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Computing the first eigenpair for problems with variable exponents

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Abstract. In this paper, we compute the first eigenpair for variable exponent eigenvalue problems. We compare the homogeneous definition of first eigenvalue with previous nonhomogeneous notions in the literature. We also highlight the symmetry-breaking phenomena.

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1. Introduction

The aim of this paper is the study of numerical solutions to the minimization problem introduced in [7]

$$\lambda_1 = \inf_{\substack{u \in W_0^{1,p(x)}(\Omega) \\ u \neq 0}} \frac{\|\nabla u\|_{p(x)}}{\|u\|_{p(x)}}. \tag{1.1}$$

Here Ω is a bounded domain in $\mathbb{R}^n$ and the variable exponent $p : \bar{\Omega} \to \mathbb{R}^+$ is a smooth function such that $1 < p^- \leq p(x) \leq p^+ < \infty$ for every $x \in \Omega$. The norm $\|\cdot\|_{p(x)}$ is the so-called *Luxemburg norm*

$$\|f\|_{p(x)} = \inf \left\{ \gamma > 0 : \int_\Omega \left| \frac{f(x)}{\gamma} \right|^{p(x)} \frac{1}{p(x)} \leq 1 \right\}. \tag{1.2}$$

If p is a constant function, the problem reduces (up to a power p) to the minimization of the quotient

$$\inf_{\substack{u \in W_0^{1,p}(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p} \tag{1.3}$$

and, as it is known, its associated Euler–Lagrange equation is

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda\,|u|^{p-2}\,u. \tag{1.4}$$

In this case, we refer the reader to [12, 13] for the theoretical aspects and to [2] for a recent numerical analysis. The special case $p = 2$ is the classical eigenvalue problem for the Laplacian $-\Delta u = \lambda u$, for which we refer the reader to [11]. In general, in these types of problems, it is crucial that some homogeneity holds, namely, if u is a minimizer, so is ωu for any nonzero real constant ω. On the contrary, the quotient

$$\frac{\int_\Omega |\nabla u|^{p(x)}}{\int_\Omega |u|^{p(x)}} \tag{1.5}$$

with variable exponents fails to possess this feature. Therefore, as we point out in the following section, its infimum over nontrivial functions of $W_0^{1,p(x)}(\Omega)$ turns out to be often equal to zero and no minimizer exists [4, 6]. A way to avoid this collapse is to impose the constraint $\int_\Omega |u|^{p(x)}\mathrm{d}x = C$. Unfortunately, doing so, minimizers obtained for different normalization constants C are difficult to compare. For a suitable $p(x)$, it could even happen that any $\lambda > 0$ is an eigenvalue for some choice of C. Thus (1.5) is not a proper generalization of (1.3), which has well-defined (variational) eigenvalues, although the full spectrum is not completely understood yet. A way to avoid this situation is to use the Rayleigh quotient (1.1), restoring the necessary homogeneity. In the integrand of (1.2), the use of the measure $p(x)^{-1}\mathrm{d}x$ simplifies the Euler–Lagrange equation. The Sobolev inequality $\|v\|_{p(x)} \le C\|\nabla v\|_{p(x)}$ shows that $\lambda_1 > 0$ (see [3]). It is easy to see that (1.1) has a nonnegative minimizer. Pick a minimizing sequence of v_j, namely $\|v_j\|_{p(x)} = 1$ and $\|\nabla v_j\|_{p(x)} = \lambda_1 + o(1)$. By Rellich theorem for variable Sobolev exponents [3], up to a subsequence, we find u such that $v_j \to u$ in $L^{p(x)}(\Omega)$ and $\nabla v_j \rightharpoonup \nabla u$ in $L^{p(x)}(\Omega)$. This yields

$$\lambda_1 \le \frac{\|\nabla u\|_{p(x)}}{\|u\|_{p(x)}} \le \lim_{j\to\infty} \frac{\|\nabla v_j\|_{p(x)}}{\|v_j\|_{p(x)}} = \lambda_1.$$

