{L^1(\Omega)} + |Du|_A(\Omega).$$

Lemma 3.5 (Norm equivalence). *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set. Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be locally bounded. Then $u \in \mathrm{BV}_A(\Omega)$ if and only if $u \in \mathrm{BV}(\Omega)$. Moreover, for every $u \in \mathrm{BV}_A(\Omega)$, there exists a positive constant $K = K(A, \Omega)$ such that*

$$K^{-1}\|u\|_{\mathrm{BV}(\Omega)} \leq \|u\|_{\mathrm{BV}_A(\Omega)} \leq K\|u\|_{\mathrm{BV}(\Omega)}.$$

Proof. Denoting by $\sup_\varphi$ the supremum over functions $\varphi \in C_c^\infty(\Omega, \mathbb{R}^N)$ with $\|\varphi\|_{L^\infty(\Omega)} \leq 1$, we get

$$\begin{aligned} |Du|(\Omega) = |\Re u|(\Omega) + |\Im u|(\Omega) &= \sup_\varphi \int_\Omega \Re u(x)\operatorname{div}\varphi(x)\,dx + \sup_\varphi \int_\Omega \Im u(x)\operatorname{div}\varphi(x)\,dx \\ &= \sup_\varphi \int_\Omega \Re u(x)\operatorname{div}\varphi(x) - A(x)\cdot\varphi(x)\Im u(x) + A(x)\cdot\varphi(x)\Im u\,dx \\ &\quad + \sup_\varphi \int_\Omega \Im u(x)\operatorname{div}\varphi(x) + A(x)\cdot\varphi(x)\Re u(x) - A(x)\cdot\varphi(x)\Re u(x)\,dx \\ &\leq \sup_\varphi \int_\Omega \Re u(x)\operatorname{div}\varphi(x) - A(x)\cdot\varphi(x)\Im u(x)\,dx + \sup_\varphi \int_\Omega A(x)\cdot\varphi(x)\Im u(x)\,dx \\ &\quad + \sup_\varphi \int_\Omega \Im u(x)\operatorname{div}\varphi(x) + A(x)\cdot\varphi(x)\Re u(x)\,dx + \sup_\varphi \int_\Omega A(x)\cdot(-\varphi)(x)\Re u(x)\,dx \\ &\leq C_{1,A,u}(\Omega) + C_{2,A,u}(\Omega) + \|A\|_{L^\infty(\Omega)}\|u\|_{L^1(\Omega)}. \end{aligned}$$

Therefore, we have that

$$\|u\|_{\mathrm{BV}(\Omega)} \le (1 + \|A\|_{L^\infty(\Omega)})\|u\|_{\mathrm{BV}_A(\Omega)}.$$

For the second inequality, we have

$$\begin{aligned} C_{1,A,u}(\Omega) &\le \sup_{\varphi} \int_\Omega \Re u(x) \operatorname{div} \varphi(x)\, dx + \sup_{\varphi} \int_\Omega A(x) \cdot (-\varphi)(x) \Im u(x)\, dx \\ &\le |D\Re u|(\Omega) + \|A\|_{L^\infty(\Omega)} \int_\Omega |\Im u|\, dx, \end{aligned}$$

and similarly for $C_{2,A,u}(\Omega)$. Therefore, we conclude

$$\|u\|_{\mathrm{BV}_A(\Omega)} \le (1 + \|A\|_{L^\infty(\Omega)})\|u\|_{\mathrm{BV}(\Omega)}.$$

Calling $K := (1 + \|A\|_{L^\infty(\Omega)})$ concludes the proof. □

Lemma 3.6 (Structure theorem for BV_A-functions). *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set, $A : \mathbb{R}^N \to \mathbb{R}^N$ locally bounded and $u \in \mathrm{BV}_A(\Omega)$. There exists a unique $\mathbb{R}^{2N}$-valued finite Radon measure $\mu_{A,u} = (\mu_{1,A,u}, \mu_{2,A,u})$ such that*

$$\begin{aligned} \int_\Omega u(x) \operatorname{div} \varphi(x) + \mathrm{i}A(x) \cdot \varphi(x) u(x)\, dx &= \int_\Omega \Re u(x) \operatorname{div} \varphi(x) - A(x) \cdot \varphi(x) \Im u(x)\, dx \\ &\quad + \mathrm{i} \int_\Omega \Im u(x) \operatorname{div} \varphi(x) + A(x) \cdot \varphi(x) \Re u(x)\, dx \\ &= \int_\Omega \varphi(x) \cdot d(\mu_{1,A,u} + \mathrm{i}\mu_{2,A,u})(x) \end{aligned}$$

for every $\varphi \in C_c^\infty(\Omega, \mathbb{R}^N)$ and

$$|Du|_A(\Omega) = |\mu_{1,A,u}|(\Omega) + |\mu_{2,A,u}|(\Omega).$$

Proof. Of course, we have

$$\left| \int_\Omega \Re u(x) \operatorname{div} \varphi(x) - A(x) \cdot \varphi(x) \Im u(x)\, dx \right| \le C_{1,A,u}(\Omega) \|\varphi\|_{L^\infty(\Omega)} \quad \text{for all } \varphi \in C_c^\infty(\Omega, \mathbb{R}^N).$$

Then a standard application of the Hahn–Banach Theorem yields the existence of a linear and continuous extension L of the functional $\Psi : C_c^\infty(\Omega, \mathbb{R}^N) \to \mathbb{R}$

$$\langle \Psi, \varphi \rangle = \int_\Omega \Re u(x) \operatorname{div} \varphi(x) - A(x) \cdot \varphi(x) \Im u(x)\, dx$$

to the normed space $(C_c(\Omega, \mathbb{R}^N), \|\cdot\|_{L^\infty(\Omega)})$ such that

$$\|L\| = \|\Psi\| = C_{1,A,u}(\Omega).$$

On the other hand, by the Riesz Representation Theorem (cf. [3, Corollary 1.55]) there exists a unique $\mathbb{R}^N$-valued finite Radon measure $\mu_{1,A,u}$ with

$$L(\varphi) = \int_\Omega \varphi(x) \cdot d\mu_{1,A,u}(x) \quad \text{for all } \varphi \in C_c(\Omega, \mathbb{R}^N),$$

and such that $|\mu_{1,A,u}|(\Omega) = \|L\|$. Thus $|\mu_{1,A,u}|(\Omega) = C_{1,A,u}(\Omega)$. The same argument can be repeated verbatim for the functional

$$\varphi \mapsto \int_\Omega \Im u(x) \operatorname{div} \varphi(x) + A(x) \cdot \varphi(x) \Re u(x)\, dx,$$

which concludes the proof. □

Lemma 3.7 (Lower semicontinuity of $|Du|_A(\Omega)$). *Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be locally bounded. Let $\Omega \subset \mathbb{R}^N$ be an open set and $\{u_k\}_{k\in\mathbb{N}} \subset \mathrm{BV}_A(\Omega)$ a sequence converging locally in $L^1(\Omega)$ to a function u. Then*

$$\liminf_{k\to\infty} |Du_k|_A(\Omega) \geq |Du|_A(\Omega).$$

Proof. Fix $\varphi \in C_c^\infty(\Omega, \mathbb{R}^N)$ with $\|\varphi\|_{L^\infty(\Omega)} \leq 1$. By the definitions of $C_{i,A,u_k}(\Omega)$, we have

$$C_{1,A,u_k}(\Omega) \geq \int_\Omega \Re u_k(x) \operatorname{div} \varphi(x) - A(x) \cdot \varphi(x) \Im u_k(x)\, dx,$$

$$C_{2,A,u_k}(\Omega) \geq \int_\Omega \Im u_k(x) \operatorname{div} \varphi(x) + A(x) \cdot \varphi(x) \Re u_k(x)\, dx.$$

By the convergence of $\{u_k\}_{k\in\mathbb{N}}$ in $L^1_{\mathrm{loc}}(\Omega, \mathbb{C})$ to u, we get

$$\liminf_{k\to\infty} C_{1,A,u_k}(\Omega) \geq \int_\Omega \Re u(x) \operatorname{div} \varphi(x) - A(x) \cdot \varphi(x) \Im u(x)\, dx,$$

$$\liminf_{k\to\infty} C_{2,A,u_k}(\Omega) \geq \int_\Omega \Im u(x) \operatorname{div} \varphi(x) + A(x) \cdot \varphi(x) \Re u(x)\, dx.$$

The assertion follows by the definition of $|Du|_A(\Omega)$ and the arbitrariness of such functions φ. □

Lemma 3.8. *The space $(\mathrm{BV}_A(\Omega), \|\cdot\|_{\mathrm{BV}_A(\Omega)})$ is a real Banach space.*

Proof. It is readily seen that $\|\cdot\|_{\mathrm{BV}_A(\Omega)}$ is a norm (to this end, it is enough to check that the map $u \mapsto |Du|_A(\Omega)$ defines a semi-norm over $\mathrm{BV}_A(\Omega)$, which is left to the reader). Let us prove that the space is complete. Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathrm{BV}_A(\Omega)$ be a Cauchy sequence, namely for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\int_\Omega |u_n - u_k|_1\, dx + |D(u_n - u_k)|_A(\Omega) < \varepsilon \quad \text{for all } n, k \geq n_0.$$

In particular, $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $(L^1(\Omega), \|\cdot\|_{L^1(\Omega)})$, which implies that there exists $u \in L^1(\Omega)$ with $\|u_n - u\|_{L^1(\Omega)} \to 0$, as $n \to \infty$. Therefore, in light of Lemma 3.7, we get

$$|D(u - u_k)|_A(\Omega) \leq \liminf_n |D(u_n - u_k)|_A(\Omega) \leq \varepsilon \quad \text{for all } k \geq n_0,$$

namely $|D(u_n - u)|_A(\Omega) \to 0$, as $n \to \infty$, which concludes the proof. □

Lemma 3.9 (Multiplication by Lipschitz functions). *Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be locally bounded and $u \in \mathrm{BV}_{A,\mathrm{loc}}(\Omega)$. Then, for every locally Lipschitz $\psi : \Omega \to \mathbb{R}$, the function $u\psi \in \mathrm{BV}_{A,\mathrm{loc}}(\Omega)$ and*

$$\mu_{1,A,\psi u} = \psi \mu_{1,A,u} - \Re u \cdot \nabla\psi \mathcal{L}^N,$$
$$\mu_{2,A,\psi u} = \psi \mu_{2,A,u} - \Im u \cdot \nabla\psi \mathcal{L}^N,$$

where $\mathcal{L}^N$ denotes the N-dimensional Lebesgue measure.

Proof. Consider $U \Subset \Omega$ open and let $\varphi \in C_c^\infty(U, \mathbb{R}^N)$ be such that $\|\varphi\|_{L^\infty(U)} \leq 1$. By Rademacher's Theorem we have $\psi \operatorname{div} \varphi = \operatorname{div}(\psi\varphi) - \varphi \cdot \nabla\psi$ a.e. in U. Therefore, up to smoothing ψ, we get

$$\begin{aligned}
&\int_U \Re(u\psi)(x) \operatorname{div} \varphi(x) - A(x) \cdot \varphi(x) \Im(u\psi)(x)\, dx \\
&\quad = \int_U \psi(x) \Re u(x) \operatorname{div} \varphi(x) - A(x) \cdot \varphi(x) \psi(x) \Im u(x)\, dx \\
&\quad = \int_U \Re u(x) \operatorname{div}(\psi\varphi)(x) - A(x) \cdot \varphi(x) \psi(x) \Im u(x)\, dx - \int_U \Re u(x) \varphi(x) \cdot \nabla\psi(x)\, dx \\
&\quad \leq C_{1,A,u}(U) \|\psi\|_{L^\infty(\overline{U})} + \mathrm{Lip}(\psi) \|u\|_{L^1(U)}.
\end{aligned}$$

A similar estimate holds for the second term, proving $u\psi \in \mathrm{BV}_{A,\mathrm{loc}}(\Omega)$. By Lemma 3.6, we have

$$\begin{aligned}\int_\Omega \varphi(x)\cdot d\mu_{1,A,u\psi} &= \int_\Omega \psi(x)\Re u(x)\operatorname{div}\varphi(x) - A(x)\cdot\varphi(x)\psi(x)\Im u(x)\,dx\\ &= \int_\Omega \Re u(x)\operatorname{div}(\psi\varphi)(x) - A(x)\cdot\varphi(x)\psi(x)\Im u(x)\,dx - \int_\Omega \Re u(x)\varphi(x)\cdot\nabla\psi(x)\,dx\\ &= \int_\Omega \varphi(x)\psi(x)d\mu_{1,A,u} - \int_\Omega \Re u(x)\varphi(x)\cdot\nabla\psi(x)\,dx\end{aligned}$$

and the assertion follows. A similar argument holds also for $\mu_{2,A,u\psi}$, and this concludes the proof. □

Let $\eta \in C_0^\infty(\mathbb{R}^N)$ be a radial nonnegative function with $\int_{\mathbb{R}^N}\eta(x)\,dx = 1$ and $\operatorname{supp}(\eta)\subset B_1(0)$. Given $\varepsilon > 0$ and $u \in L^1(\Omega;\mathbb{C})$, extended to zero out of Ω, we define the usual regularization

$$u_\varepsilon(x) := \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\eta\Big(\frac{x-y}{\varepsilon}\Big)u(y)\,dy = \frac{1}{\varepsilon^N}\int_{B(x,\varepsilon)}\eta\Big(\frac{x-y}{\varepsilon}\Big)u(y)\,dy. \tag{3.2}$$

Next we have the magnetic counterpart of the classic Anzellotti–Giaquinta Theorem [4].

