Semiclassical states for coupled nonlinear Schrödinger equations with critical frequency

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Abstract. In this paper, we are concerned with the coupled nonlinear Schrödinger system

$$\begin{cases} -\varepsilon^2 \Delta u + a(x)u = \mu_1 u^3 + \beta v^2 u & \text{in } \mathbb{R}^N, \\ -\varepsilon^2 \Delta v + b(x)v = \mu_2 v^3 + \beta u^2 v & \text{in } \mathbb{R}^N, \end{cases}$$

where $1 \leqslant N \leqslant 3$, $\mu_1, \mu_2, \beta > 0$, $a(x)$ and $b(x)$ are nonnegative continuous potentials, and $\varepsilon > 0$ is a small parameter. We show the existence of positive ground state solutions for the system above and also establish the concentration behaviour as $\varepsilon \to 0$, when $a(x)$ and $b(x)$ achieve 0 with a homogeneous behaviour or vanish in some nonempty open set with smooth boundary.

Keywords: Nonlinear Schrödinger systems, semiclassical limit, critical frequency, variational methods

1. Introduction

1.1. Background

Consider the following two-component coupled nonlinear Schrödinger equations (also known as Gross–Pitaevskii equations):

$$\begin{cases} -i\hbar \frac{\partial}{\partial_t} \Phi_1 + V_1(x)\Phi_1 = \frac{\hbar^2}{2m} \Delta \Phi_1 + \mu_1 |\Phi_1|^2 \Phi_1 + \beta |\Phi_2|^2 \Phi_1, & \text{in } \mathbb{R}^N, t > 0, \\ -i\hbar \frac{\partial}{\partial_t} \Phi_2 + V_2(x)\Phi_2 = \frac{\hbar^2}{2m} \Delta \Phi_2 + \mu_2 |\Phi_2|^2 \Phi_2 + \beta |\Phi_1|^2 \Phi_2, & \text{in } \mathbb{R}^N, t > 0, \end{cases} \quad (1.1)$$

where $N \leqslant 3$, i is the imaginary unit, $\hbar$ is the Plank constant, $\mu_1, \mu_2 > 0$ and $\beta \neq 0$ is a coupling constant. System (1.1) has applications in many physical problems, such as nonlinear optics, contin-

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uum mechanics, plasma physics and Bose–Einstein condensates theory for multispecies Bose–Einstein condensates. In the context of Bose–Einstein condensates, $\Phi_j(x, t)(j = 1, 2)$ are the corresponding condensate amplitudes; $V_j(x)$ are the traping potentials; μ_j and β are the intraspecies and interspecies scattering lengths. The physically realistic spatial dimension are $1 \leqslant N \leqslant 3$. When $N = 2$, problem (1.1) arises in the Hartree–Fock theory for a double condenstate, *i.e.*, a binary mixture of Bose–Einstein condensates in two different hyperfine states $|1\rangle$ and $|2\rangle$ (see [12,26]). The sign of β determines whether the interactions of states are repulsive or attractive. The interaction is attractive if $\beta > 0$ and repulsive if $\beta < 0$, where the two states are in strong competition.

A standing wave solution of (1.1) is of the following form

$$\Phi_1(x, t) = e^{-iE_t/\hbar}u(x) \quad \text{and} \quad \Phi_2(x, t) = e^{-iE_t/\hbar}v(x). \tag{1.2}$$

Substituting (1.2) into (1.1), then system (1.1) is reduced to the following elliptic system

$$\begin{cases} -\frac{\hbar^2}{2m}\Delta u + (V_1(x) - E)u = \mu_1 u^3 + \beta v^2 u, & \text{in } \mathbb{R}^N, \\ -\frac{\hbar^2}{2m}\Delta v + (V_2(x) - E)v = \mu_2 v^3 + \beta u^2 v, & \text{in } \mathbb{R}^N. \end{cases} \tag{1.3}$$

Renaming the parameters by

$$\varepsilon = \sqrt{\frac{\hbar^2}{2m}}, \qquad a(x) = V_1(x) - E, \qquad b(x) = V_2(x) - E,$$

we obtain the folowing elliptic system

$$\begin{cases} -\varepsilon^2\Delta u + a(x)u = \mu_1 u^3 + \beta v^2 u & \text{in } \mathbb{R}^N, \\ -\varepsilon^2\Delta v + b(x)v = \mu_2 v^3 + \beta u^2 v & \text{in } \mathbb{R}^N. \end{cases} \tag{$\mathcal{C}_\varepsilon$}$$

Before going further, we point out that the system ($\mathcal{C}_\varepsilon$) also possesses a trivial solution $(0, 0)$ and semi-trivial solutions of type $(u, 0)$ or $(0, v)$. A solution (u, v) of ($\mathcal{C}_\varepsilon$) is nontrivial if $u \not\equiv 0$ and $v \not\equiv 0$. A solution (u, v) with $u > 0$ and $v > 0$ is called a positive solution.

When $\varepsilon = 1$, $a(x)$ and $b(x)$ are positive constants, the existence, multiplicity, bifurcation and concentration behavior of positive solutions of ($\mathcal{C}_\varepsilon$) have been entensively studied, see Refs. [1,2,4,5,11, 14,18,20,21,25,34]. Sign and size of β are important in the study of constant coefficient system. Particularly, the interesting paper [25] proved that there exists $0 < \bar{\beta}_1 < \bar{\beta}_2$ such that ($\mathcal{C}_\varepsilon$) has nontrivial positive solutions for $0 < \beta < \bar{\beta}_1$ or $\beta > \bar{\beta}_2$. For the non-constant potentials case, according to Bohr's corresponding principle, the classical description of the mechanical physics should be recovered by considering the semiclassical limit, i.e. when the Planck constant goes to zero. Therefore beside proving existence of solutions to ($\mathcal{C}_\varepsilon$), it is physically relevant to study the asymptotic limit as $\varepsilon \to 0$ of such solutions.

The semiclassical limit is well understood for the nonlinear Schrödinger equation

$$-\varepsilon^2\Delta u_\varepsilon + Vu_\varepsilon = |u_\varepsilon|^{p-1}u_\varepsilon \quad \text{in } \mathbb{R}^N.$$

The semi-classical states of the above equation have different limiting behaviors depend upon $\inf_{\mathbb{R}^N} V > 0$ or $\inf_{\mathbb{R}^N} V = 0$, the latter is regarded as the critical frequency case. When $\inf_{\mathbb{R}^N} V > 0$, P. Rabinowitz

[24] proved that the above problem has a least energy solution for $\varepsilon > 0$ small in the case that $\inf_{\mathbb{R}^N} V < \liminf_{|x| \to \infty} V(x)$. In case $\inf_{\mathbb{R}^N} V > 0$, solutions concentrating at critical points of the potential V have been construted by topological and variational methods [10,13,24,31] and many others. Motivations can be dated back to Floer and Weinstein in there pioneering work [13]. However, the semi-classical solutions exhibit quite different characteristic features that the energy governed by the shape of the potential V near the zero set of V in the critical frequency case $\inf_{\mathbb{R}^N} V = 0$ [7,8].

The semi-classical case $(\mathcal{C}_\varepsilon)$ with trapping potentials $a(x)$ and $b(x)$ has been studied in some years. Lin and Wei [19] studied $(\mathcal{C}_\varepsilon)$ by analyzing least energy non-trivial vector solutions. They studied both of attractive interaction and replusive interaction. Especially, when $\beta > 0$, they showed the existence of a least energy. In [22], for small $\beta > 0$, Montefusco, Pellacci and Squassina exhibited the existence of nonnegative ground state solutions of $(\mathcal{C}_\varepsilon)$ concentrating around the local minimum points of the potentials, which are in the same region. Ikoma and Tanaka [15] also considered the case $\beta > 0$. They connected the solutions of $(\mathcal{C}_\varepsilon)$ with the limiting system with $\lambda_1 = a(x_0)$ and $\lambda_2 = b(x_0)$ for a fixed $x_0 \in \mathbb{R}^N$. Assume that there exists an open bounded set $\Lambda \subset \mathbb{R}^N$ such that

$$\inf_{x_0 \in \Lambda} m(x_0) < \inf_{x_0 \in \partial \Lambda} m(x_0),$$

where $m(x_0)$ is the ground state energy level of the limiting system. When $\beta > 0$ is relatively small, they constructed a family of solutions to $(\mathcal{C}_\varepsilon)$ which converge to a positive vector solution. For the large $\beta > 0$ case, Shi and Wang [33] showed the existence of positive ground state solution of $(\mathcal{C}_\varepsilon)$ concentrating near the minimum points of potential functions. They also showed that multiple positive concentration solutions exist when the topological structure of the set of minimum points satisfies certain condition. We also refer to [19,23,32] for the study of $(\mathcal{C}_\varepsilon)$ when $\beta < 0$.

Note that in all works mentioned above, they all assumed that $a(x)$ and $b(x)$ are positive bounded away from 0. In this paper, we consider the semi-classical in the critical frequency case

$$\inf_{x \in \mathbb{R}^N} a(x) = \inf_{x \in \mathbb{R}^N} b(x) = 0.$$

For the large $\beta > 0$ case, Chen and Zou [9] assumed $a(x)$ and $b(x)$ are non-negative and may vanish at someplace and decay to 0 at infinity. They proved the existence of positive solutions for $(\mathcal{C}_\varepsilon)$ which concentrate around local minima of the potentials as $\varepsilon \to 0$ under suitable conditions on the behaviour of $a(x)$ and $b(x)$. Lucia and Tang [17] obtained synchronized solutions which trapped near zero sets of the potentials for n-coupled system under some conditions. In [27], Tang and Wang obtained multi-scale segregated positive solutions which are trapped near the nondegenerate critical points as well as the zero set of potentials by using the Lyapunov–Scjmidt reduction method. Tang and Xie [28] studied $a(x)$ and $b(x)$ admitted several common isolated connected components by using spectral analysis. They obtained the existence synchronized positive vector solutions which are trapped in a neighbourhood of the zero set of potentials and also the local maximum points of potentials for ε small.

1.2. Motivation

In this paper, inspired by Byeon and Wang [7], we are concerned with the asymptotic behavior of positive ground state solutions of $(\mathcal{C}_\varepsilon)$ in the critical frequency case for $\beta > 0$ sufficiently large. Obviously,

problem ($\mathcal{C}_\varepsilon$) is variational. That is, our main aim is to find nontrivial solutions of problem ($\mathcal{C}_\varepsilon$) by seeking nontrivial critical points in some weighted Sobolev space of the corresponding energy functional $\mathcal{J}_\varepsilon$ given by

$$\mathcal{J}_\varepsilon(u, v) = \frac{1}{2} \int_{\mathbb{R}^N} \left[\varepsilon^2 \left(|\nabla u|^2 + |\nabla v|^2 \right) + a(x)u^2 + b(x)v^2 \right] - \frac{1}{4} \int_{\mathbb{R}^N} \left(\mu_1 v^4 + 2\beta u^2 v^2 + \mu_2 v^4 \right).$$

In the sequel, the potentials $a(x)$ and $b(x)$ may achieve 0 with homogeneous behaviour or vanish on the closure of an open set, but remain bounded away from 0 at infinity. Throughout the paper, we assume that

(f_1) $\mu_1, \mu_2, \beta > 0$.
(f_2) $a \in C(\mathbb{R}^N, [0, +\infty))$ and $0 = \inf_{x \in \mathbb{R}^N} a(x) < \liminf_{|x| \to \infty} a(x) := a_\infty < \infty$.
(f_3) $b \in C(\mathbb{R}^N, [0, +\infty))$ and $0 = \inf_{x \in \mathbb{R}^N} b(x) < \liminf_{|x| \to \infty} b(x) := b_\infty < \infty$.

One of the difficulties in the study of ($\mathcal{C}_\varepsilon$) is that it has semi-trivial solutions of type $(u, 0)$ and $(0, v)$, where $u(x)$ and $v(x)$ solve

$$-\varepsilon^2 \Delta u + a(x)u = \mu_1 u^3, \quad \text{in } \mathbb{R}^N. \tag{1.4}$$

$$-\varepsilon^2 \Delta v + b(x)v = \mu_2 v^3, \quad \text{in } \mathbb{R}^N. \tag{1.5}$$

By [7], problem (1.4) and (1.5) both admit ground state solutions for ε small enough. Here, we borrow an idea of [25] to overcome this difficulty. Denote by $U_{1\varepsilon}$ and $V_{1\varepsilon}$ a positive ground state solution of (1.4) and (1.5) respectively.

1.3. Main results

Let $\mathcal{A} = \{x \in \mathbb{R}^N | a(x) = 0\}$, $\mathcal{B} = \{x \in \mathbb{R}^N | b(x) = 0\}$ and $\Sigma = \mathcal{A} \cup \mathcal{B}$. Throughout this paper, we assume that $0 \in \Omega = \mathcal{A} \cap \mathcal{B} \neq \emptyset$. Let

$$\beta_0^\varepsilon = \max \left\{ \mu_2 \frac{\int_{\mathbb{R}^N} V_{1\varepsilon}^4}{\int_{\mathbb{R}^N} U_{1\varepsilon}^2 V_{1\varepsilon}^2}, \mu_1 \frac{\int_{\mathbb{R}^N} U_{1\varepsilon}^4}{\int_{\mathbb{R}^N} U_{1\varepsilon}^2 V_{1\varepsilon}^2} \right\}.$$

Definition 1.1 ([7]). A function $W : \mathbb{R}^N \to \mathbb{R}$ is called positive γ-homogeneous if $W(y) > 0$ for all $y \neq 0$ and $W(ty) = t^\gamma W(y)$ for any $t \in [0, +\infty)$ and $y \in \mathbb{R}^N$.

