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# Stability and instability results for standing waves of quasi-linear Schrödinger equations

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**Abstract**
We study a class of quasi-linear Schrödinger equations arising in the theory of superfluid film in plasma physics. Using gauge transforms and a derivation process we solve, under some regularity assumptions, the Cauchy problem. Then, by means of variational methods, we study the existence, the orbital stability and instability of standing waves which minimize some associated energy.

Mathematics Subject Classification: 35J40; 58E05

## 1. Introduction and main results

Several physical situations are described by generic quasi-linear equations of the form

$$\begin{cases} \mathrm{i}\phi_t + \Delta\phi + \phi\ell'(|\phi|^2)\Delta\ell(|\phi|^2) + f(|\phi|^2)\phi = 0 & \text{in } (0,\infty)\times\mathbb{R}^N, \\ \phi(0,x) = a_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $\ell$ and $f$ are given functions. Here i is the imaginary unit, $N \geqslant 1$, $\phi : \mathbb{R}^N \to \mathbb{C}$ is a complex valued function. For example, the particular case $\ell(s) = \sqrt{1+s}$ models the self-channelling of a high-power ultra short laser in matter (see [6, 13, 33]) whereas if $\ell(s) = \sqrt{s}$, equation (1.1) appears in dissipative quantum mechanics [15]. It is also used in plasma physics and fluid mechanics [14, 26], in the theory of Heisenberg ferromagnets and magnons [2] and in condensed matter theory [29]. The dynamical features are closely related to the two functions

  

$\ell$ and $f$. Only few intents have been done to develop general theories for the Cauchy problem (see nevertheless [10, 18, 31]). In this paper we focus on the particular case $\ell(s) = s$, that is

$$\begin{cases} \mathrm{i}\phi_t + \Delta\phi + \phi\Delta|\phi|^2 + f(|\phi|^2)\phi = 0 & \text{in } (0,\infty)\times\mathbb{R}^N, \\ \phi(0,x) = a_0(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.2}$$

Our first result concerns the Cauchy problem. Due to the quasi-linear term, it seems difficult to exhibit a well-posedness result in the natural energy space

$$X_{\mathbb{C}} = \left\{ u \in H^1(\mathbb{R}^N, \mathbb{C}) : \int_{\mathbb{R}^N} |u|^2 |\nabla |u||^2 \, \mathrm{d}x < \infty \right\}.$$

The local and global well posedness of the Cauchy problem (1.1) have been studied by Poppenberg in [31] in any dimension $N \geqslant 1$ and for smooth initial data, precisely belonging to the space $H^\infty$. In [10], equation (1.1) is solved locally in the function space $L^\infty(0,T; H^{s+2}(\mathbb{R}^N)) \cap C([0,T); H^s(\mathbb{R}^N))$, where $s = 2E(N/2) + 2$ (here $E(a)$ denotes the integer part of $a$) for any initial data and smooth nonlinearities $\ell$ and $f$ such that there exists a positive constant $C_\ell$ with

$$1 - 4\sigma\ell'^2(\sigma) > C_\ell \ell'^2(\sigma), \qquad \text{for all } \sigma \in \mathbb{R}_+. \tag{1.3}$$

Note that the function $\ell(\sigma) = \sigma$ does not satisfy (1.3) and, then, it is not possible to apply [10, theorem 1.1] to problem (1.2). Before stating our result, we introduce the energy functional $\mathcal{E}$ associated with (1.2), by setting

$$\mathcal{E}(\phi) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\phi|^2 \, \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^N} |\nabla|\phi|^2|^2 \, \mathrm{d}x - \int_{\mathbb{R}^N} F(|\phi|^2) \, \mathrm{d}x,$$

for all $\phi \in X_{\mathbb{C}}$, where $F(\sigma) = \int_0^\sigma f(u) \, \mathrm{d}u$. Note that $\mathcal{E}(\phi)$ can also be written

$$\mathcal{E}(\phi) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\phi|^2 \, \mathrm{d}x + \int_{\mathbb{R}^N} |\phi|^2|\nabla|\phi||^2 \, \mathrm{d}x - \int_{\mathbb{R}^N} F(|\phi|^2) \, \mathrm{d}x.$$

We prove the following.

**Theorem 1.1.** *Let $N \geqslant 1$, $s = 2E(N/2) + 2$ and assume that $a_0 \in H^{s+2}(\mathbb{R}^N)$ and $f \in C^{s+2}(\mathbb{R}^+)$. Then there exists a $T > 0$ and a unique solution to the Cauchy problem (1.2) satisfying*

$$\phi(0,x) = a_0(x),$$

$$\phi \in L^\infty([0,T[; H^{s+2}(\mathbb{R}^N)) \cap C([0,T[; H^s(\mathbb{R}^N)),$$

*and the conservation laws*

$$\|\phi(t)\|_2 = \|a_0\|_2, \tag{1.4}$$

$$\mathcal{E}(\phi(t)) = \mathcal{E}(a_0), \tag{1.5}$$

*for all $t \in [0,T[$.*

The proof of theorem 1.1 follows the approach developed in [10]. It is based on energy methods and to overcome the loss of derivatives induced by the quasi-linear term, gauge transforms are used. We rewrite equation (1.1) as a system in $(\phi, \overline{\phi})$ where $\overline{z}$ denotes the complex conjugate of $z$. Then, we differentiate the resulting equation with respect to space and time in order to linearize the quasi-linear part and we introduce a set of new unknowns (see (2.2)). A fixed-point procedure is then applied on the linearized version. Since (1.3) does not hold we need, with respect to [10], to modify the linearized version and to perform different energy estimates on the Schrödinger part of the equation.

From now on and in the rest of the paper we assume that $f$ is a power nonlinearity $f(\sigma) = \sigma^{\frac{p-1}{2}}$ for some $p > 1$. In this case (1.2) becomes

$$\begin{cases} i\phi_t + \Delta\phi + \phi\Delta|\phi|^2 + |\phi|^{p-1}\phi = 0 & \text{in } (0,\infty)\times\mathbb{R}^N, \\ \phi(0,x) = a_0(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.6}$$

For these power nonlinearities, motivated by the classical results of the Schrödinger equation

$$\begin{cases} i\phi_t + \Delta\phi + |\phi|^{p-1}\phi = 0 & \text{in } (0,\infty)\times\mathbb{R}^N, \\ \phi(0,x) = a_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.7}$$

we address the question of existence of standing waves. We also study the standing waves associated with ground states, see theorem 1.3, their orbital stability or instability.

**Remark 1.2.** Note that if $p > 1$ is an odd integer or $p > 4E(N/2) + 9$ then $f(\sigma) = \sigma^{\frac{p-1}{2}}$ belongs to $C^{s+2}(\mathbb{R}^+)$. Clearly it would be very interesting to derive a local Cauchy theory without the restrictions on the smoothness of the nonlinearity $f(\sigma)$ and the data $a_0$. It seems out of reach with the approach used to prove theorem 1.1. We also point out that, even under smoothness assumptions, we do not say anything about possible global existence. However, our theorem 1.5 regarding instability or theorem 1.9 dealing with stability provides some indications in that direction.

By standing waves, we mean solutions of the form $\phi_\omega(t,x) = u_\omega(x)e^{-i\omega t}$. Here $\omega > 0$ is a fixed parameter and $\phi_\omega(t,x)$ satisfies problem (1.6) if and only if $u_\omega$ is a solution of the equation

$$-\Delta u - u\Delta(|u|^2) + \omega u = |u|^{p-1}u, \qquad \text{in } \mathbb{R}^N. \tag{1.8}$$

For reasons explained in remark 1.7, we assume throughout the paper that $1 < p < (3N+2)/(N-2)$ if $N \geqslant 3$ and $p > 1$ if $N = 1, 2$. A function $u \in X_{\mathbb{C}}$ is called a (complex) weak solution of equation (1.8) if

$$\Re \int_{\mathbb{R}^N} \left(\nabla u \cdot \nabla\overline{\phi} + \nabla(|u|^2)\cdot\nabla(u\overline{\phi}) + \omega u\overline{\phi} - |u|^{p-1}u\overline{\phi}\right) dx = 0 \tag{1.9}$$

for all $\phi \in C_0^\infty(\mathbb{R}^N, \mathbb{C})$ (here $\Re(z)$ is the real part of $z \in \mathbb{C}$). We say that a weak solution of (1.8) is a ground state if it satisfies

$$\mathcal{E}_\omega(u) = m_\omega, \tag{1.10}$$

where

$$m_\omega = \inf\{\mathcal{E}_\omega(u) : u \text{ is a nontrivial weak solution of (1.8)}\}.$$

Here, $\mathcal{E}_\omega$ is the action associated with (1.8) and reads

$$\mathcal{E}_\omega(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{1}{4}\int_{\mathbb{R}^N} |\nabla|u|^2|^2\,dx + \frac{\omega}{2}\int_{\mathbb{R}^N} |u|^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\,dx.$$

We denote by $\mathcal{G}_\omega$ the set of weak solutions to (1.8) satisfying (1.10). It is easy to check that $u$ is a weak solution of equation (1.8) if, and only if,

$$\mathcal{E}'_\omega(u)\phi := \lim_{t\to 0^+} \frac{\mathcal{E}_\omega(u+t\phi) - \mathcal{E}_\omega(u)}{t} = 0,$$

for every direction $\phi \in C_0^\infty(\mathbb{R}^N, \mathbb{C})$.

Our second result establishes the existence of ground states to (1.8) and derive some qualitative properties of the elements of $\mathcal{G}_\omega$. Our existence result complements the ones of [1, 12, 27, 28, 32].

**Theorem 1.3.** *For all $\omega > 0$, $\mathcal{G}_\omega$ is non-void and any $u \in \mathcal{G}_\omega$ is of the form*

$$u(x) = e^{i\theta}|u(x)|, \qquad x \in \mathbb{R}^N,$$

*for some $\theta \in \mathbb{S}^1$. In particular, the elements of $\mathcal{G}_\omega$ are, up to a constant complex phase, real valued and non-negative. Furthermore any real non-negative ground state $u \in \mathcal{G}_\omega$ satisfies the following properties:*

*(i) $u > 0$ in $\mathbb{R}^N$,*
*(ii) $u$ is a radially symmetric decreasing function with respect to some point,*
*(iii) $u \in C^2(\mathbb{R}^N)$,*
*(iv) for all $\alpha \in \mathbb{N}^N$ with $|\alpha| \leqslant 2$, there exists $(c_\alpha, \delta_\alpha) \in (\mathbb{R}_+^*)^2$ such that*

$$|D^\alpha u(x)| \leqslant C_\alpha e^{-\delta_\alpha |x|}, \qquad \textit{for all } x \in \mathbb{R}^N.$$

*Moreover, in the case $N = 1$ there exists a unique non-negative solution to (1.8), up to translations. In particular, there is a unique non-negative ground state to (1.8), up to translation.*

**Remark 1.4.**

(1) Observe that if $u \in \mathcal{G}_\omega$ is real and positive any $v(x) = e^{i\theta}u(x - y)$ for $\theta \in \mathbb{S}^1$ and $y \in \mathbb{R}^N$ belongs to $\mathcal{G}_\omega$.
(2) Except when $N = 1$ we do not know if there exists a unique real positive ground state, up to translation. Regarding the existence of excited states we conjecture that, when $N \geqslant 2$, there exist, at least, infinitely many radial real solutions to (1.8), as it is the case of the semi-linear equation

$$-\Delta u + \omega u = |u|^{p-1}u, \qquad \text{in } \mathbb{R}^N \tag{1.11}$$

corresponding to (1.7).
(3) The proof of theorem 1.3 uses the so-called dual approach introduced in [12] which transforms equation (1.8) into a semi-linear one which belongs to the framework handled in [4, 5]. We also mention that, as it is apparent from its proof, the conclusions of theorem 1.3 hold for more general nonlinearities than power-type. Precisely when (1.8) is replaced by

$$-\Delta u - u\Delta(|u|^2) + \omega u = g(u), \qquad \text{in } \mathbb{R}^N \tag{1.12}$$

and $g(u) - \omega u$ satisfies the assumptions (g0)–(g3) of [12, theorem 1.2].
(a) As pointed out to us by Selvitella [34] a boots-strap argument makes it possible to show that any ground state actually belongs to $\cap_{t>0} H^t(\mathbb{R}^N)$ and, in particular, is of class $C^\infty$.

Next we establish, for $p > 1$ sufficiently large, a result of instability by blow-up.

**Theorem 1.5.** *Assume that $\omega > 0$,*

$$3 + \frac{4}{N} < p < \frac{3N+2}{N-2}$$

*and that $f(\sigma) = \sigma^{\frac{p-1}{2}} \in C^{s+2}(\mathbb{R}^+)$. Let $u \in X_{\mathbb{C}}$ be a ground state solution of*

$$-\Delta u + u\Delta|u|^2 + \omega u = |u|^{p-1}u \qquad \textit{in } \mathbb{R}^N. \tag{1.13}$$

*Then, for all $\varepsilon > 0$, there exists $a_0 \in H^{s+2}(\mathbb{R}^N)$ such that $\|a_0 - u\|_{H^{s+2}(\mathbb{R}^N)} < \varepsilon$ and the solution $\phi(t)$ of (1.6) with $\phi(0) = a_0$ blows up in finite time in the $H^{s+2}(\mathbb{R}^N)$ norm.*

**Remark 1.6.** Concerning the nonlinearity $f$, the assumptions of theorem 1.5 hold for

$$p \geqslant 9 \text{ when } N = 1, \quad p = 7, 9, 11 \text{ or } p \geqslant 13 \text{ if } N = 2,$$
$$p = 5, 7, 9 \text{ if } N = 3 \quad \text{and} \quad p = 5 \text{ if } N = 4. \tag{1.14}$$

Clearly any weakening of the smoothness assumptions in theorem 1.1 would extend the conclusion of theorem 1.5.

To prove theorem 1.5 we assume by contradiction that the solution $\phi(t)$ exists globally in $H^{s+2}(\mathbb{R}^N)$ and we show that, actually, a blow-up behaviour must occur. For this we first establish a virial type identity. Then, we introduce some sets which are invariant under the flow, in the spirit of [3]. At this point we take advantage of ideas of [23]. Namely, by introducing a constrained approach and playing between various characterizations of the ground states, we are able to derive the blow-up result without having to solve directly a minimization problem, in contrast to [3].

When $1 < p < 3 + 4/N$, we conjecture that the ground state solutions of (1.8) are orbitally stable. However, we do not manage to prove this result. Instead, we consider the stability issue for the minimizers of the problem

$$m(c) = \inf\{\mathcal{E}(u) : u \in X, \|u\|_2^2 = c\}, \tag{1.15}$$

where the energy $\mathcal{E}$ reads as

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} |u|^2 |\nabla |u||^2 \, dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1} \, dx. \tag{1.16}$$

This problem is interesting for itself but also, hopefully, could be a first step towards considering the orbital stability of ground states of (1.8) for fixed $\omega > 0$. Indeed take any solution $u$ to problem (1.15), namely $\|u\|_2^2 = c$ and $\mathcal{E}(u) = m(c)$. Then it is a classical fact that there exists a parameter $\omega^*$, depending on $c$ and $u$, such that $u$ solves equation (1.8) with $\omega = \omega^*$ (see lemma 4.6). However, to study the orbital stability of the ground states of (1.8) via the constrained approach (as it is the case in the classical paper of Cazenave–Lions [9] on (1.11)) we need to have more information on the ground states of (1.8). In particular we need to know that they share the same $L^2$ norm. Except when $N = 1$ where we have the uniqueness of the ground states, this information is not available to us. Now, when $N = 1$ we still need to know if, when $u_1$ and $u_2$ are two distinct solutions to the minimization problem (1.15), then we have $\omega_1^* = \omega_2^*$. We do not manage to show this.

Concerning problem (1.15) we show that if $p < 3+4/N$ then $m(c) > -\infty$ for any $c > 0$. In contrast, when $p > 3 + 4/N$, we have $m(c) = -\infty$ for any $c > 0$.

**Remark 1.7.** The key point to show that $m(c) > -\infty$ if $1 < p < 3 + 4/N$ is the use of the following Gagliardo–Nirenberg inequality: for some $K > 0$ depending only on $N$ and for any $u \in X_{\mathbb{C}}$

$$\int_{\mathbb{R}^N} |u|^{p+1} \, dx \leqslant K \left(\int_{\mathbb{R}^N} |u|^2 \, dx\right)^{1-\theta} \left(\int_{\mathbb{R}^N} |u|^2 |\nabla |u||^2 \, dx\right)^{\frac{\theta N}{N-2}},$$

with

$$\theta = \frac{(p-1)(N-2)}{2(N+2)}.$$

When $p < 3 + 4/N$ we have $\theta N/(N-2) < 1$ and thus the negative term in (1.16) can be controlled by the second one. Recall that the corresponding functional setting associated with (1.11) is given, on $H^1(\mathbb{R}^N)$, by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1} \, dx$$

and

$$d(c) = \inf\{I(u) : u \in H^1(\mathbb{R}^N),\ \|u\|_2^2 = c\}.$$

In this case to control the negative term, and thus to ensure that $d(c) > -\infty$, requiring that $p < 1+4/N$ is necessary. These considerations show that the exponent $3+4/N$ plays for (1.8) the role of $1+4/N$ in (1.11). The same Gagliardo–Nirenberg inequality, and the definition of $X_{\mathbb{C}}$, also permits the range of the power to be extended to $1 < p < (3N+2)/(N-2)$. The value $(3N+2)/(N-2)$ corresponds to the classical limiting Sobolev exponent $(N+2)/(N-2)$.

**Remark 1.8.** We recall that for (1.11) the ground states are stable for $1 < p < 1+4/N$ and unstable for $p \geqslant 1+4/N$ (see [3, 9]). Thus, in light of remark 1.7, not surprisingly the condition $p > 3+4/N$ appears in our theorem 1.5.

Denote by $\mathcal{G}(c)$ the set of solutions to (1.15) and observe that if $u \in \mathcal{G}(c)$, then any $v(x) = e^{i\theta}u(x-y)$ for $\theta \in \mathbb{S}^1$ and $y \in \mathbb{R}^N$ belongs to $\mathcal{G}(c)$. Our result of orbital stability is the following.

