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An asymptotic expansion for the fractional p-Laplacian and for gradient-dependent nonlocal operators

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Mean value formulas are of great importance in the theory of partial differential equations: many very useful results are drawn, for instance, from the well-known equivalence between harmonic functions and mean value properties. In the nonlocal setting of fractional harmonic functions, such an equivalence still holds, and many applications are nowadays available. The nonlinear case, corresponding to the p-Laplace operator, has also been recently investigated, whereas the validity of a nonlocal, nonlinear, counterpart remains an open problem. In this paper, we propose a formula for the *nonlocal, nonlinear mean value kernel*, by means of which we obtain an asymptotic representation formula for harmonic functions in the viscosity sense, with respect to the fractional (variational) p-Laplacian (for $p \geq 2$) and to other gradient-dependent nonlocal operators.

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1. Introduction

One of the most famous basic facts of partial differential equations is that a smooth function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is harmonic (i.e. $\Delta u = 0$) in an open set Ω if and only if

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it satisfies the mean value property, that is,

$$u(x) = \fint_{B_r(x)} u(y)dy, \quad \text{whenever } B_r(x) \Subset \Omega. \tag{1.1}$$

This remarkable characterization of harmonic functions provided a fertile ground for extensive developments and applications. Additionally, such a representation holds in some sense for harmonic functions with respect to more general differential operators. In fact, similar properties can be obtained for quasi-linear operators such as the p-Laplace operator Δ_p, in an asymptotic form. More precisely, a first result is due to Manfredi *et al.*, who proved in [29] that if $p \in (1, \infty]$, a continuous function $u : \Omega \to \mathbb{R}$ is p-harmonic in Ω if and only if (in the viscosity sense)

$$u(x) = \frac{2+n}{p+n} \fint_{B_r(x)} u(y)dy + \frac{p-2}{2p+2n}\Big(\max_{B_r(x)} u + \min_{B_r(x)} u \Big) + o(r^2), \tag{1.2}$$

as the radius r of the ball vanishes. This characterization also encouraged a series of new research, such as [30, 31], whereas other very nice results were obtained in sequel, see e.g., [15, 18, 20]. Further new and very interesting related work is contained in [4, 14].

Notice that formula (1.2) boils down to (1.1) for $p = 2$, up to a rest of order $o(r^2)$, and that it holds true in the classical sense at those points $x \in \Omega$ such that u is C^2 around x and such that the gradient of u does not vanish at x. In the case $p = \infty$ the formula fails in the classical sense, since $|x|^{4/3} - |y|^{4/3}$ is ∞-harmonic in $\mathbb{R}^2$ in the viscosity sense, but (1.2) fails to hold point-wisely. If $p \in (1, \infty)$ and $n = 2$ the characterization holds in the classical sense (see [3, 28]). Finally, the limiting case $p = 1$ was investigated in 2012 in [21].

Nonlocal operators have been under scrutiny in the past decade. The interest, not only from the purely mathematical point of view, has exponentially risen, and it was natural to ask the questions affirmatively answered in the classical case, to the respective fractional counterparts.

The investigation of the validity of a mean value property in the *nonlocal* linear case, that is for fractional harmonic functions, provided a first positive answer. Let $s \in (0, 1)$, we define formally

$$(-\Delta)^s u(x) := C(n,s) \lim_{r\to 0} \int_{\mathbb{R}^n \setminus B_r} \frac{u(x) - u(x-y)}{|y|^{n+2s}}\, dy, \quad C(n,s) = \frac{2^{2s} s\Gamma\left(\frac{n}{2} + s\right)}{\pi^{\frac{n}{2}}\Gamma(1-s)}.$$

The equivalence between s-harmonic functions (i.e. functions that satisfy $(-\Delta)^s u = 0$) and the fractional mean value property is proved in [1] (see also [9]), with the fractional mean kernel given by

$$\mathscr{M}_r^s u(x) = c(n,s) r^{2s} \int_{\mathbb{R}^n \setminus B_r} \frac{u(x-y)}{(|y|^2 - r^2)^s |y|^n}\, dy. \tag{1.3}$$

Here,

$$c(n,s) = \left[\int_{\mathbb{R}^n \setminus B_r} \frac{r^{2s}dy}{(|y|^2 - r^2)^s |y|^n}\right]^{-1} = \frac{\Gamma\left(\frac{n}{2}\right)\sin \pi s}{\pi^{n/2+1}}. \tag{1.4}$$

The formula (1.3) is far from being the outcome of a recent curiosity. It was introduced in 1967 (up to the authors knowledge) in [25], formula (1.6.2), and was recently fleshed out for its connection with the fractional Laplace operator. Different applications rose from such a formula, just to name a few [2, 10, 24]. We point out furthermore that the formula in (1.3) is consistent with the classical case, as expected: as $s \to 1^-$, the fractional Laplacian goes to the classical Laplacian, and the mean value kernel goes to the classical mean value on the boundary of the ball (see [11, 25]), that is,

$$\lim_{s\to 1^-} \mathscr{M}_r^s u(x) = \fint_{B_r(x)} u(y)dy. \tag{1.5}$$

An asymptotic expansion can be obtained also for fractional anisotropic operators (that include the case of the fractional Laplacian), as one can observe in [11]. In particular, the result is that a continuous function u is s-harmonic in the viscosity sense if and only if (1.3) holds in a viscosity sense up to a rest of order two, namely

$$u(x) = \mathscr{M}_r^s u(x) + o(r^2) \quad \text{as } r \to 0^+. \tag{1.6}$$

The goal of this paper is to extend the analysis of the nonlocal case to the fractional p-Laplace operator. Namely, the fractional (variational) p-Laplacian is the differential (in a suitable Banach space) of the convex functional

$$u \mapsto \frac{1}{p}[u]^p_{s,p} := \frac{1}{p}\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}dxdy$$

and is formally defined as

$$(-\Delta)^s_p u(x) = \lim_{\varepsilon\to 0^+} \mathcal{L}^{s,p}_\varepsilon u(x),$$

$$\mathcal{L}^{s,p}_\varepsilon u(x) := \int_{|y|>\varepsilon} \frac{|u(x)-u(x-y)|^{p-2}(u(x)-u(x-y))}{|y|^{n+sp}}\,dy.$$

Notice that this definition is consistent, up to a normalization, with the linear operator $(-\Delta)^s = (-\Delta)^s_2$. The interested reader can appeal to [8, 16, 17, 22, 23, 32] to find an extensive theory on the fractional p-Laplace operator and other very useful references.

A first issue toward our goal is to identify a reasonable version of a *nonlocal, nolinear, mean value property*. Up to the authors' knowledge, this is the first attempt to obtain similar properties in the nonlocal, nonlinear, case. Consequently, on the one hand, the argument is new, so we cannot base our results on any reference. On the other hand, intuitively one can say that a formula could be reasonable if it were consistent with the already known problems: the nonlocal case of $p = 2$

(corresponding to the case of the fractional Laplacian), and the local, nonlinear case, $s = 1$ (corresponding to the case of the classical p-Laplacian). The main result that we propose is the following.

Main Result 1. Let $p \geq 2$, $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in C(\Omega) \cap L^\infty(\mathbb{R}^n)$ be a non-constant function. Then

$$(-\Delta)_p^s u(x) = 0$$

in the viscosity sense if and only if

$$\int_{\mathbb{R}^n \setminus B_r} \left(\frac{|u(x) - u(x-y)|}{|y|^s} \right)^{p-2} \frac{u(x) - u(x-y)}{|y|^n (|y|^2 - r^2)^s} \, dy = o_r(1) \quad \text{as } r \to 0^+$$

holds in the viscosity sense for all $x \in \Omega$.

This main achievement is precisely stated in Theorem 2.7. The proof of this main theorem is based on an expansion formula for the fractional p-Laplacian for smooth functions, that we do in Theorem 2.3, obtaining by means of Taylor expansions and a very careful handling of the remainders, that

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus B_r} \left(\frac{|u(x) - u(x-y)|}{|y|^s} \right)^{p-2} \frac{u(x) - u(x-y)}{|y|^n (|y|^2 - r^2)^s} dy \\ &\quad = (-\Delta)_p^s u(x) + \mathcal{O}(r^{2-2s}) \quad \text{as } r \to 0^+. \end{aligned}$$

It is enough then to use the viscosity setting to obtain our main result.

If we try to write the formula in this first main result in a way that reflects the usual "$u(x)$ equals its mean value property", we obtain

$$\begin{aligned} u(x) = &\left[\int_{\mathbb{R}^n \setminus B_r} \left(\frac{|u(x) - u(x-y)|}{|y|^s} \right)^{p-2} \frac{dy}{|y|^n (|y|^2 - r^2)^s} \right]^{-1} \\ &\times \left[\int_{\mathbb{R}^n \setminus B_r} \left(\frac{|u(x) - u(x-y)|}{|y|^s} \right)^{p-2} \frac{u(x-y)}{|y|^n (|y|^2 - r^2)^s} \, dy + o_r(1) \right] \\ &\text{as } r \to 0^+. \end{aligned} \tag{1.7}$$

If we now compare formula (1.7) with our desired assumptions, we notice that for $p = 2$, supported by (1.4), we recover the result in (1.6). Our candidate for the role of the *nonlocal, nonlinear, mean property* gives indeed the usual formula for the fractional s-mean value property. Looking also at the validity of (1.5) in our nonlinear case, we come up with an interesting ancillary result as $s \to 1^-$. In Proposition 2.10, we obtain an asymptotic expansion for the (classical) variational p-Laplace operator, and the equivalence in the viscosity sense between p-harmonic and a p-mean value property in Theorem 2.12. The expression obtained by us has some similitudes with other formulas from the literature: compare, e.g., (2.28a) with [15, Theorem 6.1] that has recently appeared in the literature, or to [18, Theorem 1.1], and (2.28b) with [29] (see Remark 2.11 for further details).

Thus, it appears that our formula is consistent with the expressions in the literature for $s = 1$, and with the case $p = 2$. In our opinion, such a consistency suggests that formula (1.7) is a reasonable proposal for a *nonlocal, nonlinear, mean value property*.

We mention that a downside of (1.7) is that it does not allow to obtain a clean "$u(x)$ equal to its mean value property", as customary, given the dependence of $u(x)$ itself, inside the integral, of the right-hand side. This "inconvenient", nonetheless, does not disappear in the local setting: the reader can see the already mentioned works [15, 18].

In the second part of the paper, we investigate a different nonlocal version of the p-Laplace and of the infinity Laplace operators, that arise in tug-of-war games, introduced in [5, 6]. To avoid overloading the notations, we summarize the results on these two nonlocal operators as follows. We denote by $(-\Delta)^s_{p,\pm}$ the nonlocal p-Laplace (as in [5, Sec. 4]) and by $M_r^{s,p,\pm}$ the "nonlocal p-mean kernel". The precise definitions of the operator, of the mean kernel and of viscosity solutions are given in Sec. 3. The asymptotic representation formula is the content of Theorem 3.4.

Main Result 2. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in C(\Omega) \cap L^\infty(\mathbb{R}^n)$. Then

$$(-\Delta)^s_{p,\pm} u(x) = 0$$

in the viscosity sense if and only if

$$\lim_{r \to 0^+} (u(x) - M_r^{s,p,\pm} u(x)) = o(r^{2s})$$

holds for all $x \in \Omega$ in the viscosity sense.

An analogous result is stated in Theorem 3.9 for the infinity Laplacian (introduced in [6]) and the "infinity mean kernel", defined, respectively, in Sec. 3.2. The same strategy for the proof as Main Result 1 is adopted for the nonlocal p and infinity Laplacian, by coming up in Theorems 3.1 and 3.8, by means of Taylor expansions, with formulas which hold for smooth functions, and then passing to the viscosity setting.

Furthermore, we study the asymptotic properties of these gradient-dependent operators and of the mean kernels as $s \to 1^-$. As a collateral result, we are able to obtain in Proposition 3.6 an expansion for the normalized p-Laplacian (and the consequence for viscosity solutions in Theorem 3.7), which up to our knowledge, is new in the literature. In our opinion, the behavior in the limit case $s \to 1^-$ of the formula we propose, justifies here also our choice of the mean value property expression.

