STRONGLY DAMPED WAVE EQUATIONS ON $\mathbb{R}^3$ WITH CRITICAL NONLINEARITIES

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ABSTRACT: We prove the existence of a global attractor for a strongly damped semilinear wave equation in the whole space, with a quite general nonlinearity at critical growth and a nonlinear weak damping term, generalizing some previously known results.

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1. INTRODUCTION AND MAIN RESULT

We consider the following initial-boundary value problem for $u : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}$:

$$(1.1)\qquad \begin{cases} u_{tt} - \Delta u_t - \Delta u + g(x,u) + \phi(x,u_t) = f(x), & x \in \mathbb{R}^3,\ t>0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^3, \\ u_t(x,0) = v_0(x), & x \in \mathbb{R}^3, \\ \lim\limits_{|x|\to\infty} u(x,t) = 0, & \forall t \geq 0. \end{cases}$$

Well-posedness and longtime behavior for analogous equations on bounded domains have been investigated by many authors in recent years (see, e.g., [4, 5, 7, 9, 11, 15], cf. also [3], for equations with memory effects). In this situation, under suitable assumptions, the existence of global and exponential attractors of finite fractal dimensions has been proved. To the best of our knowledge, the first result on the existence of attractors on unbounded domains for equation (1.1) has been provided in [2], in presence of a cubic nonlinearity. The main obstacle here is that, due to the unboundedness of the domain, some compactness results are not available, so that, for instance, it seems hard to show the existence of compact attracting sets, that would entail right away the existence of the global attractor (cf. [1, 14]). A similar difficulty has been encountered by Feireisl [6], who considered a weakly damped wave equation on $\mathbb{R}^3$, although in that case the finite propagation speed of initial disturbances is of some help. To circumvent this problem in the present situation (here we have infinite propagation speed), in [2] a new technique has been introduced, based on a decomposition of the solution by means of suitable cut-off functions.

The aim of this paper is to improve the main result of [2]; namely, we make a nontrivial generalization of the assumptions, and, most of all, we allow the nonlinear term $g(\cdot, u)$ to reach the critical power 5, beyond which neither existence nor uniqueness of solutions are guaranteed any longer. This introduces new difficulties, for, beside the lack of compactness caused by the unbounded domain, we have to deal with a further loss of compactness due to the critical growth of g. The key ingredient is to properly match the use of the cut-off functions and the fractional powers of the restriction of the differential operator $-\Delta$ to certain bounded subsets of $\mathbb{R}^3$.

Before stating the main result of the paper, some assumptions on the functions g and ϕ are in order.

Conditions on g. Let $g : \mathbb{R}^4 \to \mathbb{R}$ be locally bounded and measurable, with $g(x, \cdot) \in C^2(\mathbb{R})$ for almost every $x \in \mathbb{R}^3$. Assume that there exist $r_0 > 0$ and $c_1, c_2, c_3 > 0$ such that

$$
\begin{aligned}
&\text{(g1)} \quad g(\cdot, 0) \in L^2(\mathbb{R}^3),\\
&\text{(g2)} \quad |g'(x, 0)| \le c_1,\\
&\text{(g3)} \quad |g''(x, s)| \le c_2(1 + |s|^3), \quad \forall s \in \mathbb{R},\\
&\text{(g4)} \quad \liminf_{|s|\to\infty} \frac{g(x, s)}{s} \ge 0, \quad \text{uniformly as } |x| \le r_0,\\
&\text{(g5)} \quad (g(x, s) - g(x, 0))s \ge c_3 s^2, \quad \forall s \in \mathbb{R},\ |x| > r_0.
\end{aligned}
$$

Here the *prime* denotes the derivative with respect to the second variable of g.

Conditions on ϕ. Let $\phi : \mathbb{R}^4 \to \mathbb{R}$ be locally bounded and measurable, and assume that there exist $c_4, c_5 > 0$ such that

$$
\begin{aligned}
&(\phi 1) \quad \phi(\cdot, 0) \in L^2(\mathbb{R}^3),\\
&(\phi 2) \quad |\phi(x, r) - \phi(x, s)| \le c_4|r - s|, \quad \forall r, s \in \mathbb{R},\\
&(\phi 3) \quad \phi(x, s)s \ge 0, \quad \forall s \in \mathbb{R},\\
&(\phi 4) \quad \phi(x, s)s \ge c_5 s^2, \quad \forall s \in \mathbb{R},\ |x| > r_0.
\end{aligned}
$$

All the above conditions are understood to hold for almost every x in $\mathbb{R}^3$.

The main result is the following

Theorem 1.1. *Assume that* (g1)-(g5) *and* (ϕ1)-(ϕ4) *hold true, and let* $f \in L^2(\mathbb{R}^3)$. *Then the* C_0*-semigroup* $S(t)$ *generated by* (1.1) *has a (unique) connected global attractor.*

As we mentioned before, the major feature of this result is that g is allowed to grow up to 5, which is the borderline exponent also for the well-posedness of the problem. Furthermore, compared with [2, Theorem 4.5] we drop the technical (but indeed somehow restrictive) assumptions (G6) and (G7) therein, and we handle a more general nonlinear damping term $\phi(x, u_t)$, in place of a linear damping of the form $\phi(x)u_t$.