**Theorem 1.9.** *Assume that*

$$1 < p < 3 + \frac{4}{N},$$

*and let $c > 0$ be such that $m(c) < 0$. Then $\mathcal{G}(c)$ is non-void and, if $f(\sigma) = \sigma^{\frac{p-1}{2}} \in C^{s+2}(\mathbb{R}^+)$, it is orbitally stable.*

**Remark 1.10.** In theorem 1.9 when we say that $\mathcal{G}(c)$ is orbitally stable we mean the following: for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any initial data $a_0 \in X_{\mathbb{C}} \cap H^{s+2}(\mathbb{R}^N)$ such that $\inf_{u\in\mathcal{G}(c)} \|a_0 - u\|_{H^1} < \delta$ the solution $\phi(t,\cdot)$ of (1.2) with initial condition $a_0$ satisfies

$$\sup_{0<t<T_0} \inf_{u\in\mathcal{G}(c)} \|\phi(t,\cdot) - u\|_{H^1} < \varepsilon,$$

where $T_0 > 0$ is the existence time for $\phi$ given by theorem 1.1. We observe that our assumptions permit one to treat the case $p = 3$ in any dimension $N \geqslant 1$.

The proof of theorem 1.9 relies, in an essential way, on the convergence of any real minimizing sequences for (1.15). This convergence result being established, the proof of orbital stability follows in a standard fashion.

**Theorem 1.11.** *Assume that $1 < p < 3+4/N$ and $c > 0$ is such that $m(c) < 0$. Then for any real minimizing sequence of (4.2), there exists a subsequence that is strongly converging in $X$, up to a translation in $\mathbb{R}^N$.*

The proof of theorem 1.11 itself relies on the use of concentration-compactness arguments. The key difficulty is to rule out a possible dichotomy. For this when one considers (1.11) it suffices to use the fact that the nonlinearity is superlinear. Here it is essential to make use of the autonomous feature of (1.8) as we need to use scaling properties.
We end this paper discussing the condition $m(c) < 0$.

**Theorem 1.12.** *The following results hold:*

*(1) If $1 < p < 1+4/N$, then $m(c) < 0$ for all $c > 0$.*
*(2) If $1+4/N \leqslant p \leqslant 3+4/N$, then there exists $c(p,N) > 0$ such that*
- *(i) If $0 < c < c(p,N)$ then $m(c) = 0$ and $m(c)$ does not admit a minimizer.*
- *(ii) If $c > c(p,N)$ then $m(c) < 0$ and $m(c)$ admits a minimizer. In addition, the map $(c(p,N),\infty) \ni \lambda \to m(\lambda)$ is strictly decreasing.*

**Remark 1.13.** We recall that dealing with (1.11) we have that $m(c) < 0$ for any $c > 0$ (see [35]) if and only if $1 < p < 1+4/N$. Theorem 1.12 reveals that the minimizing problem (1.15) has a much richer structure.

**Notations.**

(1) For a function $f : \mathbb{R}^N \to \mathbb{R}^N$ and $1 \leqslant j \leqslant N$, we denote by $\partial_j f$ the partial derivative with respect to the $j$th coordinate.
(2) $M(\mathbb{R}^N)$ is the set of measurable functions in $\mathbb{R}^N$. For any $p > 1$ we denote by $L^p(\mathbb{R}^N)$ the space of $f$ in $M(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} |f|^p \, \mathrm{d}x < \infty$.
(3) The norm $(\int_{\mathbb{R}^N} |f|^p \, \mathrm{d}x)^{1/p}$ in $L^p(\mathbb{R}^N)$ is denoted by $\| \cdot \|_p$.
(4) For $s \in \mathbb{N}$, we denote by $H^s(\mathbb{R}^N)$ the Sobolev space of functions $f$ in $L^2(\mathbb{R}^N)$ having generalized partial derivatives $\partial_i^k f$ in $L^2(\mathbb{R}^N)$, for $i = 1, \ldots, N$ and $0 \leqslant k \leqslant s$.
(5) The norm $(\int_{\mathbb{R}^N} |f|^2 \, \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla f|^2 \, \mathrm{d}x)^{1/2}$ in $H^1(\mathbb{R}^N)$ is denoted by $\| \cdot \|$ and, more generally, the norm in $H^s$ is denoted by $\| \cdot \|_{H^s}$.
(6) $\mathcal{L}^N(E)$ denotes the Lebesgue measure of a measurable set $E \subset \mathbb{R}^N$.
(7) For $R > 0$, $B(0, R)$ is the ball in $\mathbb{R}^N$ centred at zero with radius $R$.
(8) $\Re(z)$ (respectively $\Im(z)$) denotes the real part (respectively the imaginary part) of a complex number $z$.
(9) For a real number $r$, we denote by $E(r)$ the integer part of $r$.
(10) $X$ denotes the restriction of $X_{\mathbb{C}}$ to real functions.
(11) $K$, $K(p, N)$ denote various constants which are not essential in the problem and may vary from line to line.

**Organization of the paper**

In section 2, we prove theorem 1.1 concerning the well-posedness result for equation (1.2). In section 3, we establish the existence and properties of the ground states solutions of (1.8), theorem 1.3 and we prove the instability result, theorem 1.5. In section 4, we study the minimization problem (1.15). Assuming that $m(c) < 0$ we prove the existence of a minimizer and we study under which conditions $m(c) < 0$ hold. Finally, in section 5, we prove the convergence of all minimizing sequences of (1.15) and thus derive the stability result, theorem 1.9.

## 2. The Cauchy problem

This section is fully devoted to the proof of theorem 1.1.

We first rewrite equation (1.2) into a system involving $\phi$ and $\overline{\phi}$ in the following way:

$$2i \begin{pmatrix} \phi_t \\ \overline{\phi}_t \end{pmatrix} + \mathcal{A}(\phi) \begin{pmatrix} \Delta\phi \\ \Delta\overline{\phi} \end{pmatrix} - \begin{pmatrix} 2\phi|\nabla\phi|^2 + \phi f(|\phi|^2) \\ -2\overline{\phi}|\nabla\phi|^2 - \overline{\phi} f(|\phi|^2) \end{pmatrix} = 0, \tag{2.1}$$

where

$$\mathcal{A}(\phi) = \begin{pmatrix} 1 + |\phi|^2 & \phi^2 \\ -\overline{\phi}^2 & -(1 + |\phi|^2) \end{pmatrix}.$$

A direct calculation shows that $\mathcal{A}(\phi)$ is invertible and that

$$\mathcal{A}^{-1}(\phi) = \frac{1}{1 + 2|\phi|^2} \mathcal{A}(\phi).$$

In order to overcome the loss of derivatives and to linearize the quadratic term involving $\nabla\phi$, we differentiate the equation with respect to space and time variables to obtain a new system in $\phi_0, \ldots, \phi_{N+2}$, where $\phi_0 = \phi$ and

$$\forall 1 \leqslant j \leqslant N, \qquad \phi_j = \partial_j \phi, \quad \phi_{N+1} = \mathrm{e}^{g(|\phi|^2)} \phi_t, \quad \phi_{N+2} = \mathrm{e}^{q(|\phi|^2)} \Delta\phi. \tag{2.2}$$

The functions $g$ and $q$ are used as gauge transforms and their role will be explained later. We also set $\Phi^* = (\phi_j)_{j=0}^{N}$ and $\Phi = (\phi_j)_{j=0}^{N+2}$. Equation (2.1) can be rewritten as

$$2i\begin{pmatrix}(\phi_0)_t\\ (\overline{\phi}_0)_t\end{pmatrix} + \mathcal{A}(\phi_0)\begin{pmatrix}\Delta\phi_0\\ \Delta\overline{\phi}_0\end{pmatrix} + \mathcal{F}_0(\Phi^*) = 0, \tag{2.3}$$

where $\mathcal{F}_0$ is a smooth function depending only on $\Phi^*$. Differentiating equation (2.3) with respect to $x_j$ for $j = 1, \ldots, N$, we obtain

$$2i\begin{pmatrix}(\phi_j)_t\\ (\overline{\phi}_j)_t\end{pmatrix} + \mathcal{A}(\phi_0)\begin{pmatrix}\Delta\phi_j\\ \Delta\overline{\phi}_j\end{pmatrix} + \sum_{k=1}^{N} B(\phi_0,\phi_k)\begin{pmatrix}T_{kj}\phi_{N+2}\\ T_{kj}\overline{\phi}_{N+2}\end{pmatrix}$$
$$+ C(\phi_0,\phi_j)\begin{pmatrix}e^{-q(|\phi_0|^2)}\phi_{N+2}\\ e^{-q(|\phi_0|^2)}\overline{\phi}_{N+2}\end{pmatrix} + \begin{pmatrix}F(\Phi^*,\phi_j)\\ -\overline{F}(\Phi^*,\phi_j)\end{pmatrix} = 0,$$

where $B$, $C$ and $F$ are smooth functions of their arguments and especially

$$C(\phi_0,\phi_j) = \partial_j\mathcal{A}(\phi_0) = \begin{pmatrix}\phi_0\overline{\phi_j} + \overline{\phi_0}\phi_j & 2\phi_0\phi_j\\ -2\overline{\phi}_0\overline{\phi}_j & -\phi_0\overline{\phi_j} - \overline{\phi_0}\phi_j\end{pmatrix}.$$

For $i, j = 1, \ldots, N$, $T_{ij}$ is the following operator of order 0

$$T_{ij}\phi = \partial_i\partial_j\Delta^{-1}(\mathrm{e}^{-q(|u_0|^2)}\phi).$$

We can rewrite these equations as follows:

$$2i\begin{pmatrix}(\phi_j)_t\\ (\overline{\phi}_j)_t\end{pmatrix} + \mathcal{A}(\phi_0)\begin{pmatrix}\Delta\phi_j\\ \Delta\overline{\phi}_j\end{pmatrix} + \mathcal{F}_j(\Phi^*,\phi_{N+2},T\phi_{N+2}) = 0, \tag{2.4}$$

where $\mathcal{F}_j$ is a smooth function of its arguments. Differentiating equation (2.3) with respect to $t$, we derive

$$2i\begin{pmatrix}(\mathrm{e}^{-f(|\phi_0|^2)}\phi_{N+1})_t\\ (\mathrm{e}^{-f(|\phi_0|^2)}\overline{\phi}_{N+1})_t\end{pmatrix} + C(\phi_0,\mathrm{e}^{-f(|\phi_0|^2)}\phi_{N+1})\begin{pmatrix}\mathrm{e}^{-q(|\phi_0|^2)}\phi_{N+2}\\ \mathrm{e}^{-q(|\phi_0|^2)}\overline{\phi}_{N+2}\end{pmatrix} + \mathcal{A}(\phi_0)\begin{pmatrix}\Delta(\mathrm{e}^{-f(|\phi_0|^2)}\phi_{N+1})\\ \Delta(\mathrm{e}^{-f(|\phi_0|^2)}\overline{\phi}_{N+1})\end{pmatrix}$$
$$+ \sum_{k=1}^{N} B(\phi_0,\phi_k)\begin{pmatrix}\partial_k(\mathrm{e}^{-f(|\phi_0|^2)}\phi_{N+1})\\ \partial_k(\mathrm{e}^{-f(|\phi_0|^2)}\overline{\phi}_{N+1})\end{pmatrix} + \begin{pmatrix}F(\Phi^*,\mathrm{e}^{-f(|\phi_0|^2)}\phi_{N+1})\\ -\overline{F}(\Phi^*,\mathrm{e}^{-f(|\phi_0|^2)}\phi_{N+1})\end{pmatrix} = 0, \tag{2.5}$$

which can be rewritten as

$$2i\begin{pmatrix}(\phi_{N+1})_t\\ (\overline{\phi}_{N+1})_t\end{pmatrix} + \mathcal{A}(\phi_0)\begin{pmatrix}\Delta\phi_{N+1}\\ \Delta\overline{\phi}_{N+1}\end{pmatrix} + \sum_{k=1}^{N}\mathcal{D}(\phi_0,\phi_k)\begin{pmatrix}\partial_k\phi_{N+1}\\ \partial_k\overline{\phi}_{N+1}\end{pmatrix} + \mathcal{G}(\Phi,T\phi_{N+2}) = 0, \tag{2.6}$$

where $\mathcal{D}$ and $\mathcal{G}$ are smooth functions of their arguments. By applying the operator $\Delta$ on equation (2.3), we obtain

$$2i\begin{pmatrix}(\phi_{N+2})_t\\ (\overline{\phi}_{N+2})_t\end{pmatrix} + \mathcal{A}(\phi_0)\begin{pmatrix}\Delta\phi_{N+2}\\ \Delta\overline{\phi}_{N+2}\end{pmatrix} + \sum_{k=1}^{N}\mathcal{E}(\phi_0,\phi_k)\begin{pmatrix}\partial_k\phi_{N+2}\\ \partial_k\overline{\phi}_{N+2}\end{pmatrix} + \mathcal{I}(\Phi,T\phi_{N+2}) = 0, \tag{2.7}$$

where $\mathcal{E}$ and $\mathcal{I}$ are also smooth functions of their arguments. At this point, we need to make more precise the matrices $B$, $\mathcal{D}$ and $\mathcal{E}$ since they represent the quasi-linear part of the equations. A direct computation gives

$$B(\phi_0,\phi_k) = \begin{pmatrix}2\phi_0\overline{\phi}_k & 2\phi_0\phi_k\\ -2\overline{\phi}_0\overline{\phi}_k & -2\overline{\phi}_0\phi_k\end{pmatrix},$$

$$\mathcal{D}(\phi_0,\phi_k) = B(\phi_0,\phi_k) - 2f'(|\phi_0|^2)\mathcal{A}(\phi_0)\begin{pmatrix}\phi_0\overline{\phi}_k+\overline{\phi}_0\phi_k & 0\\ 0 & \phi_0\overline{\phi}_k+\overline{\phi}_0\phi_k\end{pmatrix},$$

$$\mathcal{E}(\phi_0,\phi_k) = B(\phi_0,\phi_k) + 2C(\phi_0,\phi_k) - 2\mathcal{A}(\phi_0)q'(|\phi_0|^2)\begin{pmatrix}\phi_0\overline{\phi}_k+\overline{\phi}_0\phi_k & 0\\ 0 & \phi_0\overline{\phi}_k+\overline{\phi}_0\phi_k\end{pmatrix}.$$

Usual energy estimates for Schrödinger equations require that the diagonal coefficients of $\mathcal{D}$ and $\mathcal{E}$ in equations (2.6) and (2.7) are purely imaginary. Roughly speaking, this allows one to integrate by parts the bad terms including first order derivatives of the unknown. This is why we make use of gauge transforms $g$ and $q$. Finally, in order to avoid any smallness assumption on the initial data, we need to transform slightly equation (2.3) in the following way. We multiply the equation by $\mathcal{A}^{-1}(\phi_0)$ and we split the matrix in front of the time derivatives of $\phi_0$ into

$$\mathcal{A}^{-1}(\phi_0) = \mathrm{Id} + \left(\mathcal{A}^{-1}(\phi_0) - \mathrm{Id}\right),$$

where Id is the $2\times 2$ identity matrix. Then recalling that $\partial_t\phi_0 = e^{-g(|\phi_0|^2)}\phi_{N+1}$, we rewrite equation (2.3) in

$$2i\begin{pmatrix}(\phi_0)_t\\ (\overline{\phi}_0)_t\end{pmatrix} + \begin{pmatrix}\Delta\phi_0\\ \Delta\overline{\phi}_0\end{pmatrix} + \mathcal{G}_0(\Phi) = 0, \tag{2.8}$$

where

$$\mathcal{G}_0(\Phi) = \mathcal{A}^{-1}(\phi_0)\mathcal{F}_0(\Phi^*) + i\mathrm{e}^{-g(|\phi_0|^2)}\left(\mathcal{A}^{-1}(\phi_0) - \mathrm{Id}\right)\begin{pmatrix}\phi_{N+1}\\ \overline{\phi}_{N+1}\end{pmatrix}.$$

We have then transformed equation (1.2) into the following system:

$$2i\begin{pmatrix}(\phi_0)_t\\ (\overline{\phi}_0)_t\end{pmatrix} + \begin{pmatrix}\Delta\phi_0\\ \Delta\overline{\phi}_0\end{pmatrix} + \mathcal{G}_0(\Phi) = 0, \tag{2.9}$$

for $j = 1,\dots,N$

$$2i\begin{pmatrix}(\phi_j)_t\\ (\overline{\phi}_j)_t\end{pmatrix} + \mathcal{A}(\phi_0)\begin{pmatrix}\Delta\phi_j\\ \Delta\overline{\phi}_j\end{pmatrix} + \mathcal{F}_j(\Phi^*,\phi_{N+2},T\phi_{N+2}) = 0, \tag{2.10}$$

$$2i\begin{pmatrix}(\phi_{N+1})_t\\ (\overline{\phi}_{N+1})_t\end{pmatrix} + \mathcal{A}(\phi_0)\begin{pmatrix}\Delta\phi_{N+1}\\ \Delta\overline{\phi}_{N+1}\end{pmatrix} + \sum_{k=1}^{N}\mathcal{D}(\phi_0,\phi_k)\begin{pmatrix}\partial_k\phi_{N+1}\\ \partial_k\overline{\phi}_{N+1}\end{pmatrix} + \mathcal{G}(\Phi,T\phi_{N+2}) = 0, \tag{2.11}$$

$$2i\begin{pmatrix}(\phi_{N+2})_t\\ (\overline{\phi}_{N+2})_t\end{pmatrix} + \mathcal{A}(\phi_0)\begin{pmatrix}\Delta\phi_{N+2}\\ \Delta\overline{\phi}_{N+2}\end{pmatrix} + \sum_{k=1}^{N}\mathcal{E}(\phi_0,\phi_k)\begin{pmatrix}\partial_k\phi_{N+2}\\ \partial_k\overline{\phi}_{N+2}\end{pmatrix} + \mathcal{I}(\Phi,T\phi_{N+2}) = 0. \tag{2.12}$$

