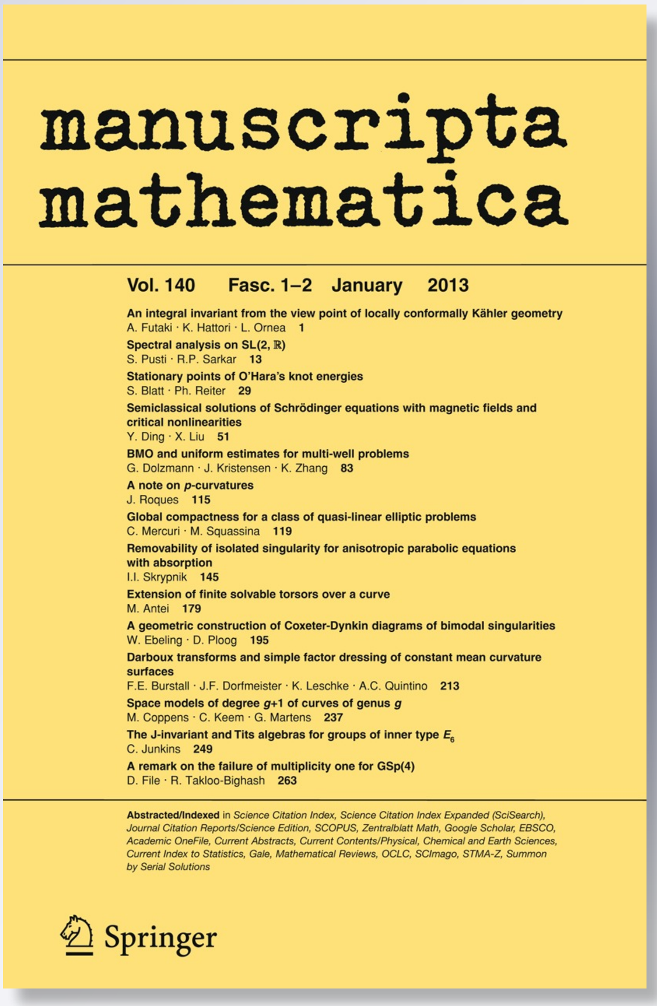
Global compactness for a class of quasi-linear elliptic problems

Carlo Mercuri & Marco Squassina

Manuscripta Mathematica

ISSN 0025-2611
Volume 140
Combined 1-2

manuscripta math. (2013) 140:119-144
DOI 10.1007/s00229-012-0533-6


manuscripta mathematica

Vol. 140 Fasc. 1–2 January 2013

An integral invariant from the view point of locally conformally Kähler geometry
A. Futaki · K. Hattori · L. Ornea 1
Spectral analysis on SL(2, ℝ)
S. Pusti · R.P. Sarkar 13
Stationary points of O'Hara's knot energies
S. Blatt · Ph. Reiter 29
Semiclassical solutions of Schrödinger equations with magnetic fields and critical nonlinearities
Y. Ding · X. Liu 51
BMO and uniform estimates for multi-well problems
G. Dolzmann · J. Kristensen · K. Zhang 83
A note on p-curvatures
J. Roques 115
Global compactness for a class of quasi-linear elliptic problems
C. Mercuri · M. Squassina 119
Removability of isolated singularity for anisotropic parabolic equations with absorption
I.I. Skrypnik 145
Extension of finite solvable torsors over a curve
M. Antei 179
A geometric construction of Coxeter-Dynkin diagrams of bimodal singularities
W. Ebeling · D. Ploog 195
Darboux transforms and simple factor dressing of constant mean curvature surfaces
F.E. Burstall · J.F. Dorfmeister · K. Leschke · A.C. Quintino 213
Space models of degree g+1 of curves of genus g
M. Coppens · C. Keem · G. Martens 237
The J-invariant and Tits algebras for groups of inner type E_6
C. Junkins 249
A remark on the failure of multiplicity one for GSp(4)
D. File · R. Takloo-Bighash 263

Abstracted/Indexed in *Science Citation Index, Science Citation Index Expanded (SciSearch), Journal Citation Reports/Science Edition, SCOPUS, Zentralblatt Math, Google Scholar, EBSCO, Academic OneFile, Current Abstracts, Current Contents/Physical, Chemical and Earth Sciences, Current Index to Statistics, Gale, Mathematical Reviews, OCLC, SCImago, STMA-Z, Summon by Serial Solutions*

Springer

Springer

manuscripta math. 140, 119–144 (2013)

Carlo Mercuri, Marco Squassina

Global compactness for a class of quasi-linear elliptic problems

Received: 20 July 2011 / Revised: 5 December 2011
Published online: 1 February 2012

Abstract. We prove a global compactness result for Palais-Smale sequences associated with a class of quasi-linear elliptic equations on exterior domains.

1. Introduction and main result

Let Ω be a smooth domain of $\mathbb{R}^N$ with a bounded complement and $N > p > m > 1$. The main goal of this paper is to obtain a global compactness result for the Palais-Smale sequences of the energy functional associated with the following quasi-linear elliptic equation

$$-\operatorname{div}(L_\xi(Du)) - \operatorname{div}(M_\xi(u, Du)) + M_s(u, Du) + V(x)|u|^{p-2}u = g(u) \quad \text{in } \Omega, \tag{1.1}$$

where $u \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$, meant as the completion of the space $\mathcal{D}(\Omega)$ of smooth functions with compact support, with respect to the norm $\|u\|_{W^{1,p}(\Omega)\cap D^{1,m}(\Omega)} = \|u\|_p + \|u\|_m$, having set $\|u\|_p := \|u\|_{W^{1,p}(\Omega)}$ and $\|u\|_m := \|Du\|_{L^m(\Omega)}$. We assume that V is a continuous function on Ω,

$$\lim_{|x|\to\infty} V(x) = V_\infty \quad \text{and} \quad \inf_{x\in\Omega} V(x) = V_0 > 0.$$

As known, lack of compactness may occur due to the lack of compact embeddings for Sobolev spaces on Ω and since the limiting equation on $\mathbb{R}^N$

$$-\operatorname{div}(L_\xi(Du)) - \operatorname{div}(M_\xi(u, Du)) + M_s(u, Du) + V_\infty|u|^{p-2}u = g(u) \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

C. Mercuri: Department of Mathematics and Computer Science, Technische Universiteit Eindhoven, Postbus 513, 5600 MB, Eindhoven, The Netherlands
e-mail: c.mercuri@tue.nl

M. Squassina(✉): Department of Computer Science, University of Verona, Strada Le Grazie 15, 37134 Verona, Italy e-mail: marco.squassina@univr.it
Mathematics Subject Classification (2000): 35D99, 35J62, 58E05, 35J70

DOI: 10.1007/s00229-012-0533-6

with $u \in W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)$, is invariant by translations. A particular case of (1.1) is

$$-\Delta_p u - \operatorname{div}(a(u)|Du|^{m-2}Du) + \frac{1}{m}a'(u)|Du|^m + V(x)|u|^{p-2}u$$
$$= |u|^{\sigma-2}u \quad \text{in } \Omega, \tag{1.3}$$

where $\Delta_p u := \operatorname{div}(|Du|^{p-2}Du)$, for a suitable function $a \in C^1(\mathbb{R}; \mathbb{R}^+)$, or the even simpler case where a is constant, namely

$$-\Delta_p u - \Delta_m u + V(x)|u|^{p-2}u = |u|^{\sigma-2}u \quad \text{in } \Omega. \tag{1.4}$$

Since the pioneering work of Benci and Cerami [3] dealing with the special case $L(\xi) = |\xi|^2/2$ and $M(s,\xi) \equiv 0$, many papers have been written on this subject, see for instance the bibliography of [16]. Quite recently, in [16], the case $L(\xi) = |\xi|^p/p$ and $M(s,\xi) \equiv 0$ was investigated. The main point in the present contribution is the fact that we allow, under suitable assumptions, a quasi-linear term $M(u, Du)$ depending on the unknown u itself. The typical tools exploited in [3,16], in addition to the point-wise convergence of the gradients, are some decomposition (splitting) results both for the energy functional and for the equation, along a given bounded Palais-Smale sequence (u_n). To this regard, the explicit dependence on u in the term $M(u, Du)$ requires a rather careful analysis, see e.g. [1,18] and references therein. We also refer the reader to the works by Filippucci [12,13] as well as the recent paper by Filippucci et al. [14] on the existence and nonexistence of large entire solutions, covering a very general class of quasi-linear equations. We shall handle our problem (1.1) under the growth condition

$$\nu|\xi|^m \le M(s,\xi) \le C|\xi|^m, \quad p-1 \le m < p-1+p/N.$$

The restriction on m, together with the sign condition (1.9) provides, thanks to the presence of L, the needed a priori regularity on the weak limit of (u_n), see Theorems 3.2 and 3.4.

Besides the aforementioned motivations, which are of mathematical interest, it is worth pointing out that in recent years, some works have been devoted to quasi-linear operators with double homogeneity, which arise from several problems of Mathematical Physics. For instance, the reaction diffusion problem $u_t = -\operatorname{div}(\mathbb{D}(u)Du) + \ell(x,u)$, where $\mathbb{D}(u) = d_p|Du|^{p-2} + d_m|Du|^{m-2}$, $d_p > 0$ and $d_m > 0$, admitting a rather wide range of applications in biophysics [11], plasma physics [19] and in the study of chemical reactions [2]. In this framework, u typically describes a concentration and $\operatorname{div}(\mathbb{D}(u)Du)$ corresponds to the diffusion with a coefficient $\mathbb{D}(u)$, whereas $\ell(x,u)$ plays the rǒle of reaction and relates to source and loss processes. We refer the interested reader to [6] and to the references therein. Furthermore, a model for elementary particles proposed by Derrick [10] yields to the study of standing wave solutions $\psi(x,t) = u(x)e^{\mathrm{i}\omega t}$ of the following nonlinear Schrödinger equation

$$\mathrm{i}\psi_t + \Delta_2\psi - b(x)\psi + \Delta_p\psi - V(x)|\psi|^{p-2}\psi + |\psi|^{\sigma-2}\psi = 0 \quad \text{in } \mathbb{R}^N,$$

for which we refer the reader e.g. to [4].