Notice that if u is a minimizer so is $|u| \ge 0$. By the maximum principle of [9], u has a fixed sign. In [7] the Euler–Lagrange equation for a minimizer is derived. Precisely, it holds that

$$\int_\Omega \left|\frac{\nabla u}{K}\right|^{p(x)-2} \left\langle \frac{\nabla u}{K}, \nabla\eta \right\rangle = \lambda_1 S \int_\Omega \left|\frac{u}{k}\right|^{p(x)-2} \frac{u}{k}\eta \quad \forall\,\eta \in C_0^\infty(\Omega), \tag{1.6}$$

where we have set

$$K = \|\nabla u\|_{p(x)}, \quad k = \|u\|_{p(x)}, \quad \lambda_1 = \frac{K}{k},$$

$$S = \left(\int_\Omega \left|\frac{u}{k}\right|^{p(x)}\right)^{-1} \int_\Omega \left|\frac{\nabla u}{K}\right|^{p(x)}.$$

More generally, $\lambda \in \mathbb{R}$ is eigenvalue if there exists $u \in W_0^{1,p(x)}(\Omega)$, $u \not\equiv 0$, such that

$$\int_\Omega \left|\frac{\nabla u}{K}\right|^{p(x)-2} \left\langle \frac{\nabla u}{K}, \nabla \eta \right\rangle = \lambda S \int_\Omega \left|\frac{u}{k}\right|^{p(x)-2} \frac{u}{k}\,\eta \quad \forall\, \eta \in C_0^\infty(\Omega). \tag{1.7}$$

It follows from the regularity theory developed in [1] that the solutions to (1.7) are continuous provided that $p(x)$ is Hölder continuous. If λ_1 is the minimum in (1.1), we have $\lambda \geq \lambda_1$ in (1.7), thus λ_1 is the first eigenvalue and a corresponding solution is the first eigenfunction. Contrary to the constant exponent case [12, 13], it is currently unknown if, in the variable exponent case, the first eigenvalue is simple, and if a given positive eigenfunction is automatically a first one. Concerning higher eigenvalues, in [15] the authors have recently proved that there is a sequence of eigenvalues of (1.7) with $\lambda_j \nearrow \infty$ and if

$$\sigma = n\left(\frac{1}{p^-} - \frac{1}{p^+}\right) < 1,$$

then there are constants $C_1, C_2 > 0$, that depend only on n and $p^\pm$, such that

$$C_1\,|\Omega|\,\lambda^{n/(1+\sigma)} \leq \#\big\{j : \lambda_j < \lambda\big\} \leq C_2\,|\Omega|\,\lambda^{n/(1-\sigma)} \quad \text{for } \lambda > 0 \text{ large}, \tag{1.8}$$

where $|\Omega|$ is the Lebesgue measure of Ω. Observe that, in the case of constant p, (1.7) reduces not exactly to (1.4), which is homogeneous of degree $p-1$, but rather to the problem (homogeneous of degree 0)

$$-\operatorname{div}\left(\frac{|\nabla u|^{p-2}\,\nabla u}{\|\nabla u\|_p^{p-1}}\right) = \lambda\,\frac{|u|^{p-2}\,u}{\|u\|_p^{p-1}}, \quad u \in W_0^{1,p}(\Omega).$$

Thus (1.8) should be compared to $C_1\,|\Omega|\,\lambda^{n/p} \leq \#\big\{j : \lambda_j < \lambda\big\} \leq C_2\,|\Omega|\,\lambda^{n/p}$, obtained in [8].