Remark 1.2. The presence of a weak damping in equation (1.1) plays an essential role in the longterm dynamics of the system. Even in the simplest possible situation, that is, the linear case, the lack of the weak damping term prevents the associated C_0-semigroup to have an exponential decay (consequently, in the nonlinear case, the existence of a bounded absorbing set is not to be expected). Indeed, let us consider

$$
\begin{cases}
u_{tt} - \Delta u_t - \Delta u + u = 0, \quad x \in \mathbb{R}^3,\ t > 0,\\
u(x, 0) = u_0(x), \quad x \in \mathbb{R}^3,\\
u_t(x, 0) = v_0(x), \quad x \in \mathbb{R}^3,\\
\lim_{|x|\to\infty} u(x, t) = 0, \quad \forall t \ge 0.
\end{cases}
$$

This linear problem generates a C_0-semigroup of contractions that *does not* decay exponentially. To see that, setting

$$U(t) = (u(t), u_t(t)), \qquad U_0 = (u_0, v_0),$$

we obtain a linear evolution equation on the Hilbert space $\mathcal{H} = H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ of the form

$$\begin{cases} U_t(t) = LU(t), \\ U(0) = U_0, \end{cases}$$

where the operator L is given by

$$L \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v \\ \Delta(u+v) - u \end{bmatrix},$$

with domain

$$\mathcal{D}(L) = \Big\{ (u,v) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \,:\, u + v \in H^2(\mathbb{R}^3) \Big\}.$$

By means of the Lumer-Phillips Theorem [12, Theorem 4.3] it is immediate to see that the above equation generates a C_0-semigroup of contractions e^{Lt} on $\mathcal{H}$. In order to prove that e^{Lt} is not exponentially stable, we show that the necessary condition $i\mathbb{R} \subset \rho(L)$ fails to hold (cf. [13]). To this aim, for $(\varphi, \psi) \in \mathcal{H}$ (actually, we are now working with the *complexification* of $\mathcal{H}$), consider the problem of finding a solution $(u,v) \in \mathcal{D}(L)$ to

$$(i - L) \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \varphi \\ \psi \end{bmatrix}.$$

Making the particular choice $\varphi = 0$ and $\psi = (1+i)h$, with $h \in L^2(\mathbb{R}^3)$, we find the equation

$$-\Delta u = h \quad \text{on } \mathbb{R}^3.$$

It is then easy to realize that the above elliptic problem is not solvable in $H^1(\mathbb{R}^3)$ for every choice of $h \in L^2(\mathbb{R}^3)$. For instance, setting

$$h(x) = \begin{cases} \frac{1}{4|x|^{5/2}} & \text{if } |x| \geq 1, \\ -\Delta\chi(x) & \text{if } |x| < 1, \end{cases}$$

for a suitable smooth function χ on the unit ball of $\mathbb{R}^3$, the function

$$u(x) = \begin{cases} \frac{1}{|x|^{1/2}} & \text{if } |x| \geq 1, \\ \chi(x) & \text{if } |x| < 1, \end{cases}$$

solves (in the distributional sense) the equation $-\Delta u = h$ on $\mathbb{R}^3$, but $u \notin L^6(\mathbb{R}^3)$, which in turn implies that $u \notin H^1(\mathbb{R}^3)$. Hence $i \notin \rho(L)$, and the proof is complete.

The plan of the paper is as follows.

– In Section 2 we give the weak formulation of the problem in a proper functional setting, and we recall well-posedness results, as well as some preliminary lemmas;

– In Section 3 we discuss the existence of a bounded absorbing set;

– In Section 4 we establish the existence of the global attractor.

2. WEAK FORMULATION AND WELL POSEDNESS

We introduce the Hilbert spaces $H = L^2(\mathbb{R}^3)$, $V = H^1(\mathbb{R}^3)$ and $V^* = H^{-1}(\mathbb{R}^3)$ (dual space of V), with the usual norms. We denote the inner product and the norm on H by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. The symbol $\langle\cdot,\cdot\rangle$ will also be employed to denote the duality product. Making the identification $H \equiv H^*$, we have the continuous and dense (but not compact) embeddings $V \hookrightarrow H \hookrightarrow V^*$. Concerning the phase-space for our problem, we consider the product Hilbert space

$$\mathcal{H} = V \times H,$$

endowed with the norm

$$\|(u,v)\|^2_{\mathcal{H}} = \|u\|^2_V + \|v\|^2 = \|u\|^2 + \|\nabla u\|^2 + \|v\|^2.$$

Definition 2.1. Let (g1)-(g3) and (ϕ1)-(ϕ2) hold true. Let $T > 0$, $(u_0, v_0) \in \mathcal{H}$, and $f \in V^*$. A function u such that

$$\begin{aligned} &u \in C([0,T],V) \\ &u_t \in C([0,T],H) \cap L^2([0,T],V) \\ &u_{tt} \in L^2([0,T],V^*) \end{aligned}$$

is a weak solution to the problem (1.1) in the time interval $[0,T]$ if

$$\langle u_{tt}, w\rangle + \langle \nabla u_t, \nabla w\rangle + \langle \nabla u, \nabla w\rangle + \langle \phi(\cdot,u_t), w\rangle + \langle g(\cdot,u), w\rangle = \langle f, w\rangle,$$

for every $w \in V$, almost every $t \in [0,T]$, and

$$u(0) = u_0, \qquad u_t(0) = v_0.$$

Well-posedness is ensured by

Theorem 2.2. *Assume that* (g1)-(g5) *and* (ϕ1)-(ϕ2) *hold true. Then, for every* $(u_0,v_0) \in \mathcal{H}$ *and every* $f \in V^*$, *problem* (1.1) *admits a unique solution on the time interval* $[0,T]$, *for every* $T > 0$. *Moreover, if* $\{(u_0^i, v_0^i)\}_{i=1,2}$ *are two sets of data satisfying* $\|(u_0^i,v_0^i)\|_{\mathcal{H}} \le R$ *for some* $R > 0$, *the corresponding solutions* u^i *to problem* (1.1) *on the time interval* $[0,T]$ *fulfill the continuous dependence estimate*

$$\|(u^1(t), u^1_t(t)) - (u^2(t), u^2_t(t))\|_{\mathcal{H}} \le C\|(u_0^1,v_0^1) - (u_0^2,v_0^2)\|_{\mathcal{H}}, \qquad \forall t \in [0,T],$$

for some $C = C(R,T) > 0$.