Lemma 3.10 (Approximation with smooth functions). *Suppose that $A : \mathbb{R}^N \to \mathbb{R}^N$ is locally Lipschitz. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set and let $u \in \mathrm{BV}_A(\Omega)$. Then there exists a sequence $\{u_k\}_{k\in\mathbb{N}} \subset C^\infty(\Omega,\mathbb{C})$ such that*

$$\lim_{k\to\infty}\int_\Omega |u_k - u|_1\,dx = 0 \quad \textit{and} \quad \lim_{k\to\infty}|Du_k|_A(\Omega) = |Du|_A(\Omega).$$

Proof. We follow closely the proof of [22, Theorem 5.3]. In light of the semicontinuity property (Lemma 3.7), it is enough to prove that, for every $\varepsilon > 0$, there exists a function $v_\varepsilon \in C^\infty(\Omega)$ such that

$$\int_\Omega |u - v_\varepsilon|_1\,dx < \varepsilon \quad \text{and} \quad |Dv_\varepsilon|_A(\Omega) < |Du|_A(\Omega) + \varepsilon. \tag{3.3}$$

Let $\{\Omega_j\}_{j\in\mathbb{N}}$ be a sequence of domains defined, for $m \in \mathbb{N}$, as follows:

$$\Omega_j := \Big\{x \in \Omega : \operatorname{dist}(x,\partial\Omega) > \frac{1}{m+j}\Big\} \cap B(0, k+m), \quad j \in \mathbb{N},$$

where $B(0, k+m)$ denotes the open ball of center 0 and radius $k+m$. Since $|Du|_A$ is a Radon measure, given $\varepsilon > 0$ we can choose $m \in \mathbb{N}$ so large that

$$|Du|_A(\Omega \setminus \Omega_0) < \varepsilon. \tag{3.4}$$

We want to stress that the sequence of open domains $\{\Omega_j\}$ is built in such a way that

$$\Omega_j \subset \Omega_{j+1} \subset \Omega \quad \text{for any } j \in \mathbb{N}, \quad \text{and} \quad \bigcup_{j=0}^\infty \Omega_j = \Omega.$$

We now define another sequence of open domains $\{U_j\}_{j\in\mathbb{N}}$, by setting

$$U_0 := \Omega_0, \quad U_j := \Omega_{j+1}\setminus\overline{\Omega}_{j-1} \quad \text{for } j \ge 1.$$

By standard results, there exists a partition of unity related to the covering $\{U_j\}_{j\in\mathbb{N}}$, which means that there exists $\{f_j\}_{j\in\mathbb{N}} \in C_c^\infty(U_j)$ such that $0 \le f_j \le 1$ for every $j \ge 0$ and $\sum_{j=0}^\infty f_j = 1$ on Ω. We stress that the last property, in particular, implies that

$$\sum_{j=0}^\infty \nabla f_j = 0 \quad \text{on } \Omega. \tag{3.5}$$

Recalling the definition of the norm $|\cdot|_1$ given by (1.4), and the classical properties of the convolution, we easily get that for every $j \ge 0$ there exists $0 < \varepsilon_j < \varepsilon$ such that

$$\operatorname{supp}((f_ju)_{\varepsilon_j}) \subset U_j, \quad \int_\Omega |(f_ju)_{\varepsilon_j} - f_ju|_1\,dx < \varepsilon 2^{-(j+1)}, \quad \int_\Omega |(u\nabla f_j)_{\varepsilon_j} - u\nabla f_j|_1\,dx < \varepsilon 2^{-(j+1)}. \tag{3.6}$$

We can now define $v_\varepsilon := \sum_{j=0}^{\infty}(uf_j)_{\varepsilon_j}$. Since the sum is locally finite, we have that $v_\varepsilon \in C^\infty(\Omega, \mathbb{C})$, and that $u = \sum_{j=0}^{\infty} uf_j$ pointwise. Let us start considering the real part of the linear functional

$$C_c^\infty(\Omega) \ni \varphi \mapsto \int_\Omega v_\varepsilon(x) \operatorname{div} \varphi(x) + \mathrm{i}A(x) \cdot \varphi(x)\, v_\varepsilon(x)\, dx.$$

We have

$$\begin{aligned}&\int_\Omega \Re v_\varepsilon(x) \operatorname{div} \varphi(x) - A(x) \cdot \varphi(x) \Im v_\varepsilon(x)\, dx \\ &\qquad = \sum_{j=0}^{\infty} \int_\Omega ((\Re uf_j) * \eta_{\varepsilon_j})(x) \operatorname{div} \varphi(x)\, dx - \sum_{j=0}^{\infty} \int_\Omega A(x) \cdot \varphi(x)((\Im uf_j) * \eta_{\varepsilon_j})(x)\, dx =: \mathcal{I} - \mathcal{II}.\end{aligned}$$

Now

$$\begin{aligned}\mathcal{I} &= \sum_{j=0}^{\infty} \frac{1}{\varepsilon_j^N} \int_\Omega \int_\Omega \Re u(y) f_j(y) \eta\Big(\frac{x-y}{\varepsilon_j}\Big) \operatorname{div} \varphi(x)\, dy\, dx \\ &= \sum_{j=0}^{\infty} \int_\Omega \Re u(y) f_j(y) \operatorname{div}(\varphi * \eta_{\varepsilon_j})(y)\, dy \\ &= \sum_{j=0}^{\infty} \int_\Omega \Re u(y) \operatorname{div}(f_j(\varphi * \eta_{\varepsilon_j}))(y)\, dy - \sum_{j=0}^{\infty} \int_\Omega \Re u(y) \nabla f_j(y) \cdot (\varphi * \eta_{\varepsilon_j})(y)\, dy \\ &= \sum_{j=0}^{\infty} \int_\Omega \Re u(y) \operatorname{div}(f_j(\varphi * \eta_{\varepsilon_j}))(y)\, dy - \sum_{j=0}^{\infty} \int_\Omega [((\Re u \nabla f_j) * \eta_{\varepsilon_j})(y) - \Re u(y) \nabla f_j(y)] \cdot \varphi(y)\, dy \\ &=: \mathcal{I}' - \mathcal{I}'',\end{aligned}$$

where in the last equality we used (3.5). For $\mathcal{II}$, we have

$$\begin{aligned}\mathcal{II} &= \sum_{j=0}^{\infty} \int_\Omega A(x) \cdot \varphi(x) \Big[\frac{1}{\varepsilon_j^N} \int_\Omega \Im u(y) f_j(y) \eta\Big(\frac{x-y}{\varepsilon_j}\Big)\, dy\Big]\, dx \\ &= \sum_{j=0}^{\infty} \frac{1}{\varepsilon_j^N} \int_\Omega \int_\Omega A(y) \cdot \varphi(x) \Im u(y) f_j(y) \eta\Big(\frac{x-y}{\varepsilon_j}\Big)\, dx\, dy \\ &\qquad + \sum_{j=0}^{\infty} \frac{1}{\varepsilon_j^N} \int_\Omega \int_\Omega (A(x) - A(y)) \cdot \varphi(x) \Im u(y) f_j(y) \eta\Big(\frac{x-y}{\varepsilon_j}\Big)\, dx\, dy \\ &= \sum_{j=0}^{\infty} \int_\Omega A(y) \cdot (f_j(\varphi * \eta_{\varepsilon_j}))(y) \Im u(y)\, dy + \sum_{j=0}^{\infty} \frac{1}{\varepsilon_j^N} \int_\Omega \int_\Omega (A(x) - A(y)) \cdot \varphi(x) \Im u(y) f_j(y) \eta\Big(\frac{x-y}{\varepsilon_j}\Big)\, dx\, dy.\end{aligned}$$

Denoting $f_j(\varphi * \eta_{\varepsilon_j}) := (f_j(\varphi_1 * \eta_{\varepsilon_j}), \dots, f_j(\varphi_n * \eta_{\varepsilon_j}))$, we note that $|f_j(\varphi * \eta_{\varepsilon_j})| \le 1$ for any $j \ge 0$, whenever $\|\varphi\|_{L^\infty(\Omega)} \le 1$. We also stress that $|\mathcal{I}''| < \varepsilon$, because of (3.6). Therefore,

$$\begin{aligned}&\Big|\int_\Omega \Re v_\varepsilon(x) \operatorname{div} \varphi(x) - A(x) \cdot \varphi(x) \Im v_\varepsilon(x)\, dx\Big| \\ &\qquad \le \Big|\sum_{j=0}^{\infty} \int_\Omega \Re u(y) \operatorname{div}(f_j(\varphi * \eta_{\varepsilon_j}))(y) - A(y) \cdot (f_j(\varphi * \eta_{\varepsilon_j}))(y) \Im u(y)\, dy\Big| \\ &\qquad\qquad + \sum_{j=0}^{\infty} \Big|\frac{1}{\varepsilon_j^N} \int_\Omega \int_\Omega (A(x) - A(y)) \cdot \varphi(x) \Im u(y) f_j(y) \eta\Big(\frac{x-y}{\varepsilon_j}\Big)\, dx\, dy\Big| + \varepsilon.\end{aligned} \tag{3.7}$$

Now,

$$\Big|\sum_{j=0}^{\infty} \int_\Omega \Re u(y) \operatorname{div}(f_j(\varphi * \eta_{\varepsilon_j}))(y) - A(y) \cdot (f_j(\varphi * \eta_{\varepsilon_j}))(y) \Im u(y)\, dy\Big|$$

can be treated as in [22, Theorem 2, Section 5.2.2.]. Indeed, recalling that by construction every point $x \in \Omega$

belongs to at most three of the sets U_j, we have

$$\begin{aligned}
&\left|\sum_{j=0}^{\infty}\int_{\Omega} \Re u(y)\operatorname{div}(f_j(\varphi * \eta_{\varepsilon_j}))(y) - A(y)\cdot(f_j(\varphi * \eta_{\varepsilon_j}))(y)\Im u(y)\,dy\right| \\
&\quad= \left|\int_{\Omega} \Re u(y)\operatorname{div}(f_0(\varphi * \eta_{\varepsilon_0}))(y) - A(y)\cdot(f_0(\varphi * \eta_{\varepsilon_0}))(y)\Im u(y)\,dy\right. \\
&\qquad\quad \left.+ \sum_{j=1}^{\infty}\int_{\Omega} \Re u(y)\operatorname{div}(f_j(\varphi * \eta_{\varepsilon_j}))(y) - A(y)\cdot(f_j(\varphi * \eta_{\varepsilon_j}))(y)\Im u(y)\,dy\right| \\
&\quad\le C_{1,A,u}(\Omega) + \sum_{j=1}^{\infty} C_{1,A,u}(U_j) \le C_{1,A,u}(\Omega) + 3C_{1,A,u}(\Omega\setminus\Omega_0) \\
&\quad\le C_{1,A,u}(\Omega) + 3\varepsilon,
\end{aligned}$$

where the last inequality follows from (3.4). It remains to estimate

$$\sum_{j=0}^{\infty}\left|\frac{1}{\varepsilon_j^N}\int_{\Omega}\int_{\Omega}(A(x)-A(y))\cdot\varphi(x)\Im u(y) f_j(y)\eta\Big(\frac{x-y}{\varepsilon_j}\Big)\,dx\,dy\right| =: \sum_{j=0}^{\infty}|\Im\Im\Im_j|.$$

Recalling that A is locally Lipschitz, $\|\varphi\|_{L^\infty(\Omega)} \le 1$ and that $\operatorname{supp}(\eta)\subset B_1(0)$, we have

$$\sum_{j=0}^{\infty}|\Im\Im\Im_j| \le \operatorname{Lip}(A,\Omega)\varepsilon\int_{\mathbb{R}^N}\eta(z)\,dz\int_{\Omega}\sum_{j=0}^{\infty} f_j(y)|\Im u(y)|\,dy = \varepsilon\operatorname{Lip}(A,\Omega)\|\Im(u)\|_{L^1(\Omega)} =: C\varepsilon.$$

Going back to (3.7), taking the supremum over φ and by the arbitrariness of $\varepsilon > 0$, we get precisely (3.3) for the real part. An analogous argument provides (3.3) also for the imaginary part and this concludes the proof. □

Definition 3.11 (Extension domains). Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be a locally bounded function. Let $\Omega \subset \mathbb{R}^N$ be an open set. We say that Ω is an extension domain if its boundary $\partial\Omega$ is bounded and for any open set $W \supset \overline{\Omega}$, there exists a linear and continuous extension operator $E : \mathrm{BV}_A(\Omega) \to \mathrm{BV}_A(\mathbb{R}^N)$ such that

$$Eu = 0 \quad \text{for almost every } x \in \mathbb{R}^N \setminus W, \quad \text{and} \quad |DEu|_A(\partial\Omega) = 0$$

for every $u \in \mathrm{BV}_A(\Omega)$.

