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The Maz'ya–Shaposhnikova limit in the magnetic setting ☆

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ABSTRACT

We prove a magnetic version of the Maz'ya–Shaposhnikova singular limit of nonlocal norms with vanishing fractional parameter. This complements a general convergence result recently obtained by authors when the parameter approaches one.

1. Introduction

About fifteen years ago, V. Maz'ya and T. Shaposhnikova proved that for any $n \geq 1$ and $p \in [1, \infty)$,

$$\lim_{s \searrow 0} s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+ps}} dxdy = \frac{4\pi^{n/2}}{p\Gamma(n/2)} \|u\|^p_{L^p(\mathbb{R}^n)},$$

whenever $u \in D^{s,p}_0(\mathbb{R}^n)$ for some $s \in (0, 1)$. Here Γ denotes the Gamma function and the space $D^{s,p}_0(\mathbb{R}^n)$ is the completion of $C^\infty_c(\mathbb{R}^n)$ with respect to the Gagliardo norm

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+ps}} dxdy.$$

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Their motivation was basically that of complementing a previous result by Bourgain–Brezis–Mironescu [4,5] providing new characterizations for functions in the Sobolev space $W^{1,p}(\Omega)$. Precisely, if $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, then for any $W^{1,p}(\Omega)$ there holds

$$\lim_{s\nearrow 1}(1-s)\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}dxdy = Q_{p,n}\int_\Omega |\nabla u|^p dx,$$

where $Q_{p,n}$ is defined by

$$Q_{p,n} = \frac{1}{p}\int_{\mathbb{S}^{n-1}} |\boldsymbol{\omega}\cdot h|^p d\mathcal{H}^{n-1}(h), \tag{1.1}$$

$\mathbb{S}^{n-1}$ being the unit sphere in $\mathbb{R}^n$ and $\boldsymbol{\omega}$ an arbitrary unit vector of $\mathbb{R}^n$. The above singular limits are natural and also admit a physical relevance in the framework of the theory of Levy processes. Also, there is a developed theory of fractional s-perimeters [6] and there have been several contributions concerning their asymptotic analysis in the limits $s \nearrow 1$ and $s \searrow 0$ [1,7,9].

One of the latest generalizations of this kind of convergence results appeared recently in [16] in the context of *magnetic Sobolev spaces* $W^{1,2}_A(\Omega)$, see [12]. In fact, a relevant role in the study of particles which interact with a magnetic field $B = \nabla\times A$, $A:\mathbb{R}^n\to\mathbb{R}^n$, is assumed by the *magnetic Laplacian* $(\nabla - \mathrm{i}A)^2$ [3,12,15], yielding to nonlinear Schrödinger equations of the type $-(\nabla-\mathrm{i}A)^2u+u=f(u)$, which have been extensively studied (see [2] and the references therein). The operator is defined weakly as the differential of the energy

$$W^{1,2}_A(\Omega)\ni u\mapsto \int_\Omega |\nabla u - \mathrm{i}A(x)u|^2dx.$$

If $A:\mathbb{R}^n\to\mathbb{R}^n$ is a smooth field and $s\in(0,1)$, a nonlocal magnetic counterpart of the magnetic laplacian,

$$(-\Delta)^s_A u(x) = c(n,s)\lim_{\varepsilon\searrow 0}\int_{B^c_\varepsilon(x)} \frac{u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)}{|x-y|^{n+2s}}dy,$$

where $c(n,s)$ is a normalization constant which behaves as follows

$$\lim_{s\searrow 0}\frac{c(n,s)}{s} = \frac{\Gamma(n/2)}{\pi^{n/2}},\qquad \lim_{s\nearrow 1}\frac{c(n,s)}{1-s} = \frac{2n\Gamma(n/2)}{\pi^{n/2}}, \tag{1.2}$$

was introduced in [8,11] for complex-valued functions, with motivations falling into the framework of the general theory of Lévy processes. Recently, the authors in [14] (see [16] for $p=2$) proved that if $A:\mathbb{R}^n\to\mathbb{R}^n$ is a C^2 vector field, then, for any $n\geq 1$, $p\in[1,\infty)$ and any Lipschitz bounded domain $\Omega\subset\mathbb{R}^n$

$$\lim_{s\nearrow 1}(1-s)\int_\Omega\int_\Omega \frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|^p_p}{|x-y|^{n+ps}}dxdy = Q_{p,n}\int_\Omega |\nabla u - \mathrm{i}A(x)u|^p_p\,dx, \tag{1.3}$$

for all $u\in W^{1,p}_A(\Omega)$, where $Q_{p,n}$ is as in (1.1) and $|z|_p := (|(\Re z_1,\dots,\Re z_n)|^p + |(\Im z_1,\dots,\Im z_n)|^p)^{1/p}$. This has provided a new nonlocal characterization of the magnetic Sobolev spaces $W^{1,p}_A(\Omega)$.