We advise the reader interested in mean value formulas for these two gradient-dependent operators here discussed to consult the very recent papers [13, 26]. Therein, the authors introduce asymptotic mean value formulas which do not depend on the gradient, making them much useful in applications.

To conclude the introduction, we mention the plan of the paper. The results relative to the fractional (variational) p-Laplace operator are the content of Sec. 2.

Section 3 contains the results on the nonlocal gradient-dependent operators, while in Appendix A we insert some very simple, basic integral asymptotics.

2. The Fractional p-Laplacian

2.1. *An asymptotic expansion*

Let $p \geq 2$. Throughout Sec. 2, we consider u to be a non-constant function. To simplify the formula in (1.7), we introduce the following notations:

$$\mathcal{D}_r^{s,p}u(x) := \int_{|y|>r} \left(\frac{|u(x)-u(x-y)|}{|y|^s} \right)^{p-2} \frac{dy}{|y|^n(|y|^2-r^2)^s}, \tag{2.1}$$

and

$$\mathscr{M}_r^{s,p}u(x) := (\mathcal{D}_r^{s,p}u(x))^{-1} \int_{|y|>r} \left(\frac{|u(x)-u(x-y)|}{|y|^s} \right)^{p-2} \frac{u(x-y)}{|y|^n(|y|^2-r^2)^s} dy. \tag{2.2}$$

To make an analogy with the local case, we may informally say that $\mathcal{D}_r^{s,p}u$ plays the "nonlocal" role of $\nabla u(x)$ (see also and Proposition 2.9, for the limit as $s \to 1^-$), and $\mathscr{M}_r^{s,p}u$ the role of a (s,p)-mean kernel. Both $\mathcal{D}_r^{s,p}$ and $\mathscr{M}_r^{s,p}$ naturally appear when we make an asymptotic expansion for smooth functions, that we do in Theorem 2.3. Notice also that for $p = 2$, $\mathscr{M}_r^{s,2}u$ is given by (1.6) (and $\mathcal{D}_r^{s,2}u(x) = c(n,s)^{-1}r^{-2s}$).

The following remark motivates working with non-constant functions, and justifies (2.2) as a good definition.

Remark 2.1. Let $u\colon \mathbb{R}^n \to \mathbb{R}$ be such that, for some $x \in \mathbb{R}^n$ there exist $z_x \in \mathbb{R}^n$ and $r_x \in (0, |x-z_x|/2)$ such that

$$u(x) \neq u(z) \quad \forall\, z \in B_{r_x}(z_x). \tag{2.3}$$

Then there exist some $c_x > 0$ such that, for all $r < r_x$, it holds that $\mathcal{D}_r^{s,p}u(x) \geq c_x$.

Notice that if u is a continuous, non-constant function, for some $x \in \mathbb{R}^n$ there exist $z_x \in \mathbb{R}^n$ and $r_x \in (0, |x-z_x|/2)$ such that (2.3) is accomplished.

The fact that $\mathcal{D}_r^{s,p}u(x)$ is bounded strictly away from zero is not difficult to see, we prove it however for completeness.

Proof. We have that

$$|y|^2 - r^2 \leq |y|^2,$$

hence

$$\mathcal{D}_r^{s,p}u(x) \geq \int_{|y|>r} \frac{|u(x)-u(x-y)|^{p-2}}{|y|^{n+sp}} dy \geq \int_{\mathcal{C}B_r(x)} \frac{|u(x)-u(y)|^{p-2}}{|x-y|^{n+sp}} dy,$$

by changing variables. For any $r < r_x/2$, $B_{r_x}(z_x) \subset \mathcal{C}B_r(x)$, thus

$$\mathcal{D}_r^{s,p}u(x) \geq \int_{B_{r_x}(z_x)} \frac{|u(x)-u(y)|^{p-2}}{|x-y|^{n+sp}} dy := c_x,$$

with c_x positive, independent of r. ☐

Remark 2.2. Notice that it is quite natural to assume that u is not constant and it is similar to what is required in the local case, namely $\nabla u(x) \neq 0$ (see the proof of [29, Theorem 2]).

Using the notations in (2.1) and (2.2), we obtain the following asymptotic property for smooth functions.

Theorem 2.3. *Let $\eta > 0, x \in \mathbb{R}^n$ and $u \in C^2(B_\eta(x)) \cap L^\infty(\mathbb{R}^n)$. Then*

$$\mathcal{D}_r^{s,p} u(x)(u(x) - \mathscr{M}_r^{s,p} u(x)) = (-\Delta)_p^s u(x) + \mathcal{O}(r^{2-2s}) \tag{2.4}$$

as $r \to 0$.

Substituting our notations, we remark that (2.4) is

$$\begin{aligned}\int_{\mathbb{R}^n \setminus B_r} &\left(\frac{|u(x) - u(x-y)|}{|y|^s}\right)^{p-2} \frac{u(x) - u(x-y)}{|y|^n (|y|^2 - r^2)^s}\, dy \\ &= (-\Delta)_p^s u(x) + \mathcal{O}(r^{2-2s}) \quad \text{as } r \to 0.\end{aligned}$$

Proof. We note that the constants may change value from line to line. We fix an arbitrary $\bar{\varepsilon}$ (not necessarily small), the corresponding $r := r(\bar{\varepsilon}) \in (0, \eta/2)$ as in (2.13), and some number $0 < \varepsilon < \min\{\bar{\varepsilon}, r\}$, to be taken arbitrarily small.

Starting from the definition, we have that

$$\begin{aligned}\mathcal{L}_\varepsilon^{s,p} u(x) = &\int_{\varepsilon < |y| < r} \frac{|u(x) - u(x-y)|^{p-2}(u(x) - u(x-y))}{|y|^{n+sp}} dy \\ &+ \int_{|y|>r} \frac{|u(x) - u(x-y)|^{p-2}(u(x) - u(x-y))}{|y|^{n+s(p-2)}} \left(\frac{1}{|y|^{2s}} - \frac{1}{(|y|^2 - r^2)^s}\right) dy \\ &+ \int_{|y|>r} \frac{|u(x) - u(x-y)|^{p-2}(u(x) - u(x-y))}{|y|^{n+s(p-2)}(|y|^2 - r^2)^s} dy.\end{aligned}$$

Thus, we obtain that

$$\begin{aligned}\mathcal{L}_\varepsilon^{s,p} u(x) &+ \int_{|y|>r} \frac{|u(x) - u(x-y)|^{p-2} u(x-y)}{|y|^{n+s(p-2)}(|y|^2 - r^2)^s} dy \\ &= u(x) \int_{|y|>r} \frac{|u(x) - u(x-y)|^{p-2}}{|y|^{n+s(p-2)}(|y|^2 - r^2)^s} dy \\ &\quad + \int_{\varepsilon < |y| < r} \frac{|u(x) - u(x-y)|^{p-2}(u(x) - u(x-y))}{|y|^{n+sp}} dy \\ &\quad + \int_{|y|>r} \frac{|u(x) - u(x-y)|^{p-2}(u(x) - u(x-y))}{|y|^{n+s(p-2)}} \left(\frac{1}{|y|^{2s}} - \frac{1}{(|y|^2 - r^2)^s}\right) dy \\ &:= u(x) \int_{|y|>r} \frac{|u(x) - u(x-y)|^{p-2}}{|y|^{n+s(p-2)}(|y|^2 - r^2)^s} dy + I_\varepsilon^s(r) + J(r).\end{aligned} \tag{2.5}$$

Since $u \in C^2(B_\eta(x))$, we can proceed as in (2.16), and by employing (2.20) and (2.22), we get that

$$\lim_{\varepsilon \to 0^+} I_\varepsilon^s(r) = \mathcal{O}(r^{p(1-s)}), \tag{2.6}$$

(see also [22, Lemma 3.6]). Looking for an estimate on $J(r)$, we split it into two parts

$$\begin{aligned} J(r) &= \int_{|y|>r} \frac{|u(x)-u(x-y)|^{p-2}(u(x)-u(x-y))}{|y|^{n+s(p-2)}} \left(\frac{1}{|y|^{2s}} - \frac{1}{(|y|^2-r^2)^s} \right) dy \\ &= r^{-sp} \int_{|y|>1} \frac{|u(x)-u(x-ry)|^{p-2}(u(x)-u(x-ry))}{|y|^{n+s(p-2)}} \\ &\quad \times \left(\frac{1}{|y|^{2s}} - \frac{1}{(|y|^2-1)^s} \right) dy \\ &= r^{-sp} \left[\int_{|y|>\frac{\eta}{r}} \frac{|u(x)-u(x-ry)|^{p-2}(u(x)-u(x-ry))}{|y|^{n+s(p-2)}} \right. \\ &\quad \times \left(\frac{1}{|y|^{2s}} - \frac{1}{(|y|^2-1)^s} \right) dy \\ &\quad + \int_{1<|y|<\frac{\eta}{r}} \frac{|u(x)-u(x-ry)|^{p-2}(u(x)-u(x-ry))}{|y|^{n+s(p-2)}} \\ &\quad \left. \times \left(\frac{1}{|y|^{2s}} - \frac{1}{(|y|^2-1)^s} \right) \right] dy \\ &=: r^{-sp}(J_1(r)+J_2(r)). \end{aligned}$$

We have that

$$|u(x)-u(x-y)|^{p-1} \le c(|u(x)|^{p-1} + |u(x-y)|^{p-1}),$$

thus we obtain the bound

$$|J_1(r)| \le C\|u\|_{L^\infty(\mathbb{R}^n)}^{p-1} \int_{\frac{\eta}{r}}^\infty \frac{dt}{t^{sp+1}} \left| 1 - \frac{1}{\left(1-\frac{1}{t^2}\right)^s} \right|.$$

The fact that

$$J_1(r) = \mathcal{O}(r^{2+sp})$$

follows from (A.1b). For J_2, by symmetry we write

$$\begin{aligned} J_2(r) = \frac{1}{2} \int_{1<|y|<\frac{\eta}{r}} & \frac{|u(x)-u(x-ry)|^{p-2}(u(x)-u(x-ry))}{|y|^{n+s(p-2)}} \\ & \times \left(\frac{1}{|y|^{2s}} - \frac{1}{(|y|^2-1)^s} \right) dy \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_{1<|y|<\frac{\eta}{r}} \frac{|u(x)-u(x+ry)|^{p-2}(u(x)-u(x+ry))}{|y|^{n+s(p-2)}} \\
&\times\left(\frac{1}{|y|^{2s}}-\frac{1}{(|y|^2-1)^s}\right)dy \\
=\,&\frac{1}{2}\int_{1<|y|<\frac{\eta}{r}} \frac{|u(x)-u(x-ry)|^{p-2}(2u(x)-u(x-ry)-u(x+ry))}{|y|^{n+s(p-2)}} \\
&\times\left(\frac{1}{|y|^{2s}}-\frac{1}{(|y|^2-1)^s}\right)dy \\
&+\frac{1}{2}\int_{1<|y|<\frac{\eta}{r}} \frac{\left(|u(x)-u(x+ry)|^{p-2}-|u(x)-u(x-ry)|^{p-2}\right)}{|y|^{n+s(p-2)}} \\
&\times(u(x)-u(x+ry))\left(\frac{1}{|y|^{2s}}-\frac{1}{(|y|^2-1)^s}\right)dy.
\end{aligned}
$$

We proceed using (2.19) and (2.21). For r small enough, we have that

$$
\begin{aligned}
|J_2(R)| &\leq Cr^p\int_1^{\frac{\eta}{r}}\rho^{p-1-s(p-2)}\left(\frac{1}{(\rho^2-1)^s}-\frac{1}{\rho^{2s}}\right)d\rho \\
&\leq Cr^{p-(p-2)(1-s)}\int_1^{\frac{\eta}{r}}\rho\left(\frac{1}{(\rho^2-1)^s}-\frac{1}{\rho^{2s}}\right)d\rho \\
&\leq Cr^{p-(p-2)(1-s)},
\end{aligned}
\tag{2.7}
$$

from (A.1a), with C depending also on η. This yields that $J_2(r)=\mathcal{O}(r^{2+sp-2s})$.