For any given positive γ-homogeneous function W, we consider the following coupled system

$$\begin{cases} -\Delta u + W(x)u = \mu_1 u^3 + \beta v^2 u & \text{in } \mathbb{R}^N, \\ -\Delta v + W(x)v = \mu_2 v^3 + \beta u^2 v & \text{in } \mathbb{R}^N, \end{cases} \tag{1.6}$$

which will play as the limit problem of ($\mathcal{C}_\varepsilon$) with homogeneous potentials and whose corresponding functional $\mathcal{J}_W(u, v) \in C^1(H_W(\mathbb{R}^N) \times H_W(\mathbb{R}^N))$ is defined by

$$\mathcal{J}_W(u, v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + W(x)u^2 + |\nabla v|^2 + W(x)v^2 - \frac{1}{4} \int_{\mathbb{R}^N} \mu_1 v^4 + 2\beta u^2 v^2 + \mu_2 v^4.$$

Here H_W is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{H_W} = \left(\int_{\mathbb{R}^N} |\nabla u|^2 + W u^2 \right)^{\frac{1}{2}}.$$

Since the potential W is coercive, one can show that, for $\beta > 0$,

$$\mathcal{E}_W := \inf\left\{ \mathcal{J}_W(u, v) \,|\, (u, v) \in H_W \times H_W \setminus \{(0, 0)\}, \langle \mathcal{J}_W'(u, v), (u, v) \rangle = 0 \right\},$$

can be attained by a positive ground state solution.

Now, we are in position to state the first result as follows.

Theorem 1.2. *Assume that (f_1)–(f_3) hold and there exists $\gamma > 0$ such that for every $x \in \Omega$, one of the following two hypotheses holds*

(H_1)

$$\lim_{z \to x} \frac{a(z)}{|z - x|^\gamma} = +\infty,$$

(H_2) *for some positive γ-homogeneous function $W \in C(\mathbb{R}^N)$, there holds that*

$$\lim_{z \to x} \frac{a(z) - W(z - x)}{|z - x|^\gamma} = 0.$$

Additionally, (H_2) holds for at least one point $x \in \Omega$, where there also holds that

$$\lim_{z \to x} \frac{b(z) - W(z - x)}{|z - x|^\gamma} = 0. \tag{1.7}$$

Then for sufficiently small $\varepsilon > 0$ and $\beta > \beta_0^\varepsilon$, system $(\mathcal{C}_\varepsilon)$ has a positive ground state solution $(U_\varepsilon, V_\varepsilon)$.

Moreover, denoting by $\{x_i; i \in \mathcal{I}\}$ all points where (H_2) and (1.7) both hold. For $\beta > \beta_0$ (see below), there exists $x_* \in \{x_i; i \in \mathcal{I}\}$ and a positive γ-homogeneous function $W_* \in C(\mathbb{R}^N)$ such that

$$\lim_{x \to x_*} \frac{a(x) - W_*(x - x_*)}{|x - x_*|^\gamma} = 0 \quad and \quad \lim_{x \to x_*} \frac{b(x) - W_*(x - x_*)}{|x - x_*|^\gamma} = 0,$$

and as $\varepsilon \to 0$,

$$\varepsilon^{-\frac{\gamma}{\gamma+2}} \left(U_\varepsilon \left(\varepsilon^{\frac{2}{\gamma+2}} \cdot + x_* \right), V_\varepsilon \left(\varepsilon^{\frac{2}{\gamma+2}} \cdot + x_* \right) \right) \to (\omega, \phi) \quad in \ H_{\mathrm{loc}}^1(\mathbb{R}^N) \times H_{\mathrm{loc}}^1(\mathbb{R}^N)$$

and

$$\lim_{\varepsilon \to 0} \frac{\mathcal{J}_\varepsilon(U_\varepsilon, V_\varepsilon)}{\varepsilon^{\frac{2}{\gamma+2}(N+2\gamma)}} = \mathcal{E}_{W_*} = \inf_{i \in \mathcal{I}} \mathcal{E}_{W_i}.$$

Here (w, ϕ) is a ground state solution of the limit problem (1.6) with $W = W_{x_*}$.

Remark 1.3. Without loss of generality, assume $0 \in \{x_i; i \in \mathcal{I}\}$. Notice that the conclusion of [7, Theorem 3.2.3] also holds for any least-energy solution. Then it follows from [7, Theorem 3.2.3] that rescaled functions $\varepsilon^{-\frac{\gamma}{\gamma+2}} U_{1\varepsilon}(\varepsilon^{\frac{2}{\gamma+2}} \cdot)$ and $\varepsilon^{-\frac{\gamma}{\gamma+2}} V_{1\varepsilon}(\varepsilon^{\frac{2}{\gamma+2}} \cdot)$ subconverge to ω_{γ,μ_1} and ω_{γ,μ_2} uniformly on $\mathbb{R}^N$ respectively as ε goes to zero. Here, for $i = 1, 2$, ω_{γ,μ_i} is a least-energy solution of the problem below

$$-\Delta u + W(x)u = \mu_i u^3 \quad \text{in } \mathbb{R}^N.$$

Therefore, $\varepsilon^{-\frac{4\gamma}{\gamma+2}} \int_{\mathbb{R}^N} U_{1\varepsilon}^4(\varepsilon^{\frac{2}{\gamma+2}} x)$ and $\varepsilon^{-\frac{4\gamma}{\gamma+2}} \int_{\mathbb{R}^N} V_{1\varepsilon}^4(\varepsilon^{\frac{2}{\gamma+2}} x)$ are bounded and

$$\lim_{\varepsilon \to 0} \varepsilon^{-\frac{4\gamma}{\gamma+2}} \int_{\mathbb{R}^N} U_{1\varepsilon}^4\left(\varepsilon^{\frac{2}{\gamma+2}} x\right) = \int_{\mathbb{R}^N} \omega_{\gamma,\mu_1}^4 > 0,$$

$$\lim_{\varepsilon \to 0} \varepsilon^{-\frac{4\gamma}{\gamma+2}} \int_{\mathbb{R}^N} V_{1\varepsilon}^4\left(\varepsilon^{\frac{2}{\gamma+2}} x\right) = \int_{\mathbb{R}^N} \omega_{\gamma,\mu_2}^4 > 0,$$

and

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \varepsilon^{-\frac{2\gamma}{\gamma+2}} U_{1\varepsilon}^2\left(\varepsilon^{\frac{2}{\gamma+2}} x\right) \cdot \varepsilon^{-\frac{2\gamma}{\gamma+2}} V_{1\varepsilon}^2\left(\varepsilon^{\frac{2}{\gamma+2}} x\right) = \int_{\mathbb{R}^N} \omega_{\gamma,\mu_1}^2 \omega_{\gamma,\mu_2}^2 > 0.$$

Then

$$0 < \lim_{\varepsilon \to 0} \beta_0^\varepsilon := \beta_0 = \max\left\{\frac{\mu_1 \int_{\mathbb{R}^N} \omega_{\gamma,\mu_1}^4}{\int_{\mathbb{R}^N} \omega_{\gamma,\mu_1}^2 \omega_{\gamma,\mu_2}^2}, \frac{\mu_2 \int_{\mathbb{R}^N} \omega_{\gamma,\mu_2}^4}{\int_{\mathbb{R}^N} \omega_{\gamma,\mu_1}^2 \omega_{\gamma,\mu_2}^2}\right\} < \infty.$$

Next we consider the case that the potential a and b vanish on the nonempty open set with smooth boundary for the nonlinear Schrödinger system. That is, we assume that $\text{int}(\mathcal{A})$ and $\text{int}(\mathcal{B})$ are nonempty with smooth boundary. For the sake of simplicity, $\text{int}(\mathcal{A})$ and $\text{int}(\mathcal{B})$ are connected.

Remark 1.4. It follows from [7, Theorem 3.1] that $\varepsilon^{-1} U_{1\varepsilon}$ subconverges pointwise to U_0 on $\text{int}(\mathcal{A})$ and to 0 on $\mathbb{R}^N \setminus \text{int}(\mathcal{A})$. Similarly, $\varepsilon^{-1} V_{1\varepsilon}$ subconverges pointwise to V_0 on $\text{int}(\mathcal{B})$ and to 0 on $\mathbb{R}^N \setminus \text{int}(\mathcal{B})$. Here U_0 is a positive ground state solution of

$$\begin{cases} -\Delta u = \mu_1 u^3, & x \in \text{int}(\mathcal{A}), \\ u \in H_0^1(\text{int}(\mathcal{A})) \end{cases}$$

and V_0 is a positive ground state solution of

$$\begin{cases} -\Delta v = \mu_2 v^3, & x \in \text{int}(\mathcal{B}), \\ v \in H_0^1(\text{int}(\mathcal{B})). \end{cases}$$

Hence $\varepsilon^{-4} \int_{\mathbb{R}^N} U_{1\varepsilon}^4$ and $\varepsilon^{-4} \int_{\mathbb{R}^N} V_{1\varepsilon}^4$ are bounded and

$$\lim_{\varepsilon \to 0} \varepsilon^{-4} \int_{\mathbb{R}^N} U_{1\varepsilon}^4 = \int_{\mathcal{A}} U_0^4 > 0,$$

$$\lim_{\varepsilon \to 0} \varepsilon^{-4} \int_{\mathbb{R}^N} V_{1\varepsilon}^4 = \int_{\mathcal{B}} V_0^4 > 0,$$

and

$$\lim_{\varepsilon \to 0} \varepsilon^{-4} \int_{\mathbb{R}^N} U_{1\varepsilon}^2 V_{1\varepsilon}^2 = \int_{\Omega} U_0^2 V_0^2 > 0.$$

Set

$$\beta_1 = \max \left\{ \mu_2 \frac{\int_{\mathcal{B}} V_0^4}{\int_{\Omega} U_0^2 V_0^2}, \mu_1 \frac{\int_{\mathcal{A}} U_0^4}{\int_{\Omega} U_0^2 V_0^2} \right\},$$

one can get that

$$\lim_{\varepsilon \to 0} \beta_0^\varepsilon = \beta_1.$$

Denote by $\text{int}(\Omega)$ the interior of subset Ω. Our second result states as following.

Theorem 1.5. *Let $\beta > \beta_1$. Assume that (f_1)–(f_3) hold and $\text{int}(\Omega)$ is a nonempty open set with smooth boundary, then, for $\varepsilon > 0$ small, system $(\mathcal{C}_\varepsilon)$ has a positive ground state solution $(U_\varepsilon, V_\varepsilon)$. Moreover, if $\mu_1, \mu_2 > 0$ small, there exists a ground state solution $(w, z) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ of the following problem*

$$\begin{cases} -\Delta w = \mu_1 w^3 + \beta z^2 w, & x \in \text{int}(\Omega), \\ -\Delta z = \mu_2 z^3 + \beta w^2 z, & x \in \text{int}(\Omega), \\ w, z \in H_0^1(\text{int}(\Omega)) \end{cases}$$

such that, as $\varepsilon \to 0$,

$$\varepsilon^{-1}(U_\varepsilon, V_\varepsilon) \to (w, z) \quad \text{in } H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N).$$

For the proof of our theorems, we shall consider an equivalent system to $(\mathcal{C}_\varepsilon)$. By the scaling $a(x) = a(\varepsilon x)$, $b(x) = b(\varepsilon x)$, we then deduce that

$$\begin{cases} -\Delta u + a(\varepsilon x)u = \mu_1 u^3 + \beta u v^2 & x \in \mathbb{R}^N, \\ -\Delta v + b(\varepsilon x)v = \mu_1 v^3 + \beta u^2 v & x \in \mathbb{R}^N. \end{cases} \tag{$\mathcal{P}_\varepsilon$}$$

Thus, in the sequel we focus only on system $(\mathcal{P}_\varepsilon)$.

The rest of this paper is organized as follows. In Section 2, we introduce a variational setting of our problem and present some preliminary results about the extended problem. Section 3 is devoted to the study of positive ground state solutions of problem $(\mathcal{P}_\varepsilon)$. Section 4 is devoted to the proof of Theorem 1.2. Section 5 is devoted to the proof of Theorem 1.5.

2. Preliminaries

To prove the main results, we use the following notations.
- $\mathbb{H} := H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ with norm

$$\left\|(u, v)\right\|_{\mathbb{H}} = \left(\|u\|_{H^1}^2 + \|v\|_{H^1}^2\right)^{1/2};$$

- $\mathbb{H}_{\text{loc}} := H_{\text{loc}}^1(\mathbb{R}^N) \times H_{\text{loc}}^1(\mathbb{R}^N)$
- $\mathbb{L}^q(\mathbb{R}^N) := L^q(\mathbb{R}^N) \times L^q(\mathbb{R}^N)$;
- $\mathbb{L}_{\text{loc}}^q(\mathbb{R}^N) := L_{\text{loc}}^q(\mathbb{R}^N) \times L_{\text{loc}}^q(\mathbb{R}^N)$;
- $\mathbb{C}_0^\infty(\mathbb{R}^N) := C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$;
- $2^* = 6$ when $N = 3$, and $2^* = +\infty$ when $N = 1, 2$.