The main goal of this paper is to complete the picture of [14] by providing a *magnetic counterpart* of the convergence result by *Maz'ya–Shaposhnikova* for vanishing fractional orders s, namely for $s\searrow 0$.

We consider a locally bounded vector potential field $A : \mathbb{R}^n \to \mathbb{R}^n$ and the space of complex valued functions $D^{s,p}_{A,0}(\mathbb{R}^n, \mathbb{C})$ defined as the completion of $C^\infty_c(\mathbb{R}^n, \mathbb{C})$ with respect to the norm

$$\|u\|_{D^{s,p}_{A,0}} = \left(\int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+ps}} dxdy \right)^{1/p}.$$

By combining Lemma 2.1 and Lemma 2.3, we shall prove the following result.

Theorem 1.1 *(Magnetic Maz'ya–Shaposhnikova). Let $n \geq 1$ and $p \in [1, \infty)$. Then for every*

$$u \in \bigcup_{0<s<1} D^{s,p}_{A,0}(\mathbb{R}^n, \mathbb{C}),$$

there holds

$$\lim_{s \searrow 0} s \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+ps}} dxdy = \frac{4\pi^{n/2}}{p\Gamma(n/2)} \|u\|^p_{L^p(\mathbb{R}^n)}.$$

In particular, while the singular limit as $s \nearrow 1$ generates the magnetic gradient $\nabla - \mathrm{i}A$, the limit for vanishing s tends to destroy the magnetic effects yielding the $L^p(\mathbb{R}^n)$-norm of the function u. We point out that, while in (1.3) the norm of complex numbers is $|\cdot|_p$, in Theorem 1.1 we use the usual norm $|\cdot| = |\cdot|_2$. In any case when $A = 0$ and u is real-valued the formulas are all consistent with the classical statements. In the case $p = 2$, combining the asymptotic formulas in (1.2) with Theorem 1.1 implies that

$$\frac{c(n,s)}{2} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y)|^2}{|x-y|^{n+2s}} dxdy \approx \|u\|^2_{L^2(\mathbb{R}^n)}, \qquad \text{as } s \searrow 0,$$

for any $u \in D^{s,2}_0(\mathbb{R}^n)$ for some $s \in (0,1)$.

As it is pointed out in [11], in place of the magnetic norm defined via the simple midpoint prescription $(x, y) \mapsto A((x + y)/2)$, other prescriptions are viable in applications such as the averaged one

$$(x, y) \mapsto \int\limits_0^1 A\left((1 - \vartheta)x + \vartheta y\right) d\vartheta =: A_\sharp(x, y).$$

If $(-\Delta)^s_A$ and $(-\Delta)^s_{A_\sharp}$ are the fractional operators associated with $A((x+y)/2)$ and $A_\sharp(x, y)$ respectively, it follows that $(-\Delta)^s_{A_\sharp}$ is Gauge covariant, which is relevant for Schrödinger operators, i.e. for all $\phi \in \mathscr{S}(\mathbb{R}^n)$

$$(-\Delta)^s_{(A+\nabla\phi)_\sharp} = e^{\mathrm{i}\phi} (-\Delta)^s_{A_\sharp} e^{-\mathrm{i}\phi},$$

see e.g. [11, Proposition 2.8]. We point out that Theorem 1.1 remains valid for the operator $A_\sharp$ and its proof carries on by trivial modifications of our arguments.

2. Proof of the main result

The proof of Theorem 1.1 follows by combining Lemma 2.1 and Lemma 2.3 below.