The proof of the above results is carried out via a standard Faedo-Galerkin approximation scheme. We agree to denote by $S(t)z_0$ the solution at time t to (1.1), with external force f and initial data $z_0 = (u_0, v_0)$ given at $t = 0$. By virtue of Theorem 2.2 the one-parameter family of operators $S(t)$ is a C_0-semigroup of (nonlinear) operators on the phase-space $\mathcal{H}$.

We conclude the section with some technical results that will be needed in the course of the investigation. The first is a Gronwall-type lemma (cf. [2, Lemma 2.7]).

Lemma 2.3. *Let* $\Phi : \mathcal{H} \to \mathbb{R}$ *be a continuous function that satisfies (in the sense of distributions) the differential inequality*

$$\frac{d}{dt}\Phi(z(t)) + \delta\|z(t)\|^2_{\mathcal{H}} \le k,$$

for some $\delta, k > 0$, *and* $z \in C(\mathbb{R}^+, \mathcal{H})$. *In addition, assume that*

$$\sup_{t\in\mathbb{R}^+} \Phi(z(t)) \ge -m \qquad \text{and} \qquad \Phi(z(0)) \le M,$$

for some $m, M \geq 0$. *Then*

$$\Phi(z(t)) \leq \sup_{\zeta \in X} \Big\{ \Phi(\zeta) : \delta \|\zeta\|_{\mathcal{H}}^2 \leq 2k \Big\}, \qquad \forall t \geq \frac{m+M}{k}.$$

Secondly, we formulate some inequalities involving the nonlinearities g and ϕ, which can be readily proved, extending similar arguments used in [10].

For $u \in V$ let us denote

$$\mathcal{G}(u) = \int_{\mathbb{R}^3} \int_0^{u(x)} g(x,\tau)\, d\tau\, dx.$$

The finiteness of $\mathcal{G}$ follows from (g1)-(g3) and the Sobolev embedding $V \hookrightarrow L^6(\mathbb{R}^3)$.

Lemma 2.4. *Assume that* (g1)-(g5) *and* (ϕ1)-(ϕ4) *hold true. Then, for every* $\nu > 0$ *there exists* $\rho(\nu) \geq 0$ *such that*

$$\mathcal{G}(u) \geq -\nu \|u\|^2 - \rho(\nu), \qquad \forall u \in V. \tag{2.1}$$

Furthermore, there exist $\alpha > 0$ *and* $\beta \geq 0$ *such that*

$$\langle g(x,u), u \rangle - \alpha \|u\|^2 \geq -\frac{1}{2}\|\nabla u\|^2 - \beta, \qquad \forall u \in V, \tag{2.2}$$

$$\langle \phi(x,\xi), \xi \rangle - \alpha \|\xi\|^2 \geq -\frac{1}{2}\|\nabla \xi\|^2, \qquad \forall \xi \in H. \tag{2.3}$$

3. BOUNDED ABSORBING SETS

The dissipative nature of our system is ensured by the following uniform-in-time energy estimate.

Theorem 3.1. *Assume that* (g1)-(g5) *and* (ϕ1)-(ϕ4) *hold true, and let* $f \in H$. *Then there exists a constant* $R_0 > 0$ *with the following property. Given any* $R \geq 0$, *there exist* $t_0 = t_0(R) \geq 0$ *such that, whenever*

$$\|z_0\|_{\mathcal{H}} \leq R,$$

it follows that

$$\|S(t)z_0\|_{\mathcal{H}} \leq R_0, \qquad \forall t \geq t_0.$$

Moreover,

$$\|S(t)z_0\|_{\mathcal{H}} \leq C_0, \qquad \forall t \in [0, t_0],$$

for some $C_0 = C_0(R) \geq 0$.

As a straightforward consequence we have

Corollary 3.2. *The set*

$$\mathcal{B}_0 = \bigcup_{t \geq 0} S(t)\mathcal{B}_{R_0},$$

where $\mathcal{B}_{R_0}$ *denotes the ball of* $\mathcal{H}$ *of radius* R_0 *centered at zero, is an invariant, bounded absorbing set for* $S(t)$ *on* $\mathcal{H}$, *that is,* $S(t)\mathcal{B}_0 \subset \mathcal{B}_0$ *for every* $t \geq 0$, *and for any bounded set* $\mathcal{B} \subset \mathcal{H}$ *there exists* $t_0 = t_0(\mathcal{B}) \geq 0$ *such that* $S(t)\mathcal{B} \subset \mathcal{B}_0$ *for every* $t \geq t_0$.