It follows that

$$
J(r)=\mathcal{O}(r^{2-2s}).
$$

Looking back at (2.5), using this and recalling (2.6), by sending $\varepsilon\to 0^+$, we obtain that

$$
(-\Delta)_p^s u(x)+\int_{|y|>r}\frac{|u(x)-u(x-y)|^{p-2}u(x-y)}{|y|^{n+s(p-2)}(|y|^2-r^2)^s}dy = u(x)\mathcal{D}_r^{s,p}u(x)+\mathcal{O}(r^{2-2s}).
$$

This concludes the proof of the theorem. □

It is a property of mean value kernels $\mathscr{M}_r u(x)$ that they converge to $u(x)$ as $r\to 0^+$ both in the local (linear and nonlinear) and in the nonlocal linear setting. In our case, due to the presence of $\mathcal{D}_r^{s,p}u$, we have this property when $\nabla u(x)\neq 0$ only for a limited range of values of p, a range depending on s and becoming larger as $s\to 1^-$. For other values of p, we were not able to obtain such a result. More precisely, we have the following proposition.

Proposition 2.4. *Let $\eta > 0$ and $x \in \mathbb{R}^n$. If $u \in C^2(B_\eta(x)) \cap L^\infty(\mathbb{R}^n)$ is such that $\nabla u(x) \neq 0$, and s, p are such that*

$$p \in \left[2, \frac{2}{1-s}\right),$$

it holds that

$$\lim_{r \to 0^+} \mathscr{M}_r^{s,p} u(x) = u(x).$$

Proof. There is some $r \in (0, \eta/4)$ such that $\nabla u(y) \neq 0$ for all $y \in B_{2r}(x)$. Then

$$\begin{aligned}\mathcal{D}_r^{s,p} u(x) &\geq \int_{B_{2r} \setminus B_r} \frac{|u(x) - u(x-y)|^{p-2}}{|y|^{n+s(p-2)}(|y|^2 - r^2)^s} dy \\ &= \int_{B_{2r} \setminus B_r} \left| \nabla u(\xi) \cdot \frac{y}{|y|} \right|^{p-2} |y|^{(p-2)(1-s)-n}(|y|^2 - r^2)^{-s} dy,\end{aligned}$$

for some $\xi \in B_{2r}(x)$. Therefore, using (2.29)

$$\mathcal{D}_r^{s,p} u(x) \geq C_{p,n} |\nabla u(\xi)|^{p-2} r^{(p-2)(1-s)-2s},$$

which for p in the given range, allows to say that

$$\lim_{r \to 0^+} \mathcal{D}_r^{s,p} u(x) = \infty.$$

From Proposition 2.1 and Theorem 2.3, we obtain

$$\lim_{r \to 0^+} (u(x) - \mathscr{M}_r^{s,p} u(x)) = \lim_{r \to 0^+} (\mathcal{D}_r^{s,p} u(x))^{-1}((-\Delta)_p^s u(x) + \mathcal{O}(r^{2-2s})),$$

and the conclusion is settled. □

2.2. *Viscosity setting*

For the viscosity setting of the (s, p)-Laplacian, see the paper [27] (and also [12, 22, 32]). As a first thing, we recall the definition of viscosity solutions.

Definition 2.5. A function $u \in L^\infty(\mathbb{R}^n)$, upper (lower) semi-continuous in $\overline{\Omega}$ is a viscosity subsolution (supersolution) in Ω of

$$(-\Delta)_p^s u = 0, \quad \text{and we write } (-\Delta)_p^s u \leq (\geq) 0$$

if for every $x \in \Omega$, any neighborhood $U = U(x) \subset \Omega$ and any $\varphi \in C^2(\overline{U})$ such that

$$\begin{aligned}&\varphi(x) = u(x), \\ &\varphi(y) > (<) u(y) \quad \text{for any } y \in U \setminus \{x\},\end{aligned} \tag{2.8}$$

if we let

$$v = \begin{cases} \varphi & \text{in } U, \\ u & \text{in } \mathbb{R}^n \setminus U, \end{cases} \tag{2.9}$$

then

$$(-\Delta)_p^s v(x) \leq \ (\geq)\, 0.$$

A viscosity solution of $(-\Delta)_p^s u = 0$ is a (continuous) function that is both a subsolution and a supersolution.

We define here what we mean when we say that an asymptotic expansion holds in the viscosity sense.

Definition 2.6. Let $u \in L^\infty(\mathbb{R}^n)$ be upper (lower) semi-continuous in Ω. We say that

$$\mathcal{D}_r^{s,p} u(x)(u(x) - \mathscr{M}_r^{s,p} u(x)) = o_r(1) \quad \text{as } r \to 0^+$$

holds in the viscosity sense if for any neighborhood $U = U(x) \subset \Omega$ and any $\varphi \in C^2(\overline{U})$ such that (2.8) holds, and if we let v be defined as in (2.9), then both

$$\liminf_{r\to 0^+} \mathcal{D}_r^{s,p} u(x)(u(x) - \mathscr{M}_r^{s,p} u(x)) \geq 0$$

and

$$\limsup_{r\to 0^+} \mathcal{D}_r^{s,p} u(x)\big(u(x) - \mathscr{M}_r^{s,p} u(x)\big) \leq 0$$

hold point-wisely.

The result for viscosity solutions is a consequence of the asymptotic expansion for smooth functions, and goes as follows.

Theorem 2.7. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in C(\Omega) \cap L^\infty(\mathbb{R}^n)$. Then*

$$(-\Delta)_p^s u(x) = 0$$

in the viscosity sense if and only if

$$\mathcal{D}_r^{s,p} u(x)\big(u(x) - \mathscr{M}_r^{s,p} u(x)\big) = o_r(1) \quad \text{as } r \to 0^+ \tag{2.10}$$

holds for all $x \in \Omega$ in the viscosity sense.

Proof. For $x \in \Omega$ and any $U(x)$ neighborhood of x, defining v as in (2.9), we have that $v \in C^2(U(x)) \cap L^\infty(\mathbb{R}^n)$. By Theorem 2.3, we have that

$$\mathcal{D}_r^{s,p} v(x)(v(x) - \mathscr{M}_r^{s,p} v(x)) = (-\Delta)_p^s v(x) + \mathcal{O}(r^{2-2s}), \tag{2.11}$$

which allows to obtain the conclusion. □

2.3. *Asymptotics as $s \to 1^-$*

We prove here that sending $s \to 1^-$, for a smooth enough function the fractional p-Laplace operator approaches the p-Laplacian, defined as

$$-\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u). \tag{2.12}$$

The result is known in the mathematical community, see [19]. We note for the interested reader that the limit case $s \to 1^-$ for fractional problems has an extensive history, see e.g., [7, 33].

We give here a complete proof of this result, on the one hand for the reader convenience and on the other hand since some estimates here introduced are heavily used throughout Sec. 2.

Theorem 2.8. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in C^2(\Omega) \cap L^\infty(\mathbb{R}^n)$. Then*

$$\lim_{s\to 1^-} (1-s)(-\Delta)_p^s u(x) = -C_{p,n}\Delta_p u(x),$$

where $C_{p,n} > 0$ for every $x \in \Omega$ such that $\nabla u(x) \neq 0$.

Proof. Since $u \in C^2(\Omega)$, for any $x \in \Omega$ we have that for any $\bar{\varepsilon} > 0$ there exists $r = r(\bar{\varepsilon}) > 0$ such that

$$\text{for any } y \in B_r(x) \subset \Omega, \quad |D^2u(x) - D^2u(x+y)| < \bar{\varepsilon}. \tag{2.13}$$

We fix an arbitrary $\bar{\varepsilon}$ (as small as we wish), the corresponding r and some number $0 < \varepsilon < \min\{\bar{\varepsilon}, r\}$, to be taken arbitrarily small.

We notice that

$$(-\Delta)_p^s u(x) = \lim_{\varepsilon\to 0} \mathcal{L}_\varepsilon^{s,p} u(x) = \mathcal{L}_r^{s,p} u(x) + \lim_{\varepsilon\to 0} \left(\mathcal{L}_\varepsilon^{s,p} u(x) - \mathcal{L}_r^{s,p} u(x)\right). \tag{2.14}$$

For the first term in this sum, we have that

$$\begin{aligned}
\mathcal{L}_r^{s,p} u(x) &= \int_{|y|>r} \frac{|u(x)-u(x-y)|^{p-2}(u(x)-u(x-y))}{|y|^{n+sp}}\, dy \\
&\leq 2^{p-1}\|u\|_{L^\infty(\mathbb{R}^n)}^{p-1}\omega_n \int_r^\infty \rho^{-1-sp}\, d\rho \\
&= C(n,p,\|u\|_{L^\infty(\mathbb{R}^n)})\frac{r^{-sp}}{sp}.
\end{aligned}$$

Notice that

$$\lim_{s\to 1^-}(1-s)\mathcal{L}_r^{s,p} u(x) = 0. \tag{2.15}$$

Now, by symmetry

$$2\Big(\mathcal{L}^{s,p}_{\varepsilon}u(x)-\mathcal{L}^{s,p}_{r}u(x)\Big)=2\int_{B_r\setminus B_\varepsilon}\frac{|u(x)-u(x-y)|^{p-2}(u(x)-u(x-y))}{|y|^{n+sp}}\,dy$$
$$=\int_{B_r\setminus B_\varepsilon}\frac{|u(x)-u(x-y)|^{p-2}(u(x)-u(x-y))}{|y|^{n+sp}}\,dy$$
$$+\int_{B_r\setminus B_\varepsilon}\frac{|u(x)-u(x+y)|^{p-2}(u(x)-u(x+y))}{|y|^{n+sp}}\,dy$$
$$=\int_{B_r\setminus B_\varepsilon}\frac{|u(x)-u(x-y)|^{p-2}(2u(x)-u(x-y)-u(x+y))}{|y|^{n+sp}}\,dy$$
$$+\int_{B_r\setminus B_\varepsilon}\frac{(|u(x)-u(x+y)|^{p-2}-|u(x)-u(x-y)|^{p-2})(u(x)-u(x+y))}{|y|^{n+sp}}\,dy$$
$$=:I_{r,\varepsilon}(x)+J_{r,\varepsilon}(x). \tag{2.16}$$

Using a Taylor expansion, there exist $\underline{\delta},\overline{\delta}\in(0,1)$ such that

$$u(x)-u(x-y)=\nabla u(x)\cdot y-\frac{1}{2}\langle D^2u(x-\underline{\delta}y)y,y\rangle,$$
$$u(x)-u(x+y)=-\nabla u(x)\cdot y-\frac{1}{2}\langle D^2u(x+\overline{\delta}y)y,y\rangle.$$

Having that $|\underline{\delta}y|,|\overline{\delta}y|\le|y|<r$, recalling (2.13), we get that $|\langle(D^2u(x)-D^2u(x-\underline{\delta}y))y,y\rangle|\le\bar{\varepsilon}|y|^2$, hence

$$2u(x)-u(x-y)-u(x+y)$$
$$=-\langle D^2u(x)y,y\rangle+\frac{1}{2}(\langle D^2u(x)y,y\rangle-\langle D^2u(x-\underline{\delta}y)y,y\rangle)$$
$$+\frac{1}{2}(\langle D^2u(x)y,y\rangle-\langle D^2u(x+\overline{\delta}y)y,y\rangle)$$
$$=-\langle D^2u(x)y,y\rangle+T_1,\quad\text{with } |T_1|\le\bar{\varepsilon}|y|^2. \tag{2.17}$$