Lemma 2.1 *(Liminf inequality). Let $n \geq 1$, $p \in [1, \infty)$ and let*

$$u \in \bigcup_{0<s<1} D^{s,p}_{A,0}(\mathbb{R}^n, \mathbb{C}).$$

Then

$$\liminf_{s \searrow 0} s \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|^p}{|x-y|^{n+ps}} dxdy \geq \frac{4\pi^{n/2}}{p\Gamma(n/2)} \|u\|^p_{L^p(\mathbb{R}^n)}.$$

Proof. If

$$\liminf_{s \searrow 0} s \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|^p}{|x-y|^{n+ps}} dxdy = \infty,$$

the assertion follows. Otherwise, there exists a sequence $\{s_k\}_{k\in\mathbb{N}} \subset (0, 1)$ with $s_k \searrow 0$ and

$$\liminf_{s \searrow 0} s \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|^p}{|x-y|^{n+ps}} dxdy = \lim_{k\to\infty} s_k \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|^p}{|x-y|^{n+ps_k}} dxdy,$$

the limit being finite. For a.e. $x, y \in \mathbb{R}^n$, by a direct consequence of the triangle inequality, we have the Diamagnetic inequality (cf. [8, Remark 3.2])

$$||u(x)| - |u(y)|| \leq |u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|. \tag{2.1}$$

In particular, since $u \in D^{s_k,p}_{A,0}(\mathbb{R}^n, \mathbb{C})$, we have $|u| \in D^{s_k,p}_0(\mathbb{R}^n)$ and, for any $k \geq 1$,

$$s_k \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{||u(x)| - |u(y)||^p}{|x-y|^{n+ps_k}} dxdy \leq s_k \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|^p}{|x-y|^{n+ps_k}} dxdy.$$

Taking the limit as $k \to \infty$ on both sides and invoking [13, Theorem 3] applied to $|u|$, yields

$$\frac{4\pi^{n/2}}{p\Gamma(\frac{n}{2})} \||u|\|^p_{L^p(\mathbb{R}^n)} \leq \lim_{k\to\infty} s_k \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|^p}{|x-y|^{n+ps_k}} dxdy,$$

which concludes the proof. □

Remark 2.2 *(Magnetic Hardy inequality).* By combining the pointwise Diamagnetic inequality (2.1) with the fractional Hardy inequality [10], for $n > ps$ the following *magnetic Hardy inequality* holds: there exists a positive constant $\mathcal{H}_{n,s,p}$ such that

$$\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^{sp}} dx \leq \mathcal{H}_{n,s,p} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|^p}{|x-y|^{n+ps}} dxdy, \tag{2.2}$$

for every $u \in D^{s,p}_{A,0}(\mathbb{R}^n, \mathbb{C})$. Similarly the following *magnetic Sobolev inequality* holds: there exists a positive constant $\mathcal{S}_{n,s,p}$ such that

$$\left(\int_{\mathbb{R}^n} |u(x)|^{\frac{np}{n-sp}} dx\right)^{\frac{n-sp}{n}} \leq \mathcal{S}_{n,s,p} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|^p}{|x-y|^{n+ps}} dxdy,$$

for every $u \in D^{s,p}_{A,0}(\mathbb{R}^n, \mathbb{C})$.

Next we state a second lemma completing the proof of Theorem 1.1 when combined with Lemma 2.1.

Lemma 2.3 *(Limsup inequality). Let $n \geq 1$, $p \in [1, \infty)$ and let*

$$u \in \bigcup_{0<s<1} D^{s,p}_{A,0}(\mathbb{R}^n, \mathbb{C}).$$

Then

$$\limsup_{s \searrow 0} s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+ps}} dxdy \leq \frac{4\pi^{n/2}}{p\Gamma(n/2)} \|u\|^p_{L^p(\mathbb{R}^n)}.$$

Proof. If $u \notin L^p(\mathbb{R}^n)$, there is nothing to prove. Hence, we may assume that $u \in L^p(\mathbb{R}^n)$. We observe that

$$\begin{aligned}
& s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+ps}} dxdy \\
&= s \int_{\mathbb{R}^n} \int_{\{|x|\leq|y|\leq 2|x|\}} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+sp}} dxdy \\
&+ s \int_{\mathbb{R}^n} \int_{\{|y|\geq 2|x|\}} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+sp}} dxdy \\
&+ s \int_{\mathbb{R}^n} \int_{\{|x|\geq|y|\}} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+sp}} dxdy \\
&= 2s \int_{\mathbb{R}^n} \int_{\{|x|\leq|y|\leq 2|x|\}} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+sp}} dxdy \\
&+ 2s \int_{\mathbb{R}^n} \int_{\{|y|\geq 2|x|\}} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+sp}} dxdy,
\end{aligned}$$

where the last equality follows noticing that since $|e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}| = 1$ then

$$\begin{aligned}
\int_{\mathbb{R}^n} \int_{\{|x|\geq|y|\}} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+sp}} dxdy &= \int_{\mathbb{R}^n} \int_{\{|y|\geq|x|\}} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+sp}} dxdy \\
&= \int_{\mathbb{R}^n} \int_{\{|y|\geq 2|x|\}} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+sp}} dxdy \\
&+ \int_{\mathbb{R}^n} \int_{\{|x|\leq|y|\leq 2|x|\}} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+sp}} dxdy.
\end{aligned}$$