For $\varepsilon > 0$, we define the rescaled weighted Sobolev space $H_{a,\varepsilon}$ by

$$H_{a,\varepsilon} = \left\{ u \in H^1\left(\mathbb{R}^N\right) \middle| \int_{\mathbb{R}^N} a(\varepsilon x) u^2 < +\infty \right\},$$

endowed with the norm

$$\|u\|_{a,\varepsilon} = \left(\int_{\mathbb{R}^N} |\nabla u|^2 + a(\varepsilon x) u^2 \right)^{\frac{1}{2}}.$$

Define $E := H_{a,\varepsilon} \times H_{b,\varepsilon}$ with norm

$$\left\|(u, v)\right\|_E = \left(\|u\|_{a,\varepsilon}^2 + \|v\|_{b,\varepsilon}^2\right)^{1/2}.$$

For any $(u, v) \in E$, set

$$F(u, v) = \int_{\mathbb{R}^N} \mu_1 u^4 + 2\beta u^2 v^2 + \mu_2 v^4$$

and the energy functional for $(\mathcal{P}_\varepsilon)$ is defined by

$$I_\varepsilon(u, v) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + a(\varepsilon x) u^2 + |\nabla v|^2 + b(\varepsilon x) v^2 - \frac{1}{4} F(u, v).$$

We first illustrate the space E can be embedded continuously into the Sobolev space $\mathbb{H}$ for fixed $\varepsilon > 0$, even though the potential a and b have a nontrivial set of zeros. Similar to [29], we have

Lemma 2.1. *Let* $a : \mathbb{R}^N \to [0, +\infty)$. *If* $\liminf_{|x| \to \infty} a(x) > 0$ *then for every* $\varepsilon > 0$, *there exists a constant* $C_\varepsilon > 0$ *such that for* $u \in H_{a,\varepsilon}(\mathbb{R}^N)$, $u \in H^1(\mathbb{R}^N)$ *and*

$$\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 \leqslant C_\varepsilon \int_{\mathbb{R}^N} |\nabla u|^2 + a(\varepsilon x) u^2.$$

Similar result holds for $H_{b,\varepsilon}$. Lemma 2.1 implies that the space E embeds into $\mathbb{H}$ continuously. Moreover the space E can be continuously embedded in $\mathbb{L}^q(\mathbb{R}^N)$ when $q \in [2, 2^*]$. It then follows that the functional I_ε is well defined in E and $I_\varepsilon \in C^1(E, \mathbb{R})$.

Definition 2.2. (u, v) is called a ground state solution of problem $(\mathcal{P}_\varepsilon)$, if $I_\varepsilon(u, v)$ is the least among all nontrival critical points of I_ε. Namely, (u, v) has the least energy among nontrivial solutions.

A natural method to search the groundstate is to minimize the functional I_ε on the Nehari manifold of problem $(\mathcal{P}_\varepsilon)$ which is defined by

$$\mathcal{N}_\varepsilon := \left\{ (u, v) \in E \setminus \{(0, 0)\} | \langle I'_\varepsilon(u, v), (u, v) \rangle = 0 \right\}.$$

The corresponding groundstate energy is described as

$$c_\varepsilon := \inf_{(u,v)\in\mathcal{N}_\varepsilon} I_\varepsilon(u, v).$$

Lemma 2.3. *For given $\varepsilon > 0$, the ground state energy c_ε is positive and $\mathcal{N}_\varepsilon$ is a manifold of class of C^1. Moreover, if $(u, v) \in \mathcal{N}_\varepsilon$ is a critical point of $I_\varepsilon|_{\mathcal{N}_\varepsilon}$, then $I'_\varepsilon(u, v) = 0$ in E^*.*

Proof. For any fixed $\varepsilon > 0$ and $(u, v) \in E \setminus \{(0, 0)\}$, we have that

$$t(u, v) \in \mathcal{N}_\varepsilon \Leftrightarrow t^2 \|(u, v)\|_E^2 = t^4 F(u, v).$$

As a consequence, for all $(u, v) \in E \setminus \{(0, 0)\}$, there exists a unique $t > 0$ such that $t(u, v) \in \mathcal{N}_\varepsilon$. Moreover, since $F(u, v)$ is homogeneous with degree 4, there is $\rho > 0$ such that

$$\|(u, v)\|_E^2 \geqslant \rho > 0, \quad \forall (u, v) \in \mathcal{N}_\varepsilon.$$

Hence, for any $(u, v) \in \mathcal{N}_\varepsilon$, we have

$$I_\varepsilon(u, v) = \frac{1}{4} \|(u, v)\|_{H_\varepsilon}^2 \geqslant \frac{1}{4}\rho > 0,$$

thus, $c_\varepsilon > 0$. We define $\mathfrak{g}_\varepsilon(u, v) = \langle I'_\varepsilon(u, v), (u, v) \rangle$ for each $(u, v) \in \mathcal{N}_\varepsilon$,

$$\langle \mathfrak{g}'_\varepsilon(u, v), (u, v) \rangle = -2 \|(u, v)\|_E^2 \leqslant -2\rho < 0, \tag{2.1}$$

which implies that $\mathcal{N}_\varepsilon$ is a smooth complete manifold of codimension one in E. Moreover, if $(u, v) \in \mathcal{N}_\varepsilon$ is a critical point of the restricted functional $I_\varepsilon|_{\mathcal{N}_\varepsilon}$, then there exists $\lambda_\varepsilon \in \mathbb{R}$, such that

$$I'_\varepsilon(u, v) = \lambda_\varepsilon \mathfrak{g}'_\varepsilon(u, v) \quad \text{in } E^*. \tag{2.2}$$

Then, we have

$$\lambda_\varepsilon \langle \mathfrak{g}'_\varepsilon(u, v), (u, v) \rangle = \langle I'_\varepsilon(u, v), (u, v) \rangle = 0$$

and deduce by (2.1) that $\lambda_\varepsilon = 0$. The conclusions follows then from (2.2). $\quad\square$

3. Existence of positive ground state solutions of ($\mathcal{P}_\varepsilon$)

Lemma 3.1. *(Mountain–Pass geometry) Assume that (f_1) holds, then the functional I_ε satisfies the following conditions.*
  (i) There exists a positive constant $r > 0$ such that $I_\varepsilon(u, v) > 0$ for $\|(u, v)\|_E = r$;
  (ii) There exists $(e_1, e_2) \in E$ with $\|(e_1, e_2)\|_E > r$ such that $I_\varepsilon(e_1, e_2) < 0$.

Proof.

$$I_\varepsilon(u, v) = \frac{1}{2} \|(u, v)\|_E^2 - F(u, v);$$

$$\geqslant \frac{1}{2} \|(u, v)\|_E^2 - C \|(u, v)\|_E^4.$$

Hence, there exists $r > 0$ such that

$$\inf_{\|(u,v)\|_E = r} I_\varepsilon(u, v) > 0.$$

For any $(u, v) \in E \setminus (0, 0)$ and $t > 0$,

$$I_\varepsilon(tu, tv) = \frac{t^2}{2} \|(u, v)\|_E^2 - \frac{t^4}{4} F(u, v),$$

which implies that $I_\varepsilon(tu, tv) \to -\infty$ as $t \to +\infty$. $\square$

That is, I_ε has a mountaion-pass structure. From Lemma 3.1, one can apply the Mountain–Pass Theorem [3] without $(PS)_c$ condition, and it follows that for any small $\varepsilon > 0$, there exists a $(PS)_c$ sequence $\{(u_n, v_n)\} \subset E$ (with $c = c_\varepsilon^*$ defined below) such that

$$I_\varepsilon(u_n, v_n) \to c_\varepsilon^* = \inf_{\gamma \in \Gamma_\varepsilon} \max_{0 \leqslant t \leqslant 1} I_\varepsilon\big(\gamma(t)\big),$$

$$I_\varepsilon'(u_n, v_n) \to 0, \quad \text{as } n \to \infty,$$

where $\Gamma_\varepsilon = \{\gamma \in C([0, 1], E) : I_\varepsilon(\gamma(0)) = 0, I_\varepsilon(\gamma(1)) < 0\}$. $\quad \square$

Lemma 3.2. $c_\varepsilon = c_\varepsilon^*$.

Proof. Define

$$c_\varepsilon^{**} := \inf_{(u,v) \in E \setminus \{(0,0)\}} \max_{t \geqslant 0} I_\varepsilon(tu, tv).$$

For any $(u, v) \in E \setminus \{(0, 0)\}$, there exists a unique $t_{u,v} > 0$ such that $(t_{u,v}u, t_{u,v}v) \in \mathcal{N}_\varepsilon$, and

$$I_\varepsilon(t_{u,v}u, t_{u,v}v) = \max_{t \geqslant 0} I_\varepsilon(tu, tv),$$

which follows that $c_\varepsilon = c_\varepsilon^{**}$. There exists $t_0 > 0$ large enough such that $I_\varepsilon(t_0 u, t_0 v) < 0$, and

$$c_\varepsilon^* \leqslant \max_{t \geqslant 0} I_\varepsilon(t t_0 u, t t_0 v),$$

$c_\varepsilon^* \leqslant c_\varepsilon^{**} = c_\varepsilon$ can be obtained. To show $c_\varepsilon^* \geqslant c_\varepsilon$, we only need to prove that for any $\gamma \in \Gamma_\varepsilon$, $\gamma([0, 1]) \cap \mathcal{N}_\varepsilon \neq \emptyset$. For any $(u, v) \in E$ with $0 < \|(u, v)\|_E \ll 1$,

$$\int |\nabla u|^2 + a(\varepsilon x)u^2 + |\nabla v|^2 + b(\varepsilon x)v^2 > \int \mu_1 u^4 + 2\beta u^2 v^2 + \mu_2 v^4.$$

Since $I_\varepsilon(\gamma(0)) = 0$, γ is continuous in $t \in [0, 1]$, $I_\varepsilon(\gamma(t)) > 0$ for $0 < t < \delta$. Thus, if set $\gamma_t = (u_t, v_t)$,

$$\int |\nabla u_t|^2 + a(\varepsilon x)u_t^2 + |\nabla v_t|^2 + b(\varepsilon x)v_t^2 > \int \mu_1 u_t^4 + 2\beta u_t^2 v_t^2 + \mu_2 v_t^4$$

holds for sufficiently small t. If $\gamma([0, 1]) \cap \mathcal{N}_\varepsilon = \emptyset$, then

$$\int |\nabla u_1|^2 + a(\varepsilon x)u_1^2 + |\nabla v_1|^2 + b(\varepsilon x)v_1^2 > \int \mu_1 u_1^4 + 2\beta u_1^2 v_1^2 + \mu_2 v_1^4.$$

This is a contradiction with $I_\varepsilon(\gamma(1)) < 0$. $\quad\square$

In order to obtain the existence of ground state solutions for $(\mathcal{P}_\varepsilon)$, we prove some compactness lemma for the functional I_ε to analysis the Palais–Smale sequence properties for the functional I_ε.

Now, we consider the limit problem

$$\begin{cases} -\Delta u + \tau u = \mu_1 u^3 + \beta v^2 u & \text{in } \mathbb{R}^N, \\ -\Delta v + \theta v = \mu_2 v^3 + \beta u^2 v & \text{in } \mathbb{R}^N, \end{cases}$$

where $(\tau, \theta) \in (0, +\infty) \times (0, \infty)$. The corresponding energy functional and Nehari manifold are defined by

$$I_{\tau\theta}(u, v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \tau u^2 + |\nabla v|^2 + \theta v^2 - \frac{1}{4} F(u, v)$$

and

$$\mathcal{N}_{\tau\theta} = \left\{ (u, v) \in \mathbb{H} \setminus \{(0, 0)\} \mid \langle I_{\tau\theta}'(u, v), (u, v) \rangle = 0 \right\}.$$

Define

$$c_{\tau\theta} = \inf_{(u,v) \in \mathcal{N}_{\tau\theta}} I_{\tau\theta}(u, v).$$

One can show that $c_{\tau\theta}$ can be achieved by some $(u, v) \in \mathbb{H} \setminus \{(0, 0)\}$ for any $\tau, \theta, \beta > 0$.

Lemma 3.3. *The map $c : (0, +\infty) \times (0, +\infty) \to \mathbb{R}$; $(\tau, \theta) \mapsto c_{\tau\theta}$, is non-decreasing in τ and θ.*

Proof. The proof is similar to the proofs of Lemma 2.3 in [33] and we omit it here. $\quad\square$

Lemma 3.4. *Assume that (f_2)–(f_3) hold and $\Omega \neq \varnothing$. For any fixed $\varepsilon > 0$, I_ε satisfies Palais–Smale condition at level $c < c_{a_\infty b_\infty}$.*

Proof. Let $c < c_{a_\infty b_\infty}$, $\{(u_n, v_n)\} \subset E$ be a sequence such that $I_\varepsilon(u_n, v_n) \to c$, and $I_\varepsilon'(u_n, v_n) \to 0$. A standard argument leads us to the fact that (u_n, v_n) is bounded in E, which implies that

$$(u_n, v_n) \rightharpoonup (u, v) \quad \text{weakly in } E, \, I_\varepsilon'(u, v) = 0, \, I_\varepsilon(u, v) \geqslant 0.$$

Set $w_n = u_n - u$, $z_n = v_n - v$, and $(w_n, z_n) \rightharpoonup 0$. By Brezis–Lieb Lemma [30], it follows that,

$$I_\varepsilon(w_n, z_n) = I_\varepsilon(u_n, v_n) - I_\varepsilon(u, v) + o(1) \leqslant c + o(1),$$
$$I_\varepsilon'(w_n, z_n) \to 0 \quad \text{as } n \to \infty.$$

There exists $t_n > 0$ such that $t_n(w_n, v_n) \in \mathcal{N}_{a_\infty, b_\infty}$, from which it follows that

$$t_n^2 \int_{\mathbb{R}^N} |\nabla w_n|^2 + a_\infty w_n^2 + |\nabla z_n|^2 + b_\infty z_n^2 = t_n^4 \int_{\mathbb{R}^N} \mu_1 w_n^4 + 2\beta w_n^2 z_n^2 + \mu_2 z_n^4, \tag{3.1}$$

Suppose that $(w_n, z_n) \nrightarrow (0, 0)$ in E, we first claim that

$$\limsup_{n \to \infty} t_n \leqslant 1. \tag{3.2}$$

Otherwise, we can assume that $t_n \geqslant 1 + \delta$ for $\delta > 0$. From $I_\varepsilon'(w_n, z_n) = o(1)$, we have that

$$\int_{\mathbb{R}^N} |\nabla w_n|^2 + a(\varepsilon x)w_n^2 + |\nabla z_n|^2 + b(\varepsilon x)z_n^2 = \int_{\mathbb{R}^N} \mu_1 w_n^4 + 2\beta w_n^2 z_n^2 + \mu_2 z_n^4 + o(1). \tag{3.3}$$