In order to state the first main result, assume $N > p > m \geq 2$ and

$$p - 1 \leq m < p - 1 + p/N, \quad p < \sigma < p^*, \tag{1.5}$$

and consider the C^2 functions $L : \mathbb{R}^N \to \mathbb{R}$ and $M : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ such that both the functions $\xi \mapsto L(\xi)$ and $\xi \mapsto M(s, \xi)$ are strictly convex and

$$\nu|\xi|^p \leq |L(\xi)| \leq C|\xi|^p, \quad |L_\xi(\xi)| \leq C|\xi|^{p-1}, \quad |L_{\xi\xi}(\xi)| \leq C|\xi|^{p-2}, \tag{1.6}$$

for all $\xi \in \mathbb{R}^N$. Furthermore, we assume

$$\nu|\xi|^m \leq M(s, \xi)| \leq C|\xi|^m, \quad |M_s(s, \xi)| \leq C|\xi|^m, \quad |M_\xi(s, \xi)| \leq C|\xi|^{m-1}, \tag{1.7}$$

$$|M_{ss}(s, \xi)| \leq C|\xi|^m, \quad |M_{s\xi}(s, \xi)| \leq C|\xi|^{m-1}, \quad |M_{\xi\xi}(s, \xi)| \leq C|\xi|^{m-2}, \tag{1.8}$$

for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and that the sign condition (cf. [18])

$$M_s(s, \xi)s \geq 0, \tag{1.9}$$

holds for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$. Also, $G : \mathbb{R} \to \mathbb{R}$ is a C^2 function with $G'(s) := g(s)$ and

$$|G'(s)| \leq C|s|^{\sigma-1}, \quad |G''(s)| \leq C|s|^{\sigma-2}, \tag{1.10}$$

for all $s \in \mathbb{R}$. We define

$$j(s, \xi) := L(\xi) + M(s, \xi) - G(s), \tag{1.11}$$

and on $W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ with $\|u\|_{W^{1,p}(\Omega) \cap D^{1,m}(\Omega)} = \|u\|_p + \|u\|_m$ the functional

$$\phi(u) := \int_\Omega j(u, Du) + \int_\Omega V(x)\frac{|u|^p}{p}.$$

Finally, on $W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)$ with $\|u\|_{W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)} = \|u\|_p + \|u\|_m$ we define

$$\phi_\infty(u) := \int_{\mathbb{R}^N} j(u, Du) + \int_{\mathbb{R}^N} V_\infty \frac{|u|^p}{p}.$$

See Sect. 2 for some properties of the functionals ϕ and ϕ_∞.
The first main global compactness type result is the following

Theorem 1.1. *Assume that (1.5)–(1.11) hold and let $(u_n) \subset W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ be a bounded sequence such that*

$$\phi(u_n) \to c \quad \phi'(u_n) \to 0 \quad in\, (W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*$$

Then, up to a subsequence, there exists a weak solution $v_0 \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ *of*

$$-\mathrm{div}(L_\xi(Du)) - \mathrm{div}(M_\xi(u, Du)) + M_s(u, Du) + V(x)|u|^{p-2}u = g(u) \quad \textit{in}\, \Omega,$$

a finite sequence $\{v_1, \dots, v_k\} \subset W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)$ *of weak solutions of*

$$-\mathrm{div}(L_\xi(Du)) - \mathrm{div}(M_\xi(u, Du)) + M_s(u, Du) + V_\infty|u|^{p-2}u = g(u) \quad \textit{in}\, \mathbb{R}^N$$

and k sequences $(y_n^i) \subset \mathbb{R}^N$ *satisfying*

$$|y_n^i| \to \infty, \quad |y_n^i - y_n^j| \to \infty, \quad i \neq j, \quad \textit{as}\, n \to \infty,$$

$$\|u_n - v_0 - \sum_{i=1}^k v_i((\cdot - y_n^i)\|_{W^{1,p}(\mathbb{R}^N)\cap D^{1,m}(\mathbb{R}^N)} \to 0, \quad \textit{as}\, n \to \infty,$$

$$\|u_n\|_p^p \to \sum_{i=0}^k \|v_i\|_p^p, \quad \|u_n\|_m^m \to \sum_{i=0}^k \|v_i\|_m^m, \quad \textit{as}\, n \to \infty,$$

as well as

$$\phi(v_0) + \sum_{i=1}^k \phi_\infty(v_i) = c.$$

Let us now come to a statement for the cases $1 < m \leq 2$ or $1 < p \leq 2$. Let us define

$$\mathfrak{L}(\xi, h) := \frac{|L_\xi(\xi+h) - L_\xi(\xi)|}{|h|^{p-1}}, \quad \text{if} \quad 1 < p < 2,$$
$$\mathfrak{G}(s, t) := \frac{|G'(s+t) - G'(s)|}{|t|^{\sigma-1}}, \quad \text{if} \quad 1 < \sigma < 2,$$
$$\mathfrak{M}(s, \xi, h) := \frac{|M_\xi(s, \xi+h) - M_\xi(s, \xi)|}{|h|^{m-1}}, \quad \text{if} \quad 1 < m < 2.$$

If either $p < 2$, $\sigma < 2$ or $m < 2$, we shall weaken the twice differentiability assumptions, by requiring $L_\xi \in C^1(\mathbb{R}^N \setminus \{0\})$, $G' \in C^1(\mathbb{R} \setminus \{0\})$, $M_\xi \in C^1(\mathbb{R} \times (\mathbb{R}^N \setminus \{0\}))$, $M_{s\xi} \in C^0(\mathbb{R} \times \mathbb{R}^N)$ and $M_{ss} \in C^0(\mathbb{R} \times \mathbb{R}^N)$. Moreover we assume the same growth conditions for L, M, G and their derivatives, replacing only the growth assumptions for $L_{\xi\xi}, M_{\xi\xi}, G''$ by the following hypotheses:

$$\sup_{h\neq 0,\, \xi\in\mathbb{R}^N} \mathfrak{L}(\xi, h) < \infty, \tag{1.12}$$

$$\sup_{t\neq 0,\, s\in\mathbb{R}} \mathfrak{G}(s, t) < \infty, \tag{1.13}$$

$$\sup_{h\neq 0,\, (s,\xi)\in\mathbb{R}\times\mathbb{R}^N} \mathfrak{M}(s, \xi, h) < \infty. \tag{1.14}$$

Conditions (1.12)–(1.13), in some more concrete situations, follow immediately by homogeneity of L_ξ and G' (see, for instance, [16, Lemma 3.1]). Similarly, (1.14) is satisfied for instance when M is of the form $M(s, \xi) = a(s)\mu(\xi)$, being $a : \mathbb{R} \to \mathbb{R}^+$ a bounded function and $\mu : \mathbb{R}^N \to \mathbb{R}^+$ a C^1 strictly convex function such that μ_ξ is homogeneous of degree $m - 1$.

Theorem 1.2. *Under the additional assumptions* (1.12)–(1.14) *in the sub-quadratic cases, the assertion of Theorem 1.1 holds true.*

As a consequence of the above results we have the following compactness criterion.

Corollary 1.3. *Assume* (2.1) *below for some* $\delta > 0$ *and* $\mu > p$. *Under the hypotheses of Theorem 1.1 or 1.2, if* $c < c^*$, *then* (u_n) *is relatively compact in* $W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ *where*

$$c^* := \min\left\{\frac{\delta}{\mu}, \frac{\mu - p}{\mu p} V_\infty\right\} \left[\frac{\min\{\nu, V_\infty\}}{C_g S_{p,\sigma}}\right]^{\frac{p}{\sigma - p}},$$

and $S_{p,\sigma}$ *and* C_g *are constants such that* $S_{p,\sigma}\|u\|_p^\sigma \geq \|u\|_{L^\sigma(\mathbb{R}^N)}^\sigma$ *and* $|g(s)| \leq C_g|s|^{\sigma-1}$.

Remark 1.4. It would be interesting to get a global compactness result in the case $L = 0$ and $p = m$, namely for the model case

$$-\operatorname{div}(a(u)|Du|^{m-2}Du) + \frac{1}{m}a'(u)|Du|^m + V(x)|u|^{m-2}u = |u|^{\sigma-2}u \quad \text{in } \Omega. \tag{1.15}$$

Notice that, even assuming a' bounded, $a'(u)|Du|^m$ is merely in $L^1(\Omega)$ for $W_0^{1,m}(\Omega)$ distributional solutions. For this situation, we refer the reader to [1,18] and to the papers by Filippucci [12,13] and Filippucci et al. [14] on existence and nonexistence of large entire solutions in very general frameworks. In this setting, we point out that the splitting properties are hard to formulate in a reasonable fashion.

Remark 1.5. The restriction between m and p in assumption (1.5) is no longer needed in the case where M is independent of the first variable s, namely $M_s \equiv 0$.

Remark 1.6. We prove the above theorems under the a-priori boundedness assumption of (u_n). This occurs in a quite large class of problems, as Proposition 2.2 shows.

Remark 1.7. With no additional effort, we could cover the case where an additional term $W(x)|u|^{m-2}u$ appears in (1.1) and the functional framework turns into $W_0^{1,p}(\Omega) \cap W_0^{1,m}(\Omega)$.

In the spirit of [15], we also get the following

Corollary 1.8. *Let* $N > p \geq m > 1$ *and assume that* $\xi \mapsto L(\xi)$ *is* p*-homogeneous,* $\xi \mapsto M(\xi)$ *is* m*-homogeneous,* $L(\xi) \geq p|\xi|^p$, $M(\xi) \geq m|\xi|^m$ *(we put* $\nu = 1$*) and set*

$$\mathbb{S}_\Omega := \inf_{\|u\|_{L^\sigma(\Omega)}=1} \int_\Omega \frac{L(Du)}{p} + \frac{M(Du)}{m} + \frac{V(x)}{p}|u|^p, \tag{1.16}$$

$$\mathbb{S}_{\mathbb{R}^N} := \inf_{\|u\|_{L^\sigma(\mathbb{R}^N)}=1} \int_{\mathbb{R}^N} \frac{|Du|^p}{p} + \frac{|u|^p}{p},$$

with $V(x) \to 1$ *as* $|x| \to \infty$. *Assume furthermore that*

$$\mathbb{S}_\Omega < \left(\frac{\sigma - p}{\sigma - m}\right)^{\frac{\sigma-p}{\sigma}} \mathbb{S}_{\mathbb{R}^N}. \tag{1.17}$$

Then (1.16) *admits a minimizer.*

Remark 1.9. We point out that, some conditions guaranteeing the nonexistence of nontrivial solutions in the star-shaped case $\Omega = \mathbb{R}^N$ can be provided. For the sake of simplicity, assume that L is p-homogeneous and that $\xi \mapsto M(s, \xi)$ is m-homogeneous. Then, in view of [17, Theorem 3], that holds for C^1 solutions by virtue of the results of [9], we have that (1.1) admits no nontrivial C^1 solution well behaved at infinity, namely satisfying condition (19) of [17], provided that there exists a number $a \in \mathbb{R}^+$ such that a.e. in $\mathbb{R}^N$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$

$$\begin{aligned}(N - p(a+1))L(\xi) + (N - m(a+1))M(s,\xi) + (asg(s) - NG(s)) \\ + \frac{(N - ap)V(x) + x \cdot DV(x)}{p}|s|^p - aM_s(s,\xi)s \geq 0,\end{aligned}$$

holding, for instance, if there exists $0 \leq a \leq \frac{N-p}{p}$ such that

$$asg(s) - NG(s) \geq 0, \quad (N - ap)V(x) + x \cdot DV(x) \geq 0, \quad M_s(s,\xi)s \leq 0,$$

for a.e. $x \in \mathbb{R}^N$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$. Also, in the more particular case where $g(s) = |s|^{\sigma-2}s$ and $V(x) = V_\infty > 0$, then the above conditions simply rephrase into

$$\sigma \geq p^*, \quad M_s(s,\xi)s \leq 0,$$

for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$. In fact, in (1.9), we consider the opposite assumption on M_s.