Lemma 3.12 (Lipschitz extension domains). *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary and $A : \mathbb{R}^N \to \mathbb{R}^N$ locally Lipschitz. Then Ω is an extension domain.*

Proof. Given an arbitrary open set $W \supset \overline{\Omega}$, by virtue of [3, Proposition 3.21] there exists a linear and continuous extension operator $E_0 : \mathrm{BV}(\Omega,\mathbb{R}) \to \mathrm{BV}(\mathbb{R}^N,\mathbb{R})$ such that $E_0 u = 0$ for almost every $x \in \mathbb{R}^N \setminus W$, and $|DE_0 u|(\partial\Omega) = 0$ for all $u \in \mathrm{BV}(\Omega)$. Given $u \in \mathrm{BV}_A(\Omega)$, we have from Lemma 3.5 that $u \in \mathrm{BV}(\Omega)$, which means that both $\Re u$ and $\Im u$ are elements of $\mathrm{BV}(\Omega,\mathbb{R})$. Let us define

$$Eu := E_0\Re u + \mathrm{i}E_0\Im u, \quad u \in \mathrm{BV}_A(\Omega).$$

Then $|DE_0\Re u|(\partial\Omega) = |DE_0\Im u|(\partial\Omega) = 0$ and there exists a positive constant C_W depending on W and Ω with

$$\|E_0\Re u\|_{\mathrm{BV}(\mathbb{R}^N)} \le C_W\|\Re u\|_{\mathrm{BV}(\Omega)}, \quad \|E_0\Im u\|_{\mathrm{BV}(\mathbb{R}^N)} \le C_W\|\Im u\|_{\mathrm{BV}(\Omega)}.$$

Taking into account Lemma 3.5, we have that

$$\begin{aligned}
\|Eu\|_{\mathrm{BV}_A(\mathbb{R}^N)} &= C_{1,A,Eu}(\mathbb{R}^N) + C_{2,A,Eu}(\mathbb{R}^N) + \|E_0\Re u\|_{L^1(\mathbb{R}^N)} + \|E_0\Im u\|_{L^1(\mathbb{R}^N)} \\
&\le |DE_0\Re u|(\mathbb{R}^N) + \|A\|_{L^\infty(W)}\|E_0\Im u\|_{L^1(\mathbb{R}^N)} + \|E_0\Re u\|_{L^1(\mathbb{R}^N)} + \|E_0\Im u\|_{L^1(\mathbb{R}^N)} \\
&\qquad + |DE_0\Im u|(\mathbb{R}^N) + \|A\|_{L^\infty(W)}\|E_0\Re u\|_{L^1(\mathbb{R}^N)} \\
&\le (1+\|A\|_{L^\infty(W)})(\|E_0\Re u\|_{\mathrm{BV}(\mathbb{R}^N)} + \|E_0\Im u\|_{\mathrm{BV}(\mathbb{R}^N)}) \\
&\le (1+\|A\|_{L^\infty(W)})C_W(\|\Re u\|_{\mathrm{BV}(\Omega)} + \|\Im u\|_{\mathrm{BV}(\Omega)}) \\
&= (1+\|A\|_{L^\infty(W)})C_W\|u\|_{\mathrm{BV}(\Omega)} \\
&\le (1+\|A\|_{L^\infty(W)})C_W K\|u\|_{\mathrm{BV}_A(\Omega)}.
\end{aligned}$$

Therefore, there exists $C = C(A, \Omega, W) > 0$ such that

$$\|Eu\|_{\mathrm{BV}_A(\mathbb{R}^N)} \leq C\|u\|_{\mathrm{BV}_A(\Omega)} \quad \text{for all } u \in \mathrm{BV}_A(\Omega).$$

We have to prove that $|DEu|_A(\partial\Omega) = 0$. We have

$$|DEu|_A(\partial\Omega) := \inf\{C_{1,A,Eu}(U) : \partial\Omega \subset U,\ U \text{ open}\} + \inf\{C_{2,A,Eu}(U) : \partial\Omega \subset U \text{ open}\}.$$

Then, for arbitrary U, U', U'' open with $\partial\Omega \subset U \subset U' \subset U'' \subset W$, we have

$$\begin{aligned}|DEu|_A(\partial\Omega) \leq |DEu|_A(U) &\leq |DE_0 \Re u|(U) + |DE_0 \Im u|(U) + \|A\|_{L^\infty(W)}\|Eu\|_{L^1(U)} \\ &\leq |DE_0 \Re u|(U) + |DE_0 \Im u|(U') + \|A\|_{L^\infty(W)}\|Eu\|_{L^1(U'')}.\end{aligned}$$

Taking the infimum over U and recalling that $|DE_0 \Re u|(\partial\Omega) = 0$ yields

$$|DEu|_A(\partial\Omega) \leq |DE_0 \Im u|(U') + \|A\|_{L^\infty(W)}\|Eu\|_{L^1(U'')}.$$

Taking the infimum over U' and recalling that $|DE_0 \Im u|(\partial\Omega) = 0$ yields

$$|DEu|_A(\partial\Omega) \leq \|A\|_{L^\infty(W)}\|Eu\|_{L^1(U'')}.$$

Finally, taking as U'' a sequence $\{U_j''\}_{j\in\mathbb{N}}$ of open sets such that $\partial\Omega \subset U_j'' \subset W$ and with $\mathcal{L}^N(U_j'') \to 0$ as $j \to \infty$, we conclude that $|DEu|_A(\partial\Omega) = 0$. □

Lemma 3.13 (Convolution). *Assume that $A : \mathbb{R}^N \to \mathbb{R}^N$ is locally Lipschitz. Suppose that $U \subset \mathbb{R}^N$ is an open set with $U \Subset \Omega$ and let $u \in \mathrm{BV}_A(\Omega)$. Then, for every sufficiently small $\varepsilon > 0$, there holds*

$$|Du_\varepsilon|_A(U) \leq |Du|_A(\Omega) + \varepsilon \operatorname{Lip}(A, \Omega)\|u\|_{L^1(\Omega)}.$$

Proof. Fix $\varphi \in C_c^1(U, \mathbb{R}^N)$ with $\|\varphi\|_{L^\infty(U)} \leq 1$. Choose $\delta > 0$ such that $\{x \in \mathbb{R}^N : d(x, U) < \delta\} \subset \Omega$. Then we have $\|\varphi_\varepsilon\|_{L^\infty(\Omega)} \leq 1$ and $\operatorname{supp}(\varphi_\varepsilon) \subset \{x \in \mathbb{R}^N : d(x, U) < \delta\}$ for all small $\varepsilon > 0$. Then

$$\begin{aligned}\int_U \Re u_\varepsilon(x) \operatorname{div} \varphi(x) - A(x) \cdot \varphi(x) \Im u_\varepsilon(x)\,dx &= \int_\Omega (\Re u)_\varepsilon(x) \operatorname{div} \varphi(x) - A(x) \cdot \varphi(x) (\Im u)_\varepsilon(x)\,dx \\ &= \int_\Omega \Re u(x) (\operatorname{div} \varphi)_\varepsilon(x) - (A(x) \cdot \varphi(x))_\varepsilon \Im u(x)\,dx \\ &= \int_\Omega \Re u(x) \operatorname{div} \varphi_\varepsilon(x) - A(x) \cdot \varphi_\varepsilon(x) \Im u(x)\,dx \\ &\quad - \int_\Omega \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \eta\Big(\frac{x-y}{\varepsilon}\Big)(A(y) - A(x)) \cdot \varphi(y)\,dy\, \Im u(x)\,dx \\ &\leq \int_\Omega \Re u(x) \operatorname{div} \varphi_\varepsilon(x) - A(x) \cdot \varphi_\varepsilon(x) \Im u(x)\,dx \\ &\quad + \int_\Omega \frac{1}{\varepsilon^N} \int_{B(x,\varepsilon)} \eta\Big(\frac{x-y}{\varepsilon}\Big)|A(y) - A(x)|\,dy |\Im u(x)|\,dx \\ &\leq C_{1,A,u}(\Omega) + \varepsilon \operatorname{Lip}(A, \Omega)\|u\|_{L^1(\Omega)}.\end{aligned}$$

Similarly, for every $\varphi \in C_c^1(U, \mathbb{R}^N)$ with $\|\varphi\|_{L^\infty(U)} \leq 1$, we get

$$\int_U \Im u_\varepsilon(x) \operatorname{div} \varphi(x) + A(x) \cdot \varphi(x) \Re u_\varepsilon(x)\,dx \leq C_{2,A,u}(\Omega) + \varepsilon \operatorname{Lip}(A, \Omega)\|u\|_{L^1(\Omega)}.$$

By the definition of $|Du|_A(\Omega)$ and taking the supremum over all φ, we get the assertion. □

Lemma 3.14 (Compactness for $\mathrm{BV}_A(\Omega)$-functions). *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary and that $A : \mathbb{R}^N \to \mathbb{R}^N$ is locally bounded. Let $\{u_k\}_{k\in\mathbb{N}}$ be a bounded sequence in $\mathrm{BV}_A(\Omega)$. Then, up to a subsequence, it converges strongly in $L^1(\Omega)$ to some function $u \in \mathrm{BV}_A(\Omega)$.*

Proof. By the approximation Lemma 3.10, for any $k \in \mathbb{N}$ there is $v_k \in \mathrm{BV}_A(\Omega) \cap C^\infty(\Omega)$ such that

$$\int_\Omega |u_k - v_k|_1\,dx < \frac{1}{k}, \quad \sup_{k\in\mathbb{N}} |Dv_k|_A(\Omega) = C, \tag{3.8}$$

for some $C > 0$. In particular, we have

$$\int_\Omega |v_k|_1\,dx \le \int_\Omega |u_k - v_k|_1\,dx + \int_\Omega |u_k|_1\,dx \le C' + 1, \quad C' := \sup_{k\in\mathbb{N}} \|u_k\|_{L^1(\Omega)}.$$

Now, Lemma 3.4 yields $v_k \in W^{1,1}_A(\Omega)$ and

$$\int_\Omega |\nabla v_k - \mathrm{i}Av_k|_1\,dx = |Dv_k|_A(\Omega).$$

Therefore, we obtain

$$\begin{aligned}\int_\Omega |\nabla v_k|_1\,dx &\le \int_\Omega |\nabla v_k - \mathrm{i}Av_k|_1\,dx + C_1\int_\Omega |Av_k|_1\,dx \\ &\le |Dv_k|_A(\Omega) + C_1\|A\|_{L^\infty(\overline{\Omega})}\|v_k\|_{L^1(\Omega)} \le C'',\end{aligned}$$

for some $C'' > 0$. Hence we infer that $\{v_k\}_{k\in\mathbb{N}}$ is a bounded sequence in $W^{1,1}(\Omega)$. Since $\partial\Omega$ is smooth, from the Rellich Compact Embedding Theorem there exists a subsequence $\{v_{k_j}\}_{j\in\mathbb{N}}$ of $\{v_k\}_{k\in\mathbb{N}}$ and $w \in L^1(\Omega)$ such that $v_{k_j} \to w$ in $L^1(\Omega)$. Then from (3.8) we get $u_{k_j} \to w$ in $L^1(\Omega)$. By the semi-continuity Lemma 3.7 we obtain

$$|Dw|_A(\Omega) \le \liminf_{k_j} |Dv_{k_j}|_A(\Omega) \le C,$$

which shows that $w \in \mathrm{BV}_A(\Omega)$ and concludes the proof. □

4 Proof of the main result

We now state two results that will be proven in the next section. In the following $Q_{p,N}$ is as in definition (1.1).

Theorem 4.1 (BV_A-case). *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary and $A : \mathbb{R}^N \to \mathbb{R}^N$ of class C^2. Let $u \in \mathrm{BV}_A(\Omega)$ and consider a sequence $\{\rho_m\}_{m\in\mathbb{N}}$ of nonnegative radial functions with*

$$\lim_{m\to\infty} \int_0^\infty \rho_m(r)r^{N-1}\,dr = 1 \tag{4.1}$$

and such that, for every $\delta > 0$,

$$\lim_{m\to\infty} \int_\delta^\infty \rho_m(r)r^{N-1}\,dr = 0. \tag{4.2}$$

Then we have

$$\lim_{m\to\infty} \int_\Omega\int_\Omega \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|_1}{|x-y|}\rho_m(x-y)\,dx\,dy = Q_{1,N}|Du|_A(\Omega).$$

Theorem 4.2 ($W^{1,p}_A(\Omega)$-case). *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary and assume that $A \in C^2(\mathbb{R}^N, \mathbb{R}^N)$. Let $p \ge 1$, $u \in W^{1,p}_A(\Omega)$ and $\{\rho_m\}_{m\in\mathbb{N}}$ as in Theorem 4.1. Then we have*

$$\lim_{m\to\infty} \int_\Omega\int_\Omega \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|_p^p}{|x-y|^p}\rho_m(x-y)\,dx\,dy = pQ_{p,N}\int_\Omega |\nabla u - \mathrm{i}Au|_p^p\,dx.$$

Remark 4.3. In the notation of Theorem 4.1, assuming (4.1) and (4.2) automatically implies that

$$\lim_{m\to\infty} \int_0^\delta \rho_m(r)r^{N-1+\beta}\,dr = 0 \quad \text{for every } \beta > 0 \text{ and for every } \delta > 0.$$

In fact, fixed $\delta > 0$, taking an arbitrary $0 < \tau < \delta$, we have

$$\int_0^\delta \rho_m(r) r^{N-1+\beta}\,dr = \int_0^\tau \rho_m(r) r^{N-1+\beta}\,dr + \int_\tau^\delta \rho_m(r) r^{N-1+\beta}\,dr$$
$$\le \tau^\beta \int_0^\tau \rho_m(r) r^{N-1}\,dr + \delta^\beta \int_\tau^\infty \rho_m(r) r^{N-1}\,dr \le C\tau^\beta + \delta^\beta \int_\tau^\infty \rho_m(r) r^{N-1}\,dr,$$

from which the assertion follows by letting $m \to \infty$ first, using (4.2), and finally letting $\tau \searrow 0$.