Also denoting $\omega=y/|y|\in\mathbb{S}^{n-1}$ and taking the Taylor expansion for the function $f(x)=|a-xb|^{p-2}$, we obtain

$$|u(x)-u(x-y)|^{p-2}=|y|^{p-2}|\nabla u(x)\cdot\omega-\frac{|y|}{2}\langle D^2u(x-\underline{\delta}y)\omega,\omega\rangle|^{p-2}$$
$$=|y|^{p-2}|\nabla u(x)\cdot\omega-\frac{|y|}{2}(\langle D^2u(x)\omega,\omega\rangle+\langle(D^2u(x-\underline{\delta}y)$$
$$-D^2u(x))\omega,\omega\rangle)|^{p-2}$$
$$=|y|^{p-2}|\nabla u(x)\cdot\omega-\frac{|y|}{2}(\langle D^2u(x)\omega,\omega\rangle+\mathcal{O}(\bar{\varepsilon}))|^{p-2}$$
$$=|y|^{p-2}|\nabla u(x)\cdot\omega|^{p-2}+T_2,\quad\text{with } |T_2|\le C|y|^{p-1}. \tag{2.18}$$

Thus, we have that

$$
\begin{aligned}
&|u(x)-u(x-y)|^{p-2}(2u(x)-u(x-y)-u(x+y)) \\
&\quad = -|y|^p|\nabla u(x)\cdot\omega|^{p-2}\langle D^2u(x)\omega,\omega\rangle + T_1|y|^{p-2}|\nabla u(x)\cdot\omega|^{p-2} + T_3, \\
&\qquad \text{with } |T_3|\le C|y|^{p+1}.
\end{aligned}
\tag{2.19}
$$

Using this and passing to hyper-spherical coordinates, with the notations in (2.16) we have that

$$
\begin{aligned}
I_{r,\varepsilon}(x) &= -\int_\varepsilon^r \rho^{p-1-sp}\,d\rho\int_{\mathbb{S}^{n-1}}|\nabla u(x)\cdot\omega|^{p-2}\langle D^2u(x)\omega,\omega\rangle\,d\omega + I^1_{r,\varepsilon}(x)+I^2_{r,\varepsilon}(x) \\
&= -\frac{r^{p(1-s)}-\varepsilon^{p(1-s)}}{p(1-s)}\int_{\mathbb{S}^{n-1}}|\nabla u(x)\cdot\omega|^{p-2}\langle D^2u(x)\omega,\omega\rangle d\omega \\
&\quad + I^1_{r,\varepsilon}(x)+I^2_{r,\varepsilon}(x).
\end{aligned}
\tag{2.20}
$$

With the above notations, we have that

$$
I^1_{r,\varepsilon}(x)\le \bar\varepsilon C\frac{r^{p(1-s)}-\varepsilon^{p(1-s)}}{p(1-s)}
$$

and that

$$
I^2_{r,\varepsilon}(x)\le C\int_{B_r\setminus B_\varepsilon}|y|^{p-sp}\,dy = C\frac{r^{p(1-s)+1}-\varepsilon^{p(1-s)+1}}{p(1-s)+1}.
$$

This means that

$$
\lim_{s\to1^-}\lim_{\varepsilon\to0^+}(1-s)(I^1_{r,\varepsilon}(x)+I^2_{r,\varepsilon}(x)) = \mathcal{O}(\bar\varepsilon).
$$

Thus, we get

$$
\lim_{s\to1^-}\lim_{\varepsilon\to0^+}(1-s)I_{r,\varepsilon}(x) = -\frac{1}{p}\int_{\mathbb{S}^{n-1}}|\nabla u(x)\cdot\omega|^{p-2}\langle D^2u(x)\omega,\omega\rangle\,d\omega + \mathcal{O}(\bar\varepsilon).
$$

Using again that $|\langle(D^2u(x)-D^2u(x-\underline{\delta}y))y,y\rangle|\le\bar\varepsilon|y|^2$, we also have that

$$
\begin{aligned}
u(x)-u(x-y) &= \nabla u(x)\cdot y - \frac12\langle D^2u(x)y,y\rangle + \frac12(\langle D^2u(x)y,y\rangle \\
&\quad - \langle D^2u(x-\underline{\delta}y)y,y\rangle) \\
&= \nabla u(x)\cdot y - \frac12\langle D^2u(x)y,y\rangle + \mathcal{O}(\bar\varepsilon)|y|^2, \\
u(x)-u(x+y) &= -\nabla u(x)\cdot y - \frac12\langle D^2u(x)y,y\rangle + \frac12(\langle D^2u(x)y,y\rangle \\
&\quad - \langle D^2u(x+\overline{\delta}y)y,y\rangle) \\
&= -\nabla u(x)\cdot y - \frac12\langle D^2u(x)y,y\rangle + \mathcal{O}(\bar\varepsilon)|y|^2.
\end{aligned}
$$

Taking the second-order expansion (i.e. taking the following order of the expansion in (2.18), with second-order remainder) we obtain

$$\begin{aligned}|u(x)-u(x+y)|^{p-2}&-|u(x)-u(x-y)|^{p-2}\\&=|y|^{p-1}(p-2)(\nabla u(x)\cdot\omega)|\nabla u(x)\cdot\omega|^{p-4}(\langle D^2u(x)\omega,\omega\rangle\\&\quad+\mathcal{O}(\bar\varepsilon))+T_4,\quad |T_4|\le C|y|^p.\end{aligned}$$

Thus,

$$\begin{aligned}(|u(x)-u(x+y)|^{p-2}&-|u(x)-u(x-y)|^{p-2})(u(x)-u(x+y))\\&=-|y|^p(p-2)|\nabla u(x)\cdot\omega)|^{p-2}(\langle D^2u(x)\omega,\omega\rangle+\mathcal{O}(\bar\varepsilon))+T_5,\quad |T_5|\le C|y|^{p+1}.\end{aligned}\tag{2.21}$$

Therefore, with the notation in (2.16), we get that

$$\begin{aligned}J_{r,\varepsilon}(x)&=-(p-2)\frac{r^{p(1-s)}-\varepsilon^{p(1-s)}}{p(1-s)}\\&\quad\times\left(\int_{\mathbb{S}^{n-1}}|\nabla u(x)\cdot\omega)|^{p-2}\langle D^2u(x)\omega,\omega\rangle d\omega+\mathcal{O}(\bar\varepsilon)\right)+\tilde J_{r,\varepsilon}(x),\end{aligned}\tag{2.22}$$

and

$$|\tilde J_{r,\varepsilon}(x)|\le C\frac{r^{p(1-s)+1}-\varepsilon^{p(1-s)+1}}{p(1-s)+1}.$$

We obtain that

$$\lim_{s\to1^-}\lim_{\varepsilon\to0^+}J_{r,\varepsilon}(x)=-\frac{p-2}{p}\int_{\mathbb{S}^{n-1}}|\nabla u(x)\cdot\omega)|^{p-2}\langle D^2u(x)\omega,\omega\rangle\,d\omega+\mathcal{O}(\bar\varepsilon).$$

Summing the limits for $I_{r,\varepsilon}(x)$ and $J_{r,\varepsilon}(x)$ we obtain

$$\begin{aligned}\lim_{s\to1^-}\lim_{\varepsilon\to0^+}&(\mathcal{L}^{s,p}_\varepsilon u(x)-\mathcal{L}^{s,p}_r u(x))\\&=-\frac{p-1}{2p}\int_{\mathbb{S}^{n-1}}|\nabla u(x)\cdot\omega)|^{p-2}\langle D^2u(x)\omega,\omega\rangle\,d\omega+\mathcal{O}(\bar\varepsilon).\end{aligned}$$

Using this and (2.15) into (2.14), it follows that

$$\begin{aligned}\lim_{s\to1^-}(1-s)(-\Delta)^s_pu(x)&=\lim_{s\to1^-}(1-s)\left(\mathcal{L}^{s,p}_r u(x)+\lim_{\varepsilon\to0^+}(\mathcal{L}^{s,p}_\varepsilon u(x)-\mathcal{L}^{s,p}_r u(x))\right)\\&=-\frac{p-1}{2p}\int_{\mathbb{S}^{n-1}}|\nabla u(x)\cdot\omega)|^{p-2}\langle D^2u(x)\omega,\omega\rangle\,d\omega+\mathcal{O}(\bar\varepsilon).\end{aligned}$$

Sending $\bar\varepsilon$ to zero, we get that

$$\lim_{s\to1^-}(1-s)(-\Delta)^s_pu(x)=-\frac{p-1}{2p}|\nabla u(x)|^{p-2}\int_{\mathbb{S}^{n-1}}|z(x)\cdot\omega|^{p-2}\langle D^2u(x)\omega,\omega\rangle\,d\omega,$$

with $z(x)=\nabla u(x)/|\nabla u(x)|$. We follow here the ideas in [19]. Let $U(x)\in\mathbb{M}^{n\times n}(\mathbb{R})$ be an orthogonal matrix, such that $z(x)=U(x)e_n$, where e_k denotes the kth vector of the canonical basis of $\mathbb{R}^n$. Changing coordinates $\omega' = U(x)\omega$ we obtain

$$\begin{aligned}\mathcal{I} :&= \int_{\mathbb{S}^{n-1}} |z(x)\cdot\omega'|^{p-2}\langle D^2u(x)\omega',\omega'\rangle\,d\omega'\\ &= \int_{\mathbb{S}^{n-1}} |e_n\cdot\omega|^{p-2}\langle U(x)^{-1}D^2u(x)U(x)\omega,\omega\rangle\,d\omega\\ &= \int_{\mathbb{S}^{n-1}} |\omega_n|^{p-2}\langle B(x)\omega,\omega\rangle\,d\omega,\end{aligned}$$

where $B(x) = U(x)^{-1}D^2u(x)U(x) \in \mathbb{M}^{n\times n}(\mathbb{R})$. Then we get that

$$\mathcal{I} = \sum_{i,j=1}^{n} b_{ij}(x)\int_{\mathbb{S}^{n-1}} |\omega_n|^{p-2}\omega_i\omega_j d\omega = \sum_{j=1}^{n} b_{jj}(x)\int_{\mathbb{S}^{n-1}} |\omega_n|^{p-2}\omega_j^2\,d\omega$$

by symmetry. Now,

$$\int_{\mathbb{S}^{n-1}} |\omega_n|^{p-2}\omega_j^2 d\omega = \begin{cases}\gamma_p & \text{if } j\neq n,\\ \gamma_p' & \text{if } j = n,\end{cases}$$

with γ_p, γ_p' two constants[a] for which $\gamma_p'/\gamma_p = p-1$, so

$$\mathcal{I} = \gamma_p\sum_{j=1}^{n} b_{jj}(x) + (\gamma_p' - \gamma_p)b_{nn}(x) = \gamma_p\left(\sum_{j=1}^{n} b_{jj}(x) + (p-2)b_{nn}(x)\right).$$

We notice that, since $U(x)$ is orthogonal and $D^2u(x)$ is symmetric,

$$\sum_{j=1}^{n} b_{jj}(x) = \mathrm{Tr}B(x) = \mathrm{Tr}(U(x)^{-1}D^2u(x)U(x)) = \mathrm{Tr}(D^2u(x)) = \Delta u(x)$$

and

$$\begin{aligned}b_{nn}(x) &= \langle U(x)^{-1}D^2u(x)U(x)e_n, e_n\rangle\\ &= \langle D^2u(x)U(x)e_n, U(x)e_n\rangle = \langle D^2u(x)z(x), z(x)\rangle\\ &= |\nabla u|^{-2}\langle D^2u(x)\nabla u(x), \nabla u(x)\rangle = \Delta_\infty u(x).\end{aligned}$$

Therefore,

$$\mathcal{I} = \gamma_p(\Delta u(x) + (p-2)\Delta_\infty u(x)), \tag{2.23}$$

and this leads to

$$\lim_{s\to 1^-}(1-s)(-\Delta)_p^s u(x) = -\frac{\gamma_p(p-1)}{2p}|\nabla u(x)|^{p-2}(\Delta u(x) + (p-2)\Delta_\infty u(x)).$$

[a]Precisely (see [19, Lemma 2.1])

$$\gamma_p = \frac{2\Gamma\left(\frac{1}{2}\right)^{n-2}\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{p-1}{2}\right)}{\Gamma\left(\frac{p+n}{2}\right)}, \quad \gamma_p' = \frac{2\Gamma\left(\frac{1}{2}\right)^{n-1}\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+n}{2}\right)}.$$

Recalling (2.12) and that

$$\Delta_p u(x) = |\nabla u(x)|^{p-2} \left(\Delta u(x) + (p-2)\Delta_\infty u(x)\right)$$

we conclude the proof of the lemma. □

Next, we study the asymptotic behavior of $\mathscr{M}_r^{s,p}$ as $s \to 1^-$.