2. Some preliminary facts

It is a standard fact that, under condition (1.6) and (1.10), the functionals

$$u \mapsto \int_\Omega L(Du), \quad u \mapsto \int_\Omega V(x)|u|^p, \quad u \mapsto \int_\Omega G(u)$$

are C^1 on $W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$. Analogously, although M depends explicitly on s, the functional

$$\mathbb{M} : W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega) \to \mathbb{R}, \quad \mathbb{M}(u) = \int_\Omega M(u, Du),$$

admits, thanks to condition (1.5), directional derivatives along any $v \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ and

$$\mathbb{M}'(u)(v) = \int_\Omega M_\xi(u, Du) \cdot Dv + \int_\Omega M_s(u, Du)v,$$

as it can be easily verified observing that $p \le \frac{p}{p-m} \le p^*$ and that, for $u \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$, by Young's inequality, for some constant C it holds

$$|M_\xi(u, Du) \cdot Dv| \le C|Du|^m + C|Dv|^m \in L^1(\Omega),$$
$$|M_s(u, Du)v| \le C|Du|^p + C|v|^{\frac{p}{p-m}} \in L^1(\Omega).$$

Furthermore, if $u_k \to u$ in $W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ as $k \to \infty$ then $\mathbb{M}'(u_k) \to \mathbb{M}'(u)$ in the dual space $(W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*$, as $k \to \infty$. Indeed, for $\|v\|_{W_0^{1,p}(\Omega)\cap D_0^{1,m}(\Omega)} \le 1$, we have

$$\begin{aligned}
&|\mathbb{M}'(u_k)(v) - \mathbb{M}'(u)(v)| \\
&\quad\le \int_\Omega |M_\xi(u_k, Du_k) - M_\xi(u, Du)||Dv| + \int_\Omega |M_s(u_k, Du_k) - M_s(u, Du)|\,|v| \\
&\quad\le \|M_\xi(u_k, Du_k) - M_\xi(u, Du)\|_{L^{m'}} \|Dv\|_{L^m} \\
&\qquad+\|M_s(u_k, Du_k) - M_s(u, Du)\|_{L^{p/m}} \|v\|_{L^{p/(p-m)}} \\
&\quad\le \|M_\xi(u_k, Du_k) - M_\xi(u, Du)\|_{L^{m'}} + \|M_s(u_k, Du_k) - M_s(u, Du)\|_{L^{p/m}}.
\end{aligned}$$

This yields the desired convergence, using (1.7) and the Dominated Convergence Theorem. Notice that the same argument carried out before applies either to integrals defined on Ω or on $\mathbb{R}^N$. Hence the following proposition is proved.

Proposition 2.1. *In the hypotheses of Theorems 1.1 and 1.2, the functionals ϕ and ϕ_∞ are C^1.*

In addition to the assumptions on L, M and g, G set in the introduction, assume now that there exist positive numbers $\delta > 0$ and $\mu > p$ such that

$$\begin{aligned}
&\mu M(s, \xi) - M_s(s, \xi)s - M_\xi(s, \xi) \cdot \xi \ge \delta|\xi|^m, \\
&\mu L(\xi) - L_\xi(\xi) \cdot \xi \ge \delta|\xi|^p, \quad sg(s) - \mu G(s) \ge 0,
\end{aligned} \tag{2.1}$$

for any $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$. This hypothesis is rather well established in the framework of quasi-linear problems (cf. [18]) and it allows an arbitrary Palais-Smale sequence (u_n) to be bounded in $W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$, as shown in the following

Proposition 2.2. *Let j be as in (1.11) and assume that (1.5) holds. Let $(u_n) \subset W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ be a sequence such that*

$$\phi(u_n) \to c \;\; \phi'(u_n) \to 0 \;\; in \; (W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*$$

Then, if condition (2.1) holds, (u_n) is bounded in $W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$.

Proof. Let $(w_n) \subset (W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*$ with $w_n \to 0$ as $n \to \infty$ be such that $\phi'(u_n)(v) = \langle w_n, v\rangle$, for every $v \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$. Whence, by choosing $v = u_n$, it follows

$$\int_\Omega j_\xi(u_n, Du_n) \cdot Du_n + \int_\Omega j_s(u_n, Du_n)u_n + \int_\Omega V(x)|u_n|^p = \langle w_n, u_n\rangle.$$

Combining this equation with $\mu\phi(u_n) = \mu c + o(1)$ as $n \to \infty$, namely

$$\int_\Omega \mu j(u_n, Du_n) + \frac{\mu}{p}\int_\Omega V(x)|u_n|^p = \mu c + o(1),$$

recalling the definition of j, and using condition (2.1), yields

$$\begin{aligned} &\frac{\mu - p}{p}\int_\Omega V(x)|u_n|^p + \delta\int_\Omega |Du_n|^p + \delta\int_\Omega |Du_n|^m \\ &\quad\le \mu c + \|w_n\|\|u_n\|_{W_0^{1,p}(\Omega)\cap D_0^{1,m}(\Omega)} + o(1), \end{aligned}$$

as $n \to \infty$, implying, due to $V \ge V_0$ that

$$\|u_n\|^p_{W^{1,p}(\Omega)} + \|u_n\|^m_{D^{1,m}(\Omega)} \le C + C\|u_n\|_{W^{1,p}(\Omega)} + C\|u_n\|_{D^{1,m}(\Omega)} + o(1),$$

as $n \to \infty$. The assertion then follows immediately. □

From now on we shall always assume to handle *bounded* Palais-Smale sequences, keeping in mind that condition (2.1) can guarantee the boundedness of such sequences.

Proposition 2.3. *Let j be as in (1.11) and assume that $1 < m < p < N$ and $p < \sigma < p^*$. Let $(u_n) \subset W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ bounded be such that*

$$\phi(u_n) \to c \ \ \phi'(u_n) \to 0 \text{ in } (W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*.$$

Then, up to a subsequence, (u_n) converges weakly to some u in $W_0^{1,p}(\Omega)\cap D_0^{1,m}(\Omega)$, $u_n(x) \to u(x)$ and $Du_n(x) \to Du(x)$ for a.e. $x \in \Omega$.

Proof. It is sufficient to justify that $Du_n(x) \to Du(x)$ for a.e. $x \in \Omega$. Given an arbitrary bounded subdomain $\omega \subset \overline{\omega} \subset \Omega$ of Ω, from the fact that $\phi'(u_n) \to 0$ in $(W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*$, we can write

$$\int_\omega a(u_n, Du_n) \cdot Dv = \langle w_n, v\rangle + \langle f_n, v\rangle + \int_\omega v\, d\mu_n, \quad \text{for all } v \in \mathcal{D}(\omega),$$

where $(w_n) \subset (W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*$ is vanishing, and hence in particular $w_n \in W^{-1,p'}(\omega)$, with $w_n \to 0$ in $W^{-1,p'}(\omega)$ as $n \to \infty$ and we have set

$$\begin{aligned} a(s,\xi) &:= L_\xi(\xi) + M_\xi(s,\xi), \quad \text{for all } (s,\xi) \in \mathbb{R}\times\mathbb{R}^N, \\ f_n &:= -V(x)|u_n|^{p-2}u_n + g(u_n) \in W^{-1,p'}(\omega), \quad n \in \mathbb{N}, \\ \mu_n &:= -M_s(u_n, Du_n) \in L^1(\omega), \quad n \in \mathbb{N}. \end{aligned}$$

Due to the strict convexity assumptions on the maps $\xi \mapsto L(\xi)$ and $\xi \mapsto M(s,\xi)$ and the growth conditions on L_ξ, M_ξ, M_s and g, all the assumptions of [7, Theorem 1] are fulfilled. Precisely,

$$|a(s,\xi)| \le |L_\xi(\xi)| + |M_\xi(s,\xi)| \le C|\xi|^{p-1} + C|\xi|^{m-1} \le C + C|\xi|^{p-1},$$

for a.e. $x \in \omega$ and all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, and

$$\begin{aligned} &f_n \to f, \quad f := -V(x)|u|^{p-2}u + g(u), \quad \text{strongly in } W^{-1,p'}(\omega), \\ &\mu_n \rightharpoonup \mu, \quad \text{weakly* in } \mathcal{M}(\omega), \text{ since } \sup_{n\in\mathbb{N}} \|M_s(u_n, Du_n)\|_{L^1(\omega)} < +\infty. \end{aligned}$$

Then, it follows that $Du_n(x) \to Du(x)$ for a.e. $x \in \omega$. Finally, a simple Cantor diagonal argument allows to recover the convergence over the whole domain Ω. □

Next we prove a regularity result for the solutions of equation (1.1).

Proposition 2.4. *Let j be as in (1.11) and assume (1.5) and (1.9). Let $u \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ be a solution of (1.1). Then*

$$u \in \bigcap_{q \ge p} L^q(\Omega), \quad u \in L^\infty(\Omega) \text{ and } \lim_{|x|\to\infty} u(x) = 0.$$

Proof. Let $k, i \in \mathbb{N}$. Then, setting $v_{k,i}(x) := (u_k(x))^i$ with $u_k(x) := \min\{u^+(x), k\}$, it follows that $v_{k,i} \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ can be used as a test function in (1.1), yielding

$$\begin{aligned} &\int_\Omega L_\xi(Du)\cdot Dv_{k,i} + \int_\Omega M_\xi(u,Du)\cdot Dv_{k,i} \\ &\quad + \int_\Omega M_s(u,Du)v_{k,i} + \int_\Omega V(x)|u|^{p-2}uv_{k,i} = \int_\Omega g(u)v_{k,i}. \end{aligned}$$

Taking into account that $Dv_{k,i}$ is equal to $iu^{i-1}Du\chi_{\{0<u<k\}}$, by convexity and positivity of the map $\xi \mapsto M(s,\xi)$ we deduce that $M_\xi(u,Du)\cdot Dv_{k,i} \ge 0$. Moreover, by the sign condition (1.9) it follows $M_s(u,Du)v_{k,i} \ge 0$ a.e. in Ω. Then, we reach

$$\int_\Omega i(u_k)^{i-1}L_\xi(Du_k)\cdot Du_k + \int_\Omega V(x)|u|^{p-2}u(u_k(x))^i \le \int_\Omega g(u)(u_k(x))^i,$$

yielding in turn, by (1.10), that for all $k, i \ge 1$

$$\nu i \int_\Omega (u_k)^{i-1}|Du_k|^p \le C\int_\Omega (u^+(x))^{\sigma-1+i}. \tag{2.2}$$

If $\hat{u}_k := \min\{u^-(x), k\}$, a similar inequality

$$\nu i \int_\Omega (\hat{u}_k)^{i-1}|D\hat{u}_k|^p \le C\int_\Omega (u^-(x))^{\sigma-1+i}, \tag{2.3}$$

can be obtained by using $\hat{v}_{k,i} := -(\hat{u}_k)^i$ as a test function in (1.1), observing that by (1.9),

$$
\begin{aligned}
M_s(u, Du)\hat{v}_{k,i} &= -M_s(u, Du)\chi_{\{-k<u<0\}}(-u)^i \geq 0, \\
M_\xi(u, Du) \cdot Dv_{k,i} &= i(-u)^{i-1}\chi_{\{-k<u<0\}}M_\xi(u, Du) \cdot Du \geq 0.
\end{aligned}
$$

Once (2.2)–(2.3) are reached, the assertion follows exactly as in [20, Lemma 2, (a) and (b)]. □

We now recall the following version of [8, Lemma 4.2] which turns out to be a rather useful tool in order to establish convergences in our setting. Roughly speaking, one needs some kind of sub-criticality in the growth conditions.