Proof of the main result (Theorem 1.1) completed. Let r_Ω denote the diameter of Ω. Then we consider a function $\psi \in C_c^\infty(\mathbb{R}^N)$, $\psi(x) = \psi_0(|x|)$ with $\psi_0(t) = 1$ for $t < r_\Omega$ and $\psi_0(t) = 0$ for $t > 2r_\Omega$. Then $\psi_0(|x-y|) = 1$ for every $x, y \in \Omega$. Let $\{s_m\}_{m\in\mathbb{N}} \subset (0,1)$ with $s_m \nearrow 1$. For a $p \ge 1$ consider the sequence of radial functions in $L^1(\mathbb{R}^N)$

$$\rho_m(|x|) := \frac{p(1-s_m)}{|x|^{N+ps_m-p}} \psi_0(|x|), \quad x \in \mathbb{R}^N,\ m \in \mathbb{N}. \tag{4.3}$$

Notice that both conditions (4.1) and (4.2) hold, since

$$\lim_{m\to\infty} \int_0^{r_\Omega} \rho_m(r) r^{N-1}\,dr = \lim_{m\to\infty} p(1-s_m) \int_0^{r_\Omega} r^{-ps_m+p-1}\,dr = \lim_{m\to\infty} r_\Omega^{p(1-s_m)} = 1$$

and

$$\lim_{m\to\infty} \int_{r_\Omega}^{2r_\Omega} \rho_m(r) r^{N-1}\,dr = \lim_{m\to\infty} p(1-s_m) \int_{r_\Omega}^{2r_\Omega} \frac{\psi_0(r)}{r^{ps_m+1-p}}\,dr \le C \lim_{m\to\infty} 1 - s_m = 0.$$

In a similar fashion, for any $\delta > 0$, there holds

$$\lim_{m\to\infty} \int_\delta^\infty \rho_m(r) r^{N-1}\,dr \le C \lim_{m\to\infty} p(1-s_m) \int_\delta^{2r_\Omega} \frac{1}{t^{ps_m+1-p}}\,dt = 0.$$

Then Theorem 1.1 follows directly from Theorems 4.1 and 4.2 using ρ_m as in (4.3). □

We first need the following:

Lemma 4.4. *Let $p \ge 1$. Then, for every $v \in \mathbb{C}^N$, it holds*

$$\lim_{m\to\infty} \int_{\mathbb{R}^N} \left| v \cdot \frac{h}{|h|} \right|_p^p \rho_m(h)\,dh = pQ_{p,N} |v|_p^p. \tag{4.4}$$

Proof. First of all we observe that, due to symmetry reasons, $Q_{p,N}$ is *independent* of the choice of the direction $\boldsymbol{\omega} \in \mathbb{S}^{N-1}$. We prove that (4.4) easily follows assuming (4.4) with $v \in \mathbb{R}^N$. Let $v = (v_1, \dots, v_N) \in \mathbb{C}^N$ and $h = (h_1, \dots, h_N) \in \mathbb{R}^N$. Then

$$\left| v \cdot \frac{h}{|h|} \right|_p^p = \left| \sum_{j=1}^N v_j \frac{h_j}{|h|} \right|_p^p = \left| \sum_{j=1}^N \Re v_j \frac{h_j}{|h|} + \mathrm{i} \sum_{j=1}^N \Im v_j \frac{h_j}{|h|} \right|_p^p$$
$$= \left| \sum_{j=1}^N \Re v_j \frac{h_j}{|h|} \right|^p + \left| \sum_{j=1}^N \Im v_j \frac{h_j}{|h|} \right|^p = \left| \Re v \cdot \frac{h}{|h|} \right|^p + \left| \Im v \cdot \frac{h}{|h|} \right|^p, \tag{4.5}$$

where we denoted by $\Re v = (\Re v_1, \dots, \Re v_N)$ and $\Im v = (\Im v_1, \dots, \Im v_N)$. Using (4.5), we get

$$\lim_{m\to\infty} \int_{\mathbb{R}^N} \left| v \cdot \frac{h}{|h|} \right|_p^p \rho_m(h)\,dh = \lim_{m\to\infty} \int_{\mathbb{R}^N} \left| \Re v \cdot \frac{h}{|h|} \right|^p \rho_m(h)\,dh + \lim_{m\to\infty} \int_{\mathbb{R}^N} \left| \Im v \cdot \frac{h}{|h|} \right|^p \rho_m(h)\,dh$$
$$= pQ_{p,N}(|\Re v|^p + |\Im v|^p) = pQ_{p,N}|v|_p^p.$$

In order to prove (4.4) with $v \in \mathbb{R}^N$, we apply the co-area formula, a change of variable and (4.1), getting

$$\begin{aligned}\lim_{m\to\infty}\int_{\mathbb{R}^N}\Big|v\cdot\frac{h}{|h|}\Big|^p\rho_m(h)\,dh &= \lim_{m\to\infty}\int_0^\infty\int_{\{|h|=R\}}\Big|v\cdot\frac{h}{|h|}\Big|^p\rho_m(h)\,d\mathcal{H}^{N-1}(h)\,dR\\ &= \lim_{m\to\infty}\int_0^\infty\rho_m(R)R^{N-1}\,dR\int_{\mathbb{S}^{N-1}}|v\cdot h|^p\,d\mathcal{H}^{N-1}(h)\\ &= |v|^p\int_{\mathbb{S}^{N-1}}\Big|\frac{v}{|v|}\cdot h\Big|^p\,d\mathcal{H}^{N-1}(h)\\ &= |v|^p\int_{\mathbb{S}^{N-1}}|\boldsymbol{\omega}\cdot h|^p\,d\mathcal{H}^{N-1}(h) = pQ_{p,N}|v|^p\end{aligned}$$

for an arbitrarily fixed $\boldsymbol{\omega} \in \mathbb{S}^{N-1}$. This concludes the proof. □

Let now $\{\rho_m\}_{m\in\mathbb{N}}$ be as in Theorem 4.1. The following is the main result for smooth functions.

Proposition 4.5 (Smooth case). *Let $\Omega \subset \mathbb{R}^N$ be a bounded set and $A \in C^2(\mathbb{R}^N, \mathbb{R}^N)$. Then*

$$\lim_{m\to\infty}\int_\Omega\int_\Omega\frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|_p^p}{|x-y|^p}\rho_m(x-y)\,dx\,dy = pQ_{p,N}\int_\Omega|\nabla u-\mathrm{i}Au|_p^p\,dx$$

for every $u \in C^2(\bar\Omega, \mathbb{C})$ and for every $p \geq 1$. In particular, if $p = 1$, then

$$\lim_{m\to\infty}\int_\Omega\int_\Omega\frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|_1}{|x-y|}\rho_m(x-y)\,dx\,dy = Q_{1,N}|Du|_A(\Omega). \tag{4.6}$$

Proof. Let $p \geq 1$. If we set $\varphi(y) := e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)$, since

$$\nabla_y\varphi(y) = e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}\Big(\nabla_y u(y) - \mathrm{i}A\Big(\frac{x+y}{2}\Big)u(y) + \frac{\mathrm{i}}{2}u(y)(x-y)\cdot\nabla_y A\Big(\frac{x+y}{2}\Big)\Big),$$

if $x, y \in \Omega$, since $u, A \in C^2(\bar\Omega)$, by Taylor's formula we get (for $y \in B(x,\rho) \subset \Omega$)

$$\frac{u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)}{|x-y|} = \frac{\varphi(x)-\varphi(y)}{|x-y|} = (\nabla u(x) - \mathrm{i}A(x)u(x))\cdot\frac{x-y}{|x-y|} + \mathcal{O}(|x-y|).$$

Then, taking into account (ii) of Lemma 5.1 below, applied with $T(x) := \nabla u(x) - \mathrm{i}A(x)u(x)$ we get

$$\Big|\frac{u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)}{|x-y|}\Big|_p^p = \Big|(\nabla u(x) - \mathrm{i}A(x)u(x))\cdot\frac{x-y}{|x-y|}\Big|_p^p + \mathcal{O}(|x-y|).$$

For $x \in \Omega$, if we set $R_x = \operatorname{dist}(x, \partial\Omega)$, then we get for some positive constant C,

$$\begin{aligned}\Psi_m(x) &:= \int_\Omega\Big|\frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|_p^p - |(\nabla u(x)-\mathrm{i}A(x)u(x))\cdot(x-y)|_p^p}{|x-y|^p}\rho_m(x-y)\Big|\,dy\\ &= \int_{B(x,R_x)}\bigg|\Big|\frac{u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)}{|x-y|}\Big|_p^p - \Big|(\nabla u(x)-\mathrm{i}A(x)u(x))\cdot\frac{x-y}{|x-y|}\Big|_p^p\bigg|\rho_m(x-y)\,dy\\ &\quad+ \int_{\Omega\setminus B(x,R_x)}\bigg|\Big|\frac{u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)}{|x-y|}\Big|_p^p - \Big|(\nabla u(x)-\mathrm{i}A(x)u(x))\cdot\frac{x-y}{|x-y|}\Big|_p^p\bigg|\rho_m(x-y)\,dy\\ &\leq C\int_{B(x,R_x)}|x-y|\rho_m(x-y)\,dy + C\int_{\Omega\setminus B(x,R_x)}\rho_m(x-y)\,dy\\ &\leq C\int_0^{R_x}\rho_m(r)r^N\,dr + C\int_{R_x}^\infty\rho_m(r)r^{N-1}\,dr,\end{aligned}$$

where to handle the second integral we used that

$$\left|\left|\frac{u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)}{|x-y|}\right|_p^p-\left|(\nabla u(x)-\mathrm{i}A(x)u(x))\cdot\frac{x-y}{|x-y|}\right|_p^p\right|\le C\quad\text{for all } x,y\in\Omega.$$

Letting $m\to\infty$ and recalling (4.2) and Remark 4.3, we get $\Psi_m(x)\to 0$ for every $x\in\Omega$. Since

$$|\Psi_m(x)|\le C\int_\Omega\rho_m(x-y)\,dy\le C\int_0^\infty\rho_m(r)r^{N-1}\,dr\le C,$$

the Dominated Convergence Theorem yields $\Psi_m\to 0$ in $L^1(\Omega)$ as $m\to\infty$. Then, to get the assertion, it is sufficient to prove that

$$\lim_{m\to\infty}\int_\Omega\int_\Omega\frac{|(\nabla u(x)-\mathrm{i}A(x)u(x))\cdot(x-y)|_p^p}{|x-y|^p}\rho_m(x-y)\,dy\,dx=pQ_{p,N}\int_\Omega|\nabla u-\mathrm{i}Au|_p^p\,dx.$$

Fixed $x\in\Omega$, by virtue of formula (4.4), we can write

$$\begin{aligned}pQ_{p,N}|\nabla u(x)-\mathrm{i}A(x)u(x)|_p^p&=\lim_{m\to\infty}\int_{\mathbb{R}^N}\left|(\nabla u(x)-\mathrm{i}A(x)u(x))\cdot\frac{h}{|h|}\right|_p^p\rho_m(h)\,dh\\&=\lim_{m\to\infty}\int_\Omega\left|(\nabla u(x)-\mathrm{i}A(x)u(x))\cdot\frac{x-y}{|x-y|}\right|_p^p\rho_m(x-y)\,dy\\&\quad+\lim_{m\to\infty}\int_{\mathbb{R}^N\setminus\Omega}\left|(\nabla u(x)-\mathrm{i}A(x)u(x))\cdot\frac{x-y}{|x-y|}\right|_p^p\rho_m(x-y)\,dy.\end{aligned}$$

To conclude the proof, it suffices to prove that

$$\lim_{m\to\infty}\int_\Omega\int_{\mathbb{R}^N\setminus\Omega}\left|(\nabla u(x)-\mathrm{i}A(x)u(x))\cdot\frac{x-y}{|x-y|}\right|_p^p\rho_m(x-y)\,dy\,dx=0.$$

For every $\lambda>0$, we denote

$$\Omega_\lambda:=\{x\in\Omega:\operatorname{dist}(x,\partial\Omega)>\lambda\},$$

and $M:=\|\nabla u-\mathrm{i}Au\|^p_{L^\infty(\Omega)}$. Then we obtain

$$\begin{aligned}&\int_\Omega\int_{\mathbb{R}^N\setminus\Omega}\left|(\nabla u(x)-\mathrm{i}A(x)u(x))\cdot\frac{x-y}{|x-y|}\right|_p^p\rho_m(x-y)\,dy\,dx\\&\quad=\int_\Omega\int_{(\mathbb{R}^N\setminus\Omega)\cap B(x,\lambda)}\left|(\nabla u(x)-\mathrm{i}A(x)u(x))\cdot\frac{x-y}{|x-y|}\right|_p^p\rho_m(x-y)\,dy\,dx\\&\qquad+\int_\Omega\int_{(\mathbb{R}^N\setminus\Omega)\cap B(x,\lambda)^c}\left|(\nabla u(x)-\mathrm{i}A(x)u(x))\cdot\frac{x-y}{|x-y|}\right|_p^p\rho_m(x-y)\,dy\,dx\\&\quad=\int_{\Omega\setminus\Omega_\lambda}\int_{(\mathbb{R}^N\setminus\Omega)\cap B(x,\lambda)}\left|(\nabla u(x)-\mathrm{i}A(x)u(x))\cdot\frac{x-y}{|x-y|}\right|_p^p\rho_m(x-y)\,dy\,dx\\&\qquad+\int_\Omega\int_{(\mathbb{R}^N\setminus\Omega)\cap B(x,\lambda)^c}\left|(\nabla u(x)-\mathrm{i}A(x)u(x))\cdot\frac{x-y}{|x-y|}\right|_p^p\rho_m(x-y)\,dy\,dx\\&\quad\le CM\int_{\Omega\setminus\Omega_\lambda}\int_{(\mathbb{R}^N\setminus\Omega)\cap B(x,\lambda)}\rho_m(x-y)\,dx\,dy+CM\int_\Omega\int_{(\mathbb{R}^N\setminus\Omega)\cap B(x,\lambda)^c}\rho_m(x-y)\,dy\,dx\\&\quad\le CM|\Omega\setminus\Omega_\lambda|\int_{\{|h|\le\lambda\}}\rho_m(h)\,dh+CM|\Omega|\int_{\{|h|>\lambda\}}\rho_m(h)\,dh,\end{aligned}$$

the assertion follows by letting $m\to\infty$, recalling formula (4.2), and finally letting $\lambda\to 0$. If $p=1$, the assertion follows recalling Lemma 3.4. □

5 Proof of Theorem 4.2

We state in the following a few elementary inequalities concerning the norm introduced in (1.4).