Using the triangle inequality for the L^p-norm on $\mathbb{R}^{2n}$ and recalling that $|e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}| = 1$, yields

$$s \int_{\mathbb{R}^n} \int_{\{|y|\geq 2|x|\}} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+sp}} dxdy$$

$$\leq \left\{ \left(s \int_{\mathbb{R}^n} \int_{\{|y|\geq 2|x|\}} \frac{|u(x)|^p}{|x-y|^{n+sp}} dxdy \right)^{1/p} + \left(s \int_{\mathbb{R}^n} \int_{\{|y|\geq 2|x|\}} \frac{|u(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{1/p} \right\}^p .$$

We claim that

$$\lim_{s\searrow 0} s \int_{\mathbb{R}^n} \int_{\{|y|\geq 2|x|\}} \frac{|u(y)|^p}{|x-y|^{n+sp}} dxdy = 0.$$

Observe that $2|x-y| \geq |y| + (|y| - 2|x|)$. Then, if $|y| \geq 2|x|$ we get $2|x-y| \geq |y|$. Now, if $\mathscr{H}^{n-1}$ denotes the $(n-1)$-dimensional Haudorff measure, it follows that

$$\begin{aligned} s^{1/p} \left(\int_{\mathbb{R}^n} \int_{\{|y|\geq 2|x|\}} \frac{|u(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{1/p} &\leq s^{1/p} \left(2^{n+sp} \int_{\mathbb{R}^n} \frac{|u(y)|^p}{|y|^{n+sp}} \Big(\int_{\{|x|\leq |y|/2\}} dx \Big) dy \right)^{1/p} \\ &= 2^s \left(\frac{s}{n} \mathscr{H}^{n-1}(\mathbb{S}^{n-1}) \right)^{1/p} \left(\int_{\mathbb{R}^n} \frac{|u(y)|^p}{|y|^{sp}} dy \right)^{1/p}, \end{aligned}$$

and the last term goes to zero as $s \searrow 0$. Notice that $y \mapsto |y|^{-s}u(y)$ remains bounded in $L^p(\mathbb{R}^n)$ as $s \searrow 0$ by the argument indicated here below. Observe now that, if $|y| \geq 2|x|$ we then get $|x-y| \geq |x|$ yielding

$$\begin{aligned} &\left(s \int_{\mathbb{R}^n} \int_{\{|y|\geq 2|x|\}} \frac{|u(x)|^p}{|x-y|^{n+sp}} dxdy \right)^{1/p} \leq \left(s \int_{\mathbb{R}^n} \int_{\{|x-y|\geq |x|\}} \frac{|u(x)|^p}{|x-y|^{n+sp}} dxdy \right)^{1/p} \\ &= \left(s \int_{\mathbb{R}^n} |u(x)|^p \int_{B(0,|x|)^c} \frac{dz}{|z|^{n+sp}} dx \right)^{1/p} = \frac{\mathscr{H}^{n-1}(\mathbb{S}^{n-1})^{1/p}}{p^{1/p}} \left(\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^{sp}} dx \right)^{1/p} . \end{aligned}$$

Moreover $|x|^{-sp}|u(x)|^p = f_s(x) + g_s(x)$, where

$$f_s(x) := \frac{|u(x)|^p}{|x|^{sp}} \mathbf{1}_{B(0,1)}(x), \qquad g_s(x) := \frac{|u(x)|^p}{|x|^{sp}} \mathbf{1}_{B(0,1)^c}(x) \leq |u(x)|^p \mathbf{1}_{B(0,1)^c}(x) \in L^1(\mathbb{R}^n),$$

and $s \mapsto f_s$ is decreasing and, moreover, by the Hardy inequality (2.2) and the assumption on u, it follows that $f_{\tilde{s}} \in L^1(\mathbb{R}^n)$ for some $\tilde{s} \in (0,1)$. Hence, by monotone and dominated convergence, we conclude that

$$\limsup_{s\searrow 0} s \int_{\mathbb{R}^n} \int_{\{|y|\geq 2|x|\}} \frac{|u(x)|^p}{|x-y|^{n+sp}} dxdy \leq \frac{\mathscr{H}^{n-1}(\mathbb{S}^{n-1})}{p} \|u\|^p_{L^p(\mathbb{R}^n)} = \frac{2\pi^{n/2}}{p\Gamma(\frac{n}{2})} \|u\|^p_{L^p(\mathbb{R}^n)}.$$