Lemma 2.5. *Let $\Omega \subset \mathbb{R}^N$ and $h : \Omega \times \mathbb{R} \times \mathbb{R}^N$ be a Carathéodory function, $p, m > 1$, $\mu \geq 1$, $p \leq \sigma \leq p^*$ and assume that, for every $\varepsilon > 0$ there exists $a_\varepsilon \in L^\mu(\Omega)$ such that*

$$
|h(x, s, \xi)| \leq a_\varepsilon(x) + \varepsilon|s|^{\sigma/\mu} + \varepsilon|\xi|^{p/\mu} + \varepsilon|\xi|^{m/\mu}, \tag{2.4}
$$

a.e. in Ω and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$. Assume that $u_n \to u$ a.e. in Ω, $Du_n \to Du$ a.e. in Ω and

$$
(u_n) \text{ is bounded in } W_0^{1,p}(\Omega), \quad (u_n) \text{ is bounded in } D_0^{1,m}(\Omega).
$$

Then $h(x, u_n, Du_n)$ converges to $h(x, u, Du)$ in $L^\mu(\Omega)$.

Proof. The proof follows as in [8, Lemma 4.2] and we shall sketch it here for self-containedness. By Fatou's Lemma, it immediately holds that $u \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$. Furthermore, there exists a positive constant C such that

$$
\begin{aligned}
|h(x, s_1, \xi_1) - h(x, s_2, \xi_2)|^\mu &\leq C(a_\varepsilon(x))^\mu + C\varepsilon^\mu|s_1|^\sigma + C\varepsilon^\mu|s_2|^\sigma \\
&\quad + C\varepsilon^\mu|\xi_1|^m + C\varepsilon^\mu|\xi_2|^m + C\varepsilon^\mu|\xi_1|^p + C\varepsilon^\mu|\xi_2|^p,
\end{aligned}
$$

a.e. in Ω and for all $(s_1, \xi_1) \in \mathbb{R} \times \mathbb{R}^N$ and $(s_2, \xi_2) \in \mathbb{R} \times \mathbb{R}^N$. Then, taking into account the boundedness of (Du_n) in $L^p(\Omega) \cap L^m(\Omega)$ and of (u_n) in $L^\sigma(\Omega)$ by interpolation being $p \leq \sigma \leq p^*$, the assertion follows by applying Fatou's Lemma to the sequence of functions $\psi_n : \Omega \to [0, +\infty]$

$$
\begin{aligned}
\psi_n(x) := &- |h(x, u_n, Du_n) - h(x, u, Du)|^\mu + C(a_\varepsilon(x))^\mu + C\varepsilon^\mu|u_n|^\sigma \\
&+ C\varepsilon^\mu|u|^\sigma + C\varepsilon^\mu|Du_n|^m + C\varepsilon^\mu|Du|^m + C\varepsilon^\mu|Du_n|^p + C\varepsilon^\mu|Du|^p,
\end{aligned}
$$

and, finally, exploiting the arbitrariness of ε. □

3. Proof of the result

3.1. Energy splitting

The next result allows to perform an energy splitting for the functional

$$J(u) = \int_\Omega j(u, Du), \quad u \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega),$$

along a bounded Palais-Smale sequence $(u_n) \subset W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$. The result is in the spirit of the classical Brezis-Lieb Lemma [5].

Lemma 3.1. *Let the integrand j be as in (1.11) and*

$$p - 1 \le m < p - 1 + p/N, \quad p \le \sigma \le p^*.$$

Let $(u_n) \subset W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ with $u_n \rightharpoonup u$, $u_n \to u$ a.e. in Ω and $Du_n \to Du$ a.e. in Ω. Then

$$\lim_{n\to\infty} \int_\Omega j(u_n - u, Du_n - Du) - j(u_n, Du_n) + j(u, Du) = 0. \tag{3.1}$$

Proof. We shall apply Lemma 2.5 to the function

$$h(x, s, \xi) := j(s - u(x), \xi - Du(x)) - j(s, \xi),$$
$$\text{for a.e. } x \in \Omega \text{ and all } (s, \xi) \in \mathbb{R} \times \mathbb{R}^N.$$

Given $x \in \Omega$, $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$, consider the C^1 map $\varphi : [0, 1] \to \mathbb{R}$ defined by setting

$$\varphi(t) := j(s - tu(x), \xi - tDu(x)), \quad \text{for all } t \in [0, 1].$$

Then, for some $\tau \in [0, 1]$ depending upon $x \in \Omega$, $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$, it holds

$$\begin{aligned}
&h(x, s, \xi) = \varphi(1) - \varphi(0) = \varphi'(\tau) \\
&= -j_s(s - \tau u(x), \xi - \tau Du(x))u(x) - j_\xi(s - \tau u(x), \xi - \tau Du(x)) \cdot Du(x) \\
&= -L_\xi(\xi - \tau Du(x)) \cdot Du(x) \\
&\quad -M_s(s - \tau u(x), \xi - \tau Du(x))u(x) \\
&\quad -M_\xi(s - \tau u(x), \xi - \tau Du(x)) \cdot Du(x) + G'(s - \tau u(x))u(x).
\end{aligned}$$

Hence, for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, it follows that

$$\begin{aligned}
|h(x, s, \xi)| &\le |L_\xi(\xi - \tau Du(x))||Du(x)| + |M_s(s - \tau u(x), \xi - \tau Du(x))||u(x)| \\
&\quad +|M_\xi(s - \tau u(x), \xi - \tau Du(x))||Du(x)| + |G'(s - \tau u(x))||u(x)| \\
&\le C(|\xi|^{p-1} + |Du(x)|^{p-1})|Du(x)| + C(|\xi|^m + |Du(x)|^m)|u(x)| \\
&\quad +C(|\xi|^{m-1} + |Du(x)|^{m-1})|Du(x)| + C(|s|^{\sigma-1} + |u(x)|^{\sigma-1})|u(x)| \\
&\le \varepsilon|\xi|^p + C_\varepsilon|Du(x)|^p + \varepsilon|\xi|^p + C_\varepsilon|Du(x)|^p + C_\varepsilon|u(x)|^{p/(p-m)} \\
&\quad +\varepsilon|\xi|^m + C_\varepsilon|Du(x)|^m + \varepsilon|s|^\sigma + C_\varepsilon|u(x)|^\sigma \\
&= a_\varepsilon(x) + \varepsilon|s|^\sigma + \varepsilon|\xi|^p + \varepsilon|\xi|^m,
\end{aligned}$$

where $a_\varepsilon : \Omega \to \mathbb{R}$ is defined a.e. by

$$a_\varepsilon(x) := C_\varepsilon |Du(x)|^p + C_\varepsilon |Du(x)|^m + C_\varepsilon |u(x)|^{p/(p-m)} + C_\varepsilon |u(x)|^\sigma.$$

Notice that, as $p-1 \le m < p-1+p/N$ it holds $p \le p/(p-m) \le p^*$, yielding $u \in L^{p/(p-m)}(\Omega)$ and in turn, $a_\varepsilon \in L^1(\Omega)$. The assertion follows directly by Lemma 2.5 with $\mu = 1$. □

We have the following splitting result

Theorem 3.2. *Let the integrand j be as in (1.11) and*

$$p-1 \le m \le p-1+p/N, \quad p < \sigma < p^*.$$

Assume that $(u_n) \subset W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ is a bounded Palais-Smale sequence for ϕ at the level $c \in \mathbb{R}$ weakly convergent to some $u \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$. Then

$$\lim_{n\to\infty} \left(\int_\Omega j(u_n - u, Du_n - Du) + \int_\Omega V_\infty \frac{|u_n - u|^p}{p} \right) = c - \int_\Omega j(u, Du) - \int_\Omega V(x) \frac{|u|^p}{p},$$

namely

$$\lim_{n\to\infty} \phi_\infty(u_n - u) = c - \phi(u),$$

being u_n and u regarded as elements of $W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)$ after extension to zero out of Ω.

Proof. In light of Proposition 2.3, up to a subsequence, (u_n) converges weakly to some function u in $W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$, $u_n(x) \to u(x)$ and $Du_n(x) \to Du(x)$ for a.e. $x \in \Omega$. Also, recalling that by assumption $V(x) \to V_\infty$ as $|x| \to \infty$, we have [5,21]

$$\lim_{n\to\infty} \int_\Omega V(x)|u_n - u|^p - V_\infty |u_n - u|^p = 0, \tag{3.2}$$

$$\lim_{n\to\infty} \int_\Omega V(x)|u_n - u|^p - V(x)|u_n|^p + V(x)|u|^p = 0. \tag{3.3}$$

Therefore, by virtue of Lemma 3.1, we conclude that

$$\begin{aligned}\lim_{n\to\infty}\phi_\infty(u_n-u) &= \lim_{n\to\infty}\left(\int_\Omega j(u_n-u, Du_n-Du)+\int_\Omega V_\infty\frac{|u_n-u|^p}{p}\right)\\ &= \lim_{n\to\infty}\left(\int_\Omega j(u_n-u, Du_n-Du)+\int_\Omega V(x)\frac{|u_n-u|^p}{p}\right)\\ &= \lim_{n\to\infty}\left(\int_\Omega j(u_n, Du_n)+\int_\Omega V(x)\frac{|u_n|^p}{p}\right)\\ &\quad -\int_\Omega j(u, Du)-\int_\Omega V(x)\frac{|u|^p}{p}\\ &= \lim_{n\to\infty}\phi(u_n)-\phi(u)=c-\phi(u),\end{aligned}$$

concluding the proof. □

Remark 3.3. In order to shed some light on the restriction (1.5) of m, it is readily seen that it is a sufficient condition for the following local compactness property to hold. Assume that ω is a smooth domain of $\mathbb{R}^n$ with finite measure. Then, if (u_h) is a bounded sequence in $W_0^{1,p}(\omega)$, there exists a subsequence (u_{h_k}) such that

$$\Upsilon(x, u_{h_k}, Du_{h_k}) \text{ converges strongly to some } \Upsilon_0 \text{ in } W^{-1,p'}(\omega),$$

where $\Upsilon(x,s,\xi)=g(s)-M_s(s,\xi)-V(x)|s|^{p-2}s$. In fact, taking into account the growth condition on g and M_s, this can be proved observing that, for every $\varepsilon>0$, there exists C_ε such that

$$|\Upsilon(x,s,\xi)|\le C_\varepsilon+\varepsilon|s|^{\frac{N(p-1)+p}{N-p}}+\varepsilon|\xi|^{p-1+p/N},$$

for a.e. $x\in\omega$ and all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$.