A different notion in the literature

We compare the minimization procedure with the Rayleigh quotient with Luxemburg norm and that without it, namely

$$\inf_{\substack{u \in W_0^{1,p(x)}(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^{p(x)}}{\int_\Omega |u|^{p(x)}}.$$

In this framework, if $\lambda \in \mathbb{R}$ and $u \in W_0^{1,p(x)}(\Omega)$, then (u, λ) is called eigenpair if $u \neq 0$ and

$$\int_\Omega |\nabla u|^{p(x)-2}\langle \nabla u, \nabla \eta\rangle = \lambda \int_\Omega |u|^{p(x)-2} u\eta \quad \forall\, \eta \in W_0^{1,p(x)}(\Omega).$$

Set $\Lambda = \{\lambda > 0 : \lambda \text{ is an eigenvalue}\}$. It is well known [12, 13] that, if the function $p(x)$ is constant, then the problem has a sequence of eigenvalues, $\sup \Lambda = +\infty$ and $\inf \Lambda > 0$. In the general case, it follows from [6] that Λ is

a nonempty infinite set and $\sup\Lambda = +\infty$. Define $\lambda_* := \inf\Lambda$. We recall that we often have $\lambda_* = 0$ (recall that $\lambda_1 > 0$ in (1.1)). Consider the following Rayleigh quotients:

$$\mu_* = \inf_{\substack{u\in W_0^{1,p(x)}(\Omega)\\ u\neq 0}} \frac{\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}}{\int_\Omega \frac{|u|^{p(x)}}{p(x)}}, \qquad \overline{\mu}_* = \inf_{\substack{u\in W_0^{1,p(x)}(\Omega)\\ u\neq 0}} \frac{\int_\Omega |\nabla u|^{p(x)}}{\int_\Omega |u|^{p(x)}}.$$

Then, in [6], the authors proved that $\lambda_* > 0 \Leftrightarrow \mu_* > 0 \Leftrightarrow \overline{\mu}_* > 0$. Furthermore, if there is an open subset $U \subset \Omega$ and a point $x_0 \in U$ such that $p(x_0) < p(x)$ (or $>$) for all $x \in \partial U$, then $\lambda_* = 0$ [6, Theorem 3.1]. In particular, if $p(x)$ has strictly local minimum (or maximum) points in Ω, then $\lambda_* = 0$. There are also statements giving some sufficient conditions for $\inf\Lambda > 0$. Let $n > 1$. If there is a vector $\ell \in \mathbb{R}^n \setminus \{0\}$ such that, for any $x \in \Omega$, the map $t \mapsto p(x + t\ell)$ is monotone on $\{t : x + t\ell \in \Omega\}$, then $\lambda_* > 0$ [6, Theorem 3.3]. If $n = 1$, then $\lambda_* > 0$ if and only if the function $p(x)$ is monotone [6, Theorem 3.2].

2. An algorithm to compute the first eigenpair

In this section we briefly describe an algorithm to approximate λ_1 in (1.1) and compute the corresponding eigenfunction. We start defining

$$\Lambda_1 := \inf_{u\in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\|\nabla u\|_{p(x)}^2}{\|u\|_{p(x)}^2} = \inf_{u\in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{R(u)}{S(u)} = \lambda_1^2.$$

It is now possible to apply the inverse power method, where the $(j+1)$th iteration is

$$\tilde{u}^{j+1} = \arg\min_u (R(u) - \nabla S(u^j)u) = \arg\min_u J(u), \tag{2.1a}$$

$$u^{j+1} = \frac{\tilde{u}^{j+1}}{S(\tilde{u}^{j+1})^{1/2}}, \tag{2.1b}$$

$$\Lambda_1^{j+1} = \frac{R(u^{j+1})}{S(u^{j+1})}, \tag{2.1c}$$

where u^j is the result of the previous iteration and, by (2.1b), has Luxemburg norm equal to 1. It is possible to show (see [5]) that the algorithm converges to a critical point of $R(u)/S(u)$, even if it is not possible in general to prove convergence to the smallest eingevalue. However, a good choice of the initial guess u^0 can reasonably assure that the result is the smallest eigenvalue. For the computation of $\nabla S(u)\eta$, for given u and η, in the so-called *inner problem* (2.1a), we observe that if $u \neq 0$, its Luxemburg norm $\gamma(u) = \|u\|_{p(x)}$ is implicitly defined by

$$F(u,\gamma) = \int_\Omega \left|\frac{u(x)}{\gamma}\right|^{p(x)} \frac{1}{p(x)} - 1 = 0. \tag{2.2}$$