Lemma 5.1. *The following properties of $|\cdot|_p$ are true:*

(i) *Let $m = N$ or $m = 1$. There exists a positive constant $C = C(p, N)$ such that $|z \cdot w|_p \le C|z|_p|w|_p$ for all $z \in \mathbb{C}^m$ and $w \in \mathbb{C}^N$.*

(ii) *If $T : \mathbb{R}^N \to \mathbb{C}^N$ is a C^1-function, there exists a positive constant C such that*

$$\left| \left| T(x) \cdot \frac{x-y}{|x-y|} + \mathcal{O}(|x-y|) \right|_p^p - \left| T(x) \cdot \frac{x-y}{|x-y|} \right|_p^p \right| \le C|x-y|$$

for all $x, y \in \Omega$, where $\mathcal{O}(|x-y|)$ denotes any continuous function $R : \mathbb{R}^{2N} \to \mathbb{C}$ such that $|R(x,y)|_p |x-y|^{-1}$ is bounded in $\Omega \times \Omega$.

Proof. To prove (i), we proceed as follows: let $z \in \mathbb{C}^N$,

$$\begin{aligned}
|z \cdot w|_p^p = \left| \sum_{j=1}^N z_j w_j \right|_p^p &= \left(\left| \sum_{j=1}^N \Re z_j \Re w_j - \Im z_j \Im w_j + \mathrm{i}(\Re z_j \Im w_j + \Im z_j \Re w_j) \right|_p \right)^p \\
&= \left| \sum_{j=1}^N \Re z_j \Re w_j - \Im z_j \Im w_j \right|^p + \left| \sum_{j=1}^N \Re z_j \Im w_j + \Im z_j \Re w_j \right|^p \\
&\le C(p) \left(\left| \sum_{j=1}^N \Re z_j \Re w_j \right|^p + \left| \sum_{j=1}^N \Im z_j \Im w_j \right|^p + \left| \sum_{j=1}^N \Re z_j \Im w_j \right|^p + \left| \sum_{j=1}^N \Im z_j \Re w_j \right|^p \right) \\
&\le C(p)(|\Re z|^p |\Re w|^p + |\Im z|^p |\Im w|^p + |\Re z|^p |\Im w|^p + |\Im z|^p |\Re w|^p) \\
&= C|z|_p^p |w|_p^p.
\end{aligned}$$

The case $m = 1$, i.e. $z \in \mathbb{C}$, works in a similar way. To prove (ii), it is sufficient to combine the inequality $|b^p - a^p| \le M(a^{p-1} + b^{p-1})|b-a|$ for

$$a := \left| T(x) \cdot \frac{x-y}{|x-y|} + \mathcal{O}(|x-y|) \right|_p, \quad b := \left| T(x) \cdot \frac{x-y}{|x-y|} \right|_p,$$

with the triangular inequality

$$\left| \left| T(x) \cdot \frac{x-y}{|x-y|} + \mathcal{O}(|x-y|) \right|_p - \left| T(x) \cdot \frac{x-y}{|x-y|} \right|_p \right| \le |\mathcal{O}(|x-y|)|_p \le C|x-y|,$$

taking into account that a, b are bounded in Ω. □

We start with the following lemma.

Lemma 5.2. *Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be locally bounded. Then, for any compact $V \subset \mathbb{R}^N$ with $\Omega \Subset V$, there exists $C = C(A, V) > 0$ such that*

$$\int_{\mathbb{R}^n} |u(y+h) - e^{\mathrm{i}h \cdot A(y+\frac{h}{2})} u(y)|_p^p \, dy \le C|h|^p \|u\|_{W_A^{1,p}(\mathbb{R}^n)}^p$$

for all $u \in W_A^{1,p}(\mathbb{R}^N)$ such that $u = 0$ on V^c and any $h \in \mathbb{R}^N$ with $|h| \le 1$.

Proof. Assume first that $u \in C_0^\infty(\mathbb{R}^N)$ with $u = 0$ on V^c. Fix $y, h \in \mathbb{R}^N$ and define

$$\varphi(t) := e^{\mathrm{i}(1-t)h \cdot A(y+\frac{h}{2})} u(y+th), \quad t \in [0,1].$$

Then we have $u(y+h) - e^{\mathrm{i}h \cdot A(y+\frac{h}{2})} u(y) = \int_0^1 \varphi'(t)\, dt$, and since

$$\varphi'(t) = e^{\mathrm{i}(1-t)h \cdot A(y+\frac{h}{2})} h \cdot \left(\nabla_y u(y+th) - \mathrm{i}A\left(y + \frac{h}{2}\right) u(y+th) \right),$$

by Hölder's inequality and recalling that $|e^{\mathrm{i}(1-t)h\cdot A(y+\frac{h}{2})}|_p \le C$, we get

$$|u(y+h)-e^{\mathrm{i}h\cdot A(y+\frac{h}{2})}u(y)|_p^p \le C|h|^p \int_0^1 \Big|\nabla_y u(y+th)-\mathrm{i}A\Big(y+\frac{h}{2}\Big)u(y+th)\Big|_p^p\,dt.$$

Therefore, integrating with respect to y over $\mathbb{R}^N$ and using Fubini's Theorem, we get

$$\begin{aligned}\int_{\mathbb{R}^N} |u(y+h)-e^{\mathrm{i}h\cdot A(y+\frac{h}{2})}u(y)|_p^p\,dy &\le C|h|^p\int_0^1 dt\int_{\mathbb{R}^n}\Big|\nabla_y u(y+th)-\mathrm{i}A\Big(y+\frac{h}{2}\Big)u(y+th)\Big|_p^p\,dy\\ &= C|h|^p\int_0^1 dt\int_{\mathbb{R}^N}\Big|\nabla_z u(z)-\mathrm{i}A\Big(z+\frac{1-2t}{2}h\Big)u(z)\Big|_p^p\,dz\\ &\le C|h|^p\int_{\mathbb{R}^n}|\nabla_z u(z)-\mathrm{i}A(z)u(z)|_p^p\,dz\\ &\quad + C|h|^p\int_V\Big|A\Big(z+\frac{1-2t}{2}h\Big)-A(z)\Big|_p^p|u(z)|_p^p\,dz.\end{aligned}$$

Then, since A is bounded on the set V, we have for some constant $C>0$

$$\begin{aligned}\int_{\mathbb{R}^N} |u(y+h)-e^{\mathrm{i}h\cdot A(y+\frac{h}{2})}u(y)|_p^p\,dy &\le C|h|^p\Big(\int_{\mathbb{R}^N}|\nabla_z u(z)-\mathrm{i}A(z)u(z)|_p^p\,dz+\int_{\mathbb{R}^n}|u(z)|_p^p\,dz\Big)\\ &= C|h|^p\|u\|^p_{W^{1,p}_A(\mathbb{R}^N)}.\end{aligned}$$

When dealing with a general u, we can argue by a density argument [27, Theorem 7.22]. □

Lemma 5.3. *Let $A:\mathbb{R}^N\to\mathbb{R}^N$ be locally bounded. Let $u\in W^{1,p}_A(\Omega)$ and $\rho\in L^1(\mathbb{R}^N)$ with $\rho\ge 0$. Then*

$$\int_\Omega\int_\Omega \frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|_p^p}{|x-y|^p}\rho(x-y)\,dx\,dy \le C\|\rho\|_{L^1}\|u\|^p_{W^{1,p}_A(\Omega)},$$

where C depends only on Ω and A.

Proof. Let $V\subset\mathbb{R}^N$ be a fixed compact set with $\Omega\Subset V$. Given $u\in W^{1,p}_A(\Omega)$, there exists $\tilde u\in W^{1,p}_A(\mathbb{R}^N)$ with $\tilde u=u$ on Ω and $\tilde u=0$ on V^c (see e.g. [37, Lemma 2.2]). By Lemma 5.2, we obtain

$$\int_{\mathbb{R}^N}|\tilde u(y+h)-e^{\mathrm{i}h\cdot A(y+\frac{h}{2})}\tilde u(y)|_p^p\,dy \le C|h|^p\|\tilde u\|^p_{W^{1,p}_A(\mathbb{R}^N)} \le C|h|^p\|u\|^p_{W^{1,p}_A(\Omega)} \tag{5.1}$$

for some positive constant C depending on Ω and A. Then, in light of (5.1), we get

$$\begin{aligned}\int_\Omega\int_\Omega \frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|_p^p}{|x-y|^p}\rho(x-y)\,dx\,dy &\le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\rho(h)\frac{|\tilde u(y+h)-e^{\mathrm{i}h\cdot A(y+\frac{h}{2})}\tilde u(y)|_p^p}{|h|^p}\,dy\,dh\\ &= \int_{\mathbb{R}^N}\frac{\rho(h)}{|h|^p}\Big(\int_{\mathbb{R}^N}|\tilde u(y+h)-e^{\mathrm{i}h\cdot A(y+\frac{h}{2})}\tilde u(y)|_p^p\,dy\Big)\,dh\\ &\le C\|\rho\|_{L^1}\|u\|^p_{W^{1,p}_A(\Omega)},\end{aligned}$$

concluding the proof. □

We can now conclude the proof of Theorem 4.2. Setting

$$F^u_m(x,y) := \frac{u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)}{|x-y|}\rho_m^{\frac{1}{p}}(x-y),\quad x,y\in\Omega,\ m\in\mathbb{N},$$

by virtue of Lemma 5.3, for all $u, v \in W^{1,p}_A(\Omega)$, we have (recall that ρ_m fulfills condition (4.1))

$$\big|\|F^u_m\|_{L^p(\Omega\times\Omega)} - \|F^v_m\|_{L^p(\Omega\times\Omega)}\big| \le \|F^u_m - F^v_m\|_{L^p(\Omega\times\Omega)} \le C\|u - v\|_{W^{1,p}_A(\Omega)}$$

for some $C > 0$ depending on Ω and A. This allows to prove the assertion for functions $u \in C^2(\bar{\Omega})$ since for every $u \in W^{1,p}_A(\Omega)$ there is a sequence $\{u_j\}_{j\in\mathbb{N}} \subset C^\infty(\overline{\Omega})$ such that $\|u_j - u\|_{W^{1,p}_A(\Omega)} \to 0$. Therefore, the assertion follows by Proposition 4.5.

6 Proof of Theorem 4.1

We first state a technical lemma.

Lemma 6.1. *Let $\Omega \subset \mathbb{R}^N$ be open and bounded and $A \in C^2(\mathbb{R}^N, \mathbb{R}^N)$ and $R > 0$. For $x, y \in \Omega$ let*

$$\psi(z) := e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2}+z)}, \quad z \in B(0, R).$$

Then there exist positive constants $D_1 = D_1(A, \Omega)$ and $D_2 = D_2(A, \Omega, R)$ such that

$$|\psi(z) - \psi(0)|_1 \le D_1|z||x - y| + D_2|z|^2|x - y| \tag{6.1}$$

for every $z \in B(0, R)$. Moreover, $\limsup_{R\to 0} D_2 < \infty$.

Proof. Recalling (1.4), we can prove (6.1) separately for the real part $\Re\psi$ and the imaginary part $\Im\psi$. To simplify the notation, for fixed $x, y \in \Omega$, let us denote

$$\vartheta(z) := (x - y) \cdot A\Big(\frac{x + y}{2} + z\Big), \quad z \in B(0, R).$$

Therefore,

$$\psi(z) = \Re\psi(z) + \mathrm{i}\Im\psi(z) = \cos(\vartheta(z)) + \mathrm{i}\sin(\vartheta(z)), \quad z \in B(0, R).$$

We start considering first the real part $\Re\psi$. By Taylor's formula with Lagrange's rest, we have

$$\Re\psi(z) - \Re\psi(0) = \nabla\Re\psi(0) \cdot z + \frac{1}{2}\nabla^2\Re\psi(\bar{t}z)z \cdot z \tag{6.2}$$

for some $\bar{t} \in [0, 1]$, where $\nabla^2\Re\psi$ stands for the Hessian matrix of $\Re\psi$. A simple computation gives

$$\partial_{z_j}\Re\psi(z) = -\sin(\vartheta(z))\partial_{z_j}\vartheta(z) = -\sin(\vartheta(z))\sum_{k=1}^{N}(x_k - y_k)\partial_{z_j}A^{(k)}\Big(\frac{x + y}{2} + z\Big)$$

for every $j = 1, \dots, N$. Therefore, we have

$$\nabla\Re\psi(0) = -\sin\Big((x - y) \cdot A\Big(\frac{x + y}{2}\Big)\Big)(x - y) \cdot \nabla A\Big(\frac{x + y}{2}\Big), \tag{6.3}$$

where ∇A denotes the Jacobian matrix of A. Another quite simple computation yields

$$\begin{aligned}(\nabla^2\Re\psi(z))_{h,j} = -\Big[&\cos(\vartheta(z))\Big((x - y) \cdot \partial_{z_h}A\Big(\frac{x + y}{2} + z\Big)\Big)\Big((x - y) \cdot \partial_{z_j}A\Big(\frac{x + y}{2} + z\Big)\Big)\\ &+ \sin(\vartheta(z))(x - y) \cdot \partial_{z_h}\partial_{z_j}A\Big(\frac{x + y}{2} + z\Big)\Big]\end{aligned} \tag{6.4}$$

for every $i, j = 1, \dots, N$.