We now apply a fixed-point theorem to system (2.9)–(2.12). Let $s$ be as in theorem 1.1 and introduce the function space

$$\mathcal{X}_T = \left\{\begin{array}{l}\Phi = (\phi_j)_{j=0}^{N+2} : \phi_j \in C([0,T];L^2(\mathbb{R}^N)) \cap L^\infty(0,T;H^s(\mathbb{R}^N)),\\ \|\Phi\|_{\mathcal{X}_T} = \displaystyle\sum_{j=0}^{N+2}\sup_{0\leqslant t\leqslant T}\|\phi_j(t)\|_{H^s(\mathbb{R}^N)} < \infty\end{array}\right\}.$$

For $M = (m_j)_{j=0}^{N+2} \in (\mathbb{R}_+^*)^{N+3}$ and $r \in \mathbb{R}_+^*$, we denote

$$\mathcal{X}_T(M,r) = \left\{\begin{array}{l}\Phi = (\phi_j)_{j=0}^{N+2} \in \mathcal{X}_T : \forall j = 0,..,N+2\ \ \|\phi_j\|_{L^\infty(0,T;H^s(\mathbb{R}^N))} \leqslant m_j\\ \|(\phi_0)_t\|_{L^\infty(0,T;H^{E(\frac{N}{2})+1}(\mathbb{R}^N))} \leqslant r \text{ and } \phi_0(0,x) = a_0(x)\end{array}\right\},$$

and let $\Psi = (\psi_j)_{j=0}^{N+2} \in \mathcal{X}_T(M, r)$. Denote $\Psi^* = (\psi_j)_{j=0}^{N}$ and consider the linearized version of system (2.9)–(2.12) as follows:

$$2i \begin{pmatrix} (\phi_0)_t \\ (\overline{\phi}_0)_t \end{pmatrix} + \begin{pmatrix} \Delta\phi_0 \\ \Delta\overline{\phi}_0 \end{pmatrix} + \mathcal{G}_0(\Psi) = 0, \tag{2.13}$$

for $j = 1, \ldots, N$

$$2i \begin{pmatrix} (\phi_j)_t \\ (\overline{\phi}_j)_t \end{pmatrix} + \mathcal{A}(\psi_0) \begin{pmatrix} \Delta\phi_j \\ \Delta\overline{\phi}_j \end{pmatrix} + \mathcal{F}_j(\Psi^*, \psi_{N+2}, T\psi_{N+2}) = 0, \tag{2.14}$$

$$2i \begin{pmatrix} (\phi_{N+1})_t \\ (\overline{\phi}_{N+1})_t \end{pmatrix} + \mathcal{A}(\psi_0) \begin{pmatrix} \Delta\phi_{N+1} \\ \Delta\overline{\phi}_{N+1} \end{pmatrix} + \sum_{k=1}^{N} \mathcal{D}(\psi_0, \psi_k) \begin{pmatrix} \partial_k\phi_{N+1} \\ \partial_k\overline{\phi}_{N+1} \end{pmatrix} + \mathcal{G}(\Psi, T\psi_{N+2}) = 0, \tag{2.15}$$

$$2i \begin{pmatrix} (\phi_{N+2})_t \\ (\overline{\phi}_{N+2})_t \end{pmatrix} + \mathcal{A}(\psi_0) \begin{pmatrix} \Delta\phi_{N+2} \\ \Delta\overline{\phi}_{N+2} \end{pmatrix} + \sum_{k=1}^{N} \mathcal{E}(\psi_0, \psi_k) \begin{pmatrix} \partial_k\phi_{N+2} \\ \partial_k\overline{\phi}_{N+2} \end{pmatrix} + \mathcal{I}(\Psi, T\psi_{N+2}) = 0. \tag{2.16}$$

Let $\mathcal{Z} = \left[L^\infty(0, T; H^s(\mathbb{R}^N)) \cap C([0, T]; L^2(\mathbb{R}^N))\right]^{N+3}$. Then the Cauchy problem (2.13)–(2.16) with the initial condition

$$\begin{aligned}
&\phi_0(0, x) = a_0(x), \text{ for } j = 1, \ldots, N, \ \phi_j(0, x) = \partial_j a_0(x), \\
&\phi_{N+1}(0, x) = \frac{1}{2i} e^{g(|a_0(x)|^2)} \left(-\mathcal{A}(\phi_0(0))\Delta a_0(x) - \mathcal{F}_0(\Psi^*(0))\right), \\
&\phi_{N+2}(0, x) = e^{q(|a_0(x)|^2)} \Delta a_0(x),
\end{aligned}$$

defines a mapping $\mathcal{S}$

$$\begin{aligned}
\mathcal{S} : \mathcal{Z} &\longrightarrow \mathcal{Z} \\
\Psi &\longmapsto \Phi.
\end{aligned}$$

For more details on the existence result for system (2.13)–(2.16), we refer to [10, 31]. In order to prove theorem 1.1, we have to find a time $T > 0$ and constants $M \in (\mathbb{R}_+^*)^{N+3}$ and $r \in \mathbb{R}_+^*$ such that $\mathcal{S}$ maps the closed ball $\mathcal{X}_T(M, r)$ into itself and is a contraction mapping under the constraint that it acts on $\mathcal{X}_T(M, r)$ in the norm $\sum_{j=0}^{N+2} \sup_{t\in[0,T]} \|\phi_j\|_{L^2}$. We begin with equation (2.16) and perform an $H^s$-estimate. Following [10], we apply the operator $(1-\Delta)^{\frac{s}{2}}$ on equation (2.16) and multiply the resulting equation by $\mathcal{A}^{-1}(\phi_0)$ to obtain, denoting $\chi = (1-\Delta)^{\frac{s}{2}}\phi_{N+2}$,

$$\begin{aligned}
2i\mathcal{A}^{-1}(\psi_0) \begin{pmatrix} (\chi)_t \\ (\overline{\chi})_t \end{pmatrix} + \begin{pmatrix} \Delta\chi \\ \Delta\overline{\chi} \end{pmatrix} + \sum_{k=1}^{N} \mathcal{L}(\psi_0, \psi_k, \partial_k\psi_0) \begin{pmatrix} \partial_k\chi \\ \partial_k\overline{\chi}_{N+2} \end{pmatrix} \\
+ \mathcal{J}_{j=0}^{s}(D^j\Psi, D^j\phi_{N+2}, T\psi_{N+2}) = 0,
\end{aligned} \tag{2.17}$$

where $D^j$ denotes any space derivation of order less than or equal to $s$ with respect to the $j$th space coordinate. The matrix $\mathcal{L}$ reads

$$\mathcal{L}(\psi_0, \psi_k, \partial_k\psi_0) = \mathcal{A}^{-1}(\psi_0) \left(\mathcal{E}(\psi_0, \psi_k) + s\partial_k\mathcal{A}(\psi_0)\right).$$

We note here that the dependence of $\mathcal{J}$ in $\phi_{N+2}$ and its derivatives is affine. We are now able to choose the gauge transform $q$. Recall that

$$\mathcal{E}(\psi_0, \psi_k) = B(\psi_0, \psi_k) + 2C(\psi_0, \psi_k) - 2\mathcal{A}(\psi_0)q'(|\psi_0|^2) \begin{pmatrix} \psi_0\overline{\psi}_k + \overline{\psi}_0\psi_k & 0 \\ 0 & \psi_0\overline{\psi}_k + \overline{\psi}_0\psi_k \end{pmatrix},$$

a direct calculation shows that for $j = 1, 2$ (denoting by $b^{11}$ and $b^{22}$ the diagonal coefficients of a $2 \times 2$ matrix $b$),

$$\Re \left(\mathcal{A}^{-1}(\psi_0) \left(B(\psi_0, \psi_k) + 2C(\psi_0, \psi_k)\right)\right)^{jj} = \frac{3}{1 + 2|\psi|^2} \left(\psi_0\overline{\psi}_k + \overline{\psi}_0\psi_k\right).$$

Then choosing

$$q(\sigma) = \tfrac{3}{4}\ln(1+2\sigma)$$

gives

$$\Re\left(\mathcal{A}^{-1}(\psi_0)\mathcal{E}(\psi_0,\psi_k)\right)^{jj} = 0.$$

Furthermore, by differentiating equation (2.17) $s$ times in space, we add in matrix $\mathcal{L}$ the term $s\mathcal{A}^{-1}(\psi_0)\partial_k\mathcal{A}(\psi_0)$ which is not eliminated by $q$. As a consequence we have to use a second gauge transform by putting $\kappa = e^{b(|\psi_0|^2)}\chi$ solution to

$$2i\mathcal{A}^{-1}(\psi_0)\begin{pmatrix}(\kappa)_t\\(\overline{\kappa})_t\end{pmatrix} + \begin{pmatrix}\Delta\kappa\\\Delta\overline{\kappa}\end{pmatrix} + \sum_{k=1}^{N}\mathcal{M}(\psi_0,\psi_k,\partial_k\psi_0)\begin{pmatrix}\partial_k\kappa\\\partial_k\overline{\kappa}_{N+2}\end{pmatrix}$$
$$+\mathcal{K}^s_{j=0}(D^j\Psi, D^j\phi_{N+2}, T\psi_{N+2}, (\psi_0)_t) = 0, \tag{2.18}$$

where

$$\mathcal{M}(\psi_0,\psi_k,\partial_k\psi_0) = \mathcal{L}(\psi_0,\psi_k,\partial_k\psi_0) - 2\begin{pmatrix}\partial_k b(|\psi_0|^2) & 0\\ 0 & \partial_k b(|\psi_0|^2)\end{pmatrix}.$$

Note that the matrix $\mathcal{K}$ also depends on $(\psi_0)_t$. Once again, an easy calculation shows that if we choose $b$ such that

$$b(\sigma) = \frac{s}{4}\ln(1+2\sigma),$$

then for $j = 1, 2$

$$\Re\left(s\mathcal{A}^{-1}(\psi_0)\partial_k\mathcal{A}(\psi_0) - 2\begin{pmatrix}\partial_k b(|\psi_0|^2) & 0\\ 0 & \partial_k b(|\psi_0|^2)\end{pmatrix}\right)^{jj} = 0.$$

We are now able to perform the suitable energy estimate on equation (2.18). Multiplying equation (2.18) by $\overline{\kappa}$, integrating over $\mathbb{R}^N$ and taking the first line of the resulting system lead to

$$i\int_{\mathbb{R}^N}\frac{1+|\psi_0|^2}{1+2|\psi_0|^2}\kappa_t\overline{\kappa}\,dx + i\int_{\mathbb{R}^N}\frac{\psi_0^2}{1+2|\psi_0|^2}\overline{\kappa}_t\overline{\kappa}\,dx + \int_{\mathbb{R}^N}\Delta\kappa\overline{\kappa}\,dx$$
$$+\sum_{k=1}^{N}\int_{\mathbb{R}^N}\mathcal{M}^{11}(\psi_0,\psi_k,\partial_k\psi_0)(\partial_k\kappa)\overline{\kappa}\,dx + \int_{\mathbb{R}^N}\mathcal{M}^{12}(\psi_0,\psi_k,\partial_k\psi_0)(\partial_k\overline{\kappa})\overline{\kappa}\,dx$$
$$+\int_{\mathbb{R}^N}\mathcal{K}^s_{j=0}(D^j\Psi, D^j\phi_{N+2}, T\psi_{N+2}, (\psi_0)_t)\overline{\kappa}\,dx. \tag{2.19}$$

We take the imaginary part of equation (2.19). We have

$$\Im\left(i\int_{\mathbb{R}^N}\frac{1+|\psi_0|^2}{1+2|\psi_0|^2}\kappa_t\overline{\kappa}\,dx + i\int_{\mathbb{R}^N}\frac{\psi_0^2}{1+2|\psi_0|^2}\overline{\kappa}_t\overline{\kappa}\,dx\right)$$
$$= \int_{\mathbb{R}^N}\frac{1+|\psi_0|^2}{2+4|\psi_0|^2}|\kappa|_t^2\,dx + \int_{\mathbb{R}^N}\left(\frac{\psi_0^2}{4(1+2|\psi_0|^2)}(\overline{\kappa}^2)_t + \frac{\overline{\psi}_0^2}{4(1+2|\psi_0|^2)}(\kappa^2)_t\right)dx$$
$$= \frac{d}{dt}\left(\int_{\mathbb{R}^N}\frac{1+|\psi_0|^2}{2+4|\psi_0|^2}|\kappa|^2\,dx + \int_{\mathbb{R}^N}\left(\frac{\psi_0^2}{4(1+2|\psi_0|^2)}\overline{\kappa}^2 + \frac{\overline{\psi}_0^2}{4(1+2|\psi_0|^2)}\kappa^2\right)dx\right)$$
$$-\int_{\mathbb{R}^N}\left(\frac{1+|\psi_0|^2}{2+4|\psi_0|^2}\right)_t|\kappa|^2\,dx$$
$$-\int_{\mathbb{R}^N}\left(\left(\frac{\psi_0^2}{4(1+2|\psi_0|^2)}\right)_t\overline{\kappa}^2 + \left(\frac{\overline{\psi}_0^2}{4(1+2|\psi_0|^2)}\right)_t\kappa^2\right)dx.$$

The other terms in equation (2.19) are classical and can be treated exactly as in [10]. The important point to notice is that since the diagonal coefficients of $\mathcal{M}$ are pure imaginary, one has for $k = 1, \ldots, N$

$$\begin{aligned}\Im\left(\int_{\mathbb{R}^N} \mathcal{M}^{11}(\psi_0, \psi_k, \partial_k\psi_0)(\partial_k\kappa)\overline{\kappa}\,\mathrm{d}x\right) &= \int_{\mathbb{R}^N} \mathcal{I}m\left(\mathcal{M}^{11}(\psi_0, \psi_k, \partial_k\psi_0)\right)\partial_k\frac{|\kappa|^2}{2}\,\mathrm{d}x,\\ &= -\int_{\mathbb{R}^N} \partial_k\left(\mathcal{I}m\left(\mathcal{M}^{11}(\psi_0, \psi_k, \partial_k\psi_0)\right)\right)\frac{|\kappa|^2}{2}\,\mathrm{d}x,\end{aligned}$$

by integration by parts. This makes it possible to overcome the loss of derivatives of this quasi-linear Schrödinger equation and brings the following estimate

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^N}\frac{1+|\psi_0|^2}{2+4|\psi_0|^2}|\kappa|^2\,\mathrm{d}x + \int_{\mathbb{R}^N}\left(\frac{\psi_0^2}{4(1+2|\psi_0|^2)}\overline{\kappa}^2 + \frac{\overline{\psi}_0^2}{4(1+2|\psi_0|^2)}\kappa^2\right)\mathrm{d}x\right)\\ &\qquad\leqslant 4\int_{\mathbb{R}^N}(|\psi_0|^2)_t|\kappa|^2\,\mathrm{d}x + C_1(M,r)\|\kappa\|_2^2,\end{aligned} \tag{2.20}$$

where $C_1(M, r)$ is a constant depending only on $M$ and $r$. To derive inequality (2.20), we have used the fact that

$$\left(\frac{1+|\psi_0|^2}{2+4|\psi_0|^2}\right)_t = \frac{(|\psi_0|^2)_t}{2+4|\psi_0|^2} - 4\left(\frac{1+|\psi_0|^2}{(2+4|\psi_0|^2)^2}\right)(|\psi_0|^2)_t,$$

$$\left(\frac{\psi_0^2}{4(1+2|\psi_0|^2)}\right)_t = \frac{(\psi_0^2)_t}{4(1+2|\psi_0|^2)} - \left(\frac{\psi_0^2}{2(1+2|\psi_0|^2)^2}\right)(|\psi_0|^2)_t,$$

$$\left(\frac{\overline{\psi}_0^2}{4(1+2|\psi_0|^2)}\right)_t = \frac{(\overline{\psi}_0^2)_t}{4(1+2|\psi_0|^2)} - \left(\frac{\overline{\psi}_0^2}{2(1+2|\psi_0|^2)^2}\right)(|\psi_0|^2)_t$$

and then

$$\begin{aligned}&\left|\int_{\mathbb{R}^N}\left(\frac{1+|\psi_0|^2}{2+4|\psi_0|^2}\right)_t|\kappa|^2\,\mathrm{d}x - \int_{\mathbb{R}^N}\left(\left(\frac{\psi_0^2}{4(1+2|\psi_0|^2)}\right)_t\overline{\kappa}^2 + \left(\frac{\overline{\psi}_0^2}{4(1+2|\psi_0|^2)}\right)_t\kappa^2\right)\mathrm{d}x\right|\\ &\qquad\leqslant 4\int_{\mathbb{R}^N}(|\psi_0|^2)_t|\kappa|^2\,\mathrm{d}x.\end{aligned}$$

Using the fact that

$$\sup_{t\in[0,T]}\|(\psi_0)_t\|_{H^{E(\frac{N}{2})+1}(\mathbb{R}^N)} \leqslant r$$

and the continuous embedding $H^{E(\frac{N}{2})+1}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$, we can find a constant $C_2(M, r)$ such that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^N}\frac{1+|\psi_0|^2}{2+4|\psi_0|^2}|\kappa|^2\,\mathrm{d}x + \int_{\mathbb{R}^N}\left(\frac{\psi_0^2}{4(1+2|\psi_0|^2)}\overline{\kappa}^2 + \frac{\overline{\psi}_0^2}{4(1+2|\psi_0|^2)}\kappa^2\right)\mathrm{d}x\right)\\ &\qquad\leqslant C_2(M,r)\|\kappa\|_2^2.\end{aligned} \tag{2.21}$$

Integrating inequality (2.21) from 0 to $t$ gives

$$\int_{\mathbb{R}^N} \frac{1+|\psi_0(t)|^2}{2+4|\psi_0(t)|^2}|\kappa(t)|^2\,\mathrm{d}x + \int_{\mathbb{R}^N}\left(\frac{\psi_0^2(t)}{4(1+2|\psi_0(t)|^2)}\overline{\kappa}^2(t) + \frac{\overline{\psi}_0^2(t)}{4(1+2|\psi_0(t)|^2)}\kappa^2(t)\right)\mathrm{d}x$$
$$\leqslant \int_{\mathbb{R}^N} \frac{1+|\psi_0(0)|^2}{2+4|\psi_0(0)|^2}|\kappa(0)|^2\,\mathrm{d}x$$
$$+\int_{\mathbb{R}^N}\left(\frac{\psi_0^2(0)}{4(1+2|\psi_0(0)|^2)}\overline{\kappa}^2(0) + \frac{\overline{\psi}_0^2(0)}{4(1+2|\psi_0(0)|^2)}\kappa^2(0)\right)\mathrm{d}x$$
$$+C_2(M,r)\int_0^t \|\kappa(s)\|_2^2\,\mathrm{d}s.$$