Lemma 2.9. *Let $u \in C^1(\Omega) \cap L^\infty(\mathbb{R}^n)$, denoting*

$$\mathscr{M}_r^p u(x) := \int_{\mathbb{S}^{n-1}} |u(x) - u(x - r\omega)|^{p-2} u(x - r\omega) d\omega \times \left(\int_{\mathbb{S}^{n-1}} |u(x) - u(x - r\omega)|^{p-2} \, d\omega \right)^{-1} \tag{2.24}$$

it holds that

$$\lim_{s \to 1^-} \mathscr{M}_r^{s,p} u(x) = \mathscr{M}_r^p u(x) \tag{2.25}$$

and that

$$\lim_{s \to 1^-} (1-s)\mathcal{D}_r^{s,p} u(x) = \frac{1}{2r^p} \int_{\mathbb{S}^{n-1}} |u(x) - u(x - r\omega)|^{p-2} \, d\omega, \tag{2.26}$$

for all $x \in \Omega$, $r > 0$ such that $B_{2r}(x) \subset \Omega$.

Proof. Let $\varepsilon \in (0, 1/2)$, to be taken arbitrarily small in the sequel. We have that

$$\begin{aligned} \mathcal{D}_r^{s,p} u(x) &= \int_{|y| > (1+\varepsilon)r} \frac{|u(x) - u(x-y)|^{p-2}}{|y|^{n+sp-2s}(|y|^2 - r^2)^s} dy \\ &\quad + \int_{r < |y| < (1+\varepsilon)r} \frac{|u(x) - u(x-y)|^{p-2}}{|y|^{n+sp-2s}(|y|^2 - r^2)^s} \, dy \\ &= I_1^{s,\varepsilon} + I_2^{s,\varepsilon}. \end{aligned}$$

Given that for $|y| > r(1+\varepsilon)$ one has that $|y|^2 - r^2 \geq \varepsilon(\varepsilon+2)(1+\varepsilon)^{-2}|y|^2$, we get

$$\begin{aligned} |I_1^{s,\varepsilon}| &\leq \frac{(1+\varepsilon^2)^s}{\varepsilon^s(\varepsilon+2)^s} c_{n,p,\|u\|_{L^\infty(\mathbb{R}^n)}} \int_{(1+\varepsilon)r}^\infty \rho^{-1-sp} \, d\rho \\ &= \frac{(1+\varepsilon^2)^s}{\varepsilon^s(\varepsilon+2)^s} c_{n,p,\|u\|_{L^\infty(\mathbb{R}^n)}} \frac{[(1+\varepsilon)r]^{-sp}}{s} \end{aligned} \tag{2.27}$$

and it follows that

$$\lim_{s \to 1^-} (1-s) I_1^{s,\varepsilon} = 0.$$

On the other hand, integrating by parts we have that

$$
\begin{aligned}
I_2^{s,\varepsilon} &= \int_{\mathbb{S}^{n-1}} \left(\int_r^{(1+\varepsilon)r} \frac{|u(x)-u(x-\rho\omega)|^{p-2}}{\rho^{1+sp-2s}(\rho^2-r^2)^s}\, d\rho \right) d\omega \\
&= \int_{\mathbb{S}^{n-1}} \left[\frac{(\rho-r)^{1-s}}{1-s} \frac{|u(x)-u(x-\rho\omega)|^{p-2}}{\rho^{1+sp-2s}(\rho+r)^s} \Bigg|_r^{(1+\varepsilon)r} \right. \\
&\qquad \left. - \int_r^{(1+\varepsilon)r} \frac{(\rho-r)^{1-s}}{1-s} \frac{d}{d\rho} \left(\frac{|u(x)-u(x-\rho\omega)|^{p-2}}{\rho^{1+sp-2s}(\rho+r)^s} \right) d\rho \right] d\omega \\
&= \int_{\mathbb{S}^{n-1}} \left[\frac{(\varepsilon r)^{1-s}}{1-s} \frac{|u(x)-u(x-(1+\varepsilon)r\omega)|^{p-2}}{[(1+\varepsilon)r]^{1+sp-2s}[(2+\varepsilon)r]^s} d\omega \right. \\
&\qquad \left. - \int_r^{(1+\varepsilon)r} \frac{(\rho-r)^{1-s}}{1-s} \frac{d}{d\rho} \left(\frac{|u(x)-u(x-\rho\omega)|^{p-2}}{\rho^{1+sp-2s}(\rho+r)^s} \right) d\rho \right].
\end{aligned}
$$

Notice that

$$
\left| \int_r^{(1+\varepsilon)r} \frac{(\rho-r)^{1-s}}{1-s} \frac{d}{d\rho} \left(\frac{|u(x)-u(x-\rho\omega)|^{p-2}}{\rho^{1+sp-2s}(\rho+r)^s} \right) d\rho \right| \le C \max\{r^{-sp}, r^{1-sp}\} \frac{\varepsilon^{2-s}}{1-s},
$$

hence

$$
\lim_{s\to 1^-} (1-s) \int_{\mathbb{S}^{n-1}} \left[\int_r^{(1+\varepsilon)r} \frac{(\rho-r)^{1-s}}{1-s} \frac{d}{d\rho} \left(\frac{|u(x)-u(x-\rho\omega)|^{p-2}}{\rho^{1+sp-2s}(\rho+r)^s} \right) d\rho \right] d\omega = \mathcal{O}(\varepsilon).
$$

Moreover,

$$
\begin{aligned}
&\lim_{s\to 1^-} (1-s) \int_{\mathbb{S}^{n-1}} \frac{(\varepsilon r)^{1-s}}{1-s} \frac{|u(x)-u(x-(1+\varepsilon)r\omega)|^{p-2}}{[(1+\varepsilon)r]^{1+sp-2s}[(2+\varepsilon)r]^s} d\omega \\
&\quad = \frac{1}{(1+\varepsilon)^{p-1}(2+\varepsilon)r^p} \int_{\mathbb{S}^{n-1}} |u(x)-u(x-(1+\varepsilon)r\omega)|^{p-2} d\omega.
\end{aligned}
$$

Finally,

$$
\lim_{\varepsilon\to 0^+} \lim_{s\to 1^-} (1-s) I_2^{s,\varepsilon} = \frac{1}{2r^p} \int_{\mathbb{S}^{n-1}} |u(x)-u(x-r\omega)|^{p-2} d\omega
$$

and one gets (2.26). In exactly the same fashion, one proves that

$$
\begin{aligned}
&\lim_{s\to 1^-} (1-s) \int_{|y|>r} \left(\frac{|u(x)-u(x-y)|}{|y|^s} \right)^{p-2} \frac{u(x-y)}{|y|^n(|y|^2-r^2)^s}\, dy \\
&\quad = \frac{1}{2r^p} \int_{\mathbb{S}^{n-1}} |u(x)-u(x-r\omega)|^{p-2} u(x-r\omega) d\omega
\end{aligned}
$$

and (2.25) can be concluded. □

Using the notations from Lemma 2.9, we obtain some equivalent asymptotic expansions for the (classical) p-Laplacian.

Proposition 2.10. *Let $u \in C^2(\Omega)$, then the following equivalent expansions hold*:

$$\int_{\partial B_r} |u(x) - u(x-y)|^{p-2} (u(x) - u(x-y))\, d\mathcal{H}^{n-1}(y) = -c_{n,p} r^p \Delta_p u(x) + o(r^p), \tag{2.28a}$$

$$(|\nabla u|^{p-2} + \mathcal{O}(r))(u(x) - \mathscr{M}_r^p u(x)) = -c_{n,p} r^2 \Delta_p u(x) + o(r^2) \tag{2.28b}$$

for all $x \in \Omega$, $r > 0$ such that $B_{2r}(x) \subset \Omega$.

Proof. We use (2.19), (2.21) and (2.23) with a Peano remainder, to obtain that

$$\begin{aligned}
&\int_{\mathbb{S}^{n-1}} |u(x) - u(x-r\omega)|^{p-2} (u(x) - u(x-r\omega))\, d\omega \\
&\quad = -\frac{p-1}{2p} r^p \int_{\mathbb{S}^{n-1}} |\nabla u(x) \cdot \omega|^{p-2} \langle D^2 u(x)\omega, \omega\rangle d\omega + o(r^p) \\
&\quad = -r^p \frac{\gamma_p (p-1)}{2p} |\nabla u(x)|^{p-2} (\Delta u(x) + (p-2)\Delta_\infty u(x)) + o(r^p) \\
&\quad = -r^p \frac{\gamma_p (p-1)}{2p} \Delta_p u(x) + o(r^p).
\end{aligned}$$

From this, (2.28a) immediately follows with a change of variables. Thus, using the notations in the (2.24),

$$\begin{aligned}
&\left(\int_{\mathbb{S}^{n-1}} |u(x) - u(x-r\omega)|^{p-2}\, d\omega\right) (u(x) - \mathscr{M}_r^p u(x)) \\
&\quad = -\frac{\gamma_p (p-1)}{2p} r^p \Delta_p u(x) + o(r^p).
\end{aligned}$$

Proving in the same way by (2.18) that

$$\int_{\mathbb{S}^{n-1}} |u(x) - u(x-r\omega)|^{p-2}\, d\omega = r^{p-2} \int_{\mathbb{S}^{n-1}} |\nabla u(x) \cdot \omega|^{p-2}\, d\omega + o(r^{p-2}),$$

and recalling that

$$\int_{\mathbb{S}^{n-1}} |\nabla u(x) \cdot \omega|^{p-2}\, d\omega = C_{n,p} |\nabla u(x)|^{p-2}, \tag{2.29}$$

we obtain

$$(|\nabla u(x)|^{p-2} + o_r(1))(u(x) - \mathscr{M}_r^p u(x)) = -\tilde{c}_{n,p} r^2 \Delta_p u(x) + o(r^2),$$

with

$$\tilde{c}_{n,p} = \frac{\gamma_p (p-1)}{p C_{n,p}} = \frac{(p-1)(p-3)}{2p(p+n-2)}.$$

This concludes the proof of the proposition. □

Remark 2.11. We compare our result in the local setting to the existing literature, pointing out that our expansion is obtained for $p \geq 2$. The formula (2.28a) is indeed the same as Theorem 6.1 in the recent paper [15] (we point out that therein the case $p \in (1,2)$ is also studied). Furthermore, the expansion (2.28a) has some similitudes to [18, Theorem 1.1] (where instead, the so-called normalized p-Laplacian

$$\Delta_p^{\mathcal{N}} u = \Delta u + (p-2)\Delta_\infty u \tag{2.30}$$

is used). Indeed, the expansion therein obtained for $n = 2$, which we rewrite for our purposes (compare the normalized p-Laplacian with (2.12)), says that

$$\begin{aligned}\int_{B_r} &|\nabla u(x)|^{p-2}(u(x) - u(x-y))dy \\ &= \int_{B_r} \left| \frac{\nabla u(x)}{|\nabla u(x)|} \cdot y \right|^{p-2} dy(-c_p r^2 \Delta_p u(x) + o(r^2)),\end{aligned}$$

so, rescaling,

$$|\nabla u(x)|^{p-2} \int_{B_1} (u(x) - u(x-ry))dy = -c_p r^p \Delta_p u(x) + o(r^p),$$

where the last line holds up to renaming the constant. On the other hand, our expansion differs from the one given in [29], again given for the normalized p-Laplacian. Rewritten for the p-Laplace as in (2.12), the very nice formula in [29] gives that

$$|\nabla u|^{p-2}(u(x) - \tilde{\mathcal{M}}_p u(x)) = -\bar{c}_{p,n} r^2 \Delta_p u(x) + o(r^2),$$

with

$$\tilde{\mathcal{M}}_p u(x) = \frac{2+n}{p+n} \fint_{B_r(x)} u(y)\, dy + \frac{p-2}{2(p+n)} \left(\max_{\overline{B_r}(x)} u(y) + \min_{\overline{B_r}(x)} u(y) \right)$$

and

$$\bar{c}_{p,n} = \frac{1}{2(p+n)}.$$

The statement (2.28b), even though it appears weaker, still allows us to conclude that in the viscosity sense, at points $x \in \mathbb{R}^n$ for which the test functions $v(x)$ satisfy $\nabla v(x) \neq 0$, if u satisfies the mean value property, then $\Delta_p u(x) = 0$.