Therefore, we can use the differentiation of implicit functions to get

$$\nabla\|u\|_{p(x)}\eta = -\frac{\nabla_u F(u,\gamma)\eta}{\partial_\gamma F(u,\gamma)} = \frac{\int_\Omega \left|\frac{u}{\|u\|_{p(x)}}\right|^{p(x)-2}\frac{u}{\|u\|_{p(x)}}\,\eta}{\int_\Omega \left|\frac{u}{\|u\|_{p(x)}}\right|^{p(x)}}$$

from which

$$\nabla S(u)\eta = 2\,\frac{\int_\Omega \left|\frac{u}{\|u\|_{p(x)}}\right|^{p(x)-2} u\eta}{\int_\Omega \left|\frac{u}{\|u\|_{p(x)}}\right|^{p(x)}}.$$

Since we are mainly interested is some particular two-dimensional domains, such as a rectangle, a disk or an annulus, we approximated the problem by the finite element method which well adapts to different geometries by constructing an appropriate discretization mesh. In particular, we used the tool FreeFem++ [10] which can handle minimization problems as (2.1a) through the function `NLCS` (nonlinear conjugate gradient method, Fletcher–Reeves implementation). Such a function requires the application of the gradient of $J(u)$ to a test function η:

$$\nabla J(u)\eta = 2\,\frac{\int_\Omega \left|\frac{\nabla u}{\|\nabla u\|_{p(x)}}\right|^{p(x)-2}\langle\nabla u,\nabla\eta\rangle}{\int_\Omega \left|\frac{\nabla u}{\|\nabla u\|_{p(x)}}\right|^{p(x)}} - 2\,\frac{\int_\Omega \left|\frac{u^j}{\|u^j\|_{p(x)}}\right|^{p(x)-2} u^j\eta}{\int_\Omega \left|\frac{u^j}{\|u^j\|_{p(x)}}\right|^{p(x)}}$$

and an initial guess which, for the $(j+1)$th iteration of the inverse power method, is u_j/Λ_1^j. The stopping criterion for the inverse power method is based on the difference of two successive approximations of Λ_1.

2.1. Some details of the algorithm

The algorithm described above requires recurrent computations of the Luxemburg norm of a function. For a given $u \neq 0$, it is the zero of the function $F(u,\cdot)$ defined in (2.2). This is a $C^2(0,+\infty)$ convex and monotonically decreasing function in γ, with $\lim_{\gamma\to 0^+} F(u,\gamma) = +\infty$ and $\lim_{\gamma\to+\infty} F(u,\gamma) = -1$. Therefore, it is possible to apply the quadratically convergent Newton's method in order to find its unique zero, starting with an initial guess γ_0 on its left-hand side (i.e., such that $F(u,\gamma_0) > 0$). As pointed out above, the inverse power method cannot guarantee the convergence to the smallest eigenvalue and relative eigenfunction. It is very reasonable to expect that if the initial guess u^0 for the method is close enough to the smallest eigenfunction, then the algorithm will converge to it. For $p \equiv 2$, the problem essentially reduces to the Helmholtz equation

$$-\Delta u = \lambda_1 u,$$

for which the eigenfunctions are well known for the domains we have in mind. Therefore, starting with $p \equiv 2$ and u^0 the eigenfunction corresponding to the smallest eigenvalue for the Helmholtz equation, we moved to the desired $p(x)$ through a standard continuation technique.

We tested our algorithm using both linear and quadratic finite elements and checked the correct order of convergence (two and three for the rectangle and one and two for the circular domains, respectively). Moreover, we checked that the results with constant p were consistent with those reported in [2]. The results in the next section were obtained with quadratic finite elements. We observed convergence of our algorithm also for some cases with $p(x) < 2$, for which the Hessian of $J(u)$ degenerates and the nonlinear conjugate gradient is not guaranteed to converge. In this case some authors add a regularization parameter to the functional $J(u)$ (see [2]). We, moreover, observed sometimes slow convergence of the nonlinear conjugate gradient. In this case, a more sophisticated method, using the Hessian of $J(u)$ or an approximation of it, could be employed. Another possibility would be to use a preconditioner, either based on a low order approximation of $J(u)$ (see again [2] and references therein) or on a linearized version of $J(u)$. The implementation of a more robust and fast algorithm, on which we are currently working, is beyond the scopes of this paper.