According to (3.1) and (3.3), we have

$$\int_{\mathbb{R}^N} (a_\infty - a(\varepsilon x))w_n^2 + (b_\infty - b(\varepsilon x))z_n^2 = (t_n^2 - 1) \int_{\mathbb{R}^N} \mu_1 w_n^4 + 2\beta w_n^2 z_n^2 + \mu_2 z_n^4 + o(1) \tag{3.4}$$

For any $\varepsilon > 0$, there exists $R = R(\varepsilon)$ such that

$$a(\varepsilon x) > a_\infty - \varepsilon \quad \text{and} \quad b(\varepsilon x) > b_\infty - \varepsilon \quad \text{for } |x| \geqslant R. \tag{3.5}$$

Since $(w_n, z_n) \to 0$ in $\mathbb{L}^2_{B_R(0)}$ and (3.5), there exists $C > 0$ such that

$$(t_n^2 - 1) \int_{\mathbb{R}^N} \mu_1 w_n^4 + 2\beta w_n^2 z_n^2 + \mu_2 z_n^4 \leqslant C\varepsilon + o(1)$$

Since $(w_n, z_n) \nrightarrow (0,0)$, there exists a sequence $\{y_n\} \in \mathbb{R}^N$ and positive constants $r, \zeta > 0$,

$$\liminf_{n \to \infty} \int_{B_r(y_n)} \left(w_n^2 + z_n^2 \right) \geqslant \zeta. \tag{3.6}$$

Set $(\tilde{w}_n(x), \tilde{z}_n(x)) = (u(x + y_n), v(x + y_n))$. Then $\{(\tilde{w}_n, \tilde{z}_n)\}$ is bounded and $(\tilde{w}_n, \tilde{z}_n) \rightharpoonup (u, v)$ in E, $(\tilde{w}_n, \tilde{z}_n) \to (u, v)$ in $\mathbb{L}^2_{\text{loc}}(\mathbb{R}^N)$. Moreover, there exists a subset $\Lambda \subset \mathbb{R}^N$ such that $(u, v) \neq (0, 0)$ *a.e.* in Λ, then it follows from Fatou's lemma that

$$0 < \left[(1 + \delta)^2 - 1 \right] \int_\Lambda \mu_1 u^4 + 2\beta u^2 v^2 + \mu_2 v^4 \leqslant C\varepsilon,$$

for any $\varepsilon > 0$, which yields a contradiction. Thus (3.2) holds. Using the fact that $t_n(w_n, v_n) \in \mathcal{N}_{a_\infty, b_\infty}$, and (3.2)

$$\begin{aligned}
c_{a_\infty b_\infty} &\leqslant I_{a_\infty b_\infty}(t_n w_n, t_n z_n) - \frac{1}{4} \langle I'_\infty (t_n(w_n, v_n)), t_n(w_n, v_n) \rangle \\
&= \frac{t_n^2}{4} \int_{\mathbb{R}^N} |\nabla w_n|^2 + a_\infty w_n^2 + |\nabla z_n|^2 + b_\infty z_n^2 \\
&\leqslant \frac{1}{4} \int_{\mathbb{R}^N} |\nabla w_n|^2 + a(\varepsilon x) w_n^2 + |\nabla z_n|^2 + b(\varepsilon x) z_n^2 + o(1) \\
&= I_\varepsilon(w_n, z_n) + o(1) \leqslant c + o(1)
\end{aligned}$$

Let $n \to \infty$ in the above inequality, we have $c_{a_\infty b_\infty} \leqslant c$ which is impossible. Hence $\limsup_{n \to \infty} t_n \leqslant 1$ cannot happen. It implies $(u_n, v_n) \to (u, v)$ in E. $\quad\square$

Lemma 3.5. $c_\varepsilon < c_{a_\infty b_\infty}$ *for sufficiently small* $\varepsilon > 0$.

Proof. Taking any $\tau \in (0, a_\infty), \theta \in (0, b_\infty)$, let (u, v) be a ground state solution for $I_{\tau\theta}$. Choose $\eta \in C_0^\infty(\mathbb{R}^N, [0, 1])$ such that $\eta(x) = 1$, for $|x| \leqslant \frac{1}{2}$, and $\eta(x) = 0$, for $|x| \geqslant 1$. For any $n \in \mathbb{N}$, define $(u_n, v_n) = \eta(\frac{\cdot}{n})(u, v)$ and $(w_n, z_n) = t_n(u_n, v_n)$ such that $(w_n, z_n) \in \mathcal{N}_{\tau\theta}$. Since $(u_n, v_n) \to (u, v)$ in $\mathbb{H}$ as $n \to \infty$, and $(u, v) \in \mathcal{N}_{\tau\theta}$, it is easy to see that $\lim_{n \to \infty} t_n = 1$. Consequently, as $n \to \infty$, we have

$$I_{\tau\theta}(w_n, z_n) = I_{\tau\theta}\big(t_n(u_n, v_n)\big) \to c_{\tau\theta} < c_{a_\infty b_\infty}.$$

Now, we have $(w_n, z_n) \to (u, v)$ in $\mathbb{H}$ and $(w_n, z_n) \in \mathcal{N}_{\tau\theta}$, which implies that

$$I_{\tau\theta}(w_n, z_n) = \max_{t \geqslant 0} I_{\tau\theta}(t w_n, t z_n).$$

By assumption (f_2)–(f_3) and $0 \in \Omega = \mathcal{A} \cap \mathcal{B}$, for any $\tau, \theta > 0$, there exists $R > 0$ small such that $a(x) < \tau, b(x) < \theta$ for $|x| < R$ and

$$a(\varepsilon x) < \tau, \qquad b(\varepsilon x) < \theta, \quad \text{for } |x| < \frac{R}{\varepsilon}.$$

Define

$$\mathbb{H}_{R/\varepsilon} := H_0^1\big(B_{R/\varepsilon}(0)\big) \times H_0^1\big(B_{R/\varepsilon}(0)\big),$$

$$c_{\tau\theta}^{R/\varepsilon} := \inf_{(u,v)\in\mathbb{H}_{R/\varepsilon}\backslash\{(0,0)\}} \max_{t\geqslant 0} I_{\tau\theta}(tu, tv),$$

then

$$c_\varepsilon = \inf_{(u,v)\in E\backslash\{(0,0)\}} \max_{t\geqslant 0} I_\varepsilon(tu, tv) \leqslant \inf_{(u,v)\in\mathbb{H}_{R/\varepsilon}\backslash\{(0,0)\}} \max_{t\geqslant 0} I_\varepsilon(tu, tv)$$

$$\leqslant \inf_{(u,v)\in\mathbb{H}_{R/\varepsilon}\backslash\{(0,0)\}} \max_{t\geqslant 0} I_{\tau\theta}(tu, tv) = c_{\tau\theta}^{R/\varepsilon}.$$

Then for any fixed n with $I_{\tau\theta}(w_n, z_n) < c_{a_\infty b_\infty}$ and sufficiently small $\varepsilon > 0$ such that $R/\varepsilon > n$, we have $(w_n, z_n) \in \mathbb{H}_{R/\varepsilon}$ and

$$c_{\tau\theta}^{R/\varepsilon} = \inf_{(u,v)\in\mathbb{H}_{R/\varepsilon}\backslash\{(0,0)\}} \max_{t\geqslant 0} I_{\tau\theta}(tu, tv) \leqslant \max_{t\geqslant 0} I_{\tau\theta}(tw_n, tz_n) = I_{\tau\theta}(w_n, z_n) < c_{a_\infty b_\infty}.$$

Then the conclusion follows $c_\varepsilon < c_{a_\infty b_\infty}$ for sufficiently small $\varepsilon > 0$. $\square$

Lemma 3.6. *Assume* (f_1) *holds, then for* $\beta > \beta_0^\varepsilon$ *and sufficiently small* $\varepsilon > 0$, $(\mathcal{P}_\varepsilon)$ *has a nontrivial ground state solution.*

Proof. From Lemmas 3.4-3.6, one sees that problem $(\mathcal{P}_\varepsilon)$ has a ground state solution $(u_\varepsilon, v_\varepsilon)$ for sufficiently small ε. Thus, $(U_\varepsilon, V_\varepsilon) = (u_\varepsilon(\frac{\cdot}{\varepsilon}), v_\varepsilon(\frac{\cdot}{\varepsilon}))$ is a ground state solution of problem $(\mathcal{C}_\varepsilon)$ for sufficiently small ε. It is sufficient to prove that for sufficiently small $\varepsilon > 0$,

$$c_\varepsilon < \min\big\{I_\varepsilon\big(U_{1\varepsilon}(\varepsilon\cdot), 0\big), I_\varepsilon(0, V_{1\varepsilon}(\varepsilon\cdot)\big\}. \tag{3.7}$$

First, as in Lemma 3.3 of [25], we know that for $\beta > 0$,

$$c_\varepsilon = \inf_{(u,v)\in E\backslash\{(0,0)\}} \mathcal{K}_\varepsilon(u, v),$$

where

$$\mathcal{K}_\varepsilon(u, v) = \frac{(\|u\|_{a,\varepsilon}^2 + \|v\|_{b,\varepsilon}^2)^2}{4F(u, v)}.$$

For $(s, t) \in \Gamma = \{(s, t) : s \geqslant 0, t \geqslant 0, (s, t) \neq (0, 0)\}$, we define a function

$$f_\varepsilon(s, t) = \mathcal{K}_\varepsilon\big(\sqrt{s}U_{1\varepsilon}(\varepsilon\cdot), \sqrt{t}V_{1\varepsilon}(\varepsilon\cdot)\big)$$

$$= \frac{(s\|U_{1\varepsilon}(\varepsilon\cdot)\|_{a,\varepsilon}^2 + t\|V_{1\varepsilon}(\varepsilon\cdot)\|_{b,\varepsilon}^2)^2}{4\int_{\mathbb{R}^N} \mu_1 s^2 U_{1\varepsilon}^4(\varepsilon x) + 2\beta st U_{1\varepsilon}^2(\varepsilon x) V_{1\varepsilon}^2(\varepsilon x) + \mu_2 t^2 V_{1\varepsilon}^4(\varepsilon x)}.$$

It follows that

$$f_\varepsilon(s, 0) = \frac{\|U_{1\varepsilon}(\varepsilon\cdot)\|_{a,\varepsilon}^4}{4 \int_{\mathbb{R}^N} \mu_1 U_{1\varepsilon}^4(\varepsilon x)} = I_\varepsilon\big(U_{1\varepsilon}(\varepsilon\cdot), 0\big),$$

$$f_\varepsilon(0, t) = \frac{\|V_{1\varepsilon}(\varepsilon\cdot)\|_{b,\varepsilon}^4}{4 \int_{\mathbb{R}^N} \mu_2 V_{1\varepsilon}^4(\varepsilon x)} = I_\varepsilon\big(0, V_{1\varepsilon}(\varepsilon\cdot)\big).$$

Thus, to guarantee (3.7) it is sufficient to prove that f_ε does not attain its minimum over Γ on the line $s = 0$ or $t = 0$. The function f_ε is a fraction of two quadratic forms in (s, t), and elementary analysis shows that the quantity

$$\frac{(as + bt)^2}{cs^2 + 2dst + et^2}, \quad a, b, c, d, e > 0$$

does not attain its minimum in Γ on the axes if and only if

$$ad - bc > 0, \qquad bd - ae > 0 \tag{3.8}$$

and then the minimum is attained at $(ad - bc, bd - ae)$. Clearly, to verify (3.7), it suffices to show

$$\beta > \mu_2 \frac{\int_{\mathbb{R}^N} V_{1\varepsilon}^4}{\int_{\mathbb{R}^N} U_{1\varepsilon}^2 V_{1\varepsilon}^2} \quad \text{and} \quad \beta > \mu_1 \frac{\int_{\mathbb{R}^N} U_{1\varepsilon}^4}{\int_{\mathbb{R}^N} U_{1\varepsilon}^2 V_{1\varepsilon}^2}.$$

Thus, if $\beta > \beta_0^\varepsilon$, then (3.7) holds sufficiently small $\varepsilon > 0$. $\quad\square$

Proposition 3.7. *Assume that* (f_1)–(f_3) *hold, then for* $\beta > \beta_0^\varepsilon$ *and small* $\varepsilon > 0$, *system* $(\mathcal{P}_\varepsilon)$ *has a positive ground state solution.*

Proof. From Lemma 3.1 the functional I_ε satisfies a mountain pass geometry. By Lemma 3.2 and Mountain Pass Theorem [3], there exists a $(PS)_{c_\varepsilon}$ sequence $\{(u_n, v_n)\} \subset E$ such that $I_\varepsilon(u_n, v_n) \to c_\varepsilon$ and $I_\varepsilon'(u_n, v_n) \to 0$. Then standard argument shows that $\{(u_n, v_n)\}$ is bounded. Up to a subsequence, Lemma 3.5 implies that there exists $(u_\varepsilon, v_\varepsilon)$ such that $(u_n, v_n) \to (u_\varepsilon, v_\varepsilon)$, $c_\varepsilon = I_\varepsilon(u_\varepsilon, v_\varepsilon)$ and $I_\varepsilon'(u_\varepsilon, v_\varepsilon) = 0$. Moreover, since $\beta > \beta_0^\varepsilon$, from Lemma 3.6, we know that $u_\varepsilon \not\equiv 0$ and $u_\varepsilon \not\equiv 0$. Thus we prove $(u_\varepsilon, v_\varepsilon) \in \mathcal{N}_\varepsilon$. Finally, we prove that $u_\varepsilon, v_\varepsilon > 0$. Since $(|u_\varepsilon|, |v_\varepsilon|) \in \mathcal{N}_\varepsilon$ and $c_\varepsilon = I_\varepsilon(|u_\varepsilon|, |v_\varepsilon|)$, we deduce that $(|u_\varepsilon|, |v_\varepsilon|)$ is a nonnegative solution of $(\mathcal{P}_\varepsilon)$. Using elliptic estimates and a Harnack inequality, we infer that $|u_\varepsilon|, |v_\varepsilon| > 0$. The proof is completed. $\quad\square$