Proof of Theorem 3.1. Given $R \geq 0$ and $z_0 \in \mathcal{H}$ with $\|z_0\|_{\mathcal{H}} \leq R$, let us denote $z(t) = S(t)z_0 = (u(t), u_t(t))$. For $\varepsilon \in [0, \frac{1}{2})$ to be fixed later, introduce the auxiliary variable $\xi = u_t + \varepsilon u$. Taking the product in H of (1.1) and ξ, and adding to both sides the term $\varepsilon\langle u, u_t\rangle$, we get

$$\begin{aligned}(3.1)\qquad &\frac{1}{2}\frac{d}{dt}\Phi + \varepsilon(1-\varepsilon)\|\nabla u\|^2 + \|\nabla\xi\|^2 - \varepsilon\|\xi\|^2 + \langle\phi(\cdot,\xi),\xi\rangle \\ &= \varepsilon\langle u, u_t\rangle - \varepsilon^2\langle u,\xi\rangle - \varepsilon\langle g(\cdot,u),u\rangle + \langle\phi(\cdot,\xi)-\phi(\cdot,u_t),\xi\rangle + \langle f,\xi\rangle,\end{aligned}$$

where the functional Φ is defined as

$$\Phi(t) = \Phi(z(t)) = \varepsilon\|u(t)\|^2 + (1-\varepsilon)\|\nabla u(t)\|^2 + \|\xi(t)\|^2 + 2\mathcal{G}(u(t)).$$

Thanks to (2.1) and to the first of the inequalities

$$(3.2)\qquad \|\xi\|^2 \geq \frac{1}{2}\|u_t\|^2 - 2\varepsilon^2\|u\|^2, \qquad \|\nabla\xi\|^2 \geq \frac{1}{2}\|\nabla u_t\|^2 - 2\varepsilon^2\|\nabla u\|^2,$$

there exist positive constants $K_1 = K_1(\varepsilon)$ and $K_2 = K_2(\varepsilon)$ such that

$$(3.3)\qquad \Phi(z(t)) \geq K_2\|z(t)\|^2_{\mathcal{H}} - K_1.$$

Besides, from (g1)-(g3), we easily find $K_3 = K_3(\varepsilon) \geq 0$ and $K_4 = K_4(\varepsilon) \geq 0$ such that

$$(3.4)\qquad \Phi(z(t)) \leq \|z(t)\|_{\mathcal{H}}\big(K_3 + K_4\|z(t)\|^5_{\mathcal{H}}\big).$$

We now proceed to the evaluation of the right–hand side of (3.1). First, we control the terms involving $g(\cdot,u)$ and $\phi(\cdot,\xi)$ by means of (2.2) and (2.3). On account of (ϕ2) there holds

$$|\langle\phi(\cdot,\xi)-\phi(\cdot,u_t),\xi\rangle| \leq \varepsilon c_4\|u\|\|\xi\|.$$

Furthermore, we have the trivial inequalities

$$\varepsilon\langle u, u_t\rangle \leq \varepsilon\|u\|\|\xi\| - \varepsilon^2\|u\|^2,$$
$$\langle f,\xi\rangle \leq \frac{1}{4\varepsilon}\|f\|^2 + \varepsilon\|\xi\|^2.$$

In order to estimate the product $\|u\|\|\xi\|$ we use again the Young inequality, which yields

$$\varepsilon(c_4+1+\varepsilon)\|u\|\|\xi\| \leq \frac{\alpha}{2}\|\xi\|^2 + \varepsilon^2\gamma\|u\|^2,$$

having set $\alpha > 0$ as in Lemma 2.4, and

$$\gamma = \frac{(2+c_4)^2}{2\alpha}.$$

Then, plugging all the above inequalities into (3.1), and taking into account (3.2), it follows that

$$\begin{aligned}(3.5)\qquad &\frac{d}{dt}\Phi + 2\varepsilon(\alpha+\varepsilon-\varepsilon\gamma-\varepsilon\alpha+4\varepsilon^2)\|u\|^2 + \varepsilon(1-4\varepsilon)\|\nabla u\|^2 \\ &+ \frac{\alpha-4\varepsilon}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u_t\|^2 \leq \frac{1}{4\varepsilon}\|f\|^2 + \varepsilon\beta.\end{aligned}$$

At this point, we fix $\varepsilon > 0$ small enough such that

$$\delta = \min\Big\{2\varepsilon(\alpha+\varepsilon-\varepsilon\gamma-\varepsilon\alpha+4\varepsilon^2),\, \frac{\alpha-4\varepsilon}{2},\, \varepsilon(1-4\varepsilon),\, \frac{1}{2}\Big\} > 0,$$

and set

$$(3.6)\qquad k = \frac{1}{4\varepsilon}\|f\|^2 + \varepsilon\beta + \omega^2,$$

being $\omega \in (0,1)$ any fixed number, so ensuring the validity of the condition $k > 0$. Then, inequality (3.5) entails

$$\frac{d}{dt}\Phi(z(t)) + \delta\|z(t)\|^2_{\mathcal{H}} + \delta\|\nabla u_t(t)\|^2 \leq k. \tag{3.7}$$