More precisely, we state the result for viscosity solutions (which follows from the asymptotic expansion for smooth functions).

Theorem 2.12. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in C(\Omega) \cap L^\infty(\mathbb{R}^n)$. Then*

$$(-\Delta)_p u(x) = 0$$

in the viscosity sense if and only if

$$\lim_{r \to 0^+} (u(x) - M_r^p u(x)) = o(r^p) \quad \textit{as } r \to 0^+$$

holds for all $x \in \Omega$ in the viscosity sense.

3. Gradient-Dependent Operators

3.1. *The "nonlocal" p-Laplacian*

In this section, we are interested in a nonlocal version of the p-Laplace operator, that arises in tug-of-war games, and that was introduced in [5].

This operator is the nonlocal version of the p-Laplacian given in a non-divergence form, and deprived of the $|\nabla u|^{p-2}$ factor (namely, the normalized p-Laplacian defined in (2.30)). So, for $p \in (1, +\infty)$, the (normalized) p-Laplace operator when $\nabla u \neq 0$ is defined as

$$\Delta_p^{\mathcal{N}} u := \Delta_{p,\pm}^{\mathcal{N}} u = \Delta u + (p-2)|\nabla u|^{-2}\langle D^2 u \nabla u, \nabla u\rangle.$$

By convention, when $\nabla u = 0$, as in [5],

$$\Delta_{p,+}^{\mathcal{N}} u := \Delta u + (p-2) \sup_{\xi \in \mathbb{S}^{n-1}} \langle D^2 u\, \xi, \xi\rangle$$

and

$$\Delta_{p,-}^{\mathcal{N}} u := \Delta u + (p-2) \inf_{\xi \in \mathbb{S}^{n-1}} \langle D^2 u\, \xi, \xi\rangle.$$

Let $s \in (1/2, 1)$ and $p \in [2, +\infty)$. In the nonlocal setting, we have the following definition given in [5, Sec. 4].

When $\nabla u(x) = 0$ we define

$$(-\Delta)_{p,+}^s u(x) := \frac{1}{\alpha_p} \sup_{\xi \in \mathbb{S}^{n-1}} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} \chi_{[c_p,1]}\left(\frac{y}{|y|}\cdot \xi\right) dy$$

and

$$(-\Delta)_{p,-}^s u(x) := \frac{1}{\alpha_p} \inf_{\xi \in \mathbb{S}^{n-1}} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} \chi_{[c_p,1]}\left(\frac{y}{|y|}\cdot \xi\right) dy.$$

When $\nabla u(x) \neq 0$ then

$$\begin{aligned}(-\Delta)_p^s u(x) \\ &:= (-\Delta)_{p,\pm}^s = \frac{1}{\alpha_p} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} \chi_{[c_p,1]}\left(\frac{y}{|y|}\cdot z(x)\right) dy\end{aligned}$$

with

$$z(x) = \frac{\nabla u(x)}{|\nabla u(x)|}.$$

Here, c_p, α_p are positive constants.

We remark that the case $p \in (1,2)$ is defined with the kernel $\chi_{[0,c_p]}(\frac{y}{|y|}\cdot z(x))$ for some $c_p > 0$, and can be treated in the same way.

In particular, for $p \in [2, +\infty)$ we consider

$$\alpha_p := \frac{1}{2}\int_{\mathbb{S}^{n-1}} (\omega \cdot e_2)^2 \chi_{[c_p,1]}(\omega \cdot e_1) d\omega,$$
$$\beta_p := \frac{1}{2}\int_{\mathbb{S}^{n-1}} (\omega \cdot e_1)^2 \chi_{[c_p,1]}(\omega \cdot e_1) d\omega - \alpha_p, \tag{3.1}$$

and

$$c_p \quad \text{such that } \frac{\beta_p}{\alpha_p} = p - 2. \tag{3.2}$$

With these constants, if $u \in C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then

$$\lim_{s \to 1^-} (1-s)\Delta_p^s u(x) = \Delta_p^{\mathcal{N}} u(x),$$

as proved in [5, Secs. 4.2.1 and 4.2.2].

We define now a (s,p)-mean kernel for the nonlocal p-Laplacian. For any $r > 0$ and $u \in L^\infty(\mathbb{R}^n)$, when $\nabla u(x) = 0$, we define

$$M_r^{s,p,+} u(x) := \frac{C_{s,p,+} r^{2s}}{2} \sup_{\xi \in \mathbb{S}^{n-1}} \int_{\mathcal{C}B_r} \frac{u(x+y) + u(x-y)}{|y|^n (|y|^2 - r^2)^s} \chi_{[c_p,1]}\left(\frac{y}{|y|} \cdot \xi\right) dy,$$

and

$$M_r^{s,p,-} u(x) := \frac{C_{s,p,-} r^{2s}}{2} \inf_{\xi \in \mathbb{S}^{n-1}} \int_{\mathcal{C}B_r} \frac{u(x+y) + u(x-y)}{|y|^n (|y|^2 - r^2)^s} \chi_{[c_p,1]}\left(\frac{y}{|y|} \cdot \xi\right) dy,$$

with

$$C_{s,p,+} = c_s \gamma_{p,+} \quad \text{with } \gamma_{p,+} := \left(\sup_{\xi \in \mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \chi_{[c_p,1]}(\omega \cdot \xi) d\omega\right)^{-1},$$

respectively

$$C_{s,p,-} = c_s \gamma_{p,-} \quad \text{with } \gamma_{p,-} := \left(\inf_{\xi \in \mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \chi_{[c_p,1]}(\omega \cdot \xi) d\omega\right)^{-1}.$$

When $\nabla u(x) \neq 0$, let

$$\begin{aligned} M_r^{s,p} u(x) :&= M_r^{s,p,\pm} u(x) \\ &= \frac{C_{s,p} r^{2s}}{2} \int_{\mathcal{C}B_r} \frac{u(x+y) + u(x-y)}{|y|^n (|y|^2 - r^2)^s} \chi_{[c_p,1]}\left(\frac{y}{|y|} \cdot z(x)\right) dy, \\ z(x) &= \frac{\nabla u(x)}{|\nabla u(x)|}, \end{aligned}$$

where[b]

$$C_{s,p} = c_s \gamma_p, \quad \text{with } c_s := \left(\int_1^\infty \frac{d\rho}{\rho(\rho^2 - 1)^s}\right)^{-1},$$
$$\gamma_p := \left(\int_{\mathbb{S}^{n-1}} \chi_{[c_p,1]}(\omega \cdot e_1) d\omega\right)^{-1}.$$

[b]It holds that $c(s) = \frac{2 \sin \pi s}{\pi}$.

We have the next asymptotic expansion for smooth functions.

Theorem 3.1. *Let $\eta > 0, x \in \mathbb{R}^n$ and let $u \in C^2(B_\eta(x)) \cap L^\infty(\mathbb{R}^n)$. Then*

$$u(x) = M_r^{s,p,\pm} u(x) + c(n,s,p) r^{2s} (-\Delta)^s_{p,\pm} u(x) + \mathcal{O}(r^2)$$

as $r \to 0^+$.

Proof. We prove the result for $\nabla u(x) \neq 0$ (the proof goes the same for $\nabla u(x) = 0$).

We fix some $\bar\varepsilon > 0$, and there exists $0 < r = r(\bar\varepsilon) \in (0, \eta/2)$ such that (2.13) is satisfied. Passing to spherical coordinates we have that

$$\begin{aligned}
C_{s,p} r^{2s} &\int_{\mathcal{C}B_r} \frac{dy}{|y|^n (|y|^2 - r^2)^s} \chi_{[c_p,1]}\left(\frac{y}{|y|} \cdot z(x)\right) \\
&= C_{s,p} \int_1^\infty \frac{d\rho}{\rho(\rho^2-1)^s} \int_{\mathbb{S}^{n-1}} \chi_{[c_p,1]}(\omega \cdot z(x))\, d\omega \\
&= C_{s,p} \int_1^\infty \frac{dy}{\rho(\rho^2-1)^s} \int_{\mathbb{S}^{n-1}} \chi_{[c_p,1]}(\omega \cdot e_1)\, d\omega = 1,
\end{aligned}$$

where the last line follows after a rotation (one takes $U \in \mathscr{M}^{n\times n}(\mathbb{R})$ an orthogonal matrix such that $U^{-1}(x) z(x) = e_1$ and changes variables).

It follows that for any $r > 0$,

$$u(x) - M_r^{s,p} u(x) = \frac{C_{s,p} r^{2s}}{2} \int_{\mathcal{C}B_r} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^n (|y|^2 - r^2)^s} \chi_{[c_p,1]}\left(\frac{y}{|y|} \cdot z(x)\right) dy.$$

Therefore, we have that

$$\begin{aligned}
u(x) - M_r^{s,p} u(x) &= \frac{C_{s,p}\alpha_p r^{2s}}{2} (-\Delta)^s_p u(x) - \frac{C_{s,p} r^{2s}}{2} \\
&\quad \int_{B_r} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} \chi_{[c_p,1]}\left(\frac{y}{|y|} \cdot z(x)\right) dy \\
&\quad + \frac{C_{s,p} r^{2s}}{2} \int_{\mathcal{C}B_r} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} \\
&\quad \times \left(\frac{|y|^{2s}}{(|y|^2 - r^2)^s} - 1\right) \chi_{[c_p,1]}\left(\frac{y}{|y|} \cdot z(x)\right) dy \\
&=: \frac{C_{s,p}\alpha_p r^{2s}}{2} (-\Delta)^s_p u(x) - I_r + J_r
\end{aligned}$$

and

$$
\begin{aligned}
J_r = {} & \frac{C_{s,p}}{2} \int_{\mathcal{C}B_1} \frac{2u(x) - u(x+ry) - u(x-ry)}{|y|^{n+2s}} \left(\frac{|y|^{2s}}{(|y|^2-1)^s} - 1 \right) \\
& \times \chi_{[c_p,1]} \left(\frac{y}{|y|} \cdot z(x) \right) dy \\
= {} & \frac{C_{s,p}}{2} \int_{B_{\frac{\eta}{r}} \setminus B_1} \frac{2u(x) - u(x+ry) - u(x-ry)}{|y|^{n+2s}} \left(\frac{|y|^{2s}}{(|y|^2-1)^s} - 1 \right) \chi_{[c_p,1]} \\
& \times \left(\frac{y}{|y|} \cdot z(x) \right) dy + \frac{C_{s,p}}{2} \int_{\mathcal{C}B_{\frac{\eta}{r}}} \frac{2u(x) - u(x+ry) - u(x-ry)}{|y|^{n+2s}} \\
& \times \left(\frac{|y|^{2s}}{(|y|^2-1)^s} - 1 \right) \chi_{[c_p,1]} \left(\frac{y}{|y|} \cdot z(x) \right) dy \\
= {} & J_r^1 + J_r^2.
\end{aligned}
$$

We obtain that

$$
\begin{aligned}
|J_r^2| & \le 4\|u\|_{L^\infty(\mathbb{R}^n)} \frac{C_{n,s,p}}{2} \int_{\frac{\eta}{r}}^\infty \frac{d\rho}{\rho^{1+2s}} \left(\frac{\rho^{2s}}{(\rho^2-1)^s} - 1 \right) \int_{\mathbb{S}^{n-1}} \chi_{[c_p,1]} \left(\omega \cdot z(x) \right) d\omega \\
& \le C_{s,p} \int_{\frac{\eta}{r}}^\infty \frac{d\rho}{\rho^{1+2s}} \left(\frac{\rho^{2s}}{(\rho^2-1)^s} - 1 \right),
\end{aligned}
$$

and using (A.1b), that

$$
J_r^2 = \mathcal{O}(r^{2+2s}).
$$

We have that

$$
\begin{aligned}
J_r^1 - I_r = \frac{C_{s,p}}{2} \Bigg[& \int_{B_{\frac{\eta}{r}} \setminus B_1} \frac{2u(x) - u(x+ry) - u(x-ry)}{|y|^n (|y|^2-1)^s} \chi_{[c_p,1]} \left(\frac{y}{|y|} \cdot z(x) \right) dy \\
& - \int_{B_{\frac{\eta}{r}}} \frac{2u(x) - u(x+ry) - u(x-ry)}{|y|^{n+2s}} \chi_{[c_p,1]} \left(\frac{y}{|y|} \cdot z(x) \right) dy \Bigg]
\end{aligned}
$$

which, by (2.17) and (A.1a), gives

$$
J_r^1 - I_r = \mathcal{O}(r^2).
$$

It follows that

$$
u(x) - M_r^{s,p} u(x) = \frac{C_{s,p}\alpha_p}{2} \, r^{2s} (-\Delta)_p^s u(x) + \mathcal{O}(r^2)
$$

for $r \to 0^+$, hence the conclusion. □

We recall the viscosity setting introduced in [5].