4. Proof of Theorem 1.2

Lemma 4.1 ([29]). *Let a function* $V \in C(\mathbb{R}^N, \mathbb{R})$ *and* $\gamma > 0$. *If* $V \geqslant 0$ *on* $\mathbb{R}^N$ *with* $\liminf_{|x| \to \infty} V(x) > 0$, *and if for each* $x \in \mathbb{R}^N$ *such that*

$$\liminf_{z \to x} \frac{V(z)}{|z - x|^\gamma} > 0,$$

then there exist $l \in \mathbb{N}$, $a_1, \ldots, a_l \in \mathbb{R}^N$, $\mu > 0$ and $\nu > 0$ such that for each $x \in \mathbb{R}^N$,

$$V(x) \geqslant \min\{\mu, \nu|x - a_1|^\gamma, \ldots, \nu|x - a_l|^\gamma\}.$$

Let us set

$$k = -\frac{\gamma}{2 + \gamma}.$$

Proposition 4.2. *Under the assumptions of Theorem 1.2, there holds*

$$\limsup_{\varepsilon \to 0} \frac{c_\varepsilon}{\varepsilon^{k(N-4)}} \leqslant m = \inf_{i \in \mathcal{I}} \mathcal{E}_{W_{x_i}}.$$

Proof. For $i \in \mathcal{I}$, $x_i \in \Omega$, let $W_i \in C(\mathbb{R}^N)$ be the positive γ-homogeneous function such that

$$\lim_{x \to x_i} \frac{a(x) - W_i(x - x_i)}{|x - x_i|^\gamma} = 0 \quad \text{and} \quad \lim_{x \to x_i} \frac{b(x) - W_i(x - x_i)}{|x - x_i|^\gamma} = 0, \tag{4.1}$$

and let $(\eta, \varphi) \in \mathbb{C}_0^\infty(\mathbb{R}^N) \setminus \{(0, 0)\}$, we define $\eta_\varepsilon(\text{resp. } \varphi_\varepsilon) : \mathbb{R}^N \to \mathbb{R}$ by

$$\left(\eta_\varepsilon(x), \varphi_\varepsilon(x)\right) = \varepsilon^{-k}\left(\eta\left(\frac{x - x_i/\varepsilon}{\varepsilon^k}\right), \varphi\left(\frac{x - x_i/\varepsilon}{\varepsilon^k}\right)\right).$$

Computing directly, we have

$$\int_{\mathbb{R}^N} |\nabla\eta_\varepsilon|^2 + \int_{\mathbb{R}^N} |\nabla\varphi_\varepsilon|^2 = \varepsilon^{k(N-4)} \int_{\mathbb{R}^N} |\nabla\eta|^2 + |\nabla\varphi|^2;$$

$$\int_{\mathbb{R}^N} a(\varepsilon x)|\eta_\varepsilon|^2 + b(\varepsilon x)|\varphi_\varepsilon|^2 = \varepsilon^{k(N-4)} \int_{\mathbb{R}^N} \frac{a(\varepsilon^{k+1}x + x_i)}{\varepsilon^{(k+1)\gamma}}|\eta|^2 + \frac{b(\varepsilon^{k+1}x + x_i)}{\varepsilon^{(k+1)\gamma}}|\varphi|^2;$$

$$\int_{\mathbb{R}^N} \mu_1\eta_\varepsilon^4 + 2\beta\eta_\varepsilon^2\varphi_\varepsilon^2 + \mu_2\varphi_\varepsilon^4 = \varepsilon^{k(N-4)} \int_{\mathbb{R}^N} \mu_1\eta^4 + 2\beta\eta^2\varphi^2 + \mu_2\varphi^4.$$

Since the function W_i is γ-homogeneous and satisfies (4.1), we have

$$\lim_{\varepsilon \to 0} \frac{a(\varepsilon^{k+1}x + x_i)}{\varepsilon^{(k+1)\gamma}} = \lim_{\varepsilon \to 0} \frac{W_i(\varepsilon^{k+1}x)}{\varepsilon^{(k+1)\gamma}} = W_i(x) \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{b(\varepsilon^{k+1}x + x_i)}{\varepsilon^{(k+1)\gamma}} = W_{x_i}(x), \tag{4.2}$$

uniformly for $x \in \text{supp}(\eta) \cap \text{supp}(\varphi)$. Thus by Lebesgue's dominated convergence theorem, we have

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{k(N-4)}} \int_{\mathbb{R}^N} a(\varepsilon x)|\eta_\varepsilon|^2 + b(\varepsilon x)|\varphi_\varepsilon|^2 = \int_{\mathbb{R}^N} W_i|\eta|^2 + W_i|\varphi|^2.$$

Choosing $t_\varepsilon > 0$ such that $\langle I_\varepsilon'(t_\varepsilon\eta_\varepsilon, t_\varepsilon\varphi_\varepsilon), (t_\varepsilon\eta_\varepsilon, t_\varepsilon\varphi_\varepsilon)\rangle = 0$, we observe that $t_\varepsilon \to t_* \in (0, +\infty)$ as $\varepsilon \to 0$. Moreover, there holds

$$\int_{\mathbb{R}^N} |\nabla\eta|^2 + W_i|\eta|^2 + |\nabla\varphi|^2 + W_i|\varphi|^2 = t_*^2 \int_{\mathbb{R}^N} \mu_1\eta^4 + 2\beta\eta^2\varphi^2 + \mu_2\varphi^4,$$

which implies

$$\limsup_{\varepsilon \to 0} \frac{c_\varepsilon}{\varepsilon^{k(N-4)}} \leqslant \lim_{\varepsilon \to 0} \frac{I_\varepsilon(t_\varepsilon \eta_\varepsilon, t_\varepsilon \varphi_\varepsilon)}{\varepsilon^{k(N-4)}} = \mathcal{J}_{W_i}(t_* \eta, t_* \varphi) = \sup_{t \in (0, +\infty)} \mathcal{J}_{W_i}(t\eta, t\varphi).$$

Then by density of $\mathbb{C}_0^\infty(\mathbb{R}^N)$ in the space $E_W := H_W \times H_W$, by taking the infimum with respect to η and φ, we have

$$\limsup_{\varepsilon \to 0} \frac{c_\varepsilon}{\varepsilon^{k(N-4)}} \leqslant \mathcal{E}_{W_i}.$$

By the arbitrary choice of $i \in \mathcal{I}$, we obtain the conclusion. The proof is completed. □

Proof of Theorem 1.2. Let $(u_\varepsilon, v_\varepsilon)$ be any positive ground state of system $(\mathcal{P}_\varepsilon)$ obtained in Proposition 3.7 and satisfying $I_\varepsilon(u_\varepsilon, v_\varepsilon) = c_\varepsilon$. Define

$$\omega_\varepsilon = \varepsilon^k u_\varepsilon(\varepsilon^k \cdot),$$
$$\phi_\varepsilon = \varepsilon^k v_\varepsilon(\varepsilon^k \cdot),$$

one has

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 = \varepsilon^{k(N-4)} \int_{\mathbb{R}^N} |\nabla \omega_\varepsilon|^2; \\
&\int_{\mathbb{R}^N} a(\varepsilon x) u_\varepsilon^2 = \varepsilon^{k(N-4)} \int_{\mathbb{R}^N} \varepsilon^{2k} a_\varepsilon(x) |\omega_\varepsilon|^2; \\
&\int_{\mathbb{R}^N} \mu_1 u_\varepsilon^4 = \varepsilon^{k(N-4)} \int_{\mathbb{R}^N} \mu_1 \omega_\varepsilon^4; \\
&\int_{\mathbb{R}^N} \beta u_\varepsilon^2 v_\varepsilon^2 = \varepsilon^{k(N-4)} \int_{\mathbb{R}^N} \beta \omega_\varepsilon^2 \phi_\varepsilon^2.
\end{aligned}
\tag{4.3}
$$

Similarly, one can get that

$$-\Delta \omega_\varepsilon + \varepsilon^{2k} a_\varepsilon(x) \omega_\varepsilon = \mu_1 \omega_\varepsilon^3 + \beta \omega_\varepsilon \phi_\varepsilon^2; \tag{4.4}$$

$$-\Delta \phi_\varepsilon + \varepsilon^{2k} b_\varepsilon(x) \phi_\varepsilon = \mu_2 \phi_\varepsilon^3 + \beta \omega_\varepsilon^2 \phi_\varepsilon, \tag{4.5}$$

where $a_\varepsilon(x) = a(\varepsilon^{k+1} x)$, $b_\varepsilon(x) = b(\varepsilon^{k+1} x)$. Consequently, there holds

$$\int_{\mathbb{R}^N} |\nabla \omega_\varepsilon|^2 + \varepsilon^{2k} a_\varepsilon(x) \omega_\varepsilon^2 + |\nabla \phi_\varepsilon|^2 + \varepsilon^{2k} b_\varepsilon(x) \phi_\varepsilon^2 = \int_{\mathbb{R}^N} \mu_1 \omega_\varepsilon^4 + 2\beta \omega_\varepsilon^2 \phi_\varepsilon^2 + \mu_2 \phi_\varepsilon^4, \tag{4.6}$$

By Proposition 4.2, we deduce that, for small $\varepsilon > 0$,

$$\int_{\mathbb{R}^N} |\nabla \omega_\varepsilon|^2 + \varepsilon^{2k} a_\varepsilon(x) \omega_\varepsilon^2 + |\nabla \phi_\varepsilon|^2 + \varepsilon^{2k} b_\varepsilon(x) \phi_\varepsilon^2 \leqslant 4m. \tag{4.7}$$

Step 1. Now we claim that there exists a sequence $\{x_\varepsilon\} \in \mathbb{R}^N$ such that

$$\liminf_{\varepsilon \to 0} \int_{B_1(x_\varepsilon)} \omega_\varepsilon^4 + \phi_\varepsilon^4 > 0.$$

It suffices to show that

$$\limsup_{\substack{\varepsilon \to 0 \\ x \in \mathbb{R}^N}} \varepsilon^{k(4-N)} \int_{B_{\varepsilon^k}(x)} u_\varepsilon^4 + v_\varepsilon^4 > 0. \tag{4.8}$$

We adopt some ideas in [29] to show that (4.8) holds. By the scaled version of the classical Sobolev embedding theorem, for every $x \in \mathbb{R}^N$

$$\left(\int_{B_{\varepsilon^k}(x)} u_\varepsilon^4 \right)^{\frac{1}{2}} \leqslant C_1 \varepsilon^{-\frac{kN}{2}} \int_{B_{\varepsilon^k}(x)} \varepsilon^{2k} |\nabla u_\varepsilon|^2 + u_\varepsilon^2,$$

where C_1 is independent of $x \in \mathbb{R}^N$ and ε. By [29, Lemma 3.4], for some constant C_2 independent of $x \in \mathbb{R}^N$ and ε small enough, we obtain

$$\left(\int_{B_{\varepsilon^k}(x)} u_\varepsilon^4 \right)^{\frac{1}{2}} \leqslant C_2 \varepsilon^{-\frac{kN}{2}+2k} \int_{B_{\varepsilon^k}(x)} |\nabla u_\varepsilon|^2 + a(\varepsilon x) u_\varepsilon^2,$$

and then

$$\left(\int_{B_{\varepsilon^k}(x)} u_\varepsilon^4 \right) \leqslant C_2 \varepsilon^{-\frac{kN}{2}+2k} \left(\int_{B_{\varepsilon^k}(x)} u_\varepsilon^4 \right)^{\frac{1}{2}} \int_{B_{\varepsilon^k}(x)} |\nabla u_\varepsilon|^2 + a(\varepsilon x) u_\varepsilon^2. \tag{4.9}$$

By integration both sides on (4.9) with respect to x over $\mathbb{R}^N$,

$$\int_{\mathbb{R}^N} u_\varepsilon^4 \leqslant C_2 \varepsilon^{-\frac{kN}{2}+2k} \left(\sup_{x \in \mathbb{R}^N} \int_{B_{\varepsilon^k}(x)} u_\varepsilon^4 \right)^{\frac{1}{2}} \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 + a(\varepsilon x) u_\varepsilon^2.$$

Similarly,

$$\int_{\mathbb{R}^N} v_\varepsilon^4 \leqslant C_2 \varepsilon^{-\frac{kN}{2}+2k} \left(\sup_{x \in \mathbb{R}^N} \int_{B_{\varepsilon^k}(x)} v_\varepsilon^4 \right)^{\frac{1}{2}} \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 + b(\varepsilon x) v_\varepsilon^2.$$

Denote $C^* = \max\{(\sup_{x \in \mathbb{R}^N} \int_{B_{\varepsilon^k}(x)} u_\varepsilon^4)^{\frac{1}{2}}, (\sup_{x \in \mathbb{R}^N} \int_{B_{\varepsilon^k}(x)} v_\varepsilon^4)^{\frac{1}{2}}\}$,

$$\int_{\mathbb{R}^N} u_\varepsilon^4 + v_\varepsilon^4 \leqslant C^* C_2 \varepsilon^{-\frac{kN}{2}+2k} \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 + a(\varepsilon x) u_\varepsilon^2 + |\nabla v_\varepsilon|^2 + b(\varepsilon x) v_\varepsilon^2$$

can be obtained. Since $(u_\varepsilon, v_\varepsilon)$ is a solution of problem $(\mathcal{P}_\varepsilon)$ satisfing

$$
\begin{aligned}
0 &< \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 + a(\varepsilon x)u_\varepsilon^2 + |\nabla v_\varepsilon|^2 + b(\varepsilon x)v_\varepsilon^2 \\
&= \int_{\mathbb{R}^N} \mu_1 u_\varepsilon^4 + 2\beta u_\varepsilon^2 v_\varepsilon^2 + \mu_2 v_\varepsilon^4 \leqslant (\mu_1 + \mu_2 + \beta) \int_{\mathbb{R}^N} u_\varepsilon^4 + v_\varepsilon^4 \\
&\leqslant C^* C_2(\mu_1 + \mu_2 + \beta)\varepsilon^{-\frac{kN}{2}+2k} \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 + a(\varepsilon x)u_\varepsilon^2 + |\nabla v_\varepsilon|^2 + b(\varepsilon x)v_\varepsilon^2,
\end{aligned}
$$

one get that

$$
(C^*)^2 \varepsilon^{-\frac{kN}{2}+2k} \geqslant \left(C_2(\mu_1 + \mu_2 + \beta)\right)^{-2}.
$$

It follows that

$$
\liminf_{\varepsilon \to 0} \sup_{x \in \mathbb{R}^N} \varepsilon^{k(4-N)} \int_{B_{\varepsilon^k}(x)} u_\varepsilon^4 > 0,
$$

or

$$
\liminf_{\varepsilon \to 0} \sup_{x \in \mathbb{R}^N} \varepsilon^{k(4-N)} \int_{B_{\varepsilon^k}(x)} v_\varepsilon^4 > 0.
$$

As a consequence, (4.8) holds.