3.2. *Equation splitting I (super-quadratic case)*

We shall assume that $m, p\ge 2$ and that conditions (1.7)–(1.8) hold. The following Theorem 3.4 and the forthcoming Theorem 3.5 (see next subsection) are in the spirit of the Brezis-Lieb Lemma [5], in a dual framework. For the particular case

$$M(s,\xi)=0 \quad\text{and}\quad L(\xi)=\frac{|\xi|^p}{p},$$

we refer the reader to [16].

Theorem 3.4. *Assume that* (1.5)–(1.11) *hold and that*

$$p - 1 \leq m < p - 1 + p/N, \quad p < \sigma < p^*.$$

Assume that $(u_n) \subset W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ *is such that* $u_n \rightharpoonup u$, $u_n \to u$ *a.e. in* Ω, $Du_n \to Du$ *a.e. in* Ω *and there is* (w_n) *in the dual space* $(W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*$ *such that* $w_n \to 0$ *as* $n \to \infty$ *and, for all* $v \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$,

$$\int_\Omega j_\xi(u_n, Du_n) \cdot Dv + \int_\Omega j_s(u_n, Du_n)v + \int_\Omega V(x)|u_n|^{p-2}u_n v = \langle w_n, v\rangle \tag{3.4}$$

Then $\phi'(u) = 0$. *Moreover, there exists a sequence* (ξ_n) *that goes to zero in* $(W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*$, *such that*

$$\begin{aligned}\langle \xi_n, v\rangle := &\int_\Omega j_s(u_n - u, Du_n - Du)v + \int_\Omega j_\xi(u_n - u, Du_n - Du) \cdot Dv \\ &- \int_\Omega j_s(u_n, Du_n)v - \int_\Omega j_\xi(u_n, Du_n) \cdot Dv \\ &+ \int_\Omega j_s(u, Du)v + \int_\Omega j_\xi(u, Du) \cdot Dv,\end{aligned} \tag{3.5}$$

for all $v \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$.
Furthermore, there exists a sequence (ζ_n) *in* $(W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*$ *such that*

$$\begin{aligned}&\int_\Omega j_\xi(u_n - u, Du_n - Du) \cdot Dv + \int_\Omega j_s(u_n - u, Du_n - Du)v \\ &\quad + \int_\Omega V_\infty|u_n - u|^{p-2}(u_n - u)v = \langle \zeta_n, v\rangle\end{aligned}$$

for all $v \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ *and* $\zeta_n \to 0$ *as* $n \to \infty$, *namely* $\phi'_\infty(u_n - u) \to 0$ *as* $n \to \infty$.

Proof. Fixed some $v \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$, let us define for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$,

$$\begin{aligned}f_v(x, s, \xi) := &\; j_s(s - u(x), \xi - Du(x))v(x) \\ &+ j_\xi(s - u(x), \xi - Du(x)) \cdot Dv(x) - j_s(s, \xi)v(x) \\ &- j_\xi(s, \xi) \cdot Dv(x).\end{aligned}$$

In order to prove 3.5 we are going to show that

$$\lim_{n\to\infty} \sup_{\|v\|_{W_0^{1,p}(\Omega)\cap D_0^{1,m}(\Omega)} \leq 1} \left| \int_\Omega f_v(x, u_n, Du_n) - f_v(x, u, Du)\right| = 0. \tag{3.6}$$

As it can be easily checked, there holds

$$
\begin{aligned}
-f_v(x,s,\xi) \; &= \int_0^1 j_{ss}(s-\tau u(x),\xi-\tau Du(x))u(x)v(x)d\tau \\
&\quad + \int_0^1 j_{s\xi}(s-\tau u(x),\xi-\tau Du(x))\cdot[Du(x)v(x)+Dv(x)u(x)]d\tau \\
&\quad + \int_0^1 [j_{\xi\xi}(s-\tau u(x),\xi-\tau Du(x))\,Du(x)]\cdot Dv(x)d\tau.
\end{aligned}
$$

Hence, by plugging the particular form of j in the above equation yields

$$
\begin{aligned}
-f_v(x,s,\xi) = a(x,s,\xi)v(x) + b(x,s)v(x) + c_1(x,s,\xi)\cdot Dv(x) \\
+ c_2(x,s,\xi)\cdot Dv(x) + d(x,\xi)\cdot Dv(x)
\end{aligned}
$$

where

$$
\begin{aligned}
a(x,s,\xi) &:= \int_0^1 [M_{ss}(s-\tau u(x),\xi-\tau Du(x))u(x) \\
&\qquad + M_{s\xi}(s-\tau u(x),\xi-\tau Du(x))\cdot Du(x)]d\tau, \\
b(x,s) &:= -\int_0^1 G''(s-\tau u(x))u(x)d\tau, \\
c_1(x,s,\xi) &:= \int_0^1 M_{\xi s}(s-\tau u(x),\xi-\tau Du(x))u(x)d\tau, \\
c_2(x,s,\xi) &:= \int_0^1 M_{\xi\xi}(s-\tau u(x),\xi-\tau Du(x))\,Du(x)d\tau, \\
d(x,\xi) &:= \int_0^1 L_{\xi\xi}(\xi-\tau Du(x))\,Du(x)d\tau.
\end{aligned}
$$

We claim that, as $n\to\infty$, it holds

$$
\begin{aligned}
a(\cdot,u_n,Du_n) \to a(\cdot,u,Du) &\quad \text{in } L^{(p^*)'}(\Omega), \\
b(\cdot,u_n) \to b(\cdot,u) &\quad \text{in } L^{\sigma'}(\Omega), \\
c_1(\cdot,u_n,Du_n) \to c_1(\cdot,u,Du) &\quad \text{in } L^{p'}(\Omega), \\
c_2(\cdot,u_n,Du_n) \to c_2(\cdot,u,Du) &\quad \text{in } L^{m'}(\Omega), \\
d(\cdot,Du_n) \to d(\cdot,Du) &\quad \text{in } L^{p'}(\Omega).
\end{aligned}
$$

Then, using Hölder's inequality and the embeddings of $W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ into $L^\sigma(\Omega)$ and $L^{p^*}(\Omega)$ we obtain

$$
\begin{aligned}
&\sup_{\|v\|_{W_0^{1,p}(\Omega)\cap D_0^{1,m}(\Omega)} \leq 1} \left| \int_\Omega f_v(x, u_n, Du_n) - f_v(x, u, Du) \right| \\
&\quad \leq C\|a(\cdot, u_n, Du_n) - a(\cdot, u, Du)\|_{L^{(p^*)'}(\Omega)} \\
&\quad\quad + C\|b(\cdot, u_n) - b(\cdot, u)\|_{L^{\sigma'}(\Omega)}, \\
&\quad\quad + C\|c_1(\cdot, u_n, Du_n) - c_1(\cdot, u, Du)\|_{L^{p'}(\Omega)}, \\
&\quad\quad + C\|c_2(\cdot, u_n, Du_n) - c_2(\cdot, u, Du)\|_{L^{m'}(\Omega)}, \\
&\quad\quad + C\|d(\cdot, Du_n) - d(\cdot, Du)\|_{L^{p'}(\Omega)},
\end{aligned}
$$

yielding the desired conclusion (3.6). It remains to prove the convergences we claimed above. For each term, we shall exploit Lemma 2.5. Since $m < p-1+p/N$, we can set

$$
\alpha := \frac{m}{p^* - 1}, \quad \beta := \frac{pN}{pN - N + p - mN}
$$

it follows $\beta > 0$ and $m < m + \alpha < p$. Young's inequality yields in turn

$$
y^{(m+\alpha)/(p^*)'} \leq C y^{m/(p^*)'} + C y^{p/(p^*)'}, \quad \text{for all } y \geq 0.
$$

Since $\beta/(p^*)' > 1$ and $(m+\alpha)/(p^*)' > 1$, by the growths of M_{ss} and $M_{s\xi}$, we have

$$
\begin{aligned}
|a(x, s, \xi)| &\leq C(|\xi|^m + |Du(x)|^m)|u(x)| + C(|\xi|^{m-1} + |Du(x)|^{m-1})|Du(x)| \\
&\leq \varepsilon|\xi|^{p/(p^*)'} + C_\varepsilon|u(x)|^{\beta/(p^*)'} + C_\varepsilon|Du(x)|^{p/(p^*)'} + \varepsilon|\xi|^{(m+\alpha)/(p^*)'} \\
&\quad + C_\varepsilon|Du(x)|^{(m+\alpha)/(p^*)'} \\
&\leq \varepsilon|\xi|^{p/(p^*)'} + \varepsilon|\xi|^{m/(p^*)'} + C_\varepsilon|u(x)|^{\beta/(p^*)'} \\
&\quad + C_\varepsilon|Du(x)|^{p/(p^*)'} + C_\varepsilon|Du(x)|^{m/(p^*)'}.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
|b(x, s)| &\leq C(|s|^{\sigma-2} + |u(x)|^{\sigma-2})|u(x)| \leq \varepsilon|s|^{\sigma/\sigma'} + C_\varepsilon|u|^{\sigma/\sigma'}, \\
|c_1(x, s, \xi)| &\leq C(|\xi|^{m-1} + |Du(x)|^{m-1})|u(x)| \\
&\leq \varepsilon|\xi|^{p/p'} + C_\varepsilon|u(x)|^{p/((p-m)p')} + C_\varepsilon|Du(x)|^{p/p'}, \\
|c_2(x, s, \xi)| &\leq C(|\xi|^{m-2} + |Du(x)|^{m-2})|Du(x)| \\
&\leq \varepsilon|\xi|^{m/m'} + C_\varepsilon|Du(x)|^{m/m'}, \\
|d(x, \xi)| &\leq C(|\xi|^{p-2} + |Du(x)|^{p-2})|Du(x)| \leq \varepsilon|\xi|^{p/p'} + C_\varepsilon|Du(x)|^{p/p'}.
\end{aligned}
$$

From the point-wise convergence of the gradients and the growth estimates of j_ξ, j_s and g that u is a week solutions to the problem, namely for all $v \in W_0^{1,p}(\Omega) \cap$

$D_0^{1,m}(\Omega)$

$$\int_\Omega L_\xi(Du)\cdot Dv + \int_\Omega M_\xi(u,Du)\cdot Dv + \int_\Omega M_s(u,Du)v + \int_\Omega V(x)|u|^{p-2}uv = \int_\Omega g(u)v. \tag{3.7}$$