Now, using (6.2) and (6.3), we get

$$|\Re\psi(z) - \Re\psi(0)| \le \Big|\nabla A\Big(\frac{x + y}{2}\Big)\Big||z||x - y| + \frac{1}{2}|z|^2|\nabla^2\Re\psi(\bar{t}z)| \quad \text{for some } \bar{t} \in [0, 1].$$

On the other hand, by (6.4) we get

$$|\nabla^2\Re\psi(\bar{t}z)| \le |x - y|\Big(C|x - y|\Big|\nabla A\Big(\frac{x + y}{2} + \bar{t}z\Big)\Big|^2 + \sum_{k=1}^{N}\Big|\nabla^2 A^{(k)}\Big(\frac{x + y}{2} + \bar{t}z\Big)\Big|\Big).$$

Therefore, (6.1) for $\mathbb{R}\psi$ follows taking

$$D_1 := \sup_{x,y\in\Omega} \left|\nabla A\left(\frac{x+y}{2}\right)\right| < \infty$$

and

$$D_2 := \frac{1}{2} \sup_{\substack{x,y\in\Omega \\ z\in B(0,R)}} \sum_{k=1}^{N} \left|\nabla^2 A^{(k)}\left(\frac{x+y}{2} + \bar{t}z\right)\right| + C|x-y|\left|\nabla A\left(\frac{x+y}{2} + \bar{t}z\right)\right|^2 < \infty.$$

The fact that $\limsup_{R\to 0} D_2 < \infty$ follows observing that D_2 decreases as R decreases. Since a similar argument holds for $\mathbb{I}\psi$, we get the assertion. □

Lemma 6.2. *Let $\Omega \subset \mathbb{R}^N$ be an open set and $A \in C^2(\mathbb{R}^N, \mathbb{R}^N)$. Let $u \in L^1(\Omega)$. Denote by u_ε its regularization as defined in* (3.2). *Define*

$$\Omega_r := \{x \in \Omega : d(x, \partial\Omega) > r\} \quad \text{for all } r > 0.$$

Then, for all $r > 0$ and $\varepsilon \in (0, r)$, there holds

$$\begin{aligned}
&\int_{\Omega_r}\int_{\Omega_r} \frac{|u_\varepsilon(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u_\varepsilon(y)|_1}{|x-y|} \rho_m(x-y)\,dx\,dy \\
&\quad\le \int_\Omega\int_\Omega \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y)|_1}{|x-y|} \rho_m(x-y)\,dx\,dy \\
&\qquad + \frac{1}{\varepsilon^N} \int_{B(0,\varepsilon)} \eta\left(\frac{z}{\varepsilon}\right) \int_\Omega\int_\Omega \frac{|e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2}+z)} u(y) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y)|_1}{|x-y|} \rho_m(x-y)\,dx\,dy\,dz
\end{aligned}$$

and

$$\lim_{\varepsilon\to 0}\lim_{m\to\infty} \frac{1}{\varepsilon^N} \int_{B(0,\varepsilon)} \eta\left(\frac{z}{\varepsilon}\right) \int_\Omega\int_\Omega \frac{|e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2}+z)} u(y) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y)|_1}{|x-y|} \rho_m(x-y)\,dx\,dy\,dz = 0.$$

Proof. Let us extend u to the whole of $\mathbb{R}^N$ by zero. To simplify the notation, let us still denote by u its extension. By definition,

$$\begin{aligned}
u_\varepsilon(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u_\varepsilon(y) &= \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \eta\left(\frac{z}{\varepsilon}\right)(u(x-z) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y-z))\,dz \\
&= \frac{1}{\varepsilon^N} \int_{B(0,\varepsilon)} \eta\left(\frac{z}{\varepsilon}\right)(u(x-z) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y-z))\,dz.
\end{aligned}$$

Thus, for every $\varepsilon \in (0, r)$, there holds

$$\begin{aligned}
&\int_{\Omega_r}\int_{\Omega_r} \frac{|u_\varepsilon(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u_\varepsilon(y)|_1}{|x-y|} \rho_m(x-y)\,dx\,dy \\
&\quad\le \frac{1}{\varepsilon^N} \int_{\Omega_r}\int_{\Omega_r}\int_{B(0,\varepsilon)} \eta\left(\frac{z}{\varepsilon}\right) \frac{|u(x-z) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y-z)|_1}{|x-y|} \rho_m(x-y)\,dz\,dx\,dy \\
&\quad\le \frac{1}{\varepsilon^N} \int_{B(0,\varepsilon)} \eta\left(\frac{z}{\varepsilon}\right) \int_\Omega\int_\Omega \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2}+z)} u(y)|_1}{|x-y|} \rho_m(x-y)\,dx\,dy\,dz \le \mathcal{I} + \mathcal{II},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{I} &:= \frac{1}{\varepsilon^N} \int_{B(0,\varepsilon)} \eta\left(\frac{z}{\varepsilon}\right) \int_\Omega\int_\Omega \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y)|_1}{|x-y|} \rho_m(x-y)\,dx\,dy\,dz \\
&= \int_\Omega\int_\Omega \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y)|_1}{|x-y|} \rho_m(x-y)\,dx\,dy
\end{aligned}$$

and

$$\mathcal{JJ} := \frac{1}{\varepsilon^N} \int\limits_{B(0,\varepsilon)} \eta\Big(\frac{z}{\varepsilon}\Big) \int\limits_{\Omega}\int\limits_{\Omega} \frac{|e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2}+z)}u(y) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|_1}{|x-y|} \rho_m(x-y)\,dx\,dy\,dz.$$

Define $\psi(z) := e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2}+z)}$. Then $|\psi(z)|_1 \le 2$ for all $z \in B(0,\varepsilon)$ and by Lemma 6.1,

$$|\psi(z) - \psi(0)|_1 \le D_1|z||x-y| + D_2|z|^2|x-y| \quad \text{for all } x, y \in \Omega,\ z \in B(0,\varepsilon),$$

for some $D_1 = D_1(A,\Omega)$ and $D_2 = D_2(A,\Omega,\varepsilon)$ which is bounded as $\varepsilon \searrow 0$. Therefore,

$$\mathcal{JJ} \le \frac{D_1}{\varepsilon^N} \int\limits_{B(0,\varepsilon)} \eta\Big(\frac{z}{\varepsilon}\Big) \int\limits_{\Omega}\int\limits_{\Omega} |u(y)|_1|z|\rho_m(x-y)\,dx\,dy\,dz + \frac{D_2}{\varepsilon^N} \int\limits_{B(0,\varepsilon)} \eta\Big(\frac{z}{\varepsilon}\Big) \int\limits_{\Omega}\int\limits_{\Omega} |u(y)|_1|z|^2\rho_m(x-y)\,dx\,dy\,dz.$$

We have

$$\begin{aligned} &\frac{D_2}{\varepsilon^N} \int\limits_{B(0,\varepsilon)} \eta\Big(\frac{z}{\varepsilon}\Big) \int\limits_{\Omega}\int\limits_{\Omega} |u(y)|_1|z|^2\rho_m(x-y)\,dx\,dy\,dz \\ &\qquad \le \frac{D_2}{\varepsilon^N} \int\limits_{B(0,\varepsilon)} \eta\Big(\frac{z}{\varepsilon}\Big)|z|^2\,dz \int\limits_{\Omega} |u(y)|_1 \Big(\int\limits_{\Omega} \rho_m(x-y)\,dx \Big)\,dy \le 2D_2|\mathbb{S}^{N-1}|\|u\|_{L^1(\Omega)}\varepsilon^2, \end{aligned}$$

since $\int_\Omega \rho_m(x-y)\,dx \le |\mathbb{S}^{N-1}| \int_0^\infty \rho_m(r)r^{N-1}\,dr \le 2|\mathbb{S}^{N-1}|$, in view of (4.1). Analogously, we have

$$\frac{D_1}{\varepsilon^N} \int\limits_{B(0,\varepsilon)} \eta\Big(\frac{z}{\varepsilon}\Big) \int\limits_{\Omega}\int\limits_{\Omega} |u(y)|_1|z|\rho_m(x-y)\,dx\,dy\,dz \le 2D_1|\mathbb{S}^{N-1}|\|u\|_{L^1(\Omega)}\varepsilon.$$

Hence, we conclude that

$$\lim_{\varepsilon\to 0}\lim_{m\to\infty} \mathcal{JJ} = 0,$$

and the assertion follows. □

Lemma 6.3. *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set. Denote by $xy_t := tx + (1-t)y$ with $t \in [0,1]$ the linear combination of $x, y \in \Omega$. There exists a positive constant $C = C(N,\Omega,A)$ such that*

$$\begin{aligned} &\int\limits_{\Omega}\int\limits_{\Omega}\int\limits_{0}^{1} \Big| \big(e^{\mathrm{i}(1-t)(x-y)\cdot A(\frac{x+y}{2})} - 1\big) \frac{x-y}{|x-y|} \cdot \Big(\nabla_y u(xy_t) - \mathrm{i}A\Big(\frac{x+y}{2}\Big)u(xy_t)\Big)\Big|_1 \rho_m(x-y)\,dt\,dx\,dy \\ &\qquad \le C\|u\|_{\mathrm{BV}_A(W)} \Big(\int\limits_0^1 r^N \rho_m(r)\,dr + \int\limits_1^\infty r^{N-1}\rho_m(r)\,dr \Big) \end{aligned}$$

for every open set $W \Supset \Omega$ and for every $u \in C^2(\mathbb{R}^N, \mathbb{C})$ such that $u = 0$ on W^c.

Proof. It is readily seen that there exists a positive constant $C = C(A,\Omega)$ such that

$$|e^{\mathrm{i}(1-t)(x-y)\cdot A(\frac{x+y}{2})} - 1|_1 \le C|x-y| \quad \text{for all } x, y \in \Omega \text{ and all } t \in [0,1]. \tag{6.5}$$

Then, by (i) of Lemma 5.1 with $p = 1$ and by (6.5), we have

$$\begin{aligned} &\int\limits_{\Omega}\int\limits_{\Omega}\int\limits_{0}^{1} \Big| \big(e^{\mathrm{i}(1-t)(x-y)\cdot A(\frac{x+y}{2})} - 1\big) \frac{x-y}{|x-y|} \cdot \Big(\nabla_y u(xy_t) - \mathrm{i}A\Big(\frac{x+y}{2}\Big)u(xy_t)\Big)\Big|_1 \rho_m(x-y)\,dt\,dx\,dy \\ &\qquad \le C \int\limits_{\Omega}\int\limits_{\Omega}\int\limits_{0}^{1} |e^{\mathrm{i}(1-t)(x-y)\cdot A(\frac{x+y}{2})} - 1|_1 \Big|\nabla_y u(xy_t) - \mathrm{i}A\Big(\frac{x+y}{2}\Big)u(xy_t)\Big|_1 \rho_m(x-y)\,dt\,dx\,dy \\ &\qquad \le C \int\limits_{\Omega}\int\limits_{\Omega}\int\limits_{0}^{1} |x-y|\rho_m(x-y) \Big|\nabla_y u(xy_t) - \mathrm{i}A\Big(\frac{x+y}{2}\Big)u(xy_t)\Big|_1\,dt\,dx\,dy \le \mathcal{J} + \mathcal{JJ}, \end{aligned}$$

where we have set

$$\mathcal{I} := C \int_\Omega \int_\Omega \int_0^1 |x-y| \rho_m(x-y) \Big| \nabla_y u(xy_t) - \mathrm{i} A(xy_t) u(xy_t) \Big|_1 \, dt \, dx \, dy,$$

$$\mathcal{II} := C \int_\Omega \int_\Omega \int_0^1 |x-y| \rho_m(x-y) \Big| A\Big(\frac{x+y}{2}\Big) - A(xy_t) \Big| |u(xy_t)|_1 \, dt \, dx \, dy$$