Step 2. We claim that

$$
\liminf_{\varepsilon \to 0} \int_{B_1(x_\varepsilon)} \omega_\varepsilon^4 > 0 \quad \text{and} \quad \liminf_{\varepsilon \to 0} \int_{B_1(x_\varepsilon)} \phi_\varepsilon^4 > 0. \tag{4.10}
$$

We argue by contradiction. Suppose that

$$
\limsup_{\varepsilon \to 0} \sup_{x \in \mathbb{R}^N} \int_{B_1(x)} \omega_\varepsilon^4 = 0,
$$

by the Lions Concentration–Compactness Principle [16], we deduce that $\omega_\varepsilon \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2^*$. Multiplying (4.4) and (4.5) by ω_ε and ϕ_ε respectively and integrating both sides with respect to x over $\mathbb{R}^N$, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla \omega_\varepsilon|^2 + \varepsilon^{2k} a_\varepsilon(x)\omega_\varepsilon^2 = o_\varepsilon(1), \\
&\int_{\mathbb{R}^N} |\nabla \phi_\varepsilon|^2 + \varepsilon^{2k} b_\varepsilon(x)\phi_\varepsilon^2 = \int_{\mathbb{R}^N} \mu_2 \phi_\varepsilon^4 + o_\varepsilon(1).
\end{aligned} \tag{4.11}
$$

Therefore, there exists $t_\varepsilon > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla \phi_\varepsilon|^2 + \varepsilon^{2k} b_\varepsilon(x) \phi_\varepsilon^2 = t_\varepsilon^2 \int_{\mathbb{R}^N} \mu_2 \phi_\varepsilon^4. \tag{4.12}$$

Combing the fact that $\liminf_{\varepsilon \to 0} \int_{B_1(x_\varepsilon)} \phi_\varepsilon^4 > 0$, it follows that $t_\varepsilon \to 1$ by (4.11) and (4.12), as $\varepsilon \to 0$. Hence

$$\begin{aligned}
\lim_{\varepsilon \to 0} \varepsilon^{k(4-N)} I_\varepsilon(u_\varepsilon, v_\varepsilon) &= \lim_{\varepsilon \to 0} \left[\frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla \phi_\varepsilon|^2 + \varepsilon^{2k} b_\varepsilon(x) \phi_\varepsilon^2 \right) - \frac{\mu_2}{4} \int_{\mathbb{R}^N} \phi_\varepsilon^4 \right] \\
&= \lim_{\varepsilon \to 0} \left[\frac{t_\varepsilon^2}{2} \int_{\mathbb{R}^N} \left(|\nabla \phi_\varepsilon|^2 + \varepsilon^{2k} b_\varepsilon(x) \phi_\varepsilon^2 \right) - \frac{\mu_2 t_\varepsilon^4}{4} \int_{\mathbb{R}^N} \phi_\varepsilon^4 \right] \\
&= \lim_{\varepsilon \to 0} \varepsilon^{k(4-N)} I_\varepsilon(0, t_\varepsilon v_\varepsilon).
\end{aligned}$$

Noting that $I_\varepsilon(0, t_\varepsilon v_\varepsilon) \geqslant I_\varepsilon(0, V_{1\varepsilon}(\varepsilon \cdot))$, one has

$$\lim_{\varepsilon \to 0} \varepsilon^{k(4-N)} c_\varepsilon \geqslant \lim_{\varepsilon \to 0} \varepsilon^{k(4-N)} I_\varepsilon(0, V_{1\varepsilon}(\varepsilon \cdot)). \tag{4.13}$$

By rescaling $U_{1\varepsilon}$ and $V_{1\varepsilon}$, f_ε obtains its minimum at $(s_\varepsilon, t_\varepsilon)$, where

$$s_\varepsilon = \mu_1 \varepsilon^{-\frac{8\gamma}{\gamma+2}} \int_{\mathbb{R}^N} U_{1\varepsilon}^4 \left(\varepsilon^{\frac{2}{\gamma+2}} x \right) \left[\beta \int_{\mathbb{R}^N} U_{1\varepsilon}^2 \left(\varepsilon^{\frac{2}{\gamma+2}} x \right) V_{1\varepsilon}^2 \left(\varepsilon^{\frac{2}{\gamma+2}} x \right) - \mu_2 \int_{\mathbb{R}^N} V_{1\varepsilon}^4 \left(\varepsilon^{\frac{2}{\gamma+2}} x \right) \right]$$

and

$$t_\varepsilon = \mu_2 \varepsilon^{-\frac{8\gamma}{\gamma+2}} \int_{\mathbb{R}^N} V_{1\varepsilon}^4 \left(\varepsilon^{\frac{2}{\gamma+2}} x \right) \left[\beta \int_{\mathbb{R}^N} U_{1\varepsilon}^2 \left(\varepsilon^{\frac{2}{\gamma+2}} x \right) V_{1\varepsilon}^2 \left(\varepsilon^{\frac{2}{\gamma+2}} x \right) - \mu_1 \int_{\mathbb{R}^N} U_{1\varepsilon}^4 \left(\varepsilon^{\frac{2}{\gamma+2}} x \right) \right].$$

It follows from Remark 1.3 that $s_\varepsilon \to s_0$, $t_\varepsilon \to t_0$ as $\varepsilon \to 0$, where

$$s_0 = \mu_1 a_0 [\beta c_0 - \mu_2 b_0], \qquad t_0 = \mu_2 b_0 [\beta c_0 - \mu_1 a_0],$$

and

$$a_0 = \int_{\mathbb{R}^N} \omega_{\gamma,\mu_1}^4, \qquad b_0 = \int_{\mathbb{R}^N} \omega_{\gamma,\mu_2}^4, \qquad c_0 = \int_{\mathbb{R}^N} \omega_{\gamma,\mu_1}^2 \omega_{\gamma,\mu_2}^2.$$

Meanwhile,

$$\varepsilon^{k(4-N)} f_\varepsilon(s, t) = f_0(s, t) + o_\varepsilon(1) \quad \text{uniformly for } (s, t) \in \Gamma, \text{ as } \varepsilon \to 0,$$

where

$$f_0(s, t) = \frac{(s\mu_1 a_0 + t\mu_2 b_0)^2}{4(\mu_1 s^2 a_0 + 2\beta st c_0 + \mu_2 t^2 b_0)}.$$

It follows that

$$f_0(s, 0) = \frac{\mu_1 a_0}{4} = \lim_{\varepsilon \to 0} \varepsilon^{k(4-N)} I_\varepsilon\big(U_{1\varepsilon}(\varepsilon \cdot), 0\big),$$

$$f_0(0, t) = \frac{\mu_2 b_0}{4} = \lim_{\varepsilon \to 0} \varepsilon^{k(4-N)} I_\varepsilon\big(0, V_{1\varepsilon}(\varepsilon \cdot)\big).$$

Moreover, for $\beta > \beta_0$,

$$f_0(s_0, t_0) = \min_{(s,t) \in \Gamma} f_0(s, t) < \min\big\{ f_0(s, 0), \, f_0(0, t) \big\}.$$

Then we observe that

$$\lim_{\varepsilon \to 0} \varepsilon^{k(4-N)} f_\varepsilon(s_\varepsilon, t_\varepsilon) < \lim_{\varepsilon \to 0} \varepsilon^{k(4-N)} \min\big\{ I_\varepsilon\big(U_{1\varepsilon}(\varepsilon \cdot), 0\big), \, I_\varepsilon\big(V_{1\varepsilon}(\varepsilon \cdot), 0\big) \big\}.$$

By Lemma 3.6,

$$\limsup_{\varepsilon \to 0} \varepsilon^{k(4-N)} c_\varepsilon < \lim_{\varepsilon \to 0} \varepsilon^{k(4-N)} \min\big\{ I_\varepsilon\big(U_{1\varepsilon}(\varepsilon \cdot), 0\big), \, I_\varepsilon\big(V_{1\varepsilon}(\varepsilon \cdot), 0\big) \big\},$$

which contradicts (4.13).

Step 3. We claim that $\varepsilon^{k+1} x_\varepsilon \to x_i$ for some $i \in \mathcal{I}$. By **Step 1–2**, for some $\tilde{\omega}, \tilde{\phi} \in H^1_{\text{loc}}(\mathbb{R}^N) \setminus \{0\}$, such that, up to a subsequence, $\omega_\varepsilon(\cdot + x_\varepsilon) \rightharpoonup \tilde{\omega}$ and $\phi_\varepsilon(\cdot + x_\varepsilon) \rightharpoonup \tilde{\phi}$ weakly in $H^1_{\text{loc}}(\mathbb{R}^N)$ as $\varepsilon \to 0$. Then if $\varepsilon^{k+1} x_\varepsilon \to \infty$ as $\varepsilon \to 0$, by Fatou's Lemma,

$$\liminf_{\varepsilon \to \infty} \int_{\mathbb{R}^N} a_\varepsilon(x)\omega_\varepsilon^2 + b_\varepsilon(x)\phi_\varepsilon^2 \geqslant \int_{\mathbb{R}^N} a_\infty \tilde{\omega}^2 + b_\infty \tilde{\phi}^2.$$

It follows from (4.7) that

$$\int_{\mathbb{R}^N} a_\infty \tilde{\omega}^2 + b_\infty \tilde{\phi}^2 = 0,$$

which implies that $a_\infty = b_\infty = 0$. This is a contradiction. As a consequence, up to a subsequence, $\varepsilon^{k+1} x_\varepsilon \to x_*$ for some $x_* \in \mathbb{R}^N$ as $\varepsilon \to 0$. Then Fatou's Lemma tells us that

$$\int_{\mathbb{R}^N} a(x_*)\tilde{\omega}^2 + b(x_*)\tilde{\phi}^2 = 0$$

and then $x_* \in \mathcal{A} \cap \mathcal{B}$

Now, we claim that $x_* = x_i$ for some $i \in \mathcal{I}$. By the assumption of Theorem 1.2, there exists some positive γ-homogeneous function $W_* \in C(\mathbb{R}^N)$, such that

$$\lim_{x \to x_*} \frac{a(x) - W_*(x - x_*)}{|x - x_*|^\gamma} = 0 \quad \text{or} \quad \infty.$$

For any $x \neq x_*$, let

$$\lambda(x - x_*) = \frac{a(x) - W_*(x - x_*)}{|x - x_*|^\gamma}, \qquad \mu(x - x_*) = \frac{b(x) - W_*(x - x_*)}{|x - x_*|^\gamma},$$

then to conclude the claim, we just need to show that $\lim_{x \to 0} \lambda(x) = 0$, which implies that $\lim_{x \to 0} \mu(x) = 0$.

According to (4.7),

$$\limsup_{\varepsilon \to 0} \inf_{x \in B_1(x_\varepsilon)} \left[\frac{a_\varepsilon(x)}{\varepsilon^{(k+1)\gamma}} + \frac{b_\varepsilon(x)}{\varepsilon^{(k+1)\gamma}} \right] < \infty.$$

Thus there exists $y_\varepsilon \in B_1(x_\varepsilon)$ such that

$$\limsup_{\varepsilon \to 0} \left[\frac{a(\varepsilon^{k+1} y_\varepsilon)}{\varepsilon^{(k+1)\gamma}} + \frac{b(\varepsilon^{k+1} y_\varepsilon)}{\varepsilon^{(k+1)\gamma}} \right] < \infty$$

and $\varepsilon^{k+1} y_\varepsilon \to x_*$ as $\varepsilon \to 0$. For ε small, by Lemma 4.1, there holds $a(\varepsilon^{k+1} y_\varepsilon) \geq c|\varepsilon^{k+1} y_\varepsilon - x_*|^\gamma$ and $b(\varepsilon^{k+1} y_\varepsilon) \geq c|\varepsilon^{k+1} y_\varepsilon - x_*|^\gamma$ for some $c > 0$. Then for ε small and some $C > 0$,

$$\left| \varepsilon^{k+1} y_\varepsilon - x_* \right| \leq C \varepsilon^{k+1},$$

from which it follows that

$$\left| x_\varepsilon - \frac{x_*}{\varepsilon^{k+1}} \right| \leq C + 1.$$

So for $R > 0$ large enough, we have $B_1(x_\varepsilon) \subset B_R(\frac{x_*}{\varepsilon^{k+1}})$. Due to (4.10), for some $\omega, \phi \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \setminus \{0\}$, such that, up to a subsequence, $\tilde{\omega}_\varepsilon := \omega_\varepsilon(\cdot + \frac{x_*}{\varepsilon^{k+1}}) \rightharpoonup \omega$ and $\tilde{\phi}_\varepsilon := \phi_\varepsilon(\cdot + \frac{x_*}{\varepsilon^{k+1}}) \rightharpoonup \phi$ weakly in $H^1_{\mathrm{loc}}(\mathbb{R}^N)$ as $\varepsilon \to 0$.