To get this, recall that $v \in L^{(p/m)'}(\Omega)$ and the sequence $(M_s(u_n, Du_n))$ is bounded in $L^{p/m}(\Omega)$ and hence it converges weakly to $M_s(u, Du)$ in $L^{p/m}(\Omega)$. Thanks to Proposition 2.4 (recall that $\beta \geq p$ if and only if $m \geq p-2+p/N$ and this is the case since $m \geq p-1$), we have $L^\beta(\Omega)$. Hence,

$$u \in L^\sigma(\Omega) \cap L^{\frac{p}{p-m}}(\Omega) \cap L^\beta(\Omega),$$

being $p \leq p/(p-m) < p^*$ and $p < \sigma < p^*$. By the previous inequalities the claim follows by Lemma 2.5 with the choice $\mu = (p^*)', \sigma', p', m'$ and p' respectively. Let us now recall a dual version of properties (3.2)–(3.3) (cf. [21]), namely there exist two sequences (μ_n) and (ν_n) in $(W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*$ which converge to zero as $n \to \infty$ and such that

$$\int_\Omega V_\infty |u_n - u|^{p-2}(u_n-u)v = \int_\Omega V(x)|u_n-u|^{p-2}(u_n-u)v + \langle \nu_n, v\rangle,$$
$$\int_\Omega V(x)|u_n-u|^{p-2}(u_n-u)v = \int_\Omega V(x)|u_n|^{p-2}u_n v - \int_\Omega V(x)|u|^{p-2}uv + \langle \mu_n, v\rangle,$$

for every $v \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$. Whence, by collecting (3.4), (3.5), (3.6), (3.7), we get

$$\begin{aligned}
&\int_\Omega j_\xi(u_n-u, Du_n-Du)\cdot Dv + \int_\Omega j_s(u_n-u, Du_n-Du)v \\
&\quad + \int_\Omega V_\infty |u_n-u|^{p-2}(u_n-u)v \\
&= \int_\Omega j_\xi(u_n, Du_n)\cdot Dv + \int_\Omega j_s(u_n, Du_n)v + \int_\Omega V(x)|u_n|^{p-2}u_n v \\
&\quad - \int_\Omega j_\xi(u, Du)\cdot Dv - \int_\Omega j_s(u, Du)v - \int_\Omega V(x)|u|^{p-2}uv \\
&\quad + \langle \xi_n + \mu_n + \nu_n, v\rangle \\
&= \langle \zeta_n, v\rangle,
\end{aligned}$$

where $\langle \zeta_n, v\rangle := \langle w_n + \xi_n + \mu_n + \nu_n, v\rangle$ and $\zeta_n \to 0$ as $n \to \infty$. This concludes the proof. □

3.3. Equation splitting II (sub-quadratic case)

We assume that (1.12)–(1.14) hold.

Theorem 3.5. *Assume* (1.9)*, let the integrand j be as in* (1.11) *and $p \le 2$ or $m \le 2$ or $\sigma \le 2$,*

$$p-1 \le m < p-1+p/N, \quad p < \sigma < p^*.$$

Assume that $(u_n) \subset W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ is such that $u_n \rightharpoonup u$, $u_n \to u$ a.e. in Ω, $Du_n \to Du$ a.e. in Ω and there exists (w_n) in $(W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*$ such that $w_n \to 0$ as $n \to \infty$ and, for every $v \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$,

$$\int_\Omega j_\xi(u_n, Du_n) \cdot Dv + \int_\Omega j_s(u_n, Du_n)v + \int_\Omega V(x)|u_n|^{p-2}u_n v = \langle w_n, v\rangle.$$

Then $\phi'(u) = 0$. Moreover, there exists a sequence $(\hat{\xi}_n)$ that goes to zero in $(W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*$, such that

$$\begin{aligned}\langle \hat{\xi}_n, v\rangle := &\int_\Omega j_s(u_n-u, Du_n-Du)v + \int_\Omega j_\xi(u_n-u, Du_n-Du)\cdot Dv \\ &-\int_\Omega j_s(u_n, Du_n)v - \int_\Omega j_\xi(u_n, Du_n)\cdot Dv + \int_\Omega j_s(u, Du)v \\ &+\int_\Omega j_\xi(u, Du)\cdot Dv,\end{aligned} \tag{3.8}$$

for all $v \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$.
Furthermore, there exists a sequence $(\hat{\zeta}_n)$ in $W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ with

$$\begin{aligned}&\int_\Omega j_\xi(u_n-u, Du_n-Du)\cdot Dv + \int_\Omega j_s(u_n-u, Du_n-Du)v \\ &+\int_\Omega V_\infty |u_n-u|^{p-2}(u_n-u)v = \langle \hat{\zeta}_n, v\rangle\end{aligned}$$

for all $v \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ and $\hat{\zeta}_n \to 0$ as $n \to \infty$, namely $\phi'_\infty(u_n-u) \to 0$ as $n \to \infty$.

Proof. Keeping in mind the argument in proof of Theorem 3.4, here we shall be more sketchy. For every $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$ we plug L, M, G into the equation

$$\begin{aligned}f_v(x, s, \xi) = &\, j_s(s-u(x), \xi - Du(x))v(x) \\ &+j_\xi(s-u(x), \xi-Du(x))\cdot Dv(x) - j_s(s,\xi)v(x) \\ &-j_\xi(s,\xi)\cdot Dv(x),\end{aligned}$$

thus obtaining

$$\begin{aligned} f_v(x,s,\xi) = &(M_s(s-u(x),\xi-Du(x))-M_s(s,\xi))v(x) \\ &-(G'(s-u(x))-G'(s))v(x) \\ &+(M_\xi(s-u(x),\xi-Du(x))-M_\xi(s,\xi))\cdot Dv(x) \\ &+(L_\xi(\xi-Du(x))-L_\xi(\xi))\cdot Dv(x) \\ = &a'v(x)+b'v(x)+c'\cdot Dv(x)+d'\cdot Dv(x). \end{aligned}$$

We write the term $M_\xi(s-u(x),\xi-Du(x))-M_\xi(s,\xi)$ in a more suitable form, namely

$$\begin{aligned} c' &= M_\xi(s-u(x),\xi-Du(x))-M_\xi(s,\xi) \\ &= \underbrace{M_\xi(s-u(x),\xi-Du(x))-M_\xi(s,\xi-Du(x))}_{c_1'(x,s,\xi)} \\ &\quad+\underbrace{M_\xi(s,\xi-Du(x))-M_\xi(s,\xi)}_{c_2'(x,s,\xi)}, \end{aligned}$$

so that

$$\begin{aligned} f_v(x,s,\xi) &= a'(x,s,\xi)v(x)+b'(x,s)v(x)+(c_1'(x,s,\xi)+c_2'(x,s,\xi))\cdot Dv(x) \\ &\quad+d'(x,\xi)\cdot Dv(x). \end{aligned}$$

The term a' admits the same growth condition of a, cf. the proof of Theorem 3.4. Also, since

$$c_1'(x,s,\xi)=-\int_0^1 M_{\xi s}(s-\tau u(x),\xi-Du(x))u(x)d\tau,$$

as for the term c_1 in the proof of Theorem 3.4 we obtain

$$|c_1'(x,s,\xi)|\le\varepsilon|\xi|^{p/p'}+C_\varepsilon|u(x)|^{p/((p-m)p')}+C_\varepsilon|Du(x)|^{p/p'}.$$

On the other hand, directly from assumptions (1.12)–(1.14) we get

$$\begin{aligned} &|b'(x,s)|\le C|u(x)|^{\sigma/\sigma'},\quad |c_2'(x,s,\xi)|\le C|Du(x)|^{m/m'}, \\ &|d'(x,\xi)|\le C|Du(x)|^{p/p'}. \end{aligned}$$

The conclusion follows then by the same argument carried out in Theorem 3.4. □

In the spirit of [21, Lemma 8.3], we have the following

Lemma 3.6. *Under the hypotheses of Theorem 1.1 or 1.2, let* $(y_n)\subset\mathbb{R}^N$ *with* $|y_n|\to\infty$,

$$\begin{aligned} &u_n(\cdot+y_n)\rightharpoonup u \quad \textit{in } W^{1,p}(\mathbb{R}^N)\cap D^{1,m}(\mathbb{R}^N), \\ &u_n(\cdot+y_n)\to u \quad \textit{a.e. in } \mathbb{R}^N, \end{aligned}$$

$$
\begin{aligned}
&Du_n(\cdot + y_n) \to Du \quad a.e.\ in\, \mathbb{R}^N,\\
&\phi_\infty(u_n) \to c,\\
&\phi'_\infty(u_n) \to 0 \quad in\,(W_0^{1,p}(\Omega)\cap D_0^{1,m}(\Omega))^*.
\end{aligned}
$$

Then $\phi'_\infty(u) = 0$ *and, setting* $v_n := u_n - u(\cdot - y_n)$*, we have*

$$
\phi_\infty(v_n) \to c - \phi_\infty(u) \tag{3.9}
$$

$$
\phi'_\infty(v_n) \to 0 \quad in\,(W_0^{1,p}(\Omega)\cap D_0^{1,m}(\Omega))^*, \tag{3.10}
$$

and $\|v_n\|_p^p = \|u_n\|_p^p - \|u\|_p^p + o(1)$ *and* $\|v_n\|_m^m = \|u_n\|_m^m - \|u\|_m^m + o(1)$ *as* $n \to \infty$.

Proof. The energy splitting (3.9) follows by Theorem 3.2 applied with $\Omega = \mathbb{R}^N$ and the sequence (u_n) replaced by $(u_n(\cdot + y_n))$. Take now $\varphi \in \mathcal{D}(\Omega)$ with $\|\varphi\|_{W_0^{1,p}(\Omega)\cap D_0^{1,m}(\Omega)} \le 1$ and define $\varphi_n := \varphi(\cdot + y_n)$. Then $\varphi_n \in \mathcal{D}(\Omega_n)$, where $\Omega_n = \Omega - \{y_n\} \subset \Omega$ for n large. For any $n \in \mathbb{N}$, we get

$$
\langle \phi'_\infty(v_n), \varphi\rangle = \langle \phi'_\infty(u_n(\cdot + y_n) - u), \varphi_n\rangle.
$$

By the splitting argument in the proof of Theorem 3.4, it follows that

$$
\langle \phi'_\infty(u_n(\cdot + y_n) - u), \varphi_n\rangle = \langle \phi'_\infty(u_n(\cdot + y_n)), \varphi_n\rangle - \langle \phi'_\infty(u), \varphi_n\rangle + \langle \zeta_n, \varphi_n\rangle,
$$

where $\zeta_n \to 0$ in the dual of $W_0^{1,p}(\Omega)\cap D_0^{1,m}(\Omega)$. If we prove that u is critical for ϕ_∞, then the right-hand side reads as $\langle \phi'_\infty(u_n), \varphi\rangle + \langle \zeta_n, \varphi_n\rangle$, and also the second limit (3.10) follows. To prove that $\phi'_\infty(u) = 0$ we observe that, for all φ in $\mathcal{D}(\mathbb{R}^N)$,

$$
\begin{aligned}
&\langle \phi'_\infty(u_n(\cdot + y_n)), \varphi\rangle \to \langle \phi'_\infty(u), \varphi\rangle,\\
&|\langle \phi'_\infty(u_n(\cdot + y_n)), \varphi\rangle| \le \|\phi'_\infty(u_n)\|_* \|\varphi\|_{W_0^{1,p}(\Omega)\cap D_0^{1,m}(\Omega)} \to 0.
\end{aligned}
$$

Indeed, defining $\hat\varphi_n := \varphi(\cdot - y_n)$, since $|y_n| \to \infty$ as $n \to \infty$, we have supp $\hat\varphi_n \subset \Omega$, for n large enough and $\|\hat\varphi_n\|_{W_0^{1,p}(\Omega)\cap D_0^{1,m}(\Omega)} = \|\varphi\|_{W^{1,p}(\mathbb{R}^N)\cap D^{1,m}(\mathbb{R}^N)}$. The last assertion follows by using Brezis-Lieb Lemma [5]. □

We can finally come to the proof of the main results.