2.2. Examples and breaking symmetry

We show in this section the results we obtained by applying the algorithm described above to three test cases, in the square, the disk and the annulus, respectively. For each test, we report the plot of the obtained eigenfunction from the x-axis, the y-axis and from the top view, respectively.

The first case (Figure 1) refers to the *unit square* $[0,1] \times [0,1]$ and $p(x,y) = 5 + 3\sin(5\pi x)$. The first plot (left) in this figure does not present

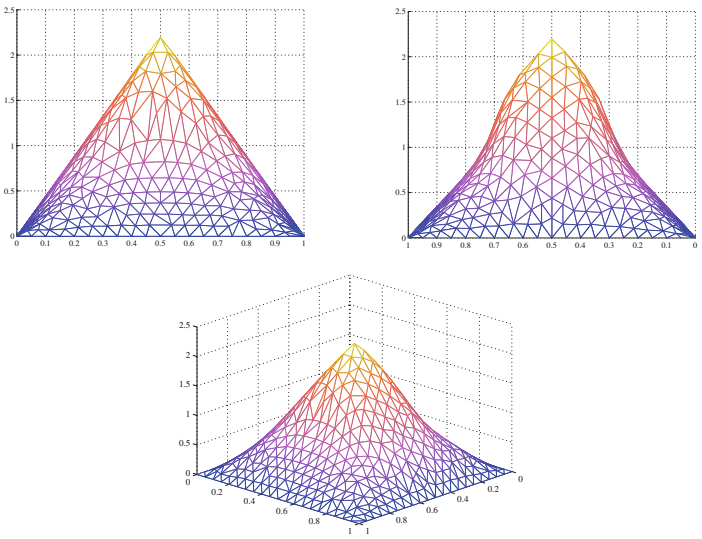

FIGURE 1. Unit square, $p(x,y) = 5 + 3\sin(5\pi x)$, x-axis (left), y-axis (right) and top views.

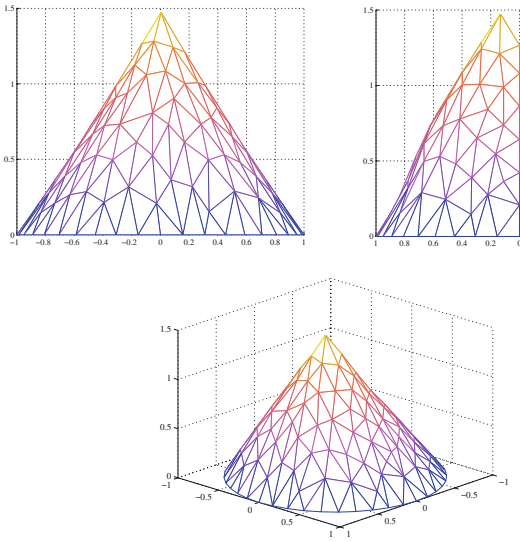

FIGURE 2. Unit disk, $p(x, y) = 11 + 9\sin(2\pi x)$, x-axis (left), y-axis (right) and top views.