If $\lim_{x \to 0} \lambda(x) = +\infty$, then for any $x \in \mathbb{R}^N$,

$$\varepsilon^{2k} a(\varepsilon^{k+1} x + x_*) = W_*(x) + \lambda(\varepsilon^{k+1} x)|x|^\gamma$$

and

$$\varepsilon^{2k} b(\varepsilon^{k+1} x + x_*) = W_*(x) + \mu(\varepsilon^{k+1} x)|x|^\gamma.$$

Thanks to (4.7) and $\mu(\cdot)$ is bounded from below in $\mathbb{R}^N$, by Rellich's theorem,

$$4m \geq \liminf_{\varepsilon \to 0} \int_{B_R(\frac{x_*}{\varepsilon^{k+1}})} \varepsilon^{2k} a_\varepsilon(x) \omega_\varepsilon^2 + \varepsilon^{2k} b_\varepsilon(x) \phi_\varepsilon^2$$

$$= \liminf_{\varepsilon \to 0} \int_{B_R(0)} \varepsilon^{2k} a(\varepsilon^{k+1} x + x_*) \tilde{\omega}_\varepsilon^2(x) + \varepsilon^{2k} b(\varepsilon^{k+1} x + x_*) \tilde{\phi}_\varepsilon^2(x)$$

$$= \liminf_{\varepsilon \to 0} \int_{B_R(0)} W_*(x)\big(\tilde{\omega}_\varepsilon^2(x) + \tilde{\phi}_\varepsilon^2(x)\big) + |x|^\gamma \big(\lambda\big(\varepsilon^{k+1}x\big)\tilde{\omega}_\varepsilon^2(x) + \mu\big(\varepsilon^{k+1}x\big)\tilde{\phi}_\varepsilon^2(x)\big)$$

$$\geqslant \liminf_{\varepsilon \to 0} \int_{B_R(0)} |x|^\gamma \big(\lambda\big(\varepsilon^{k+1}x\big)\tilde{\omega}_\varepsilon^2(x) + \mu\big(\varepsilon^{k+1}x\big)\tilde{\phi}_\varepsilon^2(x)\big) = \infty,$$

which is a contradiction. Thus,

$$\lim_{x \to x_*} \frac{a(x) - W_*(x - x_*)}{|x - x_*|^\gamma} = \lim_{x \to x_*} \frac{b(x) - W_*(x - x_*)}{|x - x_*|^\gamma} = 0.$$

By the assumption of Theorem 1.2, $x_* = x_i$ for some $i \in \mathcal{I}$.

Step 4. It is easy to know

$$\int_{\mathbb{R}^N} |\nabla \tilde{\omega}_\varepsilon|^2 + \varepsilon^{2k} a\big(\varepsilon^{k+1}x + x_*\big)\tilde{\omega}_\varepsilon^2 + |\nabla \tilde{\phi}_\varepsilon|^2 + \varepsilon^{2k} b\big(\varepsilon^{k+1}x + x_*\big)\tilde{\phi}_\varepsilon^2$$

$$= \int_{\mathbb{R}^N} \mu_1 \tilde{\omega}_\varepsilon^4 + 2\beta \tilde{\omega}_\varepsilon^2 \tilde{\phi}_\varepsilon^2 + \mu_2 \tilde{\phi}_\varepsilon^4,$$

Next, we prove (ω, ϕ) is a critical point of $\mathcal{J}_{W_*}$. By Fatou's Lemma and the lower semi-continuity of the norm,

$$\int_{B_R(0)} |\nabla \omega|^2 + W_* \omega^2 + |\nabla \phi|^2 + W_* \phi^2$$

$$\leqslant \liminf_{\varepsilon \to 0} \int_{B_R(0)} |\nabla \tilde{\omega}_\varepsilon|^2 + \varepsilon^{2k} a\big(\varepsilon^{k+1}x + x_*\big)\tilde{\omega}_\varepsilon^2 + |\nabla \tilde{\phi}_\varepsilon|^2 + \varepsilon^{2k} b\big(\varepsilon^{k+1}x + x_*\big)\tilde{\phi}_\varepsilon^2$$

$$\leqslant 4m.$$

Since R is arbitrary, we get that

$$\int_{\mathbb{R}^N} |\nabla \omega|^2 + W_* \omega^2 + |\nabla \phi|^2 + W_* \phi^2 \leqslant 4m \tag{4.14}$$

and then $\omega, \phi \in H^1(\mathbb{R}^N)$. For any $\xi, \eta \in C_0^\infty(\mathbb{R}^N)$, Multiplying equation (4.4) and (4.5) by $\xi(\cdot - \frac{x_*}{\varepsilon^{k+1}})$ and $\eta(\cdot - \frac{x_*}{\varepsilon^{k+1}})$ and making $\varepsilon \to 0$,

$$\int_{\mathbb{R}^N} \nabla \omega \nabla \xi + W_* \omega \xi + \nabla \phi \nabla \eta + W_* \phi \eta = \int_{\mathbb{R}^N} \mu_1 \omega^3 \xi + \beta \phi^2 \omega \xi + \beta \omega^2 \phi \eta + \mu_2 \phi^3 \eta,$$

which implies $\mathcal{J}'_{W_*}(\omega, \phi) = 0$ in $(H_{W_*} \times H_{W_*})^*$ and $\mathcal{J}_{W_*}(\omega, \phi) \geqslant \mathcal{E}_{W_*}$. Due to (4.14), $\mathcal{J}_{W_*}(\omega, \phi) \leqslant m = \inf_{i \in \mathcal{I}} \mathcal{E}_{W_i}$. Therefore,

$$\mathcal{J}_{W_*}(\omega, \phi) = \inf_{i \in \mathcal{I}} \mathcal{E}_{W_i}.$$

and $(\tilde{\omega}_\varepsilon, \tilde{\phi}_\varepsilon) \to (\omega, \phi)$ in $\mathbb{H}_{\text{loc}}$ as $\varepsilon \to 0$. The proof is completed. $\quad\square$

5. Proof of Theorem 1.5

This last section is devoted to the proof of Theorem 1.5, which covers the case that the potential a and b vanish on the closure of some smooth bounded open set.

Proof of Theorem 1.5. Step 1. Similarly to Theorem 1.2, for sufficiently small $\varepsilon > 0$ and $\beta > \beta_0^\varepsilon$, we prove that system $(\mathcal{C}_\varepsilon)$ has a positive ground state solution $(U_\varepsilon, V_\varepsilon)$. Define

$$E_\varepsilon := \left\{ (u, v) \in \mathbb{H}_{\text{loc}} : \int_{\mathbb{R}^N} \varepsilon^2 |\nabla u|^2 + a(x) u^2 < \infty, \int_{\mathbb{R}^N} \varepsilon^2 |\nabla v|^2 + b(x) v^2 < \infty \right\}$$

endowed with norm

$$\|(u, v)\|_\varepsilon = \left(\int_{\mathbb{R}^N} \varepsilon^2 |\nabla u|^2 + a(x) u^2 + \int_{\mathbb{R}^N} \varepsilon^2 |\nabla v|^2 + b(x) v^2 \right)^{\frac{1}{2}}.$$

Define the auxiliary functional $J_\varepsilon \in C^1(E_\varepsilon)$ by for each $(w, z) \in E_\varepsilon$,

$$J_\varepsilon(w, z) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w|^2 + \frac{a(x)}{\varepsilon^2} w^2 + |\nabla z|^2 + \frac{b(x)}{\varepsilon^2} z^2 - \frac{1}{4} F(w, z).$$

Set

$$d_\varepsilon := \inf \big\{ J_\varepsilon(w, z) : (w, z) \in E_\varepsilon \setminus \{(0, 0)\}, \langle J_\varepsilon'(w, z), (w, z) \rangle = 0 \big\}.$$

For any $(u, v) \in E_* = H_0^1(\text{int}(\Omega)) \times H_0^1(\text{int}(\Omega))$, set

$$J_*(w, z) = \frac{1}{2} \int_\Omega |\nabla w|^2 + |\nabla z|^2 - \frac{1}{4} \int_\Omega \mu_1 w^4 + 2\beta w^2 z^2 + \mu_2 z^4.$$

Similarly to Lemma 3.6, for sufficiently small $\varepsilon > 0$ and $\beta > \beta_0^\varepsilon$,

$$d_\varepsilon = \varepsilon^{N-4} c_\varepsilon < \varepsilon^{N-4} \min \big\{ I_\varepsilon \big(U_{1\varepsilon}(\varepsilon \cdot), 0 \big), I_\varepsilon(0, V_{1\varepsilon}(\varepsilon \cdot) \big\}. \tag{5.1}$$

Moreover, for sufficiently small ε, problem $(\mathcal{C}_\varepsilon)$ admits a ground state solution $(U_\varepsilon, V_\varepsilon)$, which is fully nontrivial. By virtue of the Nehari manifold method, one can show that for any $\beta > 0$, the following problem

$$\begin{cases} -\Delta w = \mu_1 w^3 + \beta z^2 w, & x \in \text{int}(\Omega), \\ -\Delta z = \mu_2 z^3 + \beta w^2 z, & x \in \text{int}(\Omega), \\ w, z \in H_0^1(\text{int}(\Omega)) \end{cases} \tag{5.2}$$

admits a ground state solution. Moreover, set

$$0 < c_* = \inf \big\{ J_*(w, z) : (w, z) \in E_* \setminus \{(0, 0)\}, \langle J_*'(w, z), (w, z) \rangle = 0 \big\},$$

it is easy to check that $d_\varepsilon = J_\varepsilon(\varepsilon^{-1}U_\varepsilon, \varepsilon^{-1}V_\varepsilon) \leqslant c_* \leqslant \tilde{c}_*$, where $\tilde{c}_*$ is the least energy of the following problem

$$\begin{cases} -\Delta w = \beta z^2 w, & x \in \text{int}(\Omega), \\ -\Delta z = \beta w^2 z, & x \in \text{int}(\Omega), \\ w, z \in H_0^1(\text{int}(\Omega)). \end{cases}$$

Step 2. Let $w_\varepsilon = \varepsilon^{-1}U_\varepsilon$ and $z_\varepsilon = \varepsilon^{-1}V_\varepsilon$. Noting that for any $\varepsilon > 0$,

$$\int_{\mathbb{R}^N} |\nabla w_\varepsilon|^2 + \frac{a(x)}{\varepsilon^2} w_\varepsilon^2 + |\nabla z_\varepsilon|^2 + \frac{b(x)}{\varepsilon^2} z_\varepsilon^2 \leqslant 4c_*. \tag{5.3}$$

It follows from [29, Lemma 2.1] that there exists $C > 0$ (independent of ε) such that

$$\int_{\mathbb{R}^N} \left(|\nabla w_\varepsilon|^2 + w_\varepsilon^2 \right) \leqslant C \int_{\mathbb{R}^N} \left(|\nabla w_\varepsilon|^2 + a(x)w_\varepsilon^2 \right),$$

which implies that $\int_{\mathbb{R}^N}(|\nabla w_\varepsilon|^2 + w_\varepsilon^2) \leqslant 4Cc_*$ if $\varepsilon \leqslant 1$. Similar result holds for z_ε. So, $(w_\varepsilon, z_\varepsilon)$ is bounded in $\mathbb{H}$ if $\varepsilon \leqslant 1$ and for some $(w, z) \in \mathbb{H}$, $(w_\varepsilon, z_\varepsilon) \rightharpoonup (w, z)$ in $\mathbb{H}$ and $(w_\varepsilon, z_\varepsilon) \to (w, z)$ in $\mathbb{L}^2_{\text{loc}}(\mathbb{R}^N)$ as $\varepsilon \to 0$.

For $R > 0$ large enough such that $a(x) \geqslant \frac{a_\infty}{2}$ and $b(x) \geqslant \frac{b_\infty}{2}$ for $|x| \geqslant R$. By (5.3),

$$\frac{1}{2} \min\{a_\infty, b_\infty\} \int_{\mathbb{R}^N \setminus B_R(0)} w_\varepsilon^2 + z_\varepsilon^2 \leqslant \int_{\mathbb{R}^N \setminus B_R(0)} a(x)w_\varepsilon^2 + b(x)z_\varepsilon^2 \leqslant 4c_*\varepsilon^2 \to 0, \quad \varepsilon \to 0.$$

So Rellich's theorem tells us that $w_\varepsilon \to w$ and $z_\varepsilon \to z$ strongly in $L^2(\mathbb{R}^N)$. On the other hand, due to $J_\varepsilon'(w_\varepsilon, z_\varepsilon) = 0$, for $\varepsilon > 0$ small and some C, C_1 (independent of ε),

$$\left\| (w_\varepsilon, z_\varepsilon) \right\|_4^2 \leqslant C \left\| (w_\varepsilon, z_\varepsilon) \right\|_{\mathbb{H}}^2 \leqslant C_1 \left\| (w_\varepsilon, z_\varepsilon) \right\|_4^4,$$

which shows that $\|(w, z)\|_4 > 0$ and $(w, z) \neq (0, 0)$.