On the other hand, by (3.3), we have

$$\sup_{t\in\mathbb{R}^+} \Phi(z(t)) \geq -K_1.$$

Furthermore, since $\|z_0\|_{\mathcal{H}} \leq R$, by (3.4) there exists $M \geq 0$ such that

$$\Phi(z(0)) \leq M.$$

We are now in the position to apply Lemma 2.3, which yields

$$\Phi(z(t)) \leq \sup_{\zeta\in\mathcal{H}} \Big\{\Phi(\zeta) : \delta\|\zeta\|^2_{\mathcal{H}} \leq 2k\Big\}, \qquad \forall t \geq t_0,$$

where $t_0 = \frac{K_1+M}{k}$. Recalling (3.3)-(3.4) we have

$$\|z(t)\|_{\mathcal{H}} \leq R_0, \qquad \forall t \geq 0, \tag{3.8}$$

where we set

$$R_0 = \frac{1}{\sqrt{K_2}}\bigg(K_3\sqrt{\frac{2k}{\delta}} + K_4\Big(\frac{2k}{\delta}\Big)^3 + K_1\bigg)^{1/2},$$

the first assertion of the theorem is proved. The second assertion immediately follows integrating (3.7) on $(0,t_0)$, with the aid of (3.3)-(3.4). □

The next result provides a uniform integral estimate.

Corollary 3.3. *Under the assumptions of Theorem 3.1, given any $R \geq 0$, there exists $\Lambda_0 = \Lambda_0(R)$ such that, whenever $\|z_0\|_{\mathcal{H}} \leq R$, the corresponding solution $S(t)z_0 = (u(t), u_t(t))$ fulfills*

$$\int_0^\infty \|\nabla u_t(\tau)\|^2 d\tau \leq \Lambda_0. \tag{3.9}$$

Proof. Setting $\varepsilon = 0$ in (3.1), in view of (2.3) there holds

$$\frac{d}{dt}\Phi(t) + \|\nabla u_t\|^2 \leq 0,$$

where

$$\Phi(t) = \|\nabla u(t)\|^2 + \|u_t(t)\|^2 + 2\mathcal{G}(u(t)) - 2\langle f, u(t)\rangle.$$

Integrating the above inequality on $(0,t)$, and recalling Theorem 3.1, we get the desired conclusion. □

Finally, we have

Corollary 3.4. *Under the assumptions of Theorem 3.1, let g satisfy the additional condition*

$$g(x,s)s \geq c_6 s^2, \qquad \forall s \in \mathbb{R},\ a.e.\ x \in \mathbb{R}^3,$$

for some $c_6 > 0$. Then there exists a constant $C > 0$ such that, for every $\omega \in (0,1)$ and for every $\|z_0\|_{\mathcal{H}} \leq R$,

$$\|S(t)z_0\|^2_{\mathcal{H}} \leq C\big(\omega + \|f\| + \|f\|^6\big), \qquad \forall t \geq t_0,$$

for some $t_0 = t_0(R, \omega, \|f\|) \geq 0$.

Proof. Repeating the proof of Theorem 3.1, with the additional hypothesis, inequalities (2.2) and (3.3) hold with $\beta = 0$ and $K_1 = 0$, respectively. Hence, (3.8) yields

$$\|S(t)z_0\|^2_{\mathcal{H}} \leq C\big(\sqrt{k} + k^3\big),$$

for all $t \geq t_0 = \frac{M}{k}$. Recalling the definition of k in (3.6), we have

$$k = \frac{1}{4\varepsilon}\|f\|^2 + \omega^2,$$

where $\omega \in (0,1)$ can be arbitrarily chosen. Therefore the assertion follows. □

4. THE UNIVERSAL ATTRACTOR

The aim of this section is to prove the existence of the global attractor for the C_0-semigroup $S(t)$ associated to (1.1), in the assumptions (g1)-(g5), (ϕ1)-(ϕ4) and $f \in H$. As pointed out in [2], due to the lack of regularization of the initial data, and due to the unboundedness of the domain, it seems out of reach to prove the existence of a compact attracting set for $S(t)$ (which is a sufficient condition in order for the attractor to exist). Thus we proceed in a different way, relying on the following abstract result (see [8], cf. also [10, Theorem A.2])

Theorem 4.1. *Let $S(t)$ be a dynamical system on a Banach space $\mathcal{H}$. Assume that the following hypotheses hold:*

(i) *there exists a bounded absorbing set $\mathcal{B}_0 \subset \mathcal{H}$; and*
(ii) *for every $\eta > 0$, there exists $t_\eta \geq 0$ and a (relatively) compact set $\mathcal{K}_\eta \subset \mathcal{H}$ such that*

$$\delta_{\mathcal{H}}(S(t_\eta)\mathcal{B}_0, \mathcal{K}_\eta) \leq \eta,$$

where $\delta_{\mathcal{H}}$ denotes the usual Hausdorff semidistance in $\mathcal{H}$.

Then the ω-limit set of $\mathcal{B}_0$ is the (connected) global attractor of $S(t)$.

On account of Corollary 3.2, in order to prove Theorem 1.1, we are left to show that condition (ii) of Theorem 4.1 applies to our case. We briefly sketch the plan of the proof. As in [2], we provide a suitable decomposition of the solution of (1.1) in three terms. Then, in Lemma 4.2 and Lemma 4.3, we show that the first two terms can be chosen arbitrarily small as $t \to \infty$, uniformly with respect to the initial data in $\mathcal{B}_0$. Besides, Lemma 4.4 proves that the remaining part of the decomposition lies in a relatively compact set. Collecting the above lemmas, we reach the conclusion.