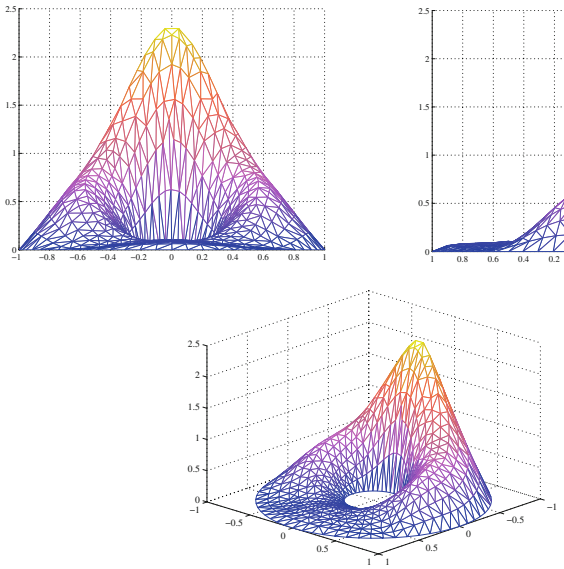

FIGURE 3. Annulus, $p(x, y) = 4 + 2\sin(2\pi x)$, x-axis (left), y-axis (right) and top views.

special features because the exponent $p(x, y)$ is independent of y. In the second plot (right) the profile is different as it feels the diffusion variation

in the x-variable. Both plots are symmetric with respect to the center of the domain, since $p(x, y)$ has a center of symmetry in $(1/2, 1/2)$; see Figure 4.

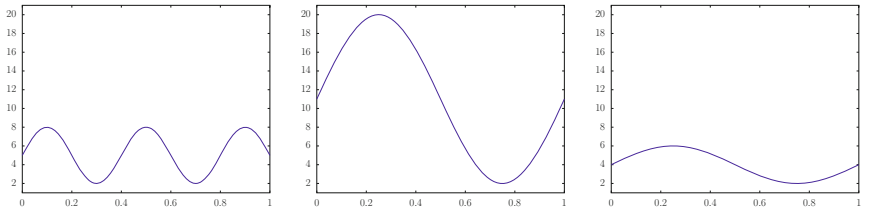

FIGURE 4. $p(x, y) = 5+3\sin(5\pi x)$, $p(x, y) = 11+9\sin(2\pi x)$ and $p(x, y) = 4 + 2\sin(2\pi x)$, respectively.

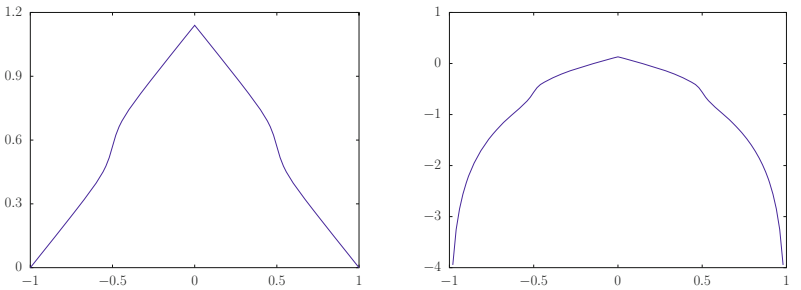

FIGURE 5. Interval $[-1, 1]$, $p(x) = 28 + 26\cos(2\pi x)$, eigenfunction (left) and its logarithm (right).

The second case (Figure 2) refers to the *unit disk* with center 0 and radius 1 and $p(x, y) = 11 + 9\sin(2\pi x)$. The maximum of $p(x, y)$ is quite high (see Figure 4) and the profile is reminiscent of the one for the limiting case $p = \infty$. The second plot is remarkable because we loose the symmetry for the center of the domain.

The third case (Figure 3) refers to the *annulus* with center 0, external radius 1 and internal radius 0.25 and $p(x, y) = 4 + 2\sin(2\pi x)$. The resulting eigenfunction is more shifted than the case in the unit square and the case in the unit disc. Even the shape of the domain influences the contour, here we see that the annulus reflects intensely the mold of the exponent. The first plot (left) maintains the center of symmetry in $(0, 0)$ and $p(x, y)$ does not depend on y.

Already in the one-dimensional case, it is evident that the logarithm of first eigenfunction is not a concave function, in general, contrary to the constant exponent case where this was proved to be true [16]. See Figure 5 for an example in this regard.

In the case where Ω has some symmetry and $p(x)$ is a radially symmetric (resp., axially symmetric with respect to some half-space) function, then some

symmetry (resp., partial symmetry) results where recently obtained in [14] for semistable solutions and mountain pass solutions.

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