for some positive constant $C = C(A, \Omega)$. Then we get

$$\begin{aligned}
\mathcal{I} &\leq C \int_\Omega \Bigg(\int_{B(y,1) \cap \Omega} |x-y| \rho_m(x-y) \Bigg(\int_0^1 |\nabla_y u(xy_t) - \mathrm{i} A(xy_t) u(xy_t)|_1 \, dt \Bigg) dx \Bigg) dy \\
&\quad + C \int_\Omega \Bigg(\int_{B(y,1)^c \cap \Omega} \rho_m(x-y) \Bigg(\int_0^1 |\nabla_y u(xy_t) - \mathrm{i} A(xy_t) u(xy_t)|_1 \, dt \Bigg) dx \Bigg) dy \\
&\leq C \int_{\mathbb{R}^N} \Bigg(\int_{B(0,1)} |z| \rho_m(z) \Bigg(\int_0^1 |\nabla_y u(y+tz) - \mathrm{i} A(y+tz) u(y+tz)|_1 \, dt \Bigg) dz \Bigg) dy \\
&\quad + C \int_{\mathbb{R}^N} \Bigg(\int_{B(0,1)^c} \rho_m(z) \Bigg(\int_0^1 |\nabla_y u(y+tz) - \mathrm{i} A(y+tz) u(y+tz)|_1 \, dt \Bigg) dz \Bigg) dy \\
&\leq C \int_{B(0,1)} |z| \rho_m(z) \Bigg(\int_{\mathbb{R}^N} \int_0^1 |\nabla_y u(y+tz) - \mathrm{i} A(y+tz) u(y+tz)|_1 \, dt \, dy \Bigg) dz \\
&\quad + C \int_{B(0,1)^c} \rho_m(z) \Bigg(\int_{\mathbb{R}^N} \int_0^1 |\nabla_y u(y+tz) - \mathrm{i} A(y+tz) u(y+tz)|_1 \, dt \, dy \Bigg) dz \\
&\leq C \Bigg(\int_W |\nabla_y u(z) - \mathrm{i} A(z) u(z)|_1 \, dz \Bigg) \Bigg(\int_0^1 r^N \rho_m(r) \, dr + \int_1^\infty r^{N-1} \rho_m(r) \, dr \Bigg),
\end{aligned}$$

where in the last inequality we used

$$\int_{\mathbb{R}^N} \int_0^1 |\nabla_y u(y+tz) - \mathrm{i} A(y+tz) u(y+tz)|_1 \, dt \, dy = \int_{\mathbb{R}^N} |\nabla_y u(z) - \mathrm{i} A(z) u(z)|_1 \, dz$$

as well as

$$\int_W |\nabla_y u(z) - \mathrm{i} A(z) u(z)|_1 \, dz = \int_{\mathbb{R}^N} |\nabla_y u(z) - \mathrm{i} A(z) u(z)|_1 \, dz.$$

On the other hand, denoting by $\mathrm{Conv}(\Omega)$ the convex hull of Ω, and arguing in a similar fashion, one obtains

$$\begin{aligned}
\mathcal{II} &\leq C \|A\|_{L^\infty(\mathrm{Conv}(\Omega))} \Bigg(\int_W |u(z)|_1 \, dz \Bigg) \Bigg(\int_0^1 r^N \rho_m(r) \, dr + \int_1^\infty r^{N-1} \rho_m(r) \, dr \Bigg) \\
&\leq C \Bigg(\int_W |u(z)|_1 \, dz \Bigg) \Bigg(\int_0^1 r^N \rho_m(r) \, dr + \int_1^\infty r^{N-1} \rho_m(r) \, dr \Bigg)
\end{aligned}$$

for some positive constant $C = C(A, \Omega)$. The desired assertion finally follows by combining the above inequalities and then using Lemma 3.4. □

The following lemma is an adaptation to our case of [20, Lemma 3] and [34, Lemma 5.2].

Lemma 6.4. *Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be locally Lipschitz and let $\Omega \subset \mathbb{R}^N$ be an open and bounded set. Then there exists a positive constant $C = C(\Omega, A)$ such that for all $r, m > 0$, $W \Supset \Omega$ (i.e. Ω is compactly contained in W) and $u \in BV_A(\Omega)$, denoting by $\overline{u} \in BV_A(\mathbb{R}^N)$ an extension of u to $\mathbb{R}^N$ such that $\overline{u} = 0$ in W^c, the following inequality holds:*

$$\begin{aligned}\int_\Omega\int_\Omega & \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|_1}{|x-y|}\rho_m(x-y)\,dx\,dy \\ &\le Q_{1,N}|D\overline{u}|_A(E_r')\int_0^r \rho_m(s)s^{N-1}\,ds + \frac{C\,\mathrm{Lip}(A, E_r')\|\overline{u}\|_{L^1(W)}}{2}\int_0^r s^N\rho_m(s)\,ds \\ &\quad + C\|\overline{u}\|_{BV_A(W')}\bigg(\int_0^1 s^N\rho_m(s)\,ds + \int_1^\infty s^{N-1}\rho_m(s)\,ds\bigg) + \frac{C\|\overline{u}\|_{L^1(W)}}{r}\int_r^\infty s^{N-1}\rho_m(s)\,ds,\end{aligned}$$

where $E_r := \Omega + B(0, r)$, W' (resp. E_r') is any bounded open set with $W' \Supset W$ (resp. $E_r' \Supset E_r$).

Proof. For any $\varepsilon \in (0, r)$, let $\overline{u}_\varepsilon$ be as in formula (3.2) for $\overline{u} : \mathbb{R}^N \to \mathbb{C}$. By a change of variables, Fubini's Theorem and Lemma 5.1, we have

$$\begin{aligned}\int_\Omega\int_\Omega & \frac{|\overline{u}_\varepsilon(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}\overline{u}_\varepsilon(y)|_1}{|x-y|}\rho_m(x-y)\,dx\,dy \\ &\le \int_\Omega\bigg(\int_{\Omega\cap B(y,r)} \frac{|\overline{u}_\varepsilon(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}\overline{u}_\varepsilon(y)|_1}{|x-y|}\rho_m(x-y)\,dx\bigg)\,dy + \frac{C\|\overline{u}\|_{L^1(W)}}{r}\int_{B(0,r)^c}\rho_m(h)\,dh,\end{aligned}$$

where $C = C(N) > 0$. Let us now define

$$\psi(t) := e^{\mathrm{i}(1-t)(x-y)\cdot A(\frac{x+y}{2})}\overline{u}_\varepsilon(tx + (1-t)y), \quad t \in [0, 1].$$

Then

$$\overline{u}_\varepsilon(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}\overline{u}_\varepsilon(y) = \psi(1) - \psi(0) = \int_0^1 \psi'(t)\,dt,$$

and since

$$\psi'(t) = e^{\mathrm{i}(1-t)(x-y)\cdot A(\frac{x+y}{2})}(x-y)\cdot\Big(\nabla_y\overline{u}_\varepsilon(tx + (1-t)y) - \mathrm{i}A\Big(\frac{x+y}{2}\Big)\overline{u}_\varepsilon(tx + (1-t)y)\Big),$$

we have

$$\int_\Omega\bigg(\int_{\Omega\cap B(y,r)} \frac{|\overline{u}_\varepsilon(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}\overline{u}_\varepsilon(y)|_1}{|x-y|}\rho_m(x-y)\,dx\bigg)\,dy \le \mathfrak{I} + \mathfrak{I}\mathfrak{I},$$

where we have set

$$\mathfrak{I} := \int_\Omega\bigg(\int_{\Omega\cap B(y,r)}\int_0^1\Big|\frac{x-y}{|x-y|}\cdot\Big(\nabla_y\overline{u}_\varepsilon(xy_t) - \mathrm{i}A\Big(\frac{x+y}{2}\Big)\overline{u}_\varepsilon(xy_t)\Big)\Big|_1\rho_m(x-y)dtdx\bigg)\,dy,$$

$$\begin{aligned}\mathfrak{I}\mathfrak{I} := \bigg|&\int_\Omega\int_{\Omega\cap B(y,r)}\int_0^1\Big|e^{\mathrm{i}(1-t)(x-y)\cdot A(\frac{x+y}{2})}\frac{x-y}{|x-y|}\cdot\Big(\nabla_y\overline{u}_\varepsilon(xy_t) - \mathrm{i}A\Big(\frac{x+y}{2}\Big)\overline{u}_\varepsilon(xy_t)\Big)\Big|_1\rho_m(x-y)\,dt\,dx\,dy \\ &- \int_\Omega\int_{\Omega\cap B(y,r)}\int_0^1\Big|\frac{x-y}{|x-y|}\cdot\Big(\nabla_y\overline{u}_\varepsilon(xy_t) - \mathrm{i}A\Big(\frac{x+y}{2}\Big)\overline{u}_\varepsilon(xy_t)\Big)\Big|_1\rho_m(x-y)\,dt\,dx\,dy\bigg|.\end{aligned}$$

Let

$$W_\varepsilon := \{x \in \mathbb{R}^N : d(x, W) < \varepsilon\};$$

we have $\overline{u}_\varepsilon = 0$ on W_ε^c and by Lemmas 6.3 and 3.13

$$\begin{aligned}\mathcal{I}\mathcal{I} &\leq C\|\overline{u}_\varepsilon\|_{\mathrm{BV}_A(W_\varepsilon)}\bigg(\int_0^1 r^N\rho_m(r)\,dr + \int_1^\infty r^{N-1}\rho_m(r)\,dr\bigg)\\ &\leq C(\|\overline{u}\|_{\mathrm{BV}_A(W')} + \varepsilon\mathrm{Lip}(A, W')\|\overline{u}\|_{L^1(W')})\bigg(\int_0^1 r^N\rho_m(r)\,dr + \int_1^\infty r^{N-1}\rho_m(r)\,dr\bigg)\end{aligned} \tag{6.6}$$

for an arbitrary open set $W' \Supset W$ and for some positive constant $C = C(N, \Omega, A)$. On the other hand, we have

$$\begin{aligned}\mathcal{I} &\leq \int_{B(0,r)}\int_0^1\int_\Omega \bigg|\bigg(\nabla_y\overline{u}_\varepsilon(y+th) - \mathrm{i}A\Big(y+\frac{h}{2}\Big)\overline{u}_\varepsilon(y+th)\bigg)\cdot\frac{h}{|h|}\bigg|_1\rho_m(h)\,dy\,dt\,dh\\ &\leq \int_{B(0,r)}\int_0^1\int_\Omega \bigg|(\nabla_y\overline{u}_\varepsilon(y+th) - \mathrm{i}A(y+th)\overline{u}_\varepsilon(y+th))\cdot\frac{h}{|h|}\bigg|_1\rho_m(h)\,dy\,dt\,dh\\ &\qquad + C\int_{B(0,r)}\int_0^1\int_\Omega \bigg|\bigg(\mathrm{i}A\Big(y+\frac{h}{2}\Big)\overline{u}_\varepsilon(y+th) - \mathrm{i}A(y+th)\overline{u}_\varepsilon(y+th)\bigg)\cdot\frac{h}{|h|}\bigg|_1\rho_m(h)\,dy\,dt\,dh\\ &\leq \int_{B(0,r)}\int_{E_r}\bigg|(\nabla_y\overline{u}_\varepsilon(z) - \mathrm{i}A(z)\overline{u}_\varepsilon(z))\cdot\frac{h}{|h|}\bigg|_1\rho_m(h)\,dz\,dh\\ &\qquad + C\int_{B(0,r)}\int_0^1\int_\Omega \bigg|\bigg(A\Big(y+\frac{h}{2}\Big) - A(y+th)\bigg)\cdot\frac{h}{|h|}\bigg|_1|\overline{u}_\varepsilon(y+th)|_1\rho_m(h)\,dy\,dt\,dh\\ &\leq \int_0^r\int_{E_r}\bigg(\int_{S^{N-1}}|(\nabla_y\overline{u}_\varepsilon(z) - \mathrm{i}A(z)\overline{u}_\varepsilon(z))\cdot w|_1\,d\mathcal{H}^{N-1}(w)\bigg)s^{N-1}\rho_m(s)\,dz\,ds\\ &\qquad + C\int_{B(0,r)}\int_0^1\int_\Omega \bigg|\bigg(A\Big(y+\frac{h}{2}\Big) - A(y+th)\bigg)\cdot\frac{h}{|h|}\bigg|_1|\overline{u}_\varepsilon(y+th)|_1\rho_m(h)\,dy\,dt\,dh.\end{aligned}$$

Taking into account that (see the final lines of the proof of Lemma 4.4)

$$\int_{S^{N-1}}|\xi\cdot w|_1\,d\mathcal{H}^{N-1}(w) = Q_{1,N}|\xi|_1 \quad \text{for any } \xi\in\mathbb{C}^N,$$

we obtain

$$\begin{aligned}\mathcal{I} &\leq Q_{1,N}\int_0^r\int_{E_r}|\nabla_y\overline{u}_\varepsilon(z) - \mathrm{i}A(z)\overline{u}_\varepsilon(z)|_1 s^{N-1}\rho_m(s)\,ds\,dz\\ &\qquad + C\int_{B(0,r)}\int_0^1\int_\Omega \bigg|\bigg(A\Big(y+\frac{h}{2}\Big) - A(y+th)\bigg)\cdot\frac{h}{|h|}\bigg|_1|\overline{u}_\varepsilon(y+th)|_1\rho_m(h)\,dy\,dt\,dh.\end{aligned}$$