For all $t \in [0, T]$, we have

$$\frac{1+|\psi_0(t)|^2}{2+4|\psi_0(t)|^2}|\kappa(t)|^2 + \frac{\psi_0^2(t)}{4(1+2|\psi_0(t)|^2)}\overline{\kappa}^2(t) + \frac{\overline{\psi}_0^2(t)}{4(1+2|\psi_0(t)|^2)}\kappa^2(t)$$
$$\geqslant \frac{1}{2+4|\psi_0(t)|^2}|\kappa(t)|^2.$$

Denoting by

$$CI_{N+2}(0) = \int_{\mathbb{R}^N} \frac{1+|\psi_0(0)|^2}{2+4|\psi_0(0)|^2}|\kappa(0)|^2\,\mathrm{d}x$$
$$+\int_{\mathbb{R}^N}\left(\frac{\psi_0^2(0)}{4(1+2|\psi_0(0)|^2)}\overline{\kappa}^2(0) + \frac{\overline{\psi}_0^2(0)}{4(1+2|\psi_0(0)|^2)}\kappa^2(0)\right)\mathrm{d}x,$$

we derive

$$\int_{\mathbb{R}^N} \frac{1}{2+4|\psi_0(t)|^2}|\kappa(t)|^2\,\mathrm{d}x \leqslant CI_{N+2}(0) + C_2(M,r)\int_0^t \|\kappa(s)\|_2^2\,\mathrm{d}s. \tag{2.22}$$

Recalling that $\psi_0 \in L^\infty(0, T; H^s(\mathbb{R}^N))$ and the continuous embedding $H^s(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ and denoting by $C_b$ the best constant of this embedding, we have

$$\|\psi_0(t)\|_{L^\infty(\mathbb{R}^N)} \leqslant C_b m_0.$$

This provides

$$\int_{\mathbb{R}^N} \frac{1}{2+4|\psi_0(t)|^2}|\kappa(t)|^2\,\mathrm{d}x \geqslant \frac{1}{2+4C_b^2 m_0^2}\int_{\mathbb{R}^N}|\kappa(t)|^2\,\mathrm{d}x$$

which gives

$$\int_{\mathbb{R}^N} |\kappa(t)|^2\,\mathrm{d}x \leqslant (2+4C_b^2m_0^2)\left(CI_{N+2}(0) + C_2(M,r)\int_0^t \|\kappa(s)\|_2^2\,\mathrm{d}s\right). \tag{2.23}$$

Since the gauge transform $b$ does not depend on $\psi_0$ and for all $t \in [0, T]$, $\|\psi_0(t)\|_{L^\infty(\mathbb{R}^N)} \leqslant m_0$, there is a constant $C(m_0)$ depending only on $m_0$ such that

$$\sup_{t\in[0,T]} \|\mathrm{e}^{-b(|\psi_0(t)|^2)}\|^2_{L^\infty(\mathbb{R}^N)} \leqslant C(m_0).$$

Recalling that $\kappa(0) = \mathrm{e}^{p(|a_0|^2)}(1-\Delta)^{\frac{s}{2}}(\mathrm{e}^{q(|a_0|^2)}\Delta a_0)$ and choosing $m_{N+2}$ such that

$$m_{N+2}^2 \geqslant 2C(m_0)(2+4C_b^2m_0^2)CI_{N+2}(0) + 1, \tag{2.24}$$

one can find a positive $T$ such that for this choice of $m_{N+2}$

$$\sup_{t\in[0,T]} \|\phi_{N+2}\|_{H^s(\mathbb{R}^N)} \leqslant m_{N+2}. \tag{2.25}$$

Note that $m_{N+2}$ depends only on the initial data $a_0$ and $m_0$. Performing the same kind of estimates on equations (2.16), one can find a positive $T$ and constant $m_{N+1}$ depending only on $a_0$ and $m_0$ satisfying

$$m_{N+1}^2 \geqslant 2C(m_0)(2+4C_b^2 m_0^2)CI_{N+1}(0)+1, \tag{2.26}$$

where

$$CI_{N+1}(0) = \int_{\mathbb{R}^N} \frac{1+|\psi_0(0)|^2}{2+4|\psi_0(0)|^2}|\nu(0)|^2\,\mathrm{d}x + \int_{\mathbb{R}^N}\left(\frac{\psi_0^2(0)}{4(1+2|\psi_0(0)|^2)}\overline{\nu}^2(0)+\frac{\overline{\psi}_0^2(0)}{4(1+2|\psi_0(0)|^2)}\nu^2(0)\right)\mathrm{d}x$$

with

$$\nu(0) = \mathrm{e}^{p(|a_0|^2)}(1-\Delta)^{\frac{s}{2}}(\mathrm{e}^{g(|a_0|^2)}\partial_t a_0),$$

such that

$$\sup_{t\in[0,T]} \|\phi_{N+1}\|_{H^s(\mathbb{R}^N)} \leqslant m_{N+1}. \tag{2.27}$$

Dealing with equation (2.14), we introduce for $j = 1,\ldots,N$

$$\mu_j(0) = (1-\Delta)^{\frac{s}{2}}\partial_j a_0$$

and

$$CI_j(0) = \int_{\mathbb{R}^N} \frac{1+|\psi_0(0)|^2}{2+4|\psi_0(0)|^2}|\mu_j(0)|^2\,\mathrm{d}x + \int_{\mathbb{R}^N}\left(\frac{\psi_0^2(0)}{4(1+2|\psi_0(0)|^2)}\overline{\mu}_j^2(0)+\frac{\overline{\psi}_0^2(0)}{4(1+2|\psi_0(0)|^2)}\mu_j^2(0)\right)\mathrm{d}x.$$

Choosing $m_j$ depending only on $a_0$ and $m_0$ such that

$$m_j^2 \geqslant 2(2+4C_b^2 m_0^2)CI_j(0)+1, \tag{2.28}$$

we derive

$$\text{for } j = 1,\ldots,N,\ \sup_{t\in[0,T]} \|\phi_j\|_{H^s(\mathbb{R}^N)} \leqslant m_j. \tag{2.29}$$

Treating now equation (2.13), we introduce

$$\xi(0) = (1-\Delta)^{\frac{s}{2}}a_0$$

and

$$CI_0(0) = \int_{\mathbb{R}^N} |\xi(0)|^2\,\mathrm{d}x.$$

It is crucial to remark here that equation (2.13) is not quasi-linear. As a consequence, we can perform a classical energy estimate on it and choose the constant $m_0$ such that

$$m_0^2 \geqslant 2CI_0(0)+1. \tag{2.30}$$

The choice of $m_0$ depends only on the initial data $a_0$.

**Remark 2.1.** If we work with equation (2.3) instead of equation (2.9) and perform the energy estimates of equation (2.16), for example, we have to choose $m_0$ such that

$$m_0^2 \geqslant 2(2+4C_b^2 m_0^2)CI_0(0)+1,$$

where

$$\widetilde{CI}_0(0) = \int_{\mathbb{R}^N} \frac{1+|\psi_0(0)|^2}{2+4|\psi_0(0)|^2}|\xi(0)|^2\,\mathrm{d}x + \int_{\mathbb{R}^N}\left(\frac{\psi_0^2(0)}{4(1+2|\psi_0(0)|^2)}\overline{\xi}^2(0) + \frac{\overline{\psi}_0^2(0)}{4(1+2|\psi_0(0)|^2)}\xi^2(0)\right)\mathrm{d}x.$$

Such a choice requires of course a smallness assumption on the initial data $a_0$.

Let us take $m_0$ as in (2.30). Then one can also find a positive $T$ such that

$$\sup_{t\in[0,T]} \|\phi_0\|_{H^s(\mathbb{R}^N)} \leqslant m_0. \tag{2.31}$$

We refer to [10] for the technical details. Due to the structure of the space $\mathcal{X}_T$, it remains to estimate $(\psi_0)_t$ in $H^{E(N/2)+1}(\mathbb{R}^N)$. This is done directly on equation (2.13) and provides that there exists a constant $C_0(M)$ depending only on $M$ such that

$$\sup_{t\in[0,T]} \|(\phi_0)_t\|_{H^{E(\frac{N}{2})+1}(\mathbb{R}^N)} \leqslant C_0(M). \tag{2.32}$$

As a conclusion, we choose constants $M$, $r$ and $T$ as follows. We first fix $m_0$ depending only on $a_0$ such that (2.30) holds. Then we take $(m_j)$, $m_{N+1}$ and $m_{N+2}$ depending only on $a_0$ and $m_0$ satisfying, respectively, (2.28), (2.26) and (2.24). Finally take $r$ such that

$$r \geqslant C_0(M)$$

and $T$ sufficiently small such that

$$C_4(M,r)T \leqslant \tfrac{1}{2},$$

and similar conditions to take into account the equations on $\phi_0$, $\phi_j$ and $\phi_{N+1}$. For such a choice of parameter, we have shown

$$\mathcal{S}\big(\mathcal{X}_T(M,r)\big) \subset \mathcal{X}_T(M,r).$$

The fact that the mapping $\mathcal{S}$ is a contraction for the suitable norm is very standard and we refer once again to [10] since the proof reads exactly the same. By the contraction mapping principle, there exists a unique solution

$$\Phi = \big(\phi_0, (\phi_j)_{j=0}^N, \phi_{N+1}, \phi_{N+2}\big)$$

to system (2.13)–(2.16). Furthermore, for each $0 \leqslant j \leqslant N+2$, the function $\phi_j$ satisfies

$$\phi_j \in L^\infty(0,T; H^s(\mathbb{R}^N)) \cap C([0,T]; L^2(\mathbb{R}^N)).$$

To conclude the proof, we have to show that the solution $\Phi$ solves system (2.9)–(2.12) and has the following regularity:

$$\Phi \in \big(L^\infty(0,T; H^{s+2}(\mathbb{R}^N)) \cap C([0,T]; H^s(\mathbb{R}^N))\big)^{N+3}.$$

This can be done exactly as in [10]. The proof of the conservation laws (1.4)–(1.5) is very standard once we have proved that $\phi$ is regular and so we omit it. At this point the proof of theorem 1.1 is completed.

## 3. Existence of ground states and orbital instability

In this section we derive the existence, as well as some qualitative properties, of the ground states solutions of (1.8). When $p > 3 + 4/N$ we shall also prove that the ground states are instable by blow-up.

We begin with the following Pohozaev-type identity.

**Lemma 3.1.** *Any* $u \in X_{\mathbb{C}}$ *solution of (1.8) satisfies* $P(u) = 0$*, where* $P : X_{\mathbb{C}} \to \mathbb{R}$ *is the function defined by*

$$P(u) = \frac{N-2}{N}\left(\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} |u|^2 |\nabla |u||^2 \,\mathrm{d}x\right) + \frac{\omega}{2}\int_{\mathbb{R}^N} |u|^2 \,\mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1} \,\mathrm{d}x.$$

**Proof.** Since the proof only uses classical arguments, we shall just sketch it and refer to [11] for further details. Let $u \in X_{\mathbb{C}}$ be a solution to equation (1.8). From [27, section 6, appendix] we learn that $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ (the proof given there extends easily to complex valued functions). We are then able to pursue as in [11, proposition 2.1]. Let $\psi \in C_0^\infty(\mathbb{R}^N)$ be such that $\psi \geqslant 0$, $\operatorname{supp}(\psi) \subset B(0,2)$ and $\psi \equiv 1$ on $B(0,1)$. For all $j \in \mathbb{N}^*$, we set $\psi_j(x) = \psi(x/j)$. Now let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of even positive functions in $L^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} \rho_n \,\mathrm{d}x = 1$ such that, for all $\kappa \in L^q(\mathbb{R}^N)$, $\rho_n * \kappa$ tends to $\kappa$ in $L^q(\mathbb{R}^N)$, as $n \to \infty$, for all $1 \leqslant q < \infty$. First, we take the convolution of (1.8) with $\rho_n$. Then, we multiply the resulting equation by $\psi_j \; x \cdot \nabla(\overline{u} * \rho_n)$, integrate over $\mathbb{R}^N$ and consider the real part of the equality. From that point, the calculus is standard consisting of various integrations by parts. Hence, we omit the details and we refer the reader to [11]. In order to conclude the proof, it is sufficient to apply the Lebesgue dominated convergence theorem. □

**Proof of theorem 1.3.** We shall distinguish between the cases $N = 1$ and $N \geqslant 2$, which require a separate treatment.

- Case $N \geqslant 2$. We divide the proof into four steps.

*Step I (existence of a solution to (1.8)).* We prove the existence of a solution $u_\omega \in X_{\mathbb{C}}$ to (1.8) satisfying conditions (i)–(iv) of theorem 1.3. Following the arguments of [12], we perform a change of unknown by setting $v = r^{-1}(u)$, where the function $r : \mathbb{R} \to \mathbb{R}$ is the unique solution to the Cauchy problem

$$r'(s) = \frac{1}{\sqrt{1 + 2r^2(s)}}, \qquad r(0) = 0. \tag{3.1}$$

Here $u \in X_{\mathbb{C}}$ is assumed to be real valued. Then, in [12] it is proved that if $v \in H^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ is a real solution to

$$-\Delta v = \frac{1}{\sqrt{1 + 2r^2(v)}}\big(|r(v)|^{p-1} r(v) - \omega r(v)\big) \tag{3.2}$$

then $u = r(v) \in X_{\mathbb{C}} \cap C^2(\mathbb{R}^N)$ and it is a real solution of (1.8). Let us set

$$k(v) := \frac{1}{\sqrt{1 + 2r^2(v)}}\big(|r(v)|^{p-1} r(v) - \omega r(v)\big) = r'(v)\big(|r(v)|^{p-1} r(v) - \omega r(v)\big),$$

and denote by $\mathcal{T}_\omega : H^1(\mathbb{R}^N) \to \mathbb{R}$ the action associated with equation (3.2), namely

$$\begin{aligned}\mathcal{T}_\omega(v) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} K(v)\,\mathrm{d}x,\\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^N} |r(v)|^{p+1}\,\mathrm{d}x + \frac{\omega}{2}\int_{\mathbb{R}^N} |r(v)|^2\,\mathrm{d}x,\end{aligned}$$

where we have set $K(t) = \int_0^t k(s)\mathrm{d}s$. Now, it is straightforward to check that $k$ satisfies assumptions (g0)–(g3) of [12]. Thus, from [12] (see also [4, 5]) we deduce the existence of a ground state $v_\omega$ of (3.2) satisfying conditions (i)–(iv) of theorem 1.3, that is $v_\omega$ solves (3.2) and minimizes the action $\mathcal{T}_\omega$ among all nontrivial solutions to (3.2). Therefore, setting $u_\omega = r(v_\omega)$, we get that $u_\omega$ solves (1.8) and satisfies conditions (i)–(iv) of theorem 1.3 (see [12, theorem 1.2]).

*Step II (existence of a ground state to (1.8)).* In this step we prove that $u_\omega$ minimizes the action $\mathcal{E}_\omega$, over the set of nontrivial solutions to the original equation (1.8). To achieve this goal, we make the following observations. Note first that, if $u = r(v)$ with $u \in X_\mathbb{C}$ real, then $\mathcal{E}_\omega(u) = \mathcal{T}_\omega(v)$. Indeed, we have

$$\begin{aligned}\mathcal{E}_\omega(r(v)) &= \frac{1}{2}\int_{\mathbb{R}^N} r'^2(v)|\nabla v|^2\,\mathrm{d}x + \int_{\mathbb{R}^N} |r(v)|^2 r'^2(v)|\nabla|v||^2\,\mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^N} |r(v)|^{p+1}\,\mathrm{d}x\\ &\quad + \frac{\omega}{2}\int_{\mathbb{R}^N} |r(v)|^2\,\mathrm{d}x\\ &= \frac{1}{2}\int_{\mathbb{R}^N} \frac{1}{1+2r^2(v)}|\nabla v|^2\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{1}{1+2r^2(v)} r(v)^2|\nabla v|^2\,\mathrm{d}x\\ &\quad - \frac{1}{p+1}\int_{\mathbb{R}^N} |r(v)|^{p+1}\,\mathrm{d}x + \frac{\omega}{2}\int_{\mathbb{R}^N} |r(v)|^2\,\mathrm{d}x\\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,\mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^N} |r(v)|^{p+1}\,\mathrm{d}x + \frac{\omega}{2}\int_{\mathbb{R}^N} |r(v)|^2\,\mathrm{d}x = \mathcal{T}_\omega(v),\end{aligned}$$

thanks to the Cauchy problem (3.1). Also, if $u \in X_\mathbb{C}$ is a solution to (1.8) we have, in light of lemma 3.1, that

$$\mathcal{E}_\omega(u) = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla u|^2 + 2|u|^2|\nabla|u||^2\,\mathrm{d}x. \tag{3.3}$$

Once these facts have been observed, take any solution $u \in X_\mathbb{C}$ to (1.8) (note that $u$ can be a complex valued function) and set $v = r^{-1}(|u|)$. Due to the well-known point-wise inequality $|\nabla|u(x)|| \leqslant |\nabla u(x)|$ for a.e. $x \in \mathbb{R}^N$, it holds that

$$\int_{\mathbb{R}^N} |\nabla|u(x)||^2\,\mathrm{d}x \leqslant \int_{\mathbb{R}^N} |\nabla u(x)|^2\,\mathrm{d}x, \tag{3.4}$$

so that $\mathcal{E}_\omega(|u|) \leqslant \mathcal{E}_\omega(u)$ (note that all the other terms in the functional $\mathcal{E}_\omega$ are invariant to the modulus). Thus, in turn, we have

$$\mathcal{E}_\omega(u) \geqslant \mathcal{E}_\omega(|u|) = \mathcal{E}_\omega(r(v)) = \mathcal{T}_\omega(v). \tag{3.5}$$

Now, let us set

$$A = \left\{v \in H^1(\mathbb{R}^N) : \tilde{P}(v) = 0\right\},$$

where $\tilde{P} : H^1(\mathbb{R}^N) \to \mathbb{R}$ is the functional defined as

$$\tilde{P}(v) = (N-2)\int_{\mathbb{R}^N} |\nabla v|^2\,\mathrm{d}x - 2N\int_{\mathbb{R}^N} K(v)\,\mathrm{d}x.$$

Clearly, for any $v \in A$, we have

$$\mathcal{T}_\omega(v) = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla v|^2\,\mathrm{d}x. \tag{3.6}$$

Also, as for the proof that $\mathcal{E}_\omega(u) = \mathcal{T}_\omega(v)$, it is readily checked that if $v = r^{-1}(u)$ with $u \in X_\mathbb{C}$ real, then $\tilde{P}(v) = P(u)$. Finally, it is well known (see, e.g., [4, 5]) that $v_\omega$ satisfies

$$v_\omega \in A, \qquad \mathcal{T}_\omega(v_\omega) = \inf_{v\in A} \mathcal{T}_\omega(v). \tag{3.7}$$

Now, if $N = 2$, it follows from the definition of $P$ in lemma 3.1 that $P(|u|) = 0$. Thus, in turn, $\tilde{P}(v) = 0$ and, using (3.5) and (3.7), it follows that

$$\mathcal{E}_\omega(u) \geqslant \mathcal{T}_\omega(v) \geqslant \mathcal{T}_\omega(v_\omega) = \mathcal{E}_\omega(u_\omega), \tag{3.8}$$

proving the desired claim. If $N \geqslant 3$, one of the following possibilities occurs.