Definition 3.2. A function $u \in L^\infty(\mathbb{R}^n)$, upper (lower) semi-continuous in $\overline{\Omega}$ is a viscosity subsolution (supersolution) in Ω of

$$(-\Delta)^s_{p,\pm} u = 0, \quad \text{and we write } (-\Delta)^s_{p,\pm} u \leq (\geq) 0$$

if for every $x \in \Omega$, any neighborhood $U = U(x) \subset \Omega$ and any $\varphi \in C^2(\overline{U})$ such that (2.8) holds if we let v as in (2.9)

$$(-\Delta)^s_{p,\pm} v(x) \leq (\geq) 0.$$

A viscosity solution of $(-\Delta)^s_{p,\pm} u = 0$ is a (continuous) function that is both a subsolution and a supersolution.

Furthermore, we define an asymptotic expansion in the viscosity sense.

Definition 3.3. Let $u \in L^\infty(\mathbb{R}^n)$ upper (lower) semi-continuous in Ω. We say that

$$\lim_{r \to 0^+} (u(x) - M_r^{s,p} u(x)) = o(r^{2s})$$

holds in the viscosity sense if for any neighborhood $U = U(x) \subset \Omega$ and any $\varphi \in C^2(\overline{U})$ such that (2.8) holds, and if we let v be defined as in (2.9), then both

$$\liminf_{r \to 0^+} \frac{u(x) - M_r^{s,p} u(x)}{r^{2s}} \geq 0$$

and

$$\limsup_{r \to 0^+} \frac{u(x) - M_r^{s,p} u(x)}{r^{2s}} \leq 0$$

hold point-wisely.

The result for viscosity solutions, which is a direct consequence of Theorem 3.1 applied to the test function v, goes as follows.

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in C(\Omega) \cap L^\infty(\mathbb{R}^n)$. Then*

$$(-\Delta)^s_{p,\pm} u(x) = 0$$

in the viscosity sense if and only if

$$\lim_{r \to 0^+} (u(x) - M_r^{s,p,\pm} u(x)) = o(r^{2s})$$

holds for all $x \in \Omega$ in the viscosity sense.

We study also the limit case as $s \to 1^-$ of this version of the (s,p)-mean kernel. We state the result only in the case $\nabla u(x) \neq 0$, remarking that an analog result holds for $M_r^{s,p,\pm}$ with the suitable $M_r^{p,\pm}$.

Proposition 3.5. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in C^1(\Omega) \cap L^\infty(\mathbb{R}^n)$. For any $r > 0$ small denoting*

$$M_r^p u(x) := \frac{\gamma_p}{2} \int_{\partial B_r} (u(x+y) - u(x-y))\, \chi_{[c_p,1]}\left(\frac{y}{|y|} \cdot z(x)\right) dy$$

it holds that

$$\lim_{s \to 1^-} M_r^{s,p} u(x) = M_r^p u(x), \tag{3.3}$$

for every $x \in \Omega, r > 0$ such that $B_{2r}(x) \subset \Omega$.

Proof. We have that

$$M_r^{s,p} u(x) = \frac{C_{s,p}}{2} \int_{\mathcal{C}B_1} \frac{u(x+ry) + u(x-ry)}{|y|^n(|y|^2-1)^s} \chi_{[c_p,1]}\left(\frac{y}{|y|} \cdot z(x)\right) dy.$$

Let $\varepsilon > 0$ be fixed (to be taken arbitrarily small). Then

$$\begin{aligned} |J_\varepsilon(x)| &:= \left| \int_{B_{1+\varepsilon}} \frac{u(x+ry) + u(x-ry)}{|y|^n(|y|^2-1)^s} \chi_{[c_p,1]}\left(\frac{y}{|y|} \cdot z(x)\right) dy \right| \\ &\le \frac{2\|u\|_{L^\infty(\mathbb{R}^n)}}{\gamma_p} \int_{1+\varepsilon}^\infty \frac{dt}{t(t^2-1)^s}, \end{aligned}$$

which from Proposition A.1 gives that

$$\lim_{s \to 1^-} C_{s,p} J_\varepsilon(x) = 0.$$

On the other hand, we have that

$$\begin{aligned} I_\varepsilon(x) &= \int_{B_{1+\varepsilon} \setminus B_1} \frac{u(x+ry) + u(x-ry)}{|y|^n(|y|^2-1)^s} \chi_{[c_p,1]}\left(\frac{y}{|y|} \cdot z(x)\right) dy \\ &= \int_{\mathbb{S}^{n-1}} \left(\int_1^{1+\varepsilon} \frac{u(x+r\rho\omega) + u(x-r\rho\omega)}{\rho(\rho^2-1)^s}\, d\rho \right) \chi_{[c_p,1]}(\omega \cdot z(x))\, d\omega \end{aligned}$$

and integrating by parts, that

$$\begin{aligned} &\int_1^{1+\varepsilon} \frac{u(x+r\rho\omega) + u(x-r\rho\omega)}{\rho(\rho^2-1)^s}\, d\rho \\ &\quad = \frac{\varepsilon^{1-s}}{1-s} \frac{u(x+r(1+\varepsilon)\omega) + u(x-r(1+\varepsilon)\omega)}{(1+\varepsilon)(2+\varepsilon)^s} - I_\varepsilon^o(x) \end{aligned}$$

with

$$I_\varepsilon^o(x) := \int_1^{1+\varepsilon} \frac{(\rho-1)^{1-s}}{1-s} \frac{d}{d\rho} \left(\frac{u(x+r\rho\omega) + u(x-r\rho\omega)}{\rho(\rho+1)^s} \right) d\rho.$$

We notice that

$$|I_\varepsilon^o(x)| \leq C\frac{\varepsilon^{2-s}}{1-s},$$

hence we get

$$\lim_{s\to 1^-} C_{s,p} I_\varepsilon^o(x) = \mathcal{O}(\varepsilon).$$

Therefore, we obtain

$$\begin{aligned}\lim_{s\to 1^-} M_r^{s,p} u(x) = \frac{\gamma_p}{(1+\varepsilon)(2+\varepsilon)} \int_{\mathbb{S}^{n-1}} (u(x+r\omega)+u(x-r\omega))\\ \times \chi_{[c_p,1]}(\omega\cdot z(x))dy + \mathcal{O}(\varepsilon),\end{aligned}$$

and (3.3) follows by sending $\varepsilon \to 0$. □

We obtain furthermore an expansion for the normalized p-Laplacian, as follows.

Proposition 3.6. *If $u \in C^2(\Omega)$, then*

$$u(x) - M_r^p u(x) = -C_p r^2 \Delta_p^{\mathcal{N}} u(x) + o(r^2).$$

Proof. Using the Taylor expansion in (2.17) with a Peano remainder, we have that

$$\begin{aligned}u(x) - M_r^p u(x) &= \frac{\gamma_p}{2}\int_{\mathbb{S}^{n-1}} (2u(x) - u(x-r\omega) - u(x+r\omega))\, \chi_{[c_p,1]}(\omega\cdot z(x))d\omega\\ &= -\frac{\gamma_p r^2}{2}\int_{\mathbb{S}^{n-1}} (\langle D^2u(x)\omega,\omega\rangle)\chi_{[c_p,1]}(\omega\cdot z(x))d\omega + o(r^2).\end{aligned}$$

As in [5, Secs. 4.2.1 and 4.2.2], it holds that

$$\int_{\mathbb{S}^{n-1}} \langle D^2u(x)\omega,\omega\rangle \chi_{[c_p,1]}\left(\omega\cdot z(x)\right) d\omega = 2\alpha_p \Delta_p^{\mathcal{N}} u(x),$$

and the conclusion immediately follows. □

An analog result holds for the suitable $M_r^{p,\pm}$, and the same we can say about the following theorem in the viscosity setting (which follows from the asymptotic expansion for smooth functions).

Theorem 3.7. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in C(\Omega)\cap L^\infty(\mathbb{R}^n)$. Then*

$$(-\Delta)_p^{\mathcal{N}} u(x) = 0$$

in the viscosity sense if and only if

$$\lim_{r\to 0^+} (u(x) - M_r^p u(x)) = o(r^2) \quad \text{as } r \to 0^+$$

holds for all $x \in \Omega$ in the viscosity sense.

3.2. *The infinity fractional Laplacian*

In this section, we deal with the infinity fractional Laplacian, arising in a nonlocal tug-of-war game, as introduced in [6]. Therein, the authors deal with viscosity solutions of a Dirichlet monotone problem and a monotone double obstacle problem, providing a comparison principle on compact sets and Hölder regularity of solutions.

The infinity Laplacian in the non-divergence form is defined by omitting the term $|\nabla u|^2$, precisely when $\nabla u(x) = 0$,

$$\Delta_{\infty,+}u(x) := \sup_{\xi\in\mathbb{S}^{n-1}} \langle D^2u(x)\xi,\xi\rangle, \quad \Delta_{\infty,-}u(x) := \inf_{\xi\in\mathbb{S}^{n-1}} \langle D^2u(x)\xi,\xi\rangle,$$

and formally

$$\Delta_\infty u(x) := \frac{\Delta_{\infty,+}u(x) + \Delta_{\infty,-}u(x)}{2},$$

whereas when $\nabla u(x) \neq 0$,

$$\Delta_\infty u(x) := \Delta_{\infty,\pm}u(x) = \langle D^2u(x)\, z(x), z(x)\rangle, \quad \text{where } z(x) = \frac{\nabla u(x)}{|\nabla u(x)|}.$$

The definition in the fractional case is well posed for $s \in (1/2, 1)$, given in [6, Definition 1.1]. Let $s \in (\frac{1}{2}, 1)$. The infinity fractional Laplacian is defined in the following way:

- If $\nabla u(x) \neq 0$ then

$$(-\Delta)^s_\infty u(x) := \int_0^\infty \frac{2u(x) - u(x+\rho z(x)) - u(x-\rho z(x))}{\rho^{1+2s}}\, d\rho, \tag{3.4}$$

  where $z(x) = \frac{\nabla u(x)}{|\nabla u(x)|} \in \mathbb{S}^{n-1}$.
- If $\nabla u(x) = 0$ then

$$(-\Delta)^s_\infty u(x) := \sup_{\omega\in\mathbb{S}^{n-1}} \inf_{\zeta\in\mathbb{S}^{n-1}} \int_0^\infty \frac{2u(x) - u(x+\rho\omega) - u(x-\rho\zeta)}{\rho^{1+2s}}\, d\rho. \tag{3.5}$$

There exist "infinity harmonic functions": it is proved in [6] that the function

$$C(x) = A|x - x_0|^{2s-1} + B$$

satisfies

$$(-\Delta)^s_\infty u(x) = 0 \quad \text{for any } x \neq x_0.$$

We denote

$$\mathcal{L}u(x,\omega,\zeta) := \int_0^\infty \frac{2u(x) - u(x+\rho\omega) - u(x-\rho\zeta)}{\rho^{1+2s}}\, d\rho$$

and for $r > 0$

$$M_r^s u(x,\omega,\zeta) := c_s r^{2s} \int_r^\infty \frac{u(x+\rho\omega)+u(x-\rho\zeta)}{(\rho^2-r^2)^s\rho} d\rho,$$

with

$$c_s := \frac{1}{2}\left(\int_1^\infty \frac{d\rho}{\rho(\rho^2-1)^s}\right)^{-1} = \frac{\sin \pi s}{\pi}.$$

We define the operators

- if $\nabla u(x) \neq 0$
$$\mathscr{M}_r^{s,\infty} u(x) := M_r^s u(x, z(x), z(x)), \qquad \text{with } z(x) = \frac{\nabla u(x)}{|\nabla u(x)|};$$
- if $\nabla u(x) = 0$
$$\mathscr{M}_r^{s,\infty} u(x) := \sup_{\omega\in\mathbb{S}^{n-1}} \inf_{\zeta\in\mathbb{S}^{n-1}} M_r^s u(x,\omega,\zeta).$$

We obtain the asymptotic mean value property for smooth functions, as follows.