We claim that $w, z \not\equiv 0$. Otherwise, without loss of generalization, if $z \equiv 0$, then $z_\varepsilon \to 0$ strongly in $L^2(\mathbb{R}^N)$ as $\varepsilon \to 0$. Noting that $\{z_\varepsilon\}$ is bounded in $H^1(\mathbb{R}^N)$, by the Gagliardo–Nirenberg–Sobolev inequality, $z_\varepsilon \to 0$ strongly in $L^4(\mathbb{R}^N)$ as $\varepsilon \to 0$ and

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} |\nabla z_\varepsilon|^2 + \frac{b(x)}{\varepsilon^2} z_\varepsilon^2 = 0.$$

Then similarly to Section 4,

$$\liminf_{\varepsilon \to 0} d_\varepsilon \geqslant \liminf_{\varepsilon \to 0} \varepsilon^{N-4} I_\varepsilon \left(U_{1\varepsilon}(\varepsilon \cdot), 0 \right),$$

which gives a contradiction by using (5.1) similarly as above.

Step 3. We show that $w, z \in H_0^1(\text{int}(\Omega))$ for $\mu_1, \mu_2 > 0$ small. By (5.3), we have

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus \Sigma} a(x)w^2 + b(x)z^2 = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus \Sigma} a(x)w_\varepsilon^2 + b(x)z_\varepsilon^2 = 0,$$

which implies that $w = z = 0$ in $\mathbb{R}^N \setminus \Sigma$. Similarly,

$$\lim_{\varepsilon \to 0} \int_{\Sigma \setminus \Omega} a(x)w^2 + b(x)z^2 = 0$$

which implies that $w = 0$ in $\text{int}(\mathcal{B}) \setminus \mathcal{A}$ and $z = 0$ in $\text{int}(\mathcal{A}) \setminus \mathcal{B}$. That is, $w \in H_0^1(\text{int}(\mathcal{A}))$ and $z \in H_0^1(\text{int}(\mathcal{B}))$.

On the one hand, one can get that $w_\varepsilon + z_\varepsilon$ satisfies, in the weak sense,

$$-\Delta(w_\varepsilon + z_\varepsilon) \leqslant \max\{\mu_1, \mu_2, \beta\}(w_\varepsilon + z_\varepsilon)^3, \quad x \in \mathbb{R}^N.$$

Recalling that $\{w_\varepsilon + z_\varepsilon\}$ is uniformly bounded in $H^1(\mathbb{R}^N)$ with respect to ε and $\mu_1, \mu_2 > 0$ small, similarly as that in [6, Proposition 2.2], one can get that $\{w_\varepsilon + z_\varepsilon\}$ is uniformly bounded in $L^\infty(\Sigma)$ with respect to ε and $\mu_1, \mu_2 > 0$ small. Moreover, $\|z\|_\infty + \|w\|_\infty \leqslant C$ for some C (independent of μ_1, μ_2). On the other hand, denote by ϕ_1 and λ_1 the first eigenfunction and eigenvalue of $(-\Delta, H_0^1(\text{int}(\mathcal{B}) \setminus \mathcal{A}))$ respectively. Since

$$-\Delta z_\varepsilon + \frac{b(x)}{\varepsilon^2} z_\varepsilon = \mu_2 z_\varepsilon^3 + \beta w_\varepsilon^2 z_\varepsilon, \quad x \in \mathbb{R}^N,$$

we obtain that

$$\lambda_1 \int_{\text{int}(\mathcal{B}) \setminus \mathcal{A}} z_\varepsilon \phi_1 = \int_{\text{int}(\mathcal{B}) \setminus \mathcal{A})} (\mu_2 z_\varepsilon^3 + \beta w_\varepsilon^2 z_\varepsilon) \phi_1.$$

Taking the limit as $\varepsilon \to 0$,

$$\int_{\text{int}(\mathcal{B}) \setminus \mathcal{A}} z \phi_1 [\lambda_1 - \mu_2 z^2] = 0.$$

So if $\mu_2 > 0$ is small such that $\lambda_1 - \mu_2 z^2(x) > 0$ a. e. in $\text{int}(\mathcal{B}) \setminus \mathcal{A}$, we know $z = 0$ in $\text{int}(\mathcal{B}) \setminus \mathcal{A}$. That is, $z \in H_0^1(\text{int}(\Omega))$. Similarly, $w \in H_0^1(\text{int}(\Omega))$.

Step 4. For any $\eta, \varphi \in H_0^1(\text{int}(\Omega))$, we have

$$\int_{\text{int}(\Omega)} \nabla w_\varepsilon \nabla \eta + \nabla z_\varepsilon \nabla \varphi - \mu_1 w_\varepsilon^3 \eta - \beta z_\varepsilon^2 w_\varepsilon \eta - \beta w_\varepsilon^2 z_\varepsilon \varphi - \mu_2 z_\varepsilon^3 \varphi = J_\varepsilon'(w_\varepsilon, z_\varepsilon)(\eta, \varphi) = 0$$

and hence

$$\int_{\text{int}(\Omega)} \nabla w \nabla \eta + \nabla z \nabla \varphi - \mu_1 w^3 \eta - \beta z^2 w \eta - \beta w^2 z \varphi - \mu_2 z^3 \varphi = 0.$$

In view of the regularity assumptions on the set Ω and by classical regularity theory, (w, z) is a fully nontrivial solution of problem (5.2) and $J_*(w, z) \geqslant c_*$.

We also have

$$
\begin{aligned}
\limsup_{\varepsilon \to 0} &\int_{\mathbb{R}^N} |\nabla w_\varepsilon|^2 + |\nabla z_\varepsilon|^2 + w_\varepsilon^2 + z_\varepsilon^2 \\
&\leqslant \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} |\nabla w_\varepsilon|^2 + \frac{a(x)}{\varepsilon^2} w_\varepsilon^2 + |\nabla z_\varepsilon|^2 + \frac{b(x)}{\varepsilon^2} z_\varepsilon^2 + w_\varepsilon^2 + z_\varepsilon^2 \\
&= \lim_{\varepsilon \to 0} F(w_\varepsilon, z_\varepsilon) + \|w\|_2^2 + \|z\|_2^2 \\
&= F(w, z) + \|w\|_2^2 + \|z\|_2^2 = \int_{\mathbb{R}^N} |\nabla w|^2 + |\nabla z|^2 + w^2 + z^2.
\end{aligned}
$$

This implies $(w_\varepsilon, z_\varepsilon) \to (w, z)$ strongly in $\mathbb{H}$ as $\varepsilon \to 0$. Due to $d_\varepsilon = J_\varepsilon(w_\varepsilon, v_\varepsilon) \leqslant c_*$, the following inequality holds

$$
J_*(w, z) \leqslant \limsup_{\varepsilon \to 0} J_\varepsilon(w_\varepsilon, z_\varepsilon) \leqslant c_*,
$$

which yields that (w, z) is a ground state of problem (5.2). The proof is completed. $\square$

Acknowledgements

References

[1] A. Ambrosetti and E. Colorado, Bound and ground states of coupled nonlinear Schrödinger equations, *C. R. Math. Acad. Sci. Paris* **342** (2006), 453–458. doi:10.1016/j.crma.2006.01.024.

[2] A. Ambrosetti and E. Colorado, Standing waves of some coupled nonlinear Schrödinger equations, *J. Lond. Math. Soc.* **75** (2007), 67–82. doi:10.1112/jlms/jdl020.

[3] A. Ambrosetti and P.H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* **14** (1973), 349–381. doi:10.1016/0022-1236(73)90051-7.

[4] T. Bartsch, N. Dancer and Z.-Q. Wang, A Liouville theorem, a-priori bounds, and bifurcating branches of positive solutions for a nonlinear elliptic system, *Calc. Var. PDE.* **37** (2010), 345–361. doi:10.1007/s00526-009-0265-y.

[5] T. Bartsch and Z.-Q. Wang, Note on ground states of nonlinear Schrödinger systems, *J. Differ. Equations* **19** (2006), 200–207.

[6] B. Barrios, E. Colorado, R. Servadei and F. Soria, A critical fractional equation with concave-convex power nonlinearities, *Ann. I.H. Poincare AN* **32** (2015), 875–900.

[7] J. Byeon and Z.-Q. Wang, Standing waves with a critical frequency for nonlinear Schrödinger equations, *Arch. Ration. Mech. Anal.* **18** (2002), 295–316. doi:10.1007/s00205-002-0225-6.

[8] J. Byeon and Z.-Q. Wang, Standing waves with a critical frequency for nonlinear Schrödinger equations. II, *Calc. Var. PDE.* **18** (2003), 207–219. doi:10.1007/s00526-002-0191-8.

[9] Z.-J. Chen and W.-M. Zou, Standing waves for coupled nonlinear Schrödinger equations with decaying potentials, *J. Math. Phys.* **54** (2013), 111505. doi:10.1063/1.4833795.

[10] M. del Pino and P.L. Felmer, Local mountain passes for semilinear elliptic problems in unbounded domains, *Calc. Var. PDE.* **11** (1996), 121–137.

[11] D.G. de Figueiredo and O. Lopes, Solitary waves for some nonlinear Schrödinger systems, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **25** (2008), 149–161. doi:10.1016/j.anihpc.2006.11.006.

[12] B. Esry, C. Greene, J. Burke and J. Bohn, Hartree–Fock theory for double condensates, *Phys. Rev. Lett.* **78** (1997), 3594–3597. doi:10.1103/PhysRevLett.78.3594.

[13] A. Floer and A. Weinstein, Nonspreading wave packets for the cubic Schrödinger equations, *J. Funct. Anal.* **69** (1986), 397–408. doi:10.1016/0022-1236(86)90096-0.

[14] N. Ikoma, Uniqueness of positive solutions for a nonlinear elliptic system, *Nonlinear Differ. Equ. Appl.* **16** (2009), 555–567. doi:10.1007/s00030-009-0017-x.

[15] N. Ikoma and K. Tanaka, A local mountain pass type result for a system of nonlinear Schrödinger equations, *Calc. Var. PDE.* **40** (2011), 449–480. doi:10.1007/s00526-010-0347-x.

[16] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case, parts 1 and 2, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **1** (1984), 109–145, 223-283.

[17] M. Lucia and Z.-W. Tang, Multi-bump bound states for a system of nonlinear Schrödinger equations, *J. Differ. Equations* **252** (2012), 3630–3657. doi:10.1016/j.jde.2011.11.017.

[18] T.-C. Lin and J.-C. Wei, Ground state of N coupled nonlinear Schrödinger equations in $\mathbf{R}^N$, $n \leqslant 3$, *Comm. Math. Phys.* **255** (2005), 629–653. doi:10.1007/s00220-005-1313-x.

[19] T.-C. Lin and J.-C. Wei, Spikes in two-component systems of nonlinear Schrödinger equations with trapping potentials, *J. Differ. Equations* **229** (2006), 538–569. doi:10.1016/j.jde.2005.12.011.

[20] Z.-L. Liu and Z.-Q. Wang, Multiple bound states of nonlinear Schrödinger systems, *Comm. Math. Phys.* **282** (2008), 721–731. doi:10.1007/s00220-008-0546-x.

[21] L. Maia, E. Montefusco and B. Pellacci, Positive solutions for a weakly coupled nonlinear Schrödinger system, *J. Differ. Equations* **229** (2006), 743–767. doi:10.1016/j.jde.2006.07.002.

[22] E. Montefusco, B. Pellacci and M. Squassina, Semiclassical states for weakly coupled nonlinear Schrödinger systems, *J. Eur. Math. Soc.* **10** (2007), 47–71.

[23] A. Pomponio, Coupled nonlinear Schrödinger systems with potentials, *J. Differ. Equations* **227** (2006), 258–281. doi:10.1016/j.jde.2005.09.002.

[24] P.H. Rabinowitz, On a class of nonlinear Schrödinger equations, *Z. Angew. Math. Phys.* **43** (1992), 270–291. doi:10.1007/BF00946631.

[25] B. Sirakov, Least energy solitary waves for a system of nonlinear Schrödinger equations in $\mathbf{R}^N$, *Comm. Math. Phys.* **271** (2007), 199–221. doi:10.1007/s00220-006-0179-x.

[26] E. Timmermans, Phase seperation of Bose–Einstein condensates, *Phys. Rev. Lett.* **81** (1998), 5718–5721. doi:10.1103/PhysRevLett.81.5718.

[27] Z.-W. Tang and L.-S. Wang, Segregated vector solutions with multi-scale spikes for nonlinear coupled elliptic systems, *J. Math. Anal. Appl.* **464** (2018), 1–31. doi:10.1016/j.jmaa.2018.02.019.

[28] Z.-W. Tang and H.-F. Xie, Multi-scale spike solutions for nonlinear coupled elliptic systems with critical frequency, *Nonlinear Differ. Equ. Appl.* **28** (2021).

[29] J. Van Schaftingen and J.-K. Xia, Standing waves with a critical frequency for nonlinear Choquard equations, *Nonlinear. Anal.* **161** (2017), 87–107. doi:10.1016/j.na.2017.05.014.

[30] M. Willem, *Minimax Theorems*, Progr. Nonlinear Differential Equations Appl., Vol. 24, Birkhäuser Boston Inc., Boston, MA, 1996.

[31] X.-F. Wang, On concertration of positive bound states of nonlinear Schrödinger equations, *Comm. Math. Phys.* **153** (1993), 229–244. doi:10.1007/BF02096642.

[32] G.-M. Wei, Existence and concentration of ground states of coupled nonlinear Schrödinger equations, *J. Math. Anal. Appl.* **332** (2007), 846–862. doi:10.1016/j.jmaa.2006.09.083.

[33] J. Wang and J.-P. Shi, Standing waves of a weakly coupled Schrödinger system with distinct potential functions, *J. Differ. Equations* **260** (2016), 1830–1864. doi:10.1016/j.jde.2015.09.052.

[34] J.-C. Wei and T. Weth, Nonradial symmetric bound states for a system of two coupled Schrödinger equations, *Rend. Lincei, Mat. Appl.* **18** (2007), 279–293.

[35] X. Zhang and J.-K. Xia, Semi-classical solutions for Schrödinger–Poisson equations with a critical frequency, *J. Differ. Equations* **265** (2018), 2121–2170. doi:10.1016/j.jde.2018.04.023.