According to [2], for every $r > r_0$, it is possible to split $-g(x,s) + f(x)$ as

$$-g(x,s) + f(x) = -g^1(x,s) - g^2(x,s) + f^1(x) + f^2(x),$$

where g^i and f^i depend on r, in such a way that the following conditions hold. First, g^1 fulfills (g2)-(g5) (possibly by redefining the constants therein), and $g^1(\cdot,0) \equiv 0$. Then, for every $s \in \mathbb{R}$ and almost every $x \in \mathbb{R}^3$,

$$g^1(x,s)s \geq c_7 s^2, \tag{4.1}$$

for some $c_7 > 0$. Finally,

$$g^2(x,s) = 0, \quad f^2(x) = 0 \quad \text{for } s \in \mathbb{R}^3,\ |x| \geq r+1,$$
$$\|f^1\| \to 0 \quad \text{as } r \to \infty.$$

At this point (cf. [6, 10]), we decompose the solution to (1.1) with initial data $(u_0, v_0) \in \mathcal{B}_0$ into the sum $u = v + w$, where v and w are the solutions to the problems

$$\begin{cases} v_{tt} - \Delta v_t - \Delta v + \phi(\cdot, v_t) + g^1(\cdot, v) = f^1 \\ v(0) = u_0 \\ v_t(0) = v_0, \end{cases} \tag{4.2}$$

and

$$\begin{cases} w_{tt} - \Delta w_t - \Delta w + \phi(\cdot, u_t) - \phi(\cdot, v_t) + g^1(\cdot, u) - g^1(\cdot, v) + g^2(\cdot, u) = f^2 \\ w(0) = 0 \\ w_t(0) = 0. \end{cases} \tag{4.3}$$

Problems (4.2)-(4.3) are easily seen to satisfy existence and continuous dependence results analogous to those of Theorem 2.2.

We first prove that the solution v becomes small as $r \to \infty$ and $t \to \infty$.

Lemma 4.2. *For every $\eta > 0$ there exist $t_\eta > 0$ and $r_\eta > r_0$ such that the solution v to problem* (4.2) *at time t_η, corresponding to $r = r_\eta$, fulfills the inequality*

$$\|(v(t_\eta), v_t(t_\eta))\|_{\mathcal{H}} \leq \frac{\eta}{2},$$

for every $(u_0, v_0) \in \mathcal{B}_0$.

Proof. Thanks to the properties of g^1, with particular reference to (4.1), we can directly apply Corollary 3.4 to the C_0-semigroup associated to (4.2). Hence there exists $C > 0$ such that, for every $\omega \in (0, 1)$,

$$\|(v(t), v_t(t))\|_{\mathcal{H}}^2 \leq C\big(\omega + \|f^1\| + \|f^1\|^6\big), \qquad \forall t \geq t_0,$$

for some $t_0 = t_0(\omega, \|f^1\|) \geq 0$. Since $\|f^1\| \to 0$ as $r \to \infty$, the thesis follows. □

In the sequel, let $\eta > 0$ be fixed. Besides, let $r_\eta > r_0$ and $t_\eta > 0$ be chosen as in the above lemma. It is then apparent that

$$\sup_{t \in [0, t_\eta]} \sup_{(u_0, v_0) \in \mathcal{B}_0} \Big\{ \|u(t)\|_V, \|u_t(t)\|, \|v(t)\|_V, \|v_t(t)\|, \|w(t)\|_V, \|w_t(t)\| \Big\} < \infty. \tag{4.4}$$

Now, following [2] again, for $\varrho > 0$ we introduce a family of smooth functions $\psi_\varrho : \mathbb{R}^3 \to [0, 1]$, such that

$$\psi_\varrho(x) = \begin{cases} 0 & \text{if } |x| \leq \varrho + 1, \\ 1 & \text{if } |x| \geq 2\varrho + 2, \end{cases}$$

and satisfying, for some $c_8 > 0$, the inequalities

$$|\nabla \psi_\varrho(x)| \leq \frac{c_8}{\varrho + 1}, \qquad |\nabla \psi_\varrho^2(x)| \leq \frac{c_8}{\varrho + 1} \psi_\varrho(x), \qquad |\Delta \psi_\varrho(x)| \leq c_8.$$

Then, for every fixed $\varrho > 0$, we write $w = \breve{w}_\varrho + \hat{w}_\varrho$, where

$$\breve{w}_\varrho(x, t) = \psi_\varrho(x) w(x, t) \qquad \text{and} \qquad \hat{w}_\varrho(x, t) = (1 - \psi_\varrho(x)) w(x, t).$$

The subsequent lemma says that $\breve{w}_\varrho(\cdot, t)$ can be made arbitrarily small as ϱ and t become sufficiently large. The proof is similar to the one of [2, Lemma 4.3], and is therefore left to the interested reader.

Lemma 4.3. *Let w be the solution to* (4.3) *corresponding to $r = r_\eta$. Then there exists $\varrho_\eta \geq r_\eta$ such that*

$$\|(\check{w}_\varrho(t_\eta), \partial_t \check{w}_\varrho(t_\eta))\|_{\mathcal{H}} \leq \frac{\eta}{2},$$

for every $\varrho \geq \varrho_\eta$, for every $(u_0, v_0) \in \mathcal{B}_0$.