Theorem 3.8. *Let $\eta > 0, x \in \mathbb{R}^n$ and let $u \in C^2(B_\eta(x)) \cap L^\infty(\mathbb{R}^n)$. Then*

$$u(x) = \mathscr{M}_r^{s,\infty} u(x) + c(s) r^{2s} (-\Delta)_\infty^s u(x) + \mathcal{O}(r^2)$$

as $r \to 0^+$.

Proof. We have that

$$u(x) - M_r^s u(x,\omega,\zeta) = c_s r^{2s} \int_r^\infty \frac{2u(x)-u(x+\rho\omega)-u(x-\rho\zeta)}{\rho(\rho^2-r^2)^s} d\rho,$$

hence

$$\begin{aligned} &u(x) - M_r^s u(x,\omega,\zeta) \\ &\quad = c_s \Bigg[r^{2s}\mathcal{L}u(x,\omega,\zeta) - \int_{B_1} \frac{2u(x)-u(x+r\rho\omega)-u(x-r\rho\zeta)}{\rho^{1+2s}} d\rho \\ &\qquad + \int_{\mathcal{C}B_1} \frac{2u(x)-u(x+r\rho\omega)-u(x-r\rho\zeta)}{\rho^{1+2s}} \left(\frac{\rho^{2s}}{(\rho^2-1)^{2s}} - 1\right) d\rho \Bigg] \\ &\quad =: c_s \left(r^{2s}\mathcal{L}u(x,\omega,\zeta) - I_r + J_r\right). \end{aligned}$$

Then

$$\begin{aligned} J_r &= \int_{B_{\frac{\eta}{r}}\setminus B_1} \frac{2u(x)-u(x+r\rho\omega)-u(x-r\rho\zeta)}{\rho^{1+2s}} \left(\frac{\rho^{2s}}{(\rho^2-1)^{2s}} - 1\right) d\rho \\ &\quad + \int_{\mathcal{C}B_{\frac{\eta}{r}}} \frac{2u(x)-u(x+r\rho\omega)-u(x-r\rho\zeta)}{\rho^{1+2s}} \left(\frac{\rho^{2s}}{(\rho^2-1)^{2s}} - 1\right) d\rho \\ &=: J_r^1 + J_r^2. \end{aligned}$$

We proceed as in the proof of Theorem 3.1, using also (2.17) and Proposition A.1, and obtain that

$$J_r^2 = \mathcal{O}(r^{2s+2s}) \quad \text{and} \quad J_r^1 - I_r = \mathcal{O}(r^2).$$

This concludes the proof of the theorem. □

The main result of this section, which follows from Theorem 3.8, is stated next.

Theorem 3.9. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in C(\Omega) \cap L^\infty(\mathbb{R}^n)$. The asymptotic expansion*

$$u(x) = \mathscr{M}_r^{s,\infty} u(x) + o(r^{2s}) \qquad \text{as } r \to 0 \tag{3.6}$$

holds for all $x \in \Omega$ in the viscosity sense if and only if

$$(-\Delta)^s_\infty u(x) = 0$$

in the viscosity sense.

We investigate also the limit case $s \to 1^-$.

Proposition 3.10. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in C^1(\Omega) \cap L^\infty(\mathbb{R}^n)$. Then*

$$\lim_{s\to 1^-} \mathscr{M}_r^s u(x) = \mathscr{M}_r^\infty u(x)$$

$$:= \begin{cases} \frac{1}{2}(u(x+rz(x)) + u(x-rz(x))) & \text{when } \nabla u(x) \neq 0, \\ \frac{1}{2}\left(\sup_{\omega\in\mathbb{S}^{n-1}} u(x+r\omega) + \inf_{\zeta\in\mathbb{S}^{n-1}} u(x-r\zeta)\right) & \text{when } \nabla u(x) = 0, \end{cases}$$

for every $x \in \Omega, r > 0$ such that $B_{2r}(x) \subset \Omega$.

Proof. For some $\varepsilon > 0$ small enough, we have that

$$\begin{aligned} M_r^s u(x,\omega,\zeta) &= c_s \left(\int_{1+\varepsilon}^\infty \frac{u(x+r\rho\omega) + u(x-r\rho\zeta)}{(\rho^2-1)^s \eta} d\rho \right. \\ &\quad \left. + \int_1^{1+\varepsilon} \frac{u(x+r\rho\omega) + u(x-r\rho\zeta)}{(\eta^2-1)^s \rho} d\rho \right) \\ &=: I_s^1 + I_s^2. \end{aligned} \tag{3.7}$$

Using Proposition A.1, we get that

$$\lim_{s\to 1} c_s I_s^1 = 0.$$

Integrating by parts in I_s^2, we have

$$\left| \int_1^{1+\varepsilon} \frac{u(x+r\rho\omega)}{(\eta^2-1)^s \rho} d\rho - \frac{\varepsilon^{1-s} u(x+r(1+\varepsilon)\omega)}{(1-s)(\varepsilon+2)^s(1+\varepsilon)} \right| \leq C \frac{\varepsilon^{2-s}}{1-s},$$

thus

$$\left| \mathcal{I}_s^2 - \frac{\varepsilon^{1-s}}{(1-s)(\varepsilon+2)^s(1+\varepsilon)}(u(x+r(1+\varepsilon)\omega)+u(x-r(1+\varepsilon)\zeta)) \right| \leq C\frac{\varepsilon^{2-s}}{1-s}.$$

We get that

$$\lim_{s\to 1^-} c_s \mathcal{I}_s^2 = \frac{1}{(\varepsilon+2)(\varepsilon+1)}(u(x+r(1+\varepsilon)y)+u(x-r(1+\varepsilon)z)) + C\varepsilon.$$

Sending $\varepsilon \to 0$ we get the conclusion. □

For completeness, we show the following, already known, result.

Proposition 3.11. *Let $u \in C^2(\Omega) \cap L^\infty(\mathbb{R}^n)$. For all $x \in \Omega$ for which $|\nabla u(x)| \neq 0$ it holds that*

$$\lim_{s\to 1^-} (1-s)(-\Delta)^s_\infty u(x) = -\Delta_\infty u(x).$$

Proof. Since $u \in C^2(\Omega)$ we have that for any $\bar{\varepsilon} > 0$ there exists $r = r(\bar{\varepsilon}) > 0$ such that (2.13) holds. We prove the result for $\nabla u(x) \neq 0$ (the other case can be proved in the same way). We have that

$$\begin{aligned}(-\Delta)^s_\infty &= \int_0^r \frac{2u(x) - u\left(x+\rho z(x)\right) - u\left(x-\rho z(x)\right)}{\rho^{1+2s}}\, d\rho \\ &\quad + \int_r^\infty \frac{2u(x) - u\left(x+\rho z(x)\right) - u\left(x-\rho z(x)\right)}{\rho^{1+2s}}\, d\rho = I_r + J_r.\end{aligned}$$

We have that

$$|J_r| \leq C\|u\|_{L^\infty(\mathbb{R}^n)} \frac{r^{-2s}}{2s} \quad \text{and} \quad \lim_{s\to 1^-}(1-s)J_r = 0.$$

On the other hand, using (2.17) we have that

$$I_r = -\int_0^r \frac{\langle D^2 u(x)z(x), z(x)\rangle}{\rho}^{1-2s} d\rho + I_r^o = -\langle D^2 u(x) z(x), z(x)\rangle \frac{r^{2-2s}}{2(1-s)} + I_r^o$$

with

$$\lim_{s\to 1^-}(1-s)I_r^o = \mathcal{O}(\bar{\varepsilon}).$$

The conclusion follows by sending $\bar{\varepsilon} \to 0$. □

We mention that the mean value property for the infinity Laplacian is settled in [29]. For the sake of completeness, we however write the very simple expansion for the infinity Laplacian.

Proposition 3.12. *If $u \in C^2(\Omega)$, then*

$$u(x) - M_r^\infty u(x) = -cr^2 \Delta_\infty u(x) + o(r^2).$$

An immediate consequence is the following theorem in the viscosity setting.

Theorem 3.13. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in C(\Omega) \cap L^\infty(\mathbb{R}^n)$. Then*

$$(-\Delta)_\infty u(x) = 0$$

in the viscosity sense if and only if

$$\lim_{r \to 0^+} (u(x) - M_r^\infty u(x)) = o(r^2) \quad \textit{as } r \to 0^+$$

holds for all $x \in \Omega$ in the viscosity sense.

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Appendix A. Useful Asymptotics

We insert in this Appendix some asymptotic results, that we use along the paper.

Proposition A.1. *Let $s \in (0,1)$. For r small enough the following hold*:

$$\int_1^{\frac{1}{r}} t \left(\frac{1}{(t^2-1)^s} - \frac{1}{t^{2s}} \right) dt = \mathcal{O}(1), \tag{A.1a}$$

$$\int_{\frac{1}{r}}^{\infty} \frac{1}{t} \left(\frac{t^{2s}}{(t^2-1)^s} - 1 \right) dt = \mathcal{O}(r^2). \tag{A.1b}$$

Furthermore,

$$\lim_{s \to 1^-} (1-s) \int_{1+r}^{\infty} \frac{dt}{t(t^2-1)^s} \, dt = 0. \tag{A.2}$$

Proof. To prove (A.1a), integrating, we have that

$$\begin{aligned}\int_1^{\frac{1}{r}} t \left(\frac{1}{(t^2-1)^s} - \frac{1}{t^{2s}} \right) dt &= \frac{1}{2(1-s)} \left(\frac{(1-r^2)^{1-s} - 1}{r^{2(1-s)}} + 1 \right) \\ &= \frac{1}{2(1-s)} (\mathcal{O}(r^{2s}) + 1).\end{aligned}$$

In a similar way, we get (A.1a). To obtain (A.1b), we notice that since $\frac{1}{t} < r < 1$, with a Taylor expansion we have

$$\frac{1}{(1-\frac{1}{t^2})^s} - 1 = s\frac{1}{t^2} + o\left(\frac{1}{t^2}\right),$$

and the conclusion is reached by integrating. Furthermore,

$$\int_{1+r}^{\infty} \frac{dt}{t(t^2-1)^s} dt = \int_{1+r}^{2} \frac{dt}{t(t^2-1)^s} dt + \int_{2}^{\infty} \frac{dt}{t(t^2-1)^s}\, dt \leq \frac{c(1-r^{1-s})}{1-s} + \frac{c}{s}.$$

Multiplying by $(1-s)$ and taking the limit, we get (A.2). □

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