(i) $P(|u|) = 0$. In this case inequality (3.8) holds exactly as in the case $N = 2$.

(ii) $P(|u|) = \tilde{P}(v) < 0$. In this case there exists a number $\theta \in (0, 1)$ such that, setting $v_\theta(x) = v(x/\theta)$, we have $\tilde{P}(v_\theta) = 0$. Now, since $v_\theta \in A$, using (3.3), (3.4), (3.6), (3.7), it follows that

$$\begin{aligned}
\mathcal{T}_\omega(v_\theta) &= \frac{1}{N}\int_{\mathbb{R}^N} |\nabla v_\theta|^2\,\mathrm{d}x = \frac{\theta^{N-2}}{N}\int_{\mathbb{R}^N} |\nabla v|^2\,\mathrm{d}x \\
&= \frac{\theta^{N-2}}{N}\int_{\mathbb{R}^N} |\nabla |u||^2 + 2|u|^2|\nabla |u||^2\,\mathrm{d}x \\
&\leqslant \frac{\theta^{N-2}}{N}\int_{\mathbb{R}^N} |\nabla u|^2 + 2|u|^2|\nabla |u||^2\,\mathrm{d}x \\
&= \frac{\theta^{N-2}}{N} N\mathcal{E}_\omega(u) = \theta^{N-2}\mathcal{E}_\omega(u) < \mathcal{E}_\omega(u).
\end{aligned}$$

Thus, we get

$$\mathcal{E}_\omega(u) > \mathcal{T}_\omega(v_\theta) \geqslant \mathcal{T}_\omega(v_\omega) = \mathcal{E}_\omega(u_\omega).$$

Then, in conclusion, we proved that for both the cases $N = 2$ and $N \geqslant 3$, $u_\omega \in X_\mathbb{C}$ indeed minimizes the action $\mathcal{E}_\omega$ over the set of nontrivial solutions to (1.8).

*Step III (real character of solutions).* First we prove that, if $u \in X_\mathbb{C}$ is a ground state solution to (1.8), then $|u| \in X$ is also a ground state. We set $v = r^{-1}(|u|)$. Observe that it holds

$$m_\omega = \mathcal{E}_\omega(u) \geqslant \mathcal{E}_\omega(|u|) \geqslant \mathcal{T}_\omega(v). \tag{3.9}$$

In the case $N = 2$, we have $\tilde{P}(v) = P(|u|) = 0$ and, thus, we conclude $\mathcal{E}_\omega(|u|) = m_\omega$ by using (3.7), (3.9) and recalling that $\mathcal{T}_\omega(v_\omega) = \mathcal{E}_\omega(u_\omega) = m_\omega$. If $N \geqslant 3$, and $\tilde{P}(v) = P(|u|) < 0$ we introduce, as before, the rescaling $v_\theta$ such that $\tilde{P}(v_\theta) = 0$. Then, we get

$$\mathcal{T}_\omega(v_\theta) < \mathcal{E}_\omega(u) = m_\omega,$$

and we immediately reach a contradiction by arguing as before. Now, let $u \in X_\mathbb{C}$ be a ground state solution of (1.8) and assume that

$$\mathcal{L}^N(\{x \in \mathbb{R}^N : |\nabla |u|(x)| < |\nabla u(x)|\}) > 0.$$

Then we get

$$\begin{aligned}
m_\omega &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla |u||^2\,\mathrm{d}x + \int_{\mathbb{R}^N} |u|^2|\nabla |u||^2\,\mathrm{d}x + \frac{\omega}{2}\int_{\mathbb{R}^N} |u|^2\,\mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\,\mathrm{d}x \\
&< \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x + \int_{\mathbb{R}^N} |u|^2|\nabla |u||^2\,\mathrm{d}x + \frac{\omega}{2}\int_{\mathbb{R}^N} |u|^2\,\mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\,\mathrm{d}x = m_\omega.
\end{aligned}$$

This is obviously not possible and, hence, we have $|\nabla|u(x)|| = |\nabla u(x)|$, for a.e. $x \in \mathbb{R}^N$. But this is true if, and only if, $\Re\, u\nabla(\Im\, u) = \Im\, u\nabla(\Re\, u)$. Whence, if this last condition holds, we get

$$\bar{u}\nabla u = \Re\, u\nabla(\Re\, u) + \Im\, u\nabla(\Im\, u), \qquad \text{a.e. in } \mathbb{R}^N,$$

which implies that $\Re\,(i\bar{u}(x)\nabla u(x)) = 0$ a.e. in $\mathbb{R}^N$. This last identity immediately gives the existence of $\theta \in \mathbb{S}^1$ such that $u(x) = \mathrm{e}^{\mathrm{i}\theta}|u(x)|$.

*Step IV (properties (i)–(iv) for any real non-negative ground state).* In light of some recent achievements [7, 30], we can prove that any real ground state solution to (1.8) is radially symmetric and radially decreasing about some point. In fact we observe first that for any given solution $u$ of (1.8), by [27, section 6, appendix], $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ and in turn $u \in C^2(\mathbb{R}^N)$ (cf [20]). Considering now the strictly increasing function $\mu : \mathbb{R} \to \mathbb{R}$ such that

$$\mu'(s) = \sqrt{1+2s^2}, \qquad \mu(0) = 0, \tag{3.10}$$

it is easy to see that $v = \mu(u)$ is a solution of (3.2). Note that $\mu$ is precisely the inverse function of the function $r$ introduced in step II, $r \circ \mu = \mu \circ r = \mathrm{Id}$. Furthermore, we claim that if $u$ is any given ground state of (1.8), then $v = \mu(u) = r^{-1}(u)$ is a ground state of (3.2). In fact, taking into account the computations in step II of the proof, for any nontrivial solution $w$ of (3.2), $r(w)$ is a (nontrivial) solution of (1.8), and we have

$$\mathcal{T}_\omega(w) = \mathcal{E}_\omega(r(w)) \geqslant m_\omega = \mathcal{E}_\omega(u) = \mathcal{E}_\omega(r(v)) = \mathcal{T}_\omega(v),$$

which yields the desired conclusion. At this point the fact that any ground state solution is radially symmetric and radially decreasing about some point is a consequence of the results of [7] applied to equation (3.2). Here let us point out that the radial symmetry (plus radial decrease) could have also been proved by arguing directly on equation (1.8) which, in fact, satisfies a scaling property being the essence of the results of [7]. Now let $u \in \mathcal{G}_\omega$ be such that $u \geqslant 0$ in $\mathbb{R}^N$. Since $u \in C^2(\mathbb{R}^N)$ we have by the maximum principle (applies to $v = \mu(u)$) that $u > 0$ on $\mathbb{R}^N$. Finally using [5, lemma 2] on equation (3.2) we immediately derive the exponential decays indicated in the statement of theorem 1.3.

• Case $N = 1$.

By taking advantage of the transformation of problem (1.8), via the dual approach, into the semi-linear equation (3.2), we know that equation (1.8) admits a unique positive and even solution (see [5, theorem 5]). Thus it just remains to prove that any solution $u$ of (1.8) is of the form $u = \mathrm{e}^{\mathrm{i}\theta}\phi$, where $\theta \in \mathbb{R}$ and $\phi > 0$ is a solution to (1.8). In fact $|u| > 0$, otherwise we would get a contradiction with the identity

$$\frac{1}{2}|u'|^2 + \frac{1}{4}|(|u|^2)'|^2 - \frac{\omega}{2}|u|^2 + \frac{1}{p+1}|u|^{p+1} = 0.$$

This identity is obtained multiplying (1.8) by the conjugate of $u'$ and by performing standard manipulations. Then, we can write down the solution in polar form, $u = \rho\mathrm{e}^{\mathrm{i}\theta}$, where $\rho, \theta \in C^2(\mathbb{R})$. By direct computation, it holds $u'' = \left[\rho\theta''+2\rho'\theta'\right]\mathrm{e}^{\mathrm{i}\theta}\mathrm{i}+\left[\rho''-\rho(\theta')^2\right]e^{\mathrm{i}\theta}$. Then, by dropping this formula into equation (1.8), exactly as in [8, proof of theorem 8.1.7(iii)], one immediately reaches (by comparison of real and imaginary parts) the following identity

$$\rho\theta'' + 2\rho'\theta' = 0, \tag{3.11}$$

namely $\theta' = K/\rho^2$, for some $K \geqslant 0$. At this point it is sufficient to follow the argument of [8, proof of theorem 8.1.7(iii)] to prove that $K = 0$ and get the desired property. Thus, when $N = 1$, theorem 1.3 holds true and the set of solutions of (1.8) is essentially unique. □

In the rest of this section we prove the instability result, theorem 1.5. We start with two preliminary results. We define the variance $\mathbb{V}(t)$, by

$$\mathbb{V}(t) = \int_{\mathbb{R}^N} |x|^2 |\phi(t,x)|^2 \, dx, \qquad t \in [0, \infty) \tag{3.12}$$

and derive a so-called virial identity in the following lemma.

**Lemma 3.2.** *Let $\phi$ be a solution of (1.6) on an interval $I = (-t_1, t_1)$. Then,*

$$\mathbb{V}''(t) = 8Q(\phi(t)), \qquad t \in I, \tag{3.13}$$

*where we have set*

$$Q(\phi) = \int_{\mathbb{R}^N} |\nabla\phi|^2 \, dx + (N+2) \int_{\mathbb{R}^N} |\phi|^2 |\nabla|\phi||^2 \, dx - \frac{N(p-1)}{2(p+1)} \int_{\mathbb{R}^N} |\phi|^{p+1} \, dx, \tag{3.14}$$

*for all $\phi \in X_{\mathbb{C}}$.*

**Proof.** We introduce the following notations:

$$z = (z^1, \ldots, z^n) \in \mathbb{C}^N; \qquad z \cdot w = \sum_{i=1}^N z^i w^i, \quad z, w \in \mathbb{C}^N;$$

$$\phi_i = \frac{\partial\phi}{\partial x_i}, \qquad \phi : \mathbb{R}^N \to \mathbb{C}.$$

Let us first prove that

$$\mathbb{V}'(t) = 4\Im \int_{\mathbb{R}^N} (x \cdot \nabla\phi)\overline{\phi} \, dx, \qquad t \in I. \tag{3.15}$$

Multiplying equation (1.6) by $2\overline{\phi}$ and taking the imaginary parts yields

$$\frac{\partial}{\partial t}|\phi|^2 = -2\Im(\overline{\phi}\Delta\phi) = -2\nabla \cdot (\Im\overline{\phi}\nabla\phi). \tag{3.16}$$

Now, multiplying (3.16) by $|x|^2$, and integrating by parts in space, we get (3.15). In order to prove (3.13), let us multiply equation (1.6) by $2x \cdot \nabla\overline{\phi}$, integrate in space on $\mathbb{R}^N$ and, finally, take the real parts yielding

$$\begin{aligned} 0 = 2\Re \int_{\mathbb{R}^N} \mathrm{i}(x \cdot \nabla\overline{\phi})\phi_t \, dx + 2\Re \int_{\mathbb{R}^N} (x \cdot \nabla\overline{\phi})\Delta\phi \, dx \\ + 2\Re \int_{\mathbb{R}^N} (x \cdot \nabla\overline{\phi})\phi\Delta|\phi|^2 \, dx + 2\Re \int_{\mathbb{R}^N} (x \cdot \nabla\overline{\phi})|\phi|^{p-1}\phi \, dx. \end{aligned}$$

We rewrite the last identity in the form

$$\mathrm{I} = \mathrm{II} + \mathrm{III}, \tag{3.17}$$

where

$$\mathrm{I} = 2\Re \int_{\mathbb{R}^N} \mathrm{i}(x \cdot \nabla\overline{\phi})\phi_t \, dx,$$

$$\mathrm{II} = -2\Re \int_{\mathbb{R}^N} (x \cdot \nabla\overline{\phi})\Delta\phi \, dx - 2\Re \int_{\mathbb{R}^N} (x \cdot \nabla\overline{\phi})\phi\Delta|\phi|^2 \, dx,$$

$$\mathrm{III} = -2\Re \int_{\mathbb{R}^N} (x \cdot \nabla\overline{\phi})|\phi|^{p-1}\phi \, dx.$$

For the first term, recalling formula (3.15) for $\mathbb{V}'$, we have

$$\begin{aligned}
\mathrm{I} &= \Re \int_{\mathbb{R}^N} \mathrm{i} \sum_{j=1}^{N} \left( x^j \overline{\phi}_j \phi_t - x^j \phi_j \overline{\phi}_t \right) \mathrm{d}x = \Re \int_{\mathbb{R}^N} \mathrm{i} \sum_{j=1}^{N} x^j \left[ (\overline{\phi}_j \phi)_t - (\phi \overline{\phi}_t)_j \right] \mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \Re \int_{\mathbb{R}^N} \mathrm{i}(x \cdot \nabla \overline{\phi}) \phi \, \mathrm{d}x + N \Re \int_{\mathbb{R}^N} \mathrm{i} \phi \overline{\phi}_t \, \mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \Im \int_{\mathbb{R}^N} (x \cdot \nabla \phi) \overline{\phi} \, \mathrm{d}x - N \int_{\mathbb{R}^N} |\nabla \phi|^2 \, \mathrm{d}x \\
&\quad + N \int_{\mathbb{R}^N} |\phi|^2 \Delta |\phi|^2 \, \mathrm{d}x + N \int_{\mathbb{R}^N} |\phi|^{p+1} \, \mathrm{d}x. \\
&= \frac{1}{4} \frac{\mathrm{d}}{\mathrm{d}t} \mathbb{V}(t) - N \int_{\mathbb{R}^N} |\nabla \phi|^2 \, \mathrm{d}x - 4N \int_{\mathbb{R}^N} |\phi|^2 |\nabla |\phi||^2 \, \mathrm{d}x + N \int_{\mathbb{R}^N} |\phi|^{p+1} \, \mathrm{d}x. \qquad (3.18)
\end{aligned}$$

A multiple integration by parts in formula II gives

$$\mathrm{II} = (2 - N) \int_{\mathbb{R}^N} |\nabla \phi|^2 \, \mathrm{d}x + 2(2 - N) \int_{\mathbb{R}^N} |\phi|^2 |\nabla |\phi||^2 \, \mathrm{d}x. \qquad (3.19)$$

As for the term III, we write it by components

$$\mathrm{III} = -\sum_{j=1}^{N} \int_{\mathbb{R}^N} x^j |\phi|^{p-1} (2 \Re \overline{\phi}_j \phi) \, \mathrm{d}x = -2 \sum_{j=1}^{N} \int_{\mathbb{R}^N} x^j \frac{\partial_j |\phi|^{p+1}}{p+1} \, \mathrm{d}x = \frac{2N}{p+1} \int_{\mathbb{R}^N} |\phi|^{p+1} \, \mathrm{d}x. \qquad (3.20)$$

Finally, recollecting (3.17), (3.18), (3.19), (3.20) and (3.15) and taking into account the definition of $Q$, the proof of (3.13) is complete. □

In our next preliminary result we establish some qualitative properties of a class of $L^2$-invariant rescaling.