Whence, taking into account Lemma 3.4 and Lemma 3.13, we finally get

$$\begin{aligned}\mathcal{I} &\leq Q_{1,N}\bigg(\int_{E_r}|\nabla_y\overline{u}_\varepsilon(z) - \mathrm{i}A(z)\overline{u}_\varepsilon(z)|_1\,dz\bigg)\int_0^r\rho_m(s)s^{N-1}\,ds\\ &\qquad + C\int_{B(0,r)}\int_0^1\int_\Omega \bigg|\bigg(A\Big(y+\frac{h}{2}\Big) - A(y+th)\bigg)\cdot\frac{h}{|h|}\bigg|_1|\overline{u}_\varepsilon(y+th)|_1\rho_m(h)\,dy\,dt\,dh\\ &\leq Q_{1,N}\bigg(|D\overline{u}|_A(E_r')\int_{B(0,r)}\rho_m(h)\,dh + \varepsilon\,\mathrm{Lip}(A, E_r')\|\overline{u}\|_{L^1(E_r')}\bigg) + \frac{C\,\mathrm{Lip}(A, E_r')\|\overline{u}\|_{L^1(W)}}{2}\int_{B(0,r)}|h|\rho_m(h)\,dh,\end{aligned} \tag{6.7}$$

where in the last inequality we used Lemma 3.13. Putting together (6.7) and (6.6), we get

$$\begin{aligned}
&\int_\Omega\int_\Omega \frac{|\overline{u}_\varepsilon(x) - e^{\mathrm{i}(x-y)A(\frac{x+y}{2})}\overline{u}_\varepsilon(y)|_1}{|x-y|}\rho_m(x-y)\,dx\,dy \\
&\quad\le Q_{1,N}\bigg(|D\overline{u}|_A(E_r')\int_0^r \rho_m(s)s^{N-1}\,ds + \varepsilon\,\mathrm{Lip}(A, E_r')\|\overline{u}\|_{L^1(E_r')}\bigg) + \frac{C\,\mathrm{Lip}(A, E_r')\|\overline{u}\|_{L^1(W)}}{2}\int_0^r s^N\rho_m(s)\,ds \\
&\qquad + C(\|\overline{u}\|_{\mathrm{BV}_A(W')} + \varepsilon\,\mathrm{Lip}(A, W')\|\overline{u}\|_{L^1(W')})\bigg(\int_0^1 s^N\rho_m(s)\,ds + \int_1^\infty s^{N-1}\rho_m(s)\,ds\bigg) \\
&\qquad + \frac{C\|\overline{u}\|_{L^1(W)}}{r}\int_r^\infty s^{N-1}\rho_m(s)\,ds.
\end{aligned}$$

The conclusion follows letting $\varepsilon \to 0^+$. □

Proof of Theorem 4.1 concluded. Fix $r > 0$, $W \Supset \Omega$ and let $\overline{u} = Eu \in \mathrm{BV}_A(\mathbb{R}^N)$ be an extension of u such that $\overline{u} = 0$ in W^c and $|D\overline{u}|_A(\partial\Omega) = 0$, according to Lemma 3.12. Using Lemma 6.2 and Lemma 6.4, for every $0 < \varepsilon < r$ we have

$$\begin{aligned}
&\int_{\Omega_r\cap B(0,\frac{1}{r})}\int_{\Omega_r\cap B(0,\frac{1}{r})} \frac{|u_\varepsilon(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u_\varepsilon(y)|_1}{|x-y|}\rho_m(x-y)\,dx\,dy \\
&\quad\le \int_\Omega\int_\Omega \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|_1}{|x-y|}\rho_m(x-y)\,dx\,dy \\
&\qquad + \frac{1}{\varepsilon^N}\int_{B(0,\varepsilon)}\eta\Big(\frac{z}{\varepsilon}\Big)\int_\Omega\int_\Omega \frac{|e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2}+z)}u(y) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|_1}{|x-y|}\rho_m(x-y)\,dx\,dy\,dz \\
&\quad\le Q_{1,N}|D\overline{u}|_A(E_r')\int_0^r \rho_m(s)s^{N-1}\,ds + \frac{C\,\mathrm{Lip}(A, E_r')\|\overline{u}\|_{L^1(W)}}{2}\int_0^r s^N\rho_m(s)\,ds \\
&\qquad + C\|\overline{u}\|_{\mathrm{BV}_A(W')}\bigg(\int_0^1 s^N\rho_m(s)\,ds + \int_1^\infty s^{N-1}\rho_m(s)\,ds\bigg) + \frac{C\|\overline{u}\|_{L^1(W)}}{r}\int_r^\infty s^{N-1}\rho_m(s)\,ds \\
&\qquad + \frac{1}{\varepsilon^N}\int_{B(0,\varepsilon)}\eta\Big(\frac{z}{\varepsilon}\Big)\int_\Omega\int_\Omega \frac{|e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2}+z)}u(y) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|_1}{|x-y|}\rho_m(x-y)\,dx\,dy\,dz.
\end{aligned}$$

Letting $m \to \infty$, using (4.6), (4.1) and (4.2) we get

$$\begin{aligned}
&Q_{1,N}|Du_\varepsilon|_A\Big(\Omega_r \cap B\Big(0, \frac{1}{r}\Big)\Big) \\
&\quad\le \lim_{m\to\infty}\int_\Omega\int_\Omega \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|_1}{|x-y|}\rho_m(x-y)\,dx\,dy \\
&\qquad + \lim_{m\to\infty}\frac{1}{\varepsilon^N}\int_{B(0,\varepsilon)}\eta\Big(\frac{z}{\varepsilon}\Big)\int_\Omega\int_\Omega \frac{|e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2}+z)}u(y) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|_1}{|x-y|}\rho_m(x-y)\,dx\,dy\,dz \\
&\quad\le Q_{1,N}|D\overline{u}|_A(E_r') + \lim_{m\to\infty}\frac{1}{\varepsilon^N}\int_{B(0,\varepsilon)}\eta\Big(\frac{z}{\varepsilon}\Big)\int_\Omega\int_\Omega \frac{|e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2}+z)}u(y) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|_1}{|x-y|}\rho_m(x-y)\,dx\,dy\,dz.
\end{aligned}$$

Letting $\varepsilon \to 0^+$, using the lower semi-continuity of the total variation and Lemma 6.2, we have

$$\begin{aligned}
Q_{1,N}|Du|_A\Big(\Omega_r \cap B\Big(0, \frac{1}{r}\Big)\Big) &\le \lim_{m\to\infty}\int_\Omega\int_\Omega \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|_1}{|x-y|}\rho_m(x-y)\,dx\,dy \\
&\le Q_{1,N}|D\overline{u}|_A(E_r'),
\end{aligned}$$

the assertion follows letting $r \searrow 0$ and observing that

$$\lim_{r\to 0^+} |Du|_A\Big(\Omega_r \cap B\Big(0, \frac{1}{r}\Big)\Big) = \lim_{r\to 0^+} |D\overline{u}|_A(E_r') = |Du|_A(\Omega).$$

Indeed, since $|Du|_A(\cdot)$ is a Radon measure, by inner regularity

$$\lim_{r\to 0^+} |Du|_A\Big(\Omega_r \cap B\Big(0, \frac{1}{r}\Big)\Big) = |Du|_A(\Omega),$$

and by outer regularity

$$\lim_{r\to 0^+} |D\overline{u}|_A(E_r') = |Du|_A(\overline{\Omega}) = |Du|_A(\Omega).$$

This concludes the proof. □

Funding: The authors are members of *Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni* (GNAMPA) of the *Istituto Nazionale di Alta Matematica* (INdAM). Eugenio Vecchi receives funding from the People Programme (Marie Curie Actions) of the European Union's Seventh Framework Programme FP7/2007-2013/ under REA grant agreement No. 607643 (Grant MaNET 'Metric Analysis for Emergent Technologies').

References

[1] L. Ambrosio, Metric space valued functions of bounded variation, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (4)* **17** (1990), no. 3, 439–478.
[2] L. Ambrosio, G. De Philippis and L. Martinazzi, Gamma-convergence of nonlocal perimeter functionals, *Manuscripta Math.* **134** (2011), no. 3–4, 377–403.
[3] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Math. Monogr., Clarendon Press, New York, 2000.
[4] G. Anzellotti and M. Giaquinta, BV functions and traces, *Rend. Semin. Mat. Univ. Padova* **60** (1978), 1–21.
[5] G. Arioli and A. Szulkin, A semilinear Schrödinger equation in the presence of a magnetic field, *Arch. Ration. Mech. Anal.* **170** (2003), no. 4, 277–295.
[6] J. Avron, I. Herbst and B. Simon, Schrödinger operators with magnetic fields. I. General interactions, *Duke Math. J.* **45** (1978), no. 4, 847–883.
[7] D. Barbieri, Approximations of Sobolev norms in Carnot groups, *Commun. Contemp. Math.* **13** (2011), no. 5, 765–794.
[8] J. Bourgain, H. Brezis and P. Mironescu, Another look at Sobolev spaces, in: *Optimal Control and Partial Differential Equations*, IOS, Amsterdam (2001), 439–455.
[9] J. Bourgain, H. Brezis and P. Mironescu, Limiting embedding theorems for $W^{s,p}$ when $s \uparrow 1$ and applications, *J. Anal. Math.* **87** (2002), 77–101.
[10] L. Brasco, E. Parini and M. Squassina, Stability of variational eigenvalues for the fractional p-Laplacian, *Discrete Contin. Dyn. Syst.* **36** (2016), no. 4, 1813–1845.
[11] H. Brezis, How to recognize constant functions. A connection with Sobolev spaces, *Uspekhi Mat. Nauk* **57** (2002), no. 4(346), 59–74.
[12] H. Brezis, New approximations of the total variation and filters in imaging, *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **26** (2015), no. 2, 223–240.
[13] H. Brezis and H.-M. Nguyen, Non-local functionals related to the total variation and connections with image processing, preprint (2016), https://arxiv.org/abs/1608.08204.
[14] H. Brezis and H.-M. Nguyen, The BBM formula revisited, *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **27** (2016), no. 4, 515–533.
[15] H. Brezis and H.-M. Nguyen, Two subtle convex nonlocal approximations of the BV-norm, *Nonlinear Anal.* **137** (2016), 222–245.
[16] L. Caffarelli, J.-M. Roquejoffre and O. Savin, Nonlocal minimal surfaces, *Comm. Pure Appl. Math.* **63** (2010), no. 9, 1111–1144.
[17] L. Caffarelli and E. Valdinoci, Regularity properties of nonlocal minimal surfaces via limiting arguments, *Adv. Math.* **248** (2013), 843–871.
[18] X. Cui, N. Lam and G. Lu, New characterizations of Sobolev spaces on the Heisenberg group, *J. Funct. Anal.* **267** (2014), no. 8, 2962–2994.

[19] P. d'Avenia and M. Squassina, Ground states for fractional magnetic operators, *ESAIM Control Optim. Calc. Var.* (2016), DOI 10.1051/cocv/2016071.
[20] J. Dávila, On an open question about functions of bounded variation, *Calc. Var. Partial Differential Equations* **15** (2002), no. 4, 519–527.
[21] S. Dipierro, A. Figalli, G. Palatucci and E. Valdinoci, Asymptotics of the s-perimeter as $s \searrow 0$, *Discrete Contin. Dyn. Syst.* **33** (2013), no. 7, 2777–2790.
[22] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, rev ed., Textb. Math., CRC Press, Boca Raton, 2015.
[23] F. Ferrari and A. Pinamonti, Characterization by asymptotic mean formulas of q-harmonic functions in Carnot groups, *Potential Anal.* **42** (2015), no. 1, 203–227.
[24] T. Ichinose, Magnetic relativistic Schrödinger operators and imaginary-time path integrals, in: *Mathematical Physics, Spectral Theory and Stochastic Analysis*, Oper. Theory Adv. Appl. 232, Birkhäuser, Basel (2013), 247–297.
[25] G. Leoni and D. Spector, Characterization of Sobolev and BV spaces, *J. Funct. Anal.* **261** (2011), no. 10, 2926–2958.
[26] G. Leoni and D. Spector, Corrigendum to "Characterization of Sobolev and BV spaces", *J. Funct. Anal.* **266** (2014), 1106–1114.
[27] E. H. Lieb and M. Loss, *Analysis*, 2nd ed., Grad. Stud. Math. 14, American Mathematical Society, Providence, 2001.
[28] M. Ludwig, Anisotropic fractional perimeters, *J. Differential Geom.* **96** (2014), no. 1, 77–93.
[29] M. Ludwig, Anisotropic fractional Sobolev norms, *Adv. Math.* **252** (2014), 150–157.
[30] V. Maz'ya and T. Shaposhnikova, On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces, *J. Funct. Anal.* **195** (2002), no. 2, 230–238.
[31] V. Maz'ya and T. Shaposhnikova, Erratum to: "On the Bourgain, Brezis and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces" [J. Funct. Anal. 195 (2002), no. 2, 230–238], *J. Funct. Anal.* **201** (2003), no. 1, 298–300.
[32] H.-M. Nguyen, Some new characterizations of Sobolev spaces, *J. Funct. Anal.* **237** (2006), no. 2, 689–720.
[33] A. Pinamonti, M. Squassina and E. Vecchi, The Maz'ya–Shaposhnikova limit in the magnetic setting, *J. Math. Anal. Appl.* **449** (2017), 1152–1159.
[34] A. C. Ponce, A new approach to Sobolev spaces and connections to Γ-convergence, *Calc. Var. Partial Differential Equations* **19** (2004), no. 3, 229–255.
[35] A. C. Ponce and D. Spector, On formulae decoupling the total variation of BV functions, *Nonlinear Anal.* **154** (2017), 241–257.
[36] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. I*, 2nd ed., Academic Press, New York, 1980.
[37] M. Squassina and B. Volzone, Bourgain–Brézis–Mironescu formula for magnetic operators, *C. R. Math. Acad. Sci. Paris* **354** (2016), no. 8, 825–831.
[38] W. P. Ziemer, *Weakly Differentiable Functions*, Grad. Texts in Math. 120, Springer, New York, 1989.