Then, we conclude from the above inequalities that

$$\limsup_{s\searrow 0} 2s \int_{\mathbb{R}^n} \int_{\{|y|\geq 2|x|\}} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+sp}} dxdy \leq \frac{4\pi^{n/2}}{p\Gamma(\frac{n}{2})} \|u\|^p_{L^p(\mathbb{R}^n)}. \tag{2.3}$$

We claim that

$$\limsup_{s\searrow 0} 2s \int_{\mathbb{R}^n} \int_{\{|x|\le|y|\le 2|x|\}} \frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|^p}{|x-y|^{n+sp}}\,dxdy = 0. \tag{2.4}$$

By assumption let $\tau\in(0,1)$ such that $u\in D^{\tau,p}_{A,0}(\mathbb{R}^n)$. Now let $N\ge 1$ and $s<\tau$. Then

$$\begin{aligned} &2s \int_{\mathbb{R}^n} \int_{\{|x|\le|y|\le 2|x|\}} \frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|^p}{|x-y|^{n+sp}}\,dxdy \\ &= 2s \int_{\mathbb{R}^n} \int_{\substack{\{|x-y|\le N\}\\ \{|x|\le|y|\le 2|x|\}}} \frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|^p}{|x-y|^{n+sp}}\,dxdy \\ &+ 2s \int_{\mathbb{R}^n} \int_{\substack{\{|x-y|> N\}\\ \{|x|\le|y|\le 2|x|\}}} \frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|^p}{|x-y|^{n+sp}}\,dxdy =: \mathcal{I}+\mathcal{II}. \end{aligned}$$

Let us consider $\mathcal{I}$ first. Since $|x-y|\le N$, it holds that

$$\frac{1}{|x-y|^{n+sp}} = \frac{|x-y|^{p(\tau-s)}}{|x-y|^{n+\tau p}} \le \frac{N^{p(\tau-s)}}{|x-y|^{n+\tau p}}.$$

Therefore $\mathcal{I}$ goes to zero as $s\searrow 0$, since

$$\mathcal{I} \le 2sN^{p(\tau-s)} \int_{\mathbb{R}^n} \int_{\substack{\{|x-y|\le N\}\\ \{|x|\le|y|\le 2|x|\}}} \frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|^p}{|x-y|^{n+\tau p}}\,dxdy.$$

Let us now move to $\mathcal{II}$. Since $|u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|^p \le 2^{p-1}\left(|u(x)|^p+|u(y)|^p\right)$, we get

$$\mathcal{II} \le 2^p s \int_{\mathbb{R}^n} \int_{\substack{\{|x-y|\ge N\}\\ \{|x|\le|y|\le 2|x|\}}} \frac{|u(x)|^p}{|x-y|^{n+sp}} dxdy + 2^p s \int_{\mathbb{R}^n} \int_{\substack{\{|x-y|\ge N\}\\ \{|x|\le|y|\le 2|x|\}}} \frac{|u(y)|^p}{|x-y|^{n+sp}} dxdy =: \mathcal{II}'+\mathcal{II}''.$$

Regarding $\mathcal{II}'$, since $|x-y|\ge N$ and $|y|\le 2|x|$, it holds

$$N \le |x-y| \le |x|+|y| \le 3|x|,$$

which implies that $|x|\ge \frac{N}{3}$. In particular, this also implies that

$$\mathcal{II}' \le 2^p s \int_{\{|x|\ge N/3\}} \left(\int_{\{|x-y|\ge N\}} \frac{|u(x)|^p}{|x-y|^{n+sp}} dy \right) dx \le C(n,p) \int_{\{|x|\ge N/3\}} |u(x)|^p\,dx.$$