**Lemma 3.3.** *Let $\psi \in X_{\mathbb{C}}$ with $Q(\psi) \leqslant 0$ and assume that*

$$p > 3 + \frac{4}{N}. \qquad (3.21)$$

*Let $\sigma > 0$ and define the rescaling $\psi^\sigma(x) = \sigma^{N/2} \psi(\sigma x)$. Then there exists $\sigma_0 \in (0, 1]$ such that the following facts hold:*

*(1) $Q(\psi^{\sigma_0}) = 0$;*
*(2) $\sigma_0 = 1$ if and only if $Q(\psi) = 0$;*
*(3) $(\partial/\partial\sigma)\mathcal{E}_\omega(\psi^\sigma) > 0$ for $\sigma \in (0, \sigma_0)$ and $(\partial/\partial\sigma)\mathcal{E}_\omega(\psi^\sigma) < 0$ for $\sigma \in (\sigma_0, \infty)$;*
*(4) $\sigma \mapsto \mathcal{E}_\omega(\psi^\sigma)$ is concave on $(\sigma_0, \infty)$;*
*(5) $(\partial/\partial\sigma)\mathcal{E}_\omega(\psi^\sigma) = Q(\psi^\sigma)/\sigma$.*

**Proof.** By direct computation, we have

$$\begin{aligned}
\mathcal{E}_\omega(\psi^\sigma) &= \frac{\sigma^2}{2} \int_{\mathbb{R}^N} |\nabla \psi|^2 \, \mathrm{d}x + \sigma^{N+2} \int_{\mathbb{R}^N} |\psi|^2 |\nabla |\psi||^2 \, \mathrm{d}x \\
&\quad + \frac{\omega}{2} \int_{\mathbb{R}^N} |u|^2 \, \mathrm{d}x - \frac{\sigma^{\frac{N(p-1)}{2}}}{p+1} \int_{\mathbb{R}^N} |\psi|^{p+1} \, \mathrm{d}x,
\end{aligned}$$

so that, using the functional $Q$ defined by (3.14), for all $\sigma > 0$, we get

$$\begin{aligned}
\frac{\partial}{\partial \sigma} \mathcal{E}_\omega(\psi^\sigma) &= \sigma \int_{\mathbb{R}^N} |\nabla \psi|^2 \, \mathrm{d}x + (N+2) \sigma^{N+1} \int_{\mathbb{R}^N} |\psi|^2 |\nabla |\psi||^2 \, \mathrm{d}x \\
&\quad - \frac{N(p-1)}{2(p+1)} \sigma^{\frac{N(p-1)}{2} - 1} \int_{\mathbb{R}^N} |\psi|^{p+1} \, \mathrm{d}x = \frac{1}{\sigma} Q(\psi^\sigma).
\end{aligned}$$

Then, taking into account (3.21), it is readily seen that there exists $\sigma_0 \in (0, 1]$ such that

$$Q(\psi^{\sigma_0}) = \sigma_0 \frac{\partial}{\partial\sigma} \mathcal{E}_\omega(\psi^\sigma)_{|\sigma=\sigma_0} = 0,$$

as well as $(\partial/\partial\sigma)\mathcal{E}_\omega(\psi^\sigma) > 0$ for $\sigma \in (0, \sigma_0)$ and $(\partial/\partial\sigma)\mathcal{E}_\omega(\psi^\sigma) < 0$ for $\sigma \in (\sigma_0, \infty)$. Furthermore, writing $\sigma = t\sigma_0$, we have

$$\begin{aligned}\frac{\partial^2}{\partial\sigma^2}\mathcal{E}_\omega(\psi^\sigma) &= \int_{\mathbb{R}^N} |\nabla\psi|^2\,\mathrm{d}x + (N+2)(N+1)t^N\sigma_0^N \int_{\mathbb{R}^N} |\psi|^2|\nabla|\psi||^2\,\mathrm{d}x \\ &\quad - \frac{N(p-1)}{2(p+1)}\left(\frac{N(p-1)}{2} - 1\right) t^{\frac{N(p-1)}{2}-2}\sigma_0^{\frac{N(p-1)}{2}-2} \int_{\mathbb{R}^N} |\psi|^{p+1}\,\mathrm{d}x, \\ &= t^N \left(\frac{1}{t^N}\int_{\mathbb{R}^N} |\nabla\psi|^2\,\mathrm{d}x + (N+2)(N+1)\sigma_0^N \int_{\mathbb{R}^N} |\psi|^2|\nabla|\psi||^2\,\mathrm{d}x\right. \\ &\quad \left. - \frac{N(p-1)}{2(p+1)}\left(\frac{N(p-1)}{2} - 1\right) t^{\frac{Np-3N-4}{2}}\sigma_0^{\frac{N(p-1)}{2}-2} \int_{\mathbb{R}^N} |\psi|^{p+1}\,\mathrm{d}x\right).\end{aligned}$$

Since, of course, we have

$$\begin{aligned}&\int_{\mathbb{R}^N} |\nabla\psi|^2\,\mathrm{d}x + (N+2)(N+1)\sigma_0^N \int_{\mathbb{R}^N} |\psi|^2|\nabla|\psi||^2\,\mathrm{d}x \\ &\qquad - \frac{N(p-1)}{2(p+1)}\left(\frac{N(p-1)}{2} - 1\right)\sigma_0^{\frac{N(p-1)}{2}-2} \int_{\mathbb{R}^N} |\psi|^{p+1}\,\mathrm{d}x \leqslant 0\end{aligned}$$

and $t > 1$, it follows that the quantity within parentheses is negative. Hence the map $\sigma \mapsto \mathcal{E}_\omega(\psi^\sigma)$ is concave on $(\sigma_0, \infty)$, concluding the proof. □

In order to establish the instability of ground states we now show in the spirit of [23] that they enjoy two additional variational characterizations. First, we have the following.

**Lemma 3.4.** *Assume that* $\omega > 0$ *and* $3 \leqslant p \leqslant (3N+2)/(N-2)$ *if* $N \geqslant 3$ *and* $3 \leqslant p$ *if* $N = 1, 2$. *Then the set of minimizers of*

$$d_\omega = \inf\{\mathcal{E}_\omega(u) : \mathcal{I}_\omega(u) = 0\}, \tag{3.22}$$

*where*

$$\mathcal{I}_\omega(u) = \int_{\mathbb{R}^N} |\nabla\phi|^2\,\mathrm{d}x + \omega\int_{\mathbb{R}^N} |\phi|^2\,\mathrm{d}x + 4\int_{\mathbb{R}^N} |\phi|^2|\nabla|\phi||^2\,\mathrm{d}x - \int_{\mathbb{R}^N} |\phi|^{p+1}\,\mathrm{d}x$$

*is exactly the set of ground state* $\mathcal{G}_\omega$. *In addition the value of the infimum is equal.*

**Proof.** First we show that if $u \in X_{\mathbb{C}}$ is a minimizer of $d_\omega$ then $|u| \in X$ is also a minimizer of $d_\omega$. Let $u \in X_{\mathbb{C}}$ with $\mathcal{I}_\omega(u) = 0$. Then $\mathcal{E}_\omega(|u|) \leqslant \mathcal{E}_\omega(u)$ as well as $\mathcal{I}_\omega(|u|) \leqslant \mathcal{I}_\omega(u) = 0$. In particular and since $p \geqslant 3$, there exists $t \in (0, 1]$ such that $\mathcal{I}_\omega(t|u|) = 0$. Observe now that, for all $v \in X_{\mathbb{C}}$ such that $\mathcal{I}_\omega(v) = 0$, it holds

$$\mathcal{E}_\omega(v) = \frac{p-1}{2(p+1)}\int_{\mathbb{R}^N} |\nabla v|^2\,\mathrm{d}x + \frac{p-3}{p+1}\int_{\mathbb{R}^N} |v|^2|\nabla|v||^2\,\mathrm{d}x + \omega\frac{p-1}{2(p+1)}\int_{\mathbb{R}^N} |v|^2\,\mathrm{d}x.$$

Thus, since $p \geqslant 3$, it is readily seen that

$$0 < \mathcal{E}_\omega(t|u|) \leqslant t^2\mathcal{E}_\omega(u).$$

In particular, if $u \in X_{\mathbb{C}}$ is a complex minimizer of $d_\omega$, then we have

$$\mathcal{E}_\omega(u) = d_\omega = \inf_{\mathcal{I}_\omega=0} \mathcal{E}_\omega(\phi) \leqslant \mathcal{E}_\omega(t|u|) \leqslant t^2\mathcal{E}_\omega(u).$$

Now, recalling that $\mathcal{E}_\omega(u) > 0$ and $t \leqslant 1$, we immediately get $t = 1$. Thus $\mathcal{I}_\omega(|u|) = \mathcal{I}_\omega(u)$ and in turn $\mathcal{E}_\omega(|u|) = \mathcal{E}_\omega(u)$ proving that $|u| \in X$ is also a minimizer. Obviously it is only possible if the set $\{x \in \mathbb{R}^N : |\nabla|u|(x)| \neq |\nabla u(x)|\}$ has zero Lebesgue measure, which in turn implies that $u = |u|e^{i\theta}$, for some $\theta \in \mathbb{S}^1$ (see, e.g., step III of the proof of theorem 1.3). Now, when $\mathcal{E}_\omega$ is considered over $X$, in [27, theorem 1.1], it is established that there exists a nontrivial solution to the minimization problem (3.22) and that this minimizer is a solution to equation (1.8) (cf [27, lemma 2.5]). Clearly, since any minimizer is of the form $u = |u|e^{i\theta}$ it is also a solution to equation (1.8). Now, any element $u \in X$ of $\mathcal{G}_\omega$ must satisfy $\mathcal{I}_\omega(u) = 0$ and thus we deduce that the set of ground states $\mathcal{G}_\omega$ and the set of minimizer of (3.22) coincide and that the values of the two infimum values are equal. □

We also have the following.

**Lemma 3.5.** *Let us set*

$$c_\omega = \inf\{\mathcal{E}_\omega(\phi) : \phi \in \mathcal{M}\}, \text{ where } \mathcal{M} = \{\phi \in X \setminus \{0\} : \ Q(\phi) = 0,\ \mathcal{I}_\omega(\phi) \leqslant 0\}.$$

*Then* $c_\omega = d_\omega\ (= m_\omega)$.

**Proof.** Let $u \in X_{\mathbb{C}}$ be a solution to (3.22). By lemma 3.4 it is a ground state solution of (1.8) and applying the virial identity (3.13) to a standing wave solution we immediately deduce that $Q(u) = 0$. By definition $\mathcal{I}_\omega(u) = 0$ and thus we have $u \in \mathcal{M}$. Hence $c_\omega \leqslant d_\omega$, since $\mathcal{E}_\omega(u) = d_\omega$. On the other hand, given $\phi \in \mathcal{M}$, either $\mathcal{I}_\omega(\phi) = 0$ (so that $\mathcal{E}_\omega(\phi) \geqslant d_\omega$) or $\mathcal{I}_\omega(\phi) < 0$. In this second case, if $\sigma > 0$ and we consider the rescaling $\phi^\sigma(x) = \sigma^{N/2}\phi(\sigma x)$, we have $\mathcal{I}_\omega(\phi^1) < 0$ and

$$\lim_{\sigma\to 0^+} \mathcal{I}_\omega(\phi^\sigma) = \lim_{\sigma\to 0^+} \left(\sigma^2 \int_{\mathbb{R}^N} |\nabla\phi|^2\,dx + \omega \int_{\mathbb{R}^N} |\phi|^2\,dx \right.$$
$$\left. + 4\sigma^{N+2} \int_{\mathbb{R}^N} |\phi|^2 |\nabla|\phi||^2\,dx - \sigma^{\frac{N(p-1)}{2}} \int_{\mathbb{R}^N} |\phi|^{p+1}\,dx \right) > 0.$$

In turn, one can find $\hat{\sigma} \in (0, 1)$ such that $\mathcal{I}_\omega(\phi^{\hat{\sigma}}) = 0$. Then, we get $\mathcal{E}_\omega(\phi^{\hat{\sigma}}) \geqslant d_\omega$. Since $Q(\phi) = 0$ and $\|\phi\|_2 = \|\phi^{\hat{\sigma}}\|_2$, from lemma 3.3 we obtain $\mathcal{E}_\omega(\phi) \geqslant \mathcal{E}_\omega(\phi^{\hat{\sigma}}) \geqslant d_\omega$. Whence $\mathcal{E}_\omega(\phi) \geqslant d_\omega$ holds true for any $\phi \in \mathcal{M}$, which yields $c_\omega \geqslant d_\omega$, proving the claim. □

**Proof of theorem 1.5.** Let $\varepsilon > 0$ be fixed and consider $u^\sigma(x) = \sigma^{N/2}u(\sigma x)$ for the ground state solution $u$. We have $\|u\|_2 = \|u^\sigma\|_2$ and by the continuity of the mapping $\sigma \mapsto \sigma^{N/2}u(\sigma x)$, it is clear that, for $\sigma$ sufficiently close to 1, $\|u - u^\sigma\|_{H^{s+2}(\mathbb{R}^N)} \leqslant \varepsilon$ (we recall that the ground state $u$ belongs to $H^{s+2}(\mathbb{R}^N)$ for all $s$). Furthermore,

$$\mathcal{E}_\omega(u^\sigma) < \mathcal{E}_\omega(u), \qquad Q(u^\sigma) < 0, \qquad \mathcal{I}_\omega(u^\sigma) < 0, \tag{3.23}$$

provided that $\sigma > 1$ is sufficiently close to 1. The first two inequalities just follow by lemma (3.3). Concerning the last one, it holds

$$\begin{aligned}
\mathcal{I}_\omega(u^\sigma) &= 2\mathcal{E}_\omega(u^\sigma) + \frac{2}{N}Q(u^\sigma) - \frac{4}{N}\int_{\mathbb{R}^N} |u^\sigma|^2|\nabla u^\sigma|^2\,dx - \frac{2}{N}\int_{\mathbb{R}^N} |\nabla u^\sigma|^2\,dx \\
&\leqslant 2\mathcal{E}_\omega(u) + \frac{2}{N}Q(u) - \mathcal{I}_\omega(u) - \frac{4\sigma^{N+2}}{N}\int_{\mathbb{R}^N} |u|^2|\nabla u|^2\,dx - \frac{2\sigma^2}{N}\int_{\mathbb{R}^N} |\nabla u|^2\,dx \\
&= \frac{4}{N}\int_{\mathbb{R}^N} |u|^2|\nabla u|^2\,dx - \frac{2}{N}\int_{\mathbb{R}^N} |\nabla u|^2\,dx \\
&\quad - \frac{4\sigma^{N+2}}{N}\int_{\mathbb{R}^N} |u|^2|\nabla u|^2\,dx - \frac{2\sigma^2}{N}\int_{\mathbb{R}^N} |\nabla u|^2\,dx \\
&= \frac{4}{N}(1-\sigma^{N+2})\int_{\mathbb{R}^N} |u|^2|\nabla u|^2\,dx + \frac{2}{N}(1-\sigma^2)\int_{\mathbb{R}^N} |\nabla u|^2\,dx < 0.
\end{aligned}$$

Now fixing a $\sigma > 1$ such that (3.23) holds, let us set $v := u^\sigma \in H^{s+2}(\mathbb{R}^N)$. Hence,

$$\mathcal{E}_\omega(v) < \mathcal{E}_\omega(u), \qquad Q(v) < 0, \qquad \mathcal{I}_\omega(v) < 0. \tag{3.24}$$

Assume now that $\phi(t)$ is the solution of (1.6) with initial data $\phi(0) = v$. Then, we claim that

$$\mathcal{E}_\omega(\phi(t)) < \mathcal{E}_\omega(u), \quad Q(\phi(t)) < 0, \quad \mathcal{I}_\omega(\phi(t)) < 0, \qquad \text{for all } t \in [0, T_{\max}), \tag{3.25}$$

$T_{\max} \in (0, \infty]$ being the maximal existence time. First, due to the conservation of the energy and (3.24), we get

$$\mathcal{E}_\omega(\phi(t)) = \mathcal{E}_\omega(v) < \mathcal{E}_\omega(u), \qquad \text{for all } t \in [0, T_{\max}).$$

In turn, it follows immediately that $\mathcal{I}_\omega(\phi(t)) \neq 0$ for all $t \in [0, T_{\max})$. Hence $\mathcal{I}_\omega(\phi(t)) < 0$ for all $t \in [0, T_{\max})$ since it is negative for $t = 0$. Similarly, $Q(\phi(t)) \neq 0$ for all $t \in [0, T_{\max})$, otherwise if $Q(\phi(t_0)) = 0$ for some $t_0 \in [0, T_{\max})$, we would have $\phi(t_0) \in \mathcal{M}$, yielding $\mathcal{E}_\omega(\phi(t_0)) \geqslant \mathcal{E}_\omega(u)$ which contradicts the first inequality of (3.25). Hence $Q(\phi(t)) < 0$ for all $t \in [0, T_{\max})$ as it is negative for $t = 0$, concluding the proof of (3.25).

Let now $\psi = \phi(t)$ be the solution to (1.6) at a fixed time $t \in (0, T_{\max})$ and let $\psi^\sigma$ be the usual $L^2$-invariant rescaling. We know that $Q(\psi) < 0$. Hence there exists $\tilde{\sigma} \in (0, 1)$ such that $Q(\psi^{\tilde{\sigma}}) = 0$. If $\mathcal{I}_\omega(\psi^{\tilde{\sigma}}) \leqslant 0$ we do not change the value of $\tilde{\sigma}$, otherwise we pick $\hat{\sigma} \in (\tilde{\sigma}, 1)$ such that $\mathcal{I}_\omega(\psi^{\hat{\sigma}}) = 0$. In any case, one obtains $\mathcal{E}_\omega(\psi^{\tilde{\sigma}}) \geqslant d_\omega$ and $Q(\psi^{\tilde{\sigma}}) \leqslant 0$. Therefore, by lemma 3.3

$$\begin{aligned} \mathcal{E}_\omega(v) = \mathcal{E}_\omega(\psi) &\geqslant \mathcal{E}_\omega(\psi^{\tilde{\sigma}}) + (1 - \tilde{\sigma}) \frac{\partial}{\partial \sigma} \mathcal{E}_\omega(\psi^\sigma)_{|\sigma=1} \\ &= \mathcal{E}_\omega(\psi^{\tilde{\sigma}}) + (1 - \tilde{\sigma}) Q(\psi) > d_\omega + Q(\psi). \end{aligned}$$

Putting $\varrho_0 := d_\omega - \mathcal{E}_\omega(v) > 0$, concluding we have

$$Q(\phi(t)) \leqslant -\varrho_0, \qquad \text{for all } t \in [0, T_{\max}).$$

Finally, assuming that $T_{\max} = +\infty$ and using the virial identity of lemma 3.2, we obtain

$$0 < \mathbb{V}(t) \leqslant \mathbb{V}(0) + \mathbb{V}'(0)t - 4\varrho_0 t^2$$

which yields a contradiction taking $t$ sufficiently large. Then $0 < T_{\max} < +\infty$ and the solution blows up in finite time. This concludes the proof. □

## 4. Stationary solutions with prescribed $L^2$ norm

In this section we study the minimization problem (1.15). We prove the existence of a minimizer when $1 < p < 3 + 4/N$ and $m(c) < 0$. We also discuss the condition $m(c) < 0$ and we prove theorem 1.12. Consider the (complex) minimization problem

$$\text{minimize} \quad \mathcal{E} \text{ on } \|u\|_2^2 = c, \tag{4.1}$$

where $c$ is a positive number. We have the following result.