In order to state the next lemma, which provides the compact part in the decomposition of the solution, some definitions are needed. Let $B \subset \mathbb{R}^3$ be a smooth bounded domain. Define the operator A on $L^2(B)$ by

$$A = -\Delta \qquad \text{with domain} \qquad \mathcal{D}(A) = H_0^1(B) \cap H^2(B),$$

and consider the family of Hilbert spaces $\mathcal{D}(A^{s/2})$, $s \in \mathbb{R}$, with inner products and norms given by

$$\langle \cdot, \cdot \rangle_{\mathcal{D}(A^{s/2})} = \langle A^{s/2} \cdot, A^{s/2} \cdot \rangle, \qquad \| \cdot \|_{\mathcal{D}(A^{s/2})} = \|A^{s/2} \cdot \|.$$

Then, $\mathcal{D}(A^0) = L^2(B)$, $\mathcal{D}(A^{1/2}) = H^1(B)$, and the compact and dense injections $\mathcal{D}(A^{s/2}) \hookrightarrow \mathcal{D}(A^{r/2})$ hold for all $s > r$. Finally, introduce the Hilbert space

$$\mathcal{H}_s(B) = \mathcal{D}(A^{(1+s)/2}) \times \mathcal{D}(A^{s/2}),$$

endowed with the usual inner product.

Lemma 4.4. *Let w be the solution to* (4.3) *corresponding to $r = r_\eta$. Let $\varrho > 0$ be fixed, and consider the ball $B_\varrho = \{x \in \mathbb{R}^3 : |x| \leq 2\varrho + 3\}$. Then there exists a constant $K_{\eta,\varrho} > 0$ such that*

$$\|(\hat{w}_\varrho(t_\eta), \partial_t \hat{w}_\varrho(t_\eta))\|_{\mathcal{H}_{1/4}(B_\varrho)} \leq K_{\eta,\varrho},$$

for every $(u_0, v_0) \in \mathcal{B}_0$.

Proof. Let us first notice that $\hat{w}_\varrho(t)$ vanishes for $|x| \geq 2\varrho + 2$, hence its restriction on B_ϱ belongs to $H_0^1(B_\varrho)$ for all $t > 0$. Along the proof we shall omit the dependence on ϱ, writing $\hat{w}$ in place of $\hat{w}_\varrho$. With the capital letter A we shall denote the operator $-\Delta$ with domain in $H_0^1(B_\varrho)$. Multiplying (4.3) by $(1 - \psi_\varrho)$, and taking into account the identities

$$\begin{aligned} -(1 - \psi_\varrho)\Delta w(t) &= A\hat{w}(t) - 2\nabla\psi_\varrho \cdot \nabla w(t) - \Delta\psi_\varrho w(t), \\ -(1 - \psi_\varrho)\Delta w_t(t) &= A\hat{w}_t(t) - 2\nabla\psi_\varrho \cdot \nabla w_t(t) - \Delta\psi_\varrho w_t(t), \end{aligned}$$

we obtain

$$\begin{aligned} &\hat{w}_{tt} + A\hat{w}_t + A\hat{w} + (1 - \psi_\varrho)[\phi(\cdot, u_t) - \phi(\cdot, v_t)] \\ &\quad + (1 - \psi_\varrho)[g^1(\cdot, u) - g^1(\cdot, v)] + (1 - \psi_\varrho)g^2(\cdot, u) \\ &= (1 - \psi_\varrho)f^2 + 2\nabla\psi_\varrho \cdot [\nabla w(t) + \nabla w_t(t)] + \Delta\psi_\varrho[w(t) + w_t(t)], \end{aligned}$$

with $\hat{w}(0) = \hat{w}_t(0) = 0$. Taking the product in H of this equality and $A^{1/4}\hat{w}_t$, we get

$$\frac{1}{2}\frac{d}{dt}\Psi + \|A^{5/8}\hat{w}_t\|^2_{L^2(B_\varrho)} = \sum_{i=1}^{6} I_i,$$

where

$$\Psi(t) = \|A^{5/8}\hat{w}(t)\|^2_{L^2(B_\varrho)} + \|A^{1/8}\hat{w}_t(t)\|^2_{L^2(B_\varrho)},$$

and

$$
\begin{aligned}
I_1 &= -\langle (1-\psi_\varrho)[\phi(\cdot,u_t)-\phi(\cdot,v_t)], A^{1/4}\hat{w}_t\rangle,\\
I_2 &= -\langle (1-\psi_\varrho)[g^1(\cdot,u)-g^2(\cdot,v)], A^{1/4}\hat{w}_t\rangle,\\
I_3 &= -\langle (1-\psi_\varrho)g^2(\cdot,u), A^{1/4}\hat{w}_t\rangle,\\
I_4 &= \langle (1-\psi_\varrho)f^2, A^{1/4}\hat{w}_t\rangle,\\
I_5 &= \langle 2\nabla\psi_\varrho\cdot[\nabla w(t)+\nabla w_t(t)], A^{1/4}\hat{w}_t\rangle,\\
I_6 &= \langle \Delta\psi_\varrho[w(t)+w_t(t)], A^{1/4}\hat{w}_t\rangle.
\end{aligned}
$$

Throughout the rest of the proof, $c>0$ will denote a generic constant, which may depend also on ϱ. By virtue of (g2)-(g3), (4.4), the generalized Hölder inequality, and the continuous embeddings $\mathcal{D}(A^{5/8})\hookrightarrow L^{12}(B_\varrho)$ and $\mathcal{D}(A^{3/8})\hookrightarrow L^4(B_\varrho)$, we infer that

$$
\begin{aligned}
I_1 &\le c\int_{B_\varrho} |(1-\psi_\varrho)w|\big(1+|u|^4+|v|^4\big)|A^{1/4}\hat{w}_t|dx\\
&\le c\big(1+\|u\|_{L^6}^4+\|v\|_{L^6}^4\big)\|\hat{w}\|_{L^{12}(B_\varrho)}\|A^{1/4}\hat{w}_t\|_{L^4(B_\varrho)}\\
&\le c\big(1+\|u\|_V^4+\|v\|_V^4\big)\|A^{5/8}\hat{w}\|_{L^2(B_\varrho)}\|A^{5/8}\hat{w}_t\|_{L^2(B_\varrho)}\\
&\le c\|A^{5/8}\hat{w}\|^2_{L^2(B_\varrho)}+\frac{1}{6}\|A^{5/8}\hat{w}_t\|^2_{L^2(B_\varrho)}.
\end{aligned}
$$