For $\mathcal{II}''$, since as before $|x-y|\ge N$ and $|x|\le|y|$, we have

$$N \le |x-y| \le |x|+|y| \le 2|y|,$$

which implies $|y| \ge \dfrac{N}{2} \ge \dfrac{N}{3}$. Therefore, we get

$$\mathcal{II}'' \leq 2^p s \int\limits_{\{|y|\geq N/3\}} |u(y)|^p \left(\int\limits_{\{|z|\geq N\}} \frac{1}{|z|^{n+sp}} dz \right) dy \leq C(n,p) \int\limits_{\{|y|\geq N/3\}} |u(y)|^p \, dy.$$

Combining the estimates for $\mathcal{II}'$ and $\mathcal{II}''$, we get

$$\mathcal{II} \leq C(n,p) \int\limits_{\{|x|\geq N/3\}} |u(x)|^p dx,$$

which is a bound independent of s. Now, going back to

$$\limsup_{s\searrow 0} 2s \int\limits_{\mathbb{R}^n} \int\limits_{\{|x|<|y|<2|x|\}} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y)|^p}{|x-y|^{n+sp}} \, dxdy \leq 2C(n,p) \|u\|^p_{L^p(B(0,N/3)^c)},$$

and (2.4) follows letting $N \to \infty$, since $u \in L^p(\mathbb{R}^n)$. Collecting (2.3) and (2.4), the assertion follows. □

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References

[1] L. Ambrosio, G. De Philippis, L. Martinazzi, Γ-convergence of nonlocal perimeter functionals, Manuscripta Math. 134 (2011) 377–403.
[2] G. Arioli, A. Szulkin, A semilinear Schrödinger equation in the presence of a magnetic field, Arch. Ration. Mech. Anal. 170 (2003) 277–295.
[3] J. Avron, I. Herbst, B. Simon, Schrödinger operators with magnetic fields. I. General interactions, Duke Math. J. 45 (1978) 847–883.
[4] J. Bourgain, H. Brezis, P. Mironescu, Another look at Sobolev spaces, in: J.L. Menaldi, E. Rofman, A. Sulem (Eds.), Optimal Control and Partial Differential Equations. A Volume in Honor of Professor Alain Bensoussan's 60th Birthday, IOS Press, Amsterdam, 2001, pp. 439–455.
[5] J. Bourgain, H. Brezis, P. Mironescu, Limiting embedding theorems for $W^{s,p}$ when $s \uparrow 1$ and applications, J. Anal. Math. 87 (2002) 77–101.
[6] L. Caffarelli, J.-M. Roquejoffre, O. Savin, Nonlocal minimal surfaces, Comm. Pure Appl. Math. 63 (2010) 1111–1144.
[7] L. Caffarelli, E. Valdinoci, Regularity properties of nonlocal minimal surfaces via limiting arguments, Adv. Math. 248 (2013) 843–871.
[8] P. d'Avenia, M. Squassina, Ground states for fractional magnetic operators, ESAIM Control Optim. Calc. Var. (2017), http://dx.doi.org/10.1051/cocv/2016071, in press.
[9] S. Dipierro, A. Figalli, G. Palatucci, E. Valdinoci, Asymptotics of the s-perimeter as $s \to 0$, Discrete Contin. Dyn. Syst. 33 (2013) 2777–2790.
[10] R.L. Frank, R. Seiringer, Non-linear ground state representation and sharp Hardy inequalities, J. Funct. Anal. 255 (2008) 3407–3430.
[11] T. Ichinose, Magnetic relativistic Schrödinger operators and imaginary-time path integrals, in: Mathematical Physics, Spectral Theory and Stochastic Analysis, in: Oper. Theory Adv. Appl., vol. 232, Birkhäuser/Springer, Basel, 2013, pp. 247–297.
[12] E. Lieb, M. Loss, Analysis, Grad. Stud. Math., vol. 14, 2001.
[13] V. Maz'ya, T. Shaposhnikova, On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces, J. Funct. Anal. 195 (2002) 230–238.
[14] A. Pinamonti, M. Squassina, E. Vecchi, Magnetic BV functions and the Bourgain–Brezis–Mironescu formula, preprint, https://arxiv.org/abs/1609.09714.
[15] M. Reed, B. Simon, Methods of Modern Mathematical Physics, I. Functional Analysis, Academic Press, Inc., New York, 1980.
[16] M. Squassina, B. Volzone, Bourgain–Brezis–Mironescu formula for magnetic operators, C. R. Math. Acad. Sci. Paris 354 (2016) 825–831.