**Proposition 4.1.** *Let $v$ be a solution to the minimization problem (4.1). Then*

$$v(x) = \mathrm{e}^{\mathrm{i}\theta} |v(|x|)|, \qquad x \in \mathbb{R}^N,$$

*for some $\theta \in \mathbb{S}^1$. In particular, the solutions of problem (4.1) are, up to a constant complex phase, real-valued positive and radially symmetric.*

**Proof.** The proof has some similarities to the final part of the proof of theorem 1.3 so we will be brief here. Let $X$ denote again the restriction of $X_{\mathbb{C}}$ to real-valued functions. We set

$$\sigma_{\mathbb{C}} = \inf\left\{\mathcal{E}(v) : v \in X_{\mathbb{C}},\ \|v\|_2^2 = c\right\}, \qquad \sigma_{\mathbb{R}} = \inf\left\{\mathcal{E}(v) : v \in X,\ \|v\|_2^2 = c\right\}.$$

Let us prove that $\sigma_{\mathbb{C}} = \sigma_{\mathbb{R}}$. Trivially one has $\sigma_{\mathbb{C}} \leqslant \sigma_{\mathbb{R}}$, since $X \subset X_{\mathbb{C}}$. Moreover, if $v \in X_{\mathbb{C}}$, we see using (3.4) that $\mathcal{E}(|v|) \leqslant \mathcal{E}(v)$. In particular, we conclude that $\sigma_{\mathbb{R}} \leqslant \sigma_{\mathbb{C}}$, yielding the desired equality $\sigma_{\mathbb{C}} = \sigma_{\mathbb{R}}$. Now let $v \in X_{\mathbb{C}}$ be a solution to $\sigma_{\mathbb{C}}$ and assume by contradiction that the Lebesgue measure $\mathcal{L}^N$ of the set $\{x \in \mathbb{R}^N : |\nabla|v|(x)| < |\nabla v(x)|\}$ is positive. Then, of course, $\||v|\|_2^2 = \|v\|_2^2 = c$, and

$$\begin{aligned}\sigma_{\mathbb{R}} &\leqslant \frac{1}{2}\int_{\mathbb{R}^N} |\nabla|v||^2\,\mathrm{d}x + \int_{\mathbb{R}^N} |v|^2|\nabla|v||^2\,\mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^N} |v|^{p+1}\,\mathrm{d}x \\ &< \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,\mathrm{d}x + \int_{\mathbb{R}^N} |v|^2|\nabla|v||^2\,\mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^N} |v|^{p+1}\,\mathrm{d}x = \sigma_{\mathbb{C}},\end{aligned}$$

contradicting equality $\sigma_{\mathbb{C}} = \sigma_{\mathbb{R}}$. Hence, we have $|\nabla|v(x)|| = |\nabla v(x)|$ for a.e. $x \in \mathbb{R}^N$ and as in the proof of theorem 1.3 this gives the existence of $\theta \in \mathbb{S}^1$ such that $v = \mathrm{e}^{\mathrm{i}\theta}|v|$. Finally the result of radial symmetry is a direct consequence of [30, theorem 2]. □

From proposition 4.1 we deduce that it is sufficient to study the (real) minimization problem

$$\text{minimize } \mathcal{E} \text{ on } \|u\|_2^2 = c \qquad \text{with } u \in X \tag{4.2}$$

for a positive number $c$. We set

$$m(c) = \inf\{\mathcal{E}(u) : u \in X,\ \|u\|_2^2 = c\}. \tag{4.3}$$

**Lemma 4.2.** *We have the following.*

*(1) Assume that $1 < p < 3+4/N$. Then $m(c) > -\infty$ for any $c > 0$. In addition if $(u_n) \subset X$ is any minimizing sequence for problem (4.2) then $(u_n)$ is bounded in $X$ and the sequence*

$$\int_{\mathbb{R}^N} |u_n|^2|\nabla u_n|^2\,\mathrm{d}x = \frac{1}{4}\int_{\mathbb{R}^N} |\nabla(u_n^2)|^2\,\mathrm{d}x \tag{4.4}$$

*is bounded in $\mathbb{R}$.*

*(2) In the case $p = 3+4/N$ the same conclusions hold provided that $c > 0$ is sufficiently small.*

*(3) Assume that $3+4/N < p < 4N/(N-2)$. Then $m(c) = -\infty$ for any $c > 0$.*

**Proof.** Note that using Hölder and Sobolev inequalities we have for

$$\theta = \frac{(p-1)(N-2)}{2(N+2)}$$

and some $K > 0$ depending only on $N$, that for any $u \in X$

$$\begin{aligned}\int_{\mathbb{R}^N} |u|^{p+1}\,\mathrm{d}x &\leqslant \left(\int_{\mathbb{R}^N} |u|^2\,\mathrm{d}x\right)^{1-\theta}\left(\int_{\mathbb{R}^N} |u|^{\frac{4N}{N-2}}\,\mathrm{d}x\right)^{\theta} \\ &\leqslant K\left(\int_{\mathbb{R}^N} |u|^2\,\mathrm{d}x\right)^{1-\theta}\left(\int_{\mathbb{R}^N} |u|^2|\nabla u|^2\,\mathrm{d}x\right)^{\frac{\theta N}{N-2}}.\end{aligned} \tag{4.5}$$

Here we have used the fact that

$$\int_{\mathbb{R}^N} |u|^{\frac{4N}{N-2}}\,\mathrm{d}x = \int_{\mathbb{R}^N} |u^2|^{\frac{2N}{N-2}}\,\mathrm{d}x, \qquad \int_{\mathbb{R}^N} |\nabla(u^2)|^2\,\mathrm{d}x = 4\int_{\mathbb{R}^N} |u|^2|\nabla u|^2\,\mathrm{d}x.$$

From (4.5) we get that

$$\mathcal{E}(u) \geqslant \int_{\mathbb{R}^N} |u|^2|\nabla u|^2\,\mathrm{d}x - \frac{1}{p+1} K c^{1-\theta} \left( \int_{\mathbb{R}^N} |u|^2|\nabla u|^2\,\mathrm{d}x \right)^{\frac{\theta N}{N-2}}.$$

If we assume that $p < 3+4/N$, we see that $\theta N/(N-2) < 1$ and thus the sequence (4.4) is bounded in $\mathbb{R}$. From (4.5) we then get that $(\|u_n\|_{p+1})$ is bounded and thus also that $(\|\nabla u_n\|_2)$ is bounded. This proves point (1). In the limit case $p = 3+4/N$ we still reach the boundedness result for any positive $c$ such that $Kc^{1-\theta} < p+1$, where $K, \theta > 0$ are the numbers introduced in the proof. Now for point (3) we fix $c > 0$ and take $u \in X$ such that $\|u\|_2^2 = c$. Then, considering the scaling,

$$\sigma \mapsto u^\sigma(x) = \sigma^{\frac{N}{2}} u(\sigma x),$$

we get, for all $\sigma > 0$,

$$\int_{\mathbb{R}^N} |u^\sigma|^2\,\mathrm{d}x = \int_{\mathbb{R}^N} |u|^2\,\mathrm{d}x = c, \qquad \int_{\mathbb{R}^N} |\nabla u^\sigma|^2\,\mathrm{d}x = \sigma^2 \int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x,$$

$$\int_{\mathbb{R}^N} |u^\sigma|^{p+1}\,\mathrm{d}x = \sigma^{\frac{N(p-1)}{2}} \int_{\mathbb{R}^N} |u|^{p+1}\,\mathrm{d}x, \qquad \int_{\mathbb{R}^N} |u^\sigma|^2|\nabla u^\sigma|^2\,\mathrm{d}x = \sigma^{(N+2)} \int_{\mathbb{R}^N} |u|^2|\nabla u|^2\,\mathrm{d}x.$$

Thus $\|u^\sigma\|_2^2 = c$ for all $\sigma > 0$ and

$$\mathcal{E}(u^\sigma) = \frac{\sigma^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x + \sigma^{(N+2)} \int_{\mathbb{R}^N} |u|^2|\nabla u|^2\,\mathrm{d}x - \frac{\sigma^{\frac{N(p-1)}{2}}}{p+1} \int_{\mathbb{R}^N} |u|^{p+1}\,\mathrm{d}x.$$

Now just note that in the range $3+4/N < p < 4N/(N-2)$ the dominant term is

$$\frac{\sigma^{\frac{N(p-1)}{2}}}{p+1} \int_{\mathbb{R}^N} |u|^{p+1}\,\mathrm{d}x.$$

Thus $\mathcal{E}(u^\sigma) \to -\infty$ as $\sigma \to +\infty$. This concludes the proof of (3). □

Concerning the existence of a minimizer we first show the following.

**Lemma 4.3.** *Assume that $1 < p < 3+4/N$. The following facts hold.*

*(1) If $u_n \rightharpoonup u$ in $X$ then setting*

$$T(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x + \int_{\mathbb{R}^N} |u|^2|\nabla u|^2\,\mathrm{d}x,$$

*we have*

$$T(u) \leqslant \liminf_{n\to\infty} T(u_n).$$

*(2) For any $u \in X$ there exists a Schwarz symmetric function $u^* \in X$ satisfying*

$$T(u^*) \leqslant T(u), \quad \int_{\mathbb{R}^N} |u^*|^2\,\mathrm{d}x = \int_{\mathbb{R}^N} |u|^2\,\mathrm{d}x, \quad \int_{\mathbb{R}^N} |u^*|^{p+1}\,\mathrm{d}x = \int_{\mathbb{R}^N} |u|^{p+1}\,\mathrm{d}x.$$

*(3) Let $(u_n) \subset X$ be a minimizing sequence for (4.2) of Schwartz symmetric functions satisfying $u_n \rightharpoonup u$ in $X$. Then we have*

$$\mathcal{E}(u) \leqslant \liminf_{n\to\infty} \mathcal{E}(u_n) = m(c). \tag{4.6}$$

**Proof.** Point (1) is standard. Defining $j : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by $j(s,\xi) = \frac{1}{2}\xi^2 + s^2\xi^2$, then $\{\xi \to j(s,\xi)\}$ is convex and thus the result follows from classical results of Ioffe (see, e.g., [16, 17]). Concerning assertion (2) all we need is $T(u^*) \leqslant T(u)$, which follows by standard rearrangement inequalities. For point (3), we claim that

$$\int_{\mathbb{R}^N} |u_n|^{p+1}\,\mathrm{d}x \to \int_{\mathbb{R}^N} |u|^{p+1}\,\mathrm{d}x \tag{4.7}$$

as $n \to \infty$. In fact, since $(u_n) \subset X$ is minimizing we have, by lemma 4.2, point (1) that $\nabla(u_n^2)$ is uniformly bounded in $L^2(\mathbb{R}^N)$ and thus by the Sobolev embedding $\sup_{n\in\mathbb{N}} \|u_n^2\|_{\frac{2N}{N-2}} < \infty$, which gives $\sup_{n\in\mathbb{N}} \|u_n\|_{\frac{4N}{N-2}} < \infty$. Now, using the fact that $(u_n) \subset X$ consists of radial decreasing functions, from the radial lemma A.IV of [5], we deduce that $(u_n)$ has a uniform decay at infinity (with respect to both $n \in \mathbb{N}$ and $|x|$) and this shows, by standard argument, that (4.7) holds. Now we conclude observing that, from point (1), $T(u) \leqslant \liminf_{n\to\infty} T(u_n)$. □

We now prove the existence of a minimizer for problem (4.2).

**Lemma 4.4.** *Assume that $1 < p < 3 + 4/N$ and let $c > 0$ be such that $m(c) < 0$. Then problem (4.2) admits a minimizer which is Schwartz symmetric.*

**Proof.** Let $(u_n)$ be a minimizing sequence for (4.2). By lemma 4.3 we know that $(u_n) \subset X$ can be replaced by a minimizing sequence $(u_n^*) \subset X$ of Schwarz symmetric functions such that $u_n^* \rightharpoonup u^*$ and

$$\mathcal{E}(u^*) \leqslant \liminf_{n\to\infty} \mathcal{E}(u_n^*) = m(c). \tag{4.8}$$

We still denote $u^*$ by $u$. To conclude we just need to prove that $\|u\|_2^2 = c$. Since $\mathcal{E}(u) \leqslant m(c) < 0$ necessarily $u \neq 0$. Assume thus that $0 < \|u\|_2^2 = \lambda < c$ and consider the scaling $v(x) = u(\sigma^{-\frac{1}{N}}x)$ for $\sigma > 1$. Then $\|v\|_2^2 = \sigma\lambda$ and for $\sigma = c/\lambda$ we have $\|v\|_2^2 = c$. Now we also get that

$$\mathcal{E}(v) = \sigma^{1-\frac{2}{N}}\left[\int_{\mathbb{R}^N} \frac{1}{2}|\nabla u|^2 + |\nabla u|^2|u|^2\,\mathrm{d}x\right] - \frac{\sigma}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\,\mathrm{d}x.$$

Thus, since $\sigma > 1$ and $\mathcal{E}(u) < 0$ we conclude that $\mathcal{E}(v) < \mathcal{E}(u)$, which is a contradiction. This proves that $\|u\|_2^2 = c$ and thus (4.2) admits a minimizer. Finally, observe that, since $\|u_n^*\|_{p+1} \to \|u^*\|_{p+1}$ as $n \to \infty$, necessarily $\|\nabla u_n^*\|_2 \to \|\nabla u^*\|_2$ as $n \to \infty$ and we deduce that the Schwarz symmetric sequence strongly converges to $u^* \in X$. □

We now start to discuss the condition $m(c) < 0$.

**Lemma 4.5.** *We have the following.*

*(1) Assume that $1 < p < 1 + 4/N$. Then $m(c) < 0$ for any $c > 0$.*
*(2) Assume that $1 + 4/N \leqslant p < 3 + 4/N$. Then $m(c) \leqslant 0$ for any $c > 0$. This inequality also holds if $p = 3 + 4/N$ and $c > 0$ is small.*
*(3) Assume that $1 + 4/N \leqslant p < 3 + 4/N$. Then there exists a $c > 0$, sufficiently large, such that $m(c) < 0$.*

**Proof.** For points (1) and (2) we use the scaling introduced in the proof of lemma 4.2, point (3). When $p < 1 + 4/N$ we see that the dominant term, as $\sigma \to 0^+$, is

$$\frac{\sigma^{\frac{N(p-1)}{2}}}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\,\mathrm{d}x,$$

and this proves point (1). For point (2), since $\mathcal{E}(u^\sigma) \to 0$ as $\sigma \to 0^+$, we directly have that $m(c) \leqslant 0$ for any $c > 0$. Now for point (3) we consider, for a fixed $R > 0$, the radial function $w_R \in H^1(\mathbb{R}^N)$ defined by

$$w_R(r) := \begin{cases} 1 & \text{if } r \leqslant R, \\ 1 + R - r & \text{if } R \leqslant r \leqslant R+1, \\ 0 & \text{if } r \geqslant R+1. \end{cases}$$

Integrating in radial coordinates, we have

$$\int_{\mathbb{R}^N} |w_R(|x|)|^2 \, \mathrm{d}x = C_N R^N + \varepsilon_1(R^{N-1}),$$

where $\varepsilon_1(R^{N-1})/R^N \to 0$, as $R \to \infty$. Also

$$\int_{\mathbb{R}^N} |w_R(|x|)|^{p+1} \, \mathrm{d}x = C_N R^N + \varepsilon_2(R^{N-1}), \qquad \int_{\mathbb{R}^N} |\nabla w_R(|x|)|^2 \, \mathrm{d}x = \varepsilon_3(R^{N-1}),$$

and

$$\int_{\mathbb{R}^N} |w_R(|x|)|^2 |\nabla w_R(|x|)|^2 \, \mathrm{d}x = \varepsilon_4(R^{N-1}),$$

where $\varepsilon_i(R^{N-1})/R^N \to 0$, as $R \to \infty$, for any $i = 2, 3, 4$. Thus letting $R \to \infty$ we have $\|w_R\|_2^2 \to +\infty$ and $\mathcal{E}(w_R) \to -\infty$. This proves point (3). □

In preparation for the proof of theorem 1.12 we also show the following.

**Lemma 4.6.** *Assume that $1 < p < 3 + 4/N$ and that $u_c \in X$ is a minimizer of (4.2) for some $c > 0$. Then $u_c \in X$ weakly satisfies*

$$-\Delta u_c - \lambda_c u_c - u_c \Delta |u_c|^2 = |u_c|^{p-1} u_c \tag{4.9}$$

*with the Lagrange multiplier $\lambda_c \in \mathbb{R}$ being strictly negative.*

**Proof.** It is standard to show that $u_c \in X$ satisfies (4.9) for $\lambda_c \in \mathbb{R}$ being the associated Lagrange multiplier, namely

$$\mathcal{E}'(u_c) = \lambda_c u_c. \tag{4.10}$$

Now applying Pohozaev identity to (4.10) yields

$$\frac{1}{p+1} \int_{\mathbb{R}^N} |u_c|^{p+1} \, \mathrm{d}x = \frac{N-2}{N} \left[ \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_c|^2 \, \mathrm{d}x + \int_{\mathbb{R}^N} |u_c|^2 |\nabla u_c|^2 \, \mathrm{d}x \right] - \frac{\lambda_c}{2} \int_{\mathbb{R}^N} |u_c|^2 \, \mathrm{d}x.$$

Thus, we obtain

$$\mathcal{E}(u_c) = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla u_c|^2 + 2|u_c|^2 |\nabla u_c|^2 \, \mathrm{d}x + \frac{\lambda_c}{2} \int_{\mathbb{R}^N} |u_c|^2 \, \mathrm{d}x.$$

Since $\mathcal{E}(u_c) \leqslant 0$, see lemma 4.5, we deduce that $\lambda_c < 0$. □

We can now give the following.