4. Proof of Theorems 1.1 and 1.2 completed

We follow the scheme of the proof given in [21, p. 121]. Let $(u_n) \subset W_0^{1,p}(\Omega)\cap D_0^{1,m}(\Omega)$ be a bounded Palais-Smale sequence for ϕ at the level $c \in \mathbb{R}$. Hence, there exists a sequence (w_n) in the dual of $W_0^{1,p}(\Omega)\cap D_0^{1,m}(\Omega)$ such that $w_n \to 0$ and $\phi(u_n) \to c$ as $n \to \infty$ and, for all $v \in W_0^{1,p}(\Omega)\cap D_0^{1,m}(\Omega)$, we have

$$
\begin{aligned}
&\int_\Omega L_\xi(Du_n)\cdot Dv + \int_\Omega M_\xi(u_n, Du_n)\cdot Dv + \int_\Omega M_s(u_n, Du_n)v\\
&\quad + \int_\Omega V(x)|u_n|^{p-2}u_n v = \int_\Omega g(u_n)v + \langle w_n, v\rangle.
\end{aligned}
$$

Since (u_n) is bounded in $W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$, up to a subsequence, it converges weakly to some function $v_0 \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ and, by virtue of Proposition 2.3, (u_n) and (Du_n) converge to v_0 and Dv_0 a.e. in Ω, respectively. In turn (see also the proof of Theorem 3.4) it follows

$$\int_\Omega L_\xi(Dv_0) \cdot Dv + \int_\Omega M_\xi(v_0, Dv_0) \cdot Dv + \int_\Omega M_s(v_0, Dv_0)v$$
$$+ \int_\Omega V(x)|v_0|^{p-2}v_0 v = \int_\Omega g(v_0)v,$$

for any $v \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$. By combining Theorem 3.2 and Theorem 3.4, setting $u_n^1 := u_n - v_0$ and, after extending by zero outside Ω, we get

$$\phi_\infty(u_n^1) \to c - \phi(v_0), \quad n \to \infty, \tag{4.1}$$

$$\int_{\mathbb{R}^N} L_\xi(Du_n^1) \cdot Dv + \int_{\mathbb{R}^N} M_\xi(u_n^1, Du_n^1) \cdot Dv + \int_{\mathbb{R}^N} M_s(u_n^1, Du_n^1)v$$
$$+ \int_{\mathbb{R}^N} V_\infty |u_n^1|^{p-2}u_n^1 v = \int_{\mathbb{R}^N} g(u_n^1)v + \langle w_n^1, v\rangle. \tag{4.2}$$

where (w_n^1) is a sequence in the dual of $W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ with $w_n^1 \to 0$ as $n \to \infty$. In turn, it follows that (u_n^1) is Palais-Smale sequence for ϕ_∞ at the energy level $c - \phi(v_0)$. In addition,

$$\|u_n^1\|_p^p = \|u_n\|_p^p - \|v_0\|_p^p + o(1), \quad \|u_n^1\|_m^m = \|u_n\|_m^m - \|v_0\|_m^m + o(1), \quad \text{as } n \to \infty,$$

by the Brezis-Lieb Lemma [5]. Let us now define

$$\varpi := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |u_n^1|^p.$$

If it is the case that $\varpi = 0$, then, according to [15, Lemma I.1], (u_n^1) converges to zero in $L^r(\mathbb{R}^N)$ for every $r \in (p, p^*)$. Then, one obtains that

$$\lim_{n\to\infty} \int_\Omega g(u_n^1)u_n^1 = 0, \qquad \int_\Omega M_s(u_n^1, Du_n^1)u_n^1 \geq 0,$$

where the inequality follows by the sign condition (1.9). In turn, testing equation (4.2) with $v = u_n^1$, by the coercivity and convexity of $\xi \mapsto L(\xi), M(s, \xi)$, we have

$$\limsup_{n\to\infty} \left[\nu \int_{\mathbb{R}^N} |Du_n^1|^p + \nu \int_{\mathbb{R}^N} |Du_n^1|^m + V_\infty \int_{\mathbb{R}^N} |u_n^1|^p \right]$$
$$\leq \limsup_{n\to\infty} \left[\int_{\mathbb{R}^N} L_\xi(Du_n^1) \cdot Du_n^1 + \int_{\mathbb{R}^N} M_\xi(u_n^1, Du_n^1) \cdot Du_n^1 + \int_{\mathbb{R}^N} V_\infty |u_n^1|^p \right]$$
$$\leq 0,$$

yielding that (u_n^1) strongly converges to zero in $W^{1,p}(\mathbb{R}^N)\cap D^{1,m}(\mathbb{R}^N)$, concluding the proof in this case. If, on the contrary, it holds $\varpi > 0$, then, there exists an unbounded sequence $(y_n^1)\subset\mathbb{R}^N$ with $\int_{B(y_n^1,1)}|u_n^1|^p > \varpi/2$. Whence, let us consider $v_n^1 := u_n^1(\cdot + y_n^1)$, which, up to a subsequence, converges weakly and pointwise to some $v_1\in W^{1,p}(\mathbb{R}^N)\cap D^{1,m}(\mathbb{R}^N)$, which is nontrivial, due to the inequality $\int_{B(0,1)}|v_1|^p\geq\varpi/2$. Notice that, of course,

$$\lim_{n\to\infty}\phi_\infty(v_n^1)=\lim_{n\to\infty}\phi_\infty(u_n^1)=c-\phi(v_0).$$

Moreover, since $|y_n^1|\to\infty$ and Ω is an exterior domain, for all $\varphi\in\mathcal{D}(\mathbb{R}^N)$ we have $\varphi(\cdot - y_n^1)\in\mathcal{D}(\Omega)$ for $n\in\mathbb{N}$ large enough. Whence, in light of equation (4.2), for every $\varphi\in\mathcal{D}(\mathbb{R}^N)$ we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} L_\xi(Dv_n^1)\cdot D\varphi+\int_{\mathbb{R}^N} M_\xi(v_n^1,Dv_n^1)\cdot D\varphi+\int_{\mathbb{R}^N} M_s(v_n^1,Dv_n^1)\varphi\\
&\quad+\int_{\mathbb{R}^N} V_\infty|v_n^1|^{p-2}(v_n^1)\varphi-\int_{\mathbb{R}^N} g(v_n^1)\varphi=\int_{\mathbb{R}^N} L_\xi(Du_n^1)\cdot D\varphi(\cdot - y_n^1)\\
&\quad+\int_{\mathbb{R}^N} M_\xi(u_n^1,Du_n^1)\cdot D\varphi(\cdot - y_n^1)\\
&\quad+\int_{\mathbb{R}^N} M_s(u_n^1,Du_n^1)\varphi(\cdot - y_n^1)+\int_{\mathbb{R}^N} V_\infty|u_n^1|^{p-2}(u_n^1)\varphi(\cdot - y_n^1)\\
&\quad-\int_{\mathbb{R}^N} g(u_n^1)\varphi(\cdot - y_n^1)=\langle w_n^1,\varphi(\cdot + y_n^1)\rangle.
\end{aligned}$$

Defining the form $\langle\hat{w}_n^1,\varphi\rangle := \langle w_n^1,\varphi(\cdot - y_n^1)\rangle$ for all $\varphi\in\mathcal{D}(\mathbb{R}^N)$, we conclude that

$$\begin{aligned}
&\int_{\mathbb{R}^N} L_\xi(Dv_n^1)\cdot D\varphi+\int_{\mathbb{R}^N} M_\xi(v_n^1,Dv_n^1)\cdot D\varphi+\int_{\mathbb{R}^N} M_s(v_n^1,Dv_n^1)\varphi\\
&\quad+\int_{\mathbb{R}^N} V_\infty|v_n^1|^{p-2}(v_n^1)\varphi-\int_{\mathbb{R}^N} g(v_n^1)\varphi=\langle\hat{w}_n^1,\varphi\rangle,\quad\forall\varphi\in\mathcal{D}(\mathbb{R}^N).
\end{aligned}$$

Since $(\hat{w}_n^1)$ converges to zero in the dual of $W^{1,p}(\mathbb{R}^N)\cap D^{1,m}(\mathbb{R}^N)$, it follows by Proposition 2.3 (with $V=V_\infty$ and $\Omega=\mathbb{R}^N$) that the gradients Dv_n^1 converge point-wise to Dv_1, namely

$$Dv_n^1(x)\to Dv_1(x),\quad\text{a.e. in }\mathbb{R}^N. \tag{4.3}$$

Setting $u_n^2 := u_n^1 - v_1(\cdot - y_n^1)$, in light of (4.1)–(4.2) and (4.3), we can apply Lemma 3.6 to the sequence (v_n^1), getting

$$\lim_{n\to\infty}\phi_\infty(u_n^2)=c-\phi(v_0)-\phi_\infty(v_1),$$

as well as $\phi_\infty(v_1) = 0$ and, furthermore, for every $v \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$, we have

$$\int_{\mathbb{R}^N} L_\xi(Du_n^2) \cdot Dv + \int_{\mathbb{R}^N} M_\xi(u_n^2, Du_n^2) \cdot Dv + \int_{\mathbb{R}^N} M_s(u_n^2, Du_n^2)v + \int_{\mathbb{R}^N} V_\infty |u_n^2|^{p-2} u_n^2 v - \int_{\mathbb{R}^N} g(u_n^2)v = \langle \zeta_n^2, v \rangle,$$

where (ζ_n^2) goes to zero in the dual of $W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$. In turn, $(u_n^2) \subset W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)$ is a Palais-Smale sequence for ϕ_∞ at the energy level $c - \phi(v_0) - \phi(v_1)$. Arguing on (u_n^2) as it was done for (u_n^1), either u_n^2 goes to zero strongly in $W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)$ or we can generate a new (u_n^3). By iterating the above procedure, one obtains diverging sequences (y_n^i), $i = 1, \dots, k-1$, solutions v_i on $\mathbb{R}^N$ to the limiting problem, $i = 1, \dots, k-1$ and a sequence

$$u_n^k = u_n - v_0 - v_1(\cdot - y_n^1) - v_2(\cdot - y_n^2) - \cdots - v_{k-1}(\cdot - y_n^{k-1}),$$