Moreover, on account of (ϕ2) and (4.4),

$$
I_2 \le c_4\int_{B_\varrho}|\hat{w}_t||A^{1/4}\hat{w}_t|\,dx \le c_4\|\hat{w}_t\|_{L^{4/3}(B_\varrho)}\|A^{1/4}\hat{w}_t\|_{L^4(B_\varrho)} \le c+\frac{1}{6}\|A^{5/8}\hat{w}_t\|^2_{L^2(B_\varrho)}.
$$

In a similar fashion, recalling the properties of ψ_ϱ, we have

$$
\begin{aligned}
I_3 &\le c\int_{B_\varrho}|u||A^{1/4}\hat{w}_t|\,dx \le c+\frac{1}{6}\|A^{5/8}\hat{w}_t\|^2_{L^2(B_\varrho)},\\
I_4 &\le c+\frac{1}{6}\|A^{5/8}\hat{w}_t\|^2_{L^2(B_\varrho)},\\
I_5 &\le c\int_{B_\varrho}|\nabla w(t)+\nabla w_t(t)||A^{1/4}\hat{w}_t|\,dx \le c+c\|\nabla w_t\|^2+\frac{1}{6}\|A^{5/8}\hat{w}_t\|^2_{L^2(B_\varrho)},\\
I_6 &\le c\int_{B_\varrho}|w(t)+w_t(t)||A^{1/4}\hat{w}_t|\,dx \le c+\frac{1}{6}\|A^{5/8}\hat{w}_t\|^2_{L^2(B_\varrho)}.
\end{aligned}
$$

Collecting the estimates for I_1-I_6, we end up with the differential inequality

$$
\frac{d}{dt}\Psi \le c\Psi + c\|\nabla w_t\|^2 + c.
$$

Let $t_\eta>0$ be as in Lemma 4.2. Taking into account (3.9) (which holds with w in place of u as well), from the Gronwall Lemma on $[0,t_\eta]$, recalling that $\Psi(0)=0$, we obtain

$$
\Psi(t_\eta) \le ce^{ct_\eta}\left[t_\eta+\int_0^\infty \|\nabla w_t(y)\|^2\,dy\right] \le ce^{ct_\eta}(t_\eta+\Lambda_0),
$$

for some Λ_0 independent of $(u_0,v_0)\in\mathcal{B}_0$. Thus, there exists $K_{\eta,\varrho}>0$ such that

$$
\|(\hat{w}(t_\eta),\hat{w}_t(t_\eta))\|_{\mathcal{H}_{1/4}(B_\varrho)} \le K_{\eta,\varrho},
$$

for every $(u_0,v_0)\in\mathcal{B}_0$. □

Note that, in contrast with [2, Lemma 4.2], since we are dealing with the critical case, we cannot use Aw_t as a test function, but instead we are forced to use a fractional power of the Laplace operator, that is still sufficient to provide the required compactness. In fact, taking into account the compact embedding

$$\mathcal{D}(A^{5/8}) \times \mathcal{D}(A^{1/8}) \hookrightarrow \mathcal{D}(A^{1/2}) \times \mathcal{D}(A^0) = H^1(B_\varrho) \times L^2(B_\varrho),$$

it turns out that, for all $\varrho > 0$, the set

$$\mathcal{K}_{\eta,\varrho} = \bigcup_{(u_0,v_0)\in\mathcal{B}_0} \big(\hat{w}_\varrho(t_\eta), \partial_t \hat{w}_\varrho(t_\eta)\big) \tag{4.5}$$

is relatively compact in $H^1(B_\varrho) \times L^2(B_\varrho)$. Since $\hat{w}_\varrho$ is identically null outside B_ϱ, the set $\mathcal{K}_{\eta,\varrho}$ is relatively compact also in $\mathcal{H}$. By virtue of the above three lemmas, we can finally prove that condition (ii) of Theorem 4.1 is fulfilled. Indeed, in correspondence to any $\eta > 0$, we fix r_η as in Lemma 4.2, and we choose ϱ_η as in Lemma 4.3. Then, by (4.5), we construct the relatively compact set $\mathcal{K}_\eta = \mathcal{K}_{\eta,\varrho_\eta}$. Finally, we decompose the solution to (1.1) at time t_η with initial data $(u_0, v_0) \in \mathcal{B}_0$ as

$$S(t_\eta)(u_0, v_0) = v(t_\eta) + \hat{w}_{\varrho_\eta}(t_\eta) + \check{w}_{\varrho_\eta}(t_\eta).$$

Since by construction $\check{w}_{\varrho_\eta}(t_\eta) \in \mathcal{K}_\eta$, from Lemma 4.2 and Lemma 4.3 we get at once the desired inequality $\delta_{\mathcal{H}}(S(t_\eta)\mathcal{B}_0, \mathcal{K}_\eta) \le \eta$.

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