**Proof of theorem 1.12.** The proof of (1) is lemma 4.5, point (1). To show (2)-(i) we assume by contradiction that there exists a sequence $(c_n) \subset \mathbb{R}^+$ with $c_n \to 0$ as $n \to \infty$ and $(u_n) \subset X$ such that $m(c_n)$ is reached by $u_n \in X$. Then we know, from lemma 4.5, point (2), that $\mathcal{E}(u_n) \leqslant 0$, for all $n \in \mathbb{N}$ and using (4.5), we get

$$\int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 \, \mathrm{d}x \leqslant K \left( \int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 \, \mathrm{d}x \right)^{\frac{\theta N}{N-2}} ||u_n||_2^{2-2\theta}. \tag{4.11}$$

If $p = 3+4/N$ we have $\theta N/(N-2) = 1$ and $2-2\theta = 4/N > 0$. Thus, since $||u_n||_2 \to 0$, we immediately get a contradiction from (4.11). Now if $p < 3+4/N$, we have $\theta N/(N-2) < 1$ and thus,

$$\int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 \, dx \to 0, \qquad \text{as } n \to \infty. \tag{4.12}$$

Still using (4.5), we see from (4.12) that $\|u_n\|_{p+1} \to 0$ as $n \to \infty$. In turn, also

$$\|\nabla u_n\|_2 \to 0, \qquad \text{as } n \to \infty, \tag{4.13}$$

since $\mathcal{E}(u_n) \leqslant 0$ implies that

$$\|\nabla u_n\|_2^2 \leqslant \frac{2}{p+1} \|u_n\|_{p+1}^{p+1}, \qquad \text{for all } n \in \mathbb{N}.$$

At this point we distinguish two cases. First assume that $1+4/N \leqslant p \leqslant (N+2)/(N-2)$ if $N \geqslant 3$, $1+4/N \leqslant p$ if $N = 1, 2$. By Hölder and Sobolev inequalities we have

$$\|u_n\|_{p+1}^{p+1} \leqslant K(p, N) \|\nabla u_n\|_2^{\frac{N}{2}(p-1)} ||u_n||_2^{p+1-\frac{N}{2}(p-1)}.$$

Since $\mathcal{E}(u_n) \leqslant 0$ it follows that

$$\|\nabla u_n\|_2^2 \leqslant \frac{2}{p+1} \int_{\mathbb{R}^N} |u_n|^{p+1} \, dx \leqslant K(p, N) \|\nabla u_n\|_2^{\frac{N}{2}(p-1)} ||u_n||_2^{p+1-\frac{N}{2}(p-1)}. \tag{4.14}$$

If $p = 1+4/N$ we have $N/2(p-1) = 2$ and $p+1-N/2(p-1) > 0$. Thus we get directly a contradiction since $||u_n||_2 \to 0$. If $1+4/N < p \leqslant (N+2)/(N-2)$ for $N \geqslant 3$ and $1+4/N < p$ for $N = 1, 2$ we have $N/2(p-1) > 2$ and $p+1-\frac{N}{2}(p-1) \geqslant 0$. Thus there exists a $d > 0$ such that $||\nabla u_n||_2 \geqslant d$ for all $n \in \mathbb{N}$, yielding a contradiction with (4.13).

Now we treat the remaining case $(N+2)/(N-2) < p < 3+4/N$ with $N \geqslant 3$. First, let us show that for any $q \geqslant 4N/(N-2)$ the sequence $(u_n) \subset X$ belongs to $L^q(\mathbb{R}^N)$ and it is uniformly bounded in $L^q(\mathbb{R}^N)$. For this we follow a Moser's iteration argument presented in the proof of [27, lemma 5.10]. Since $u_n \in X$ is a minimizer for (4.2) with $c = c_n$ we know, by lemma 4.6, that $u_n \in X$ weakly satisfies (4.9). Namely that

$$\int_{\mathbb{R}^N} (1+2|u_n|^2)\nabla u_n \cdot \nabla \phi + 2u_n |\nabla u_n|^2 \phi - \lambda_n u_n \phi - |u_n|^{p-1} u_n \phi \, dx = 0,$$

where $\lambda_n < 0$ is the Lagrange parameter and $\phi \in C_0^\infty(\mathbb{R}^N, \mathbb{R})$. By an approximation argument, it is easily seen that we can take as test functions any function in $X$ which satisfies

$$\int_{\mathbb{R}^N} u^2 |\nabla \phi|^2 \, dx < \infty \qquad \text{and} \qquad \int_{\mathbb{R}^N} |\nabla u|^2 \phi^2 \, dx < \infty.$$

In particular, setting $q_0 = 4N/(N-2)$, we can choose as test function, for any $M > 0$ and any fixed $n \in \mathbb{N}$, $\phi_n^M = |u_n^M|^{q_0-p-1} u_n^M$, where $u_n^M = u_n$ when $|u_n(x)| \leqslant M$ and $u_n^M = \pm M$ when $u_n \geqslant \pm M$. We then have, since $|u_n^M| \leqslant |u_n|$ and $|\nabla u_n^M| \leqslant |\nabla u_n|$ for any $n \in \mathbb{N}$, $M > 0$, and using the fact that $\lambda_n < 0$, that

$$(q_0 - p) \int_{\mathbb{R}^N} (1+2|u_n^M|^2) |u_n^M|^{q_0-p-1} |\nabla u_n^M|^2 \, dx \leqslant \int_{\mathbb{R}^N} |u_n|^{q_0} \, dx.$$

Since $q_0 - p > 1$ we have, in particular,

$$2 \int_{\mathbb{R}^N} |u_n^M|^{q_0-p+1} |\nabla u_n^M|^2 \, dx \leqslant \int_{\mathbb{R}^N} |u_n|^{q_0} \, dx.$$

Finally, for $n \in \mathbb{N}$ fixed, letting $M \to +\infty$ we obtain that

$$2 \int_{\mathbb{R}^N} |u_n|^{q_0-p+1} |\nabla u_n|^2 \, dx \leqslant \int_{\mathbb{R}^N} |u_n|^{q_0} \, dx. \tag{4.15}$$

Now, note that, by Sobolev inequality,

$$2\int_{\mathbb{R}^N} |u_n|^{q_0-p+1}|\nabla u_n|^2\,\mathrm{d}x = L(p,N)\int_{\mathbb{R}^N} \left|\nabla |u_n|^r\right|^2 \mathrm{d}x \geqslant \tilde{L}(p,N)\|u_n^r\|^2_{\frac{2N}{N-2}},$$

for some constants $L, \tilde{L} > 0$, and where

$$r = \frac{q_0 - p + 3}{2}.$$

Thus $(u_n) \subset L^{\frac{2Nr}{N-2}}(\mathbb{R}^N)$ and since, by (4.12), $(u_n) \subset L^{q_0}(\mathbb{R}^N)$ is bounded, by (4.15), $(u_n) \subset L^{\frac{2Nr}{N-2}}(\mathbb{R}^N)$ is also bounded. Since $p < (3N+2)/(N-2)$ it follows that

$$\frac{2Nr}{N-2} > q_0$$

and the Moser iteration can be continued further on. Thus, we obtain that $(u_n) \subset L^q(\mathbb{R}^N)$ for any $q \geqslant q_0$ with $(u_n) \subset L^q(\mathbb{R}^N)$ bounded. At this point, by Hölder and Sobolev inequalities we can write

$$\|u_n\|_{p+1}^{p+1} \leqslant C(p,N)\|\nabla u_n\|_2^{\alpha}\,\|u_n\|^{\beta}_{(p-1)N} \tag{4.16}$$

with

$$\alpha = \frac{2N(p-1) - 2(p+1)}{(p-1)(N-2) - 2} \qquad \text{and} \qquad \beta = N(p-1)\frac{(N-2)(p+1) - 2N}{(p-1)N(N-2) - 2N}.$$

Now as in (4.14), using the fact that $\mathcal{E}(u_n) \leqslant 0$, we get that

$$\|\nabla u_n\|_2^2 \leqslant K(p,N)\|\nabla u_n\|_2^{\alpha}\,\|u_n\|^{\beta}_{(p-1)N}.$$

As $p > 1$ we have $\alpha > 2$ and since $(u_n) \subset L^{(p-1)N}(\mathbb{R}^N)$ is bounded we obtain again a contradiction with (4.13). Note that, in (4.16), the coefficient $(p-1)N$ has no particular meaning, we just choose it sufficiently large in order to ensure that, in turn, $\alpha > 2$. This proves point (i) since if $m(c) < 0$ a minimizer always exists by lemma 4.4.

For the proof of point (2)-(ii) we know from lemma 4.5, point (3), that there exists a $c > 0$ such that $m(c) < 0$. Now let $d > 0$ be such that $m(d) < 0$ and $u \in X$ be an associated minimizer. We consider the scaling $v(x) = u(\sigma^{-1/N}x)$ used in the proof of lemma 4.4. For $\sigma > 1$ we have $\|v\|_2^2 > d$ and $\mathcal{E}(v) < \mathcal{E}(u)$. This proves the claim. We also point out that very likely the function $\{c \to m(c)\}$ is continuous for $c > 0$ so that also $m(c(p,N)) = 0$. However we do not pursue this further. □

## 5. Orbital stability

In this section we prove the orbital stability result, theorem 1.9. Its proof crucially relies on the relative compactness of any minimizing sequence as expressed by theorem 1.11.

**Proof of theorem 1.11.** Let $(u_n) \subset X$ be any minimizing sequence for problem (1.15). To prove its relative convergence, up to translation, we use Lions's compactness-concentration principle (cf [24, 25]), applied to the sequence

$$\rho_n(x) = u_n^2(x), \qquad n \in \mathbb{N}.$$

First we prove that the vanishing, namely

$$\sup_{y\in\mathbb{R}^N}\int_{y+B_R} |u_n|^2\,\mathrm{d}x \to 0 \qquad \text{for all } R > 0,$$

cannot occur. By lemma 4.2, we know that

$$(u_n) \subset X \text{ is bounded and } \int_{\mathbb{R}^N} |\nabla(u_n^2)|\,\mathrm{d}x \text{ is bounded in } \mathbb{R}. \tag{5.1}$$

We apply [25, lemma I.1] to the sequence $\rho_n$. Indeed, $\rho_n$ is bounded in $L^1(\mathbb{R}^N)$ and $\nabla\rho_n$ is bounded in $L^2(\mathbb{R}^N)$. Then for every $1 \leqslant \alpha \leqslant 2N/(N-2)$, $\rho_n \to 0$ in $L^\alpha(\mathbb{R}^N)$, as $n$ goes to $\infty$. Taking $\alpha = p + 1/2$ (this choice is valid since $1 < p < 3 + 4/N$) provides

$$\|\rho_n\|_{\frac{p+1}{2}} = \|u_n\|_{p+1}^2 \to 0 \qquad \text{as } n \to \infty,$$

and then $\liminf_{n\to\infty} \mathcal{E}(u_n) \geqslant 0$, which contradicts the fact that $m(c) < 0$. Now, we claim that there exists a subsequence $u_{n_k}$ (that we still denote by $(u_n)$) such that either compactness occurs or dichotomy occurs in the following sense: there exists $\alpha \in (0, c)$ such that, for all $\varepsilon > 0$, there exists $k_0 \geqslant 1$ and two sequences $(u_n^1)$, $(u_n^2)$ bounded in $X$ such that, for all $k \geqslant k_0$,

$$\|u_n - (u_n^1 + u_n^2)\|_{L^{p+1}} \leqslant \delta(\varepsilon), \quad 1 < p < 3 + \tfrac{4}{N}, \qquad \text{with } \delta(\varepsilon) \to 0 \text{ for } \varepsilon \to 0, \tag{5.2}$$

$$\left|\int_{\mathbb{R}^N} (u_n^1)^2\,\mathrm{d}x - \alpha\right| \leqslant \varepsilon, \qquad \left|\int_{\mathbb{R}^N} (u_n^1)^2\,\mathrm{d}x - (c-\alpha)\right| \leqslant \varepsilon,$$

$$\operatorname{dist}(\operatorname{supp} u_n^1, \operatorname{supp} u_n^2) \to \infty, \quad \text{as } n \to \infty,$$

$$\liminf_{k\to\infty} \int_{\mathbb{R}^N} (|\nabla u_n|^2 - |\nabla u_n^1|^2 - |\nabla u_n^2|^2)\,\mathrm{d}x \geqslant 0, \tag{5.3}$$

$$\liminf_{k\to\infty} \int_{\mathbb{R}^N} (|\nabla(u_n)^2|^2 - |\nabla(u_n^1)^2|^2 - |\nabla(u_n^2)^2|^2)\,\mathrm{d}x \geqslant 0. \tag{5.4}$$

We point out that only inequalities (5.2) and (5.4) have to be proved, the other inequalities are already contained in [24, lemma III.1]. Because of (5.1) and taking into account inequality (4.5) there exists a positive constant $K$ such that, for all $n \in \mathbb{N}$,

$$\int_{\mathbb{R}^N} |u_n|^{p+1}\,\mathrm{d}x \leqslant K\left(\int_{\mathbb{R}^N} |u_n|^2\,\mathrm{d}x\right)^{1-\theta}, \qquad \theta = \frac{(p-1)(N-2)}{2(N+2)}. \tag{5.5}$$

Thus, inequality (5.2) follows from the corresponding inequality for the $L^2$ norm given in [24, lemma III.1]. Now inequality (5.4) can be obtained by arguing as for the proof of (5.3) in [24]. Indeed, note that if $\varphi_R$ is a given smooth cut-off function, $0 \leqslant \varphi_R \leqslant 1$, $\varphi_R = 1$ on $B(0, R)$, $\varphi_R = 0$ outside $B(0, 2R)$ and $|\nabla\varphi_R| \leqslant 1/R$, and $v_n$ is a sequence in $X$ satisfying the boundedness condition (5.5), then we have

$$\begin{aligned}|\nabla(\varphi_R v_n)^2|^2 - \varphi_R^4|\nabla v_n^2|^2 &= 4\varphi_R^3 v_n^2 \nabla\varphi_R \cdot \nabla v_n^2 + 4\varphi_R^2|\nabla\varphi_R|^2 v_n^4 \\ &\leqslant 2\varphi_R^3|\nabla\varphi_R| v_n^4 + 2\varphi_R^3|\nabla\varphi_R||\nabla v_n^2|^2 + 4\varphi_R^2|\nabla\varphi_R|^2 v_n^4,\end{aligned}$$

for all $n \geqslant 1$, yielding

$$\left|\int_{\mathbb{R}^N} |\nabla(\varphi_R v_n)^2|^2\,\mathrm{d}x - \int_{\mathbb{R}^N} \varphi_R^4|\nabla v_n^2|^2\,\mathrm{d}x\right| \leqslant \frac{C}{R}, \qquad \text{for all } n \geqslant 1,$$

for some positive constant $C$ independent of $n$. This last inequality is therefore sufficient to obtain inequality (5.4).

Now, it is standard to see that if the dichotomy property holds (with the inequalities indicated above), then sending $\varepsilon$ to zero, the following inequality holds true

$$m(c) \geqslant m(\alpha) + m(c - \alpha).$$

To conclude we now prove that instead we have, for any $c_1, c_2 > 0$ such that $c_1 + c_2 = c$,

$$m(c) < m(c_1) + m(c_2). \tag{5.6}$$

In light of [24, lemma II.1], to show that (5.6) holds, it is sufficient to prove that, for any $d > 0$ such that $m(d) < 0$,

$$m(\lambda d) < \lambda m(d), \qquad \text{for any } \lambda > 1. \tag{5.7}$$

To prove inequality (5.7) we observe that if $u_d \in X$ is a minimizer of $m(d)$, then setting $v(x) = u_d(\lambda^{-\frac{1}{N}}x)$ we have $\|v\|_2^2 = \lambda d$ and

$$\begin{aligned}\mathcal{E}(v) &= \lambda^{1-\frac{2}{N}}\left[\int_{\mathbb{R}^N}\frac{1}{2}|\nabla u_d|^2 + |u_d|^2|\nabla u_d|^2\,\mathrm{d}x\right] - \lambda\int_{\mathbb{R}^N}|u_d|^{p+1}\,\mathrm{d}x \\ &= \lambda\left[\lambda^{-\frac{2}{N}}\int_{\mathbb{R}^N}\left(\frac{1}{2}|\nabla u_d|^2 + |u_d|^2|\nabla u_d|^2\,\mathrm{d}x\right) - \int_{\mathbb{R}^N}|u_d|^{p+1}\,\mathrm{d}x\right] < \lambda m(d).\end{aligned}$$

Thus $\mathcal{E}(v) < \lambda m(d)$ which lead to $m(\lambda d) < \lambda m(d)$, proving the claim.

Since we ruled out both vanishing and dichotomy, we have compactness for $\rho_n$, namely we know that there exists a sequence $(y_n) \subset \mathbb{R}^N$ such that, for any $\varepsilon > 0$, there is $R > 0$ with

$$\int_{y_n+B_R}|u_n|^2\,\mathrm{d}x \geqslant c - \varepsilon. \tag{5.8}$$

We then denote $\tilde{u}_n = u_n(\cdot + y_n)$ and clearly from inequality (5.8) we have $\tilde{u}_n \to \tilde{u}$ strongly in $L^2(\mathbb{R}^N)$, as $n \to \infty$. By (5.5) we then see that $\tilde{u}_n \to \tilde{u}$ strongly in $L^p(\mathbb{R}^N)$. At this point, taking into account point (1) of lemma 4.3, and since $\tilde{u}_n \rightharpoonup \tilde{u}$ in $X$, we get that $\mathcal{E}(\tilde{u}) \leqslant \liminf \mathcal{E}(\tilde{u_n}) = m(c)$. This proves that $\tilde{u} \in X$ minimizes (4.2) and then, necessarily, $\nabla\tilde{u}_n \to \nabla u$ in $L^2(\mathbb{R}^N)$, as $n \to \infty$, proving the strong convergence of $\tilde{u}_n$ to $\tilde{u}$ in $X$. This concludes the proof. □

Now we can state the following proof:

**Proof of theorem 1.9.** First note that if $(u_n)$ is a minimizing sequence for (4.2), then $(|u_n|)$ is also a minimizing sequence and is real. Then by theorem 1.11, there exists a subsequence $(|u_{n_k}|)$ of $(|u_n|)$ and a sequence $(y_{n_k}) \subset \mathbb{R}^N$ such that $(|u_{n_k}(\cdot - y_{n_k})|)$ converges strongly in $H^1(\mathbb{R}^N)$ towards $u$ where $u$ is real and solves (4.2). Then the result follows by standard considerations (see, for example, [9]). □

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