such that (recall again Lemma 3.6) as $n \to \infty$

$$\begin{aligned} \|u_n^k\|_p^p &= \|u_n\|_p^p - \|v_0\|_p^p - \|v_1\|_p^p - \cdots - \|v_{k-1}\|_p^p + o(1), \\ \|u_n^k\|_m^m &= \|u_n\|_m^m - \|v_0\|_m^m - \|v_1\|_m^m - \cdots - \|v_{k-1}\|_m^m + o(1), \end{aligned} \tag{4.4}$$

as well as $\phi'_\infty(u_n^k) \to 0$ in $(W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*$ and

$$\phi_\infty(u_n^k) \to c - \phi(v_0) - \sum_{j=1}^{k-1} \phi_\infty(v_j).$$

Notice that the iteration is forced to end up after a finite number $k \geq 1$ of steps. Indeed, for every nontrivial critical point $v \in W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)$ of ϕ_∞ we have,

$$\int_{\mathbb{R}^N} L_\xi(Dv) \cdot Dv + \int_{\mathbb{R}^N} M_\xi(v, Dv) \cdot Dv + \int_{\mathbb{R}^N} M_s(v, Dv)v + \int_{\mathbb{R}^N} V_\infty |v|^p = \int_{\mathbb{R}^N} g(v)v,$$

yielding by the sign condition, the coercivity-convexity conditions and the growth of g,

$$\min\{\nu, V_\infty\}\|v\|_p^p + \|Dv\|_{L^m(\mathbb{R}^N)}^m \leq C_g \|v\|_{L^\sigma(\mathbb{R}^N)}^\sigma \leq C_g S_{p,\sigma} \|v\|_p^\sigma, \tag{4.5}$$

so that, due to $\sigma > p$, it holds

$$\|v\|_p^p \geq \left[\frac{\min\{\nu, V_\infty\}}{C_g S_{p,\sigma}}\right]^{\frac{p}{\sigma - p}} =: \Gamma_\infty > 0, \tag{4.6}$$

thus yielding from (4.4)

$$\|u_n^k\|_p^p \le \|u_n\|_p^p - \|v_0\|_p^p - (k-1)\Gamma_\infty + o(1).$$

By boundedness of (u_n), k has to be finite. Hence $u_n^k \to 0$ strongly in $W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)$ at some finite index $k \in \mathbb{N}$. This concludes the proof. □

5. Proof of Corollary 1.3

As a byproduct of the proof of the Theorems 1.1 and 1.2, since the p norm is bounded away from zero on the set of nontrivial critical points of ϕ_∞, cf. (4.5),we can estimate ϕ_∞ from below on that set. In order to do so, we use condition (2.1). For any nontrivial critical point of the functional ϕ_∞, we have (see the proof of Proposition 2.2)

$$\mu\phi_\infty(v) \ge \delta \int_\Omega |Dv|^p + \frac{\mu-p}{p} V_\infty \int_{\mathbb{R}^N} |v|^p \ge \min\left\{\delta, \frac{\mu-p}{p} V_\infty\right\} \|v\|_p^p.$$

An analogous argument applies to ϕ, yielding for any nontrivial critical point

$$\mu\phi(u) \ge \delta \int_\Omega |Du|^p + \frac{\mu-p}{p} V_0 \int_\Omega |u|^p \ge \min\left\{\delta, \frac{\mu-p}{p} V_0\right\} \|u\|_p^p.$$

Now notice that, recalling (4.6) and a similar variant for the norm of the critical points of ϕ in place of ϕ_∞, setting also

$$e_\infty := \min\left\{\frac{\delta}{\mu}, \frac{\mu-p}{\mu p} V_\infty\right\} \Gamma_\infty, \quad e_0 := \min\left\{\frac{\delta}{\mu}, \frac{\mu-p}{\mu p} V_0\right\} \Gamma_0,$$

$$\Gamma_0 := \left[\frac{\min\{\nu, V_0\}}{C_g S_{p,\sigma}}\right]^{\frac{p}{\sigma-p}} > 0,$$

from Theorems 1.1 or 1.2 we have $c \ge \ell e_0 + k e_\infty$ for some $\ell \in \{0, 1\}$ and non-negative integer k. Condition $c < c^* := e_\infty$ implies necessarily $k < 1$, namely $k = 0$. This provides the desired compactness result, using Theorems 1.1 or 1.2.
□

6. Proof of Corollary 1.8

Defining the functionals $J, Q : W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega) \to \mathbb{R}$ by

$$J(u) := \frac{1}{p}\int_\Omega L(Du) + \frac{1}{m}\int_\Omega M(Du) + \frac{1}{p}\int_\Omega V(x)|u|^p, \quad Q(u) := \frac{\mathbb{S}_\Omega}{\sigma}\int_\Omega |u|^\sigma,$$

and given a minimization sequence (u_n) for problem (1.16), by Ekeland's variational principle, without loss of generality we can replace it by a new minimization sequence, still denoted by (u_n), for which there exists a sequence $(\lambda_n) \subset \mathbb{R}$ such that for all $v \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$

$$J'(u_n)(v) - \lambda_n Q'(u_n)(v) = \langle w_n, v \rangle, \quad \text{with } w_n \to 0 \text{ in the dual of } W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega).$$

Taking into account the homogeneity of L and M, choosing $v = u_n$ this means

$$\int_\Omega L(Du_n) + \int_\Omega M(Du_n) + \int_\Omega V(x)|u_n|^p - \mathbb{S}_\Omega \lambda_n \int_\Omega |u_n|^\sigma = \langle w_n, u_n \rangle.$$

Since $\|u_n\|_{L^\sigma(\Omega)=1}$ for all n and $\int_\Omega L(Du_n)/p + M(Du_n)/m + V(x)|u_n|^p/p \to \mathbb{S}_\Omega$ as $n \to \infty$, this means that (u_n) is a Palais-Smale sequence for the functional $I(u) := J(u) - \lambda Q(u)$ for some $\lambda \in$ [m, p], at an energy level

$$c \le \frac{\sigma - m}{\sigma} \mathbb{S}_\Omega. \tag{6.1}$$

From Corollary 1.3 (applied with $L(Du)$ replaced by $L(Du)/p$, $M(u, Du)$ replaced by $M(Du)/m$ and $G(s) \equiv \frac{\mathbb{S}_\Omega}{\sigma}\lambda|s|^\sigma$), the compactness of (u_n) holds provided that (in the notations of Corollary 1.3)

$$c < \min\left\{\frac{\delta}{\mu}, \frac{\mu - p}{\mu p} V_\infty\right\} \left[\frac{\min\{\nu, V_\infty\}}{C_g S_{p,\sigma}}\right]^{\frac{p}{\sigma - p}}.$$

In our case, we can take $\mu = \sigma$, $\delta = \frac{\sigma - p}{p}$, $C_g = p\mathbb{S}\Omega$, $v_\infty = 1$, $v = 1$, $S_{p,\sigma} = p\mathbb{S}_{\mathbb{R}^N}^{-\sigma/p}$, yielding

$$C < \frac{\sigma - p}{\sigma} \mathbb{S}_{\mathbb{R}^N}^{\frac{\sigma}{\sigma - p}} / \mathbb{S}_\Omega^{\frac{p}{\sigma - p}}.$$

Hence, finally, by combining this conclusion with (6.1) the compactness (and in turn the solvability of the minimization problem) holds under condition (1.17), concluding the proof. □

Acknowledgments. The authors would like to thank the anonymous Referee for his/her careful reading of the manuscript and for useful suggestions. This article was supported by Miur project: "Variational and Topological Methods in the Study of Nonlinear Phenomena".

References

[1] Arcoya, D., Boccardo, L.: Critical points for multiple integrals of the calculus of variations. Arch. Ration. Mech. Anal. **134**, 249–274 (1996)
[2] Aris, R.: Mathematical modeling techniques. Res. Notes in Math., vol. 24. Pitman, Boston (1979)
[3] Benci, V., Cerami, G.: Positive solutions of some nonlinear elliptic problems in exterior domains. Arch. Ration. Mech. Anal. **99**, 283–300 (1987)

[4] Benci, V., D'Avenia, P., Fortunato, D., Pisani, L.: Solitons in several space dimensions: Derrick's problem and infinitely many solutions. Arch. Ration. Mech. Anal. **154**, 297–324 (2000)

[5] Brezis, H., Lieb, E.: A relation between pointwise convergence of functions and convergence of functionals. Proc. Am. Math. Soc. **88**, 486–490 (1983)

[6] Cherfils, L., Ilyasov, Y.: On the stationary solutions of generalized reaction diffusion equations with $p\&q$-Laplacian. Commun. Pure Appl. Anal. **4**, 9–22 (2005)

[7] Dal Maso, G., Murat, F.: Almost everywhere convergence of gradients of solutions to nonlinear elliptic systems. Nonlinear Anal. **31**, 405–412 (1998)

[8] Degiovanni, M., Lancelotti, S.: Linking over cones and nontrivial solutions for p-Laplace equations with p-superlinear nonlinearity. Ann. I. H. Poincaré **24**, 907–919 (2007)

[9] Degiovanni, M., Musesti, A., Squassina, M.: On the regularity of solutions in the Pucci-Serrin identity. Calc. Var. Partial Differ. Equ. **18**, 317–334 (2003)

[10] Derrick, G.H.: Comments on nonlinear wave equations as models for elementary particles. J. Math. Phys. **5**, 1252–1254 (1964)

[11] Fife, P.: Mathematical aspects of reacting and diffusing systems. Lecture Notes in Biomathematics, vol. 28. Springer, Berlin (1979)

[12] Filippucci, R.: Existence of global solutions of elliptic systems. J. Math. Anal. Appl. **293**, 677–692 (2004)

[13] Filippucci, R.: Entire radial solutions of elliptic systems and inequalities of the mean curvature type. J Math. Anal. Appl. **334**, 604–620 (2007)

[14] Filippucci, R., Pucci, P., Rigoli, M.: Nonlinear weighted p-Laplacian elliptic inequalities with gradient terms. Commun. Contemp. Math. **12**, 501–535 (2010)

[15] Lions, P.L.: The concentration-compactness principle in the calculus of variations. The locally compact case. Ann. Inst. Henri Poincar 'e Anal. Non Lineaire **1**, 109–145 and 223–283 (1984)

[16] Mercuri, C., Willem, M.: A global compactness result for the p-Laplacian involving critical nonlinearities. Discrete Contin. Dyn. Syst. **28**, 469–493 (2010)

[17] Pucci, P., Serrin, J.: A general variational identity. Indiana Univ. Math. J. **35**, 681–703 (1986)

[18] Squassina, M.: Existence, multiplicity, perturbation, and concentration results for a class of quasi-linear elliptic problems. Electron. J. Differ. Equ. Monogr. **7**, +213 (2006)

[19] Wilhelmsson, H.: Explosive instabilities of reaction-diffusion equations. Phys. Rev. A **36**, 965–966 (1987)

[20] Yu, L.S.: Nonlinear p-Laplacian problems on unbounded domains. Proc. AMS **115**, 1037–1045 (1992)

[21] Willem, M.: Minimax Theorems. Birkhauser (1996)