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Critical and subcritical fractional problems with vanishing potentials

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We investigate a class of nonlinear nonautonomous scalar field equations with fractional diffusion, critical power nonlinearity and a subcritical term. The involved potentials are allowed for vanishing behavior at infinity. The problem is set on the whole space and compactness issues have to be tackled.

Keywords: Fractional Laplacian; nonautonomous equations; vanishing potentials; variational methods.

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1. Introduction and Main Results

We consider existence of solutions for the following class of equations

$$(-\Delta)^{\frac{s}{2}}u + V(x)u = K(x)f(u) + \lambda|u|^{2_s^*-2}u \quad \text{in } \mathbb{R}^N. \tag{1.1}$$

Here $\lambda \geq 0$ is a parameter, $s \in (0,2)$, $2_s^* = 2N/(N-s)$, $N > s$, $(-\Delta)^{\frac{s}{2}}$ is fractional Laplacian V, K are positive functions and f is a continuous function with quasi-critical growth. Recently, a great attention has been focused on the study of nonlinear problems involving fractional Laplacian, in view of real-world applications. For instance, this type of operators arise in thin obstacle problems, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science and water waves, see [22]. The fractional Laplacian $(-\Delta)^{\frac{s}{2}}$ with $s \in (0,2)$ of a function $\phi : \mathbb{R}^N \to \mathbb{R}$ is defined by

$$\mathcal{F}((-\Delta)^{\frac{s}{2}}\phi)(\xi) = |\xi|^s \mathcal{F}(\phi)(\xi), \quad \text{for } s \in (0,2),$$

where $\mathcal{F}$ is the Fourier transform. We are going explore problem (1.1) with *zero mass* potential, that is when $V(x) \to 0$, as $|x| \to \infty$. This class was studied by several researchers in the local case $s = 2$, e.g., in [1, 2, 4, 7, 8, 10, 11, 24, 28] and reference therein, where the main feature is to impose restrictions on V, K to get some compact embedding into a weighted L^p space. Recently Alves and Souto, in [3], in addition to improving all the former restrictions on the potentials, handled subcritical nonlinearities f which do not satisfy the so-called Ambrosetti–Rabinowitz condition, namely,

$$\text{(AR)} \quad \text{there exists } \vartheta \in (2, 2_s^*) \text{ with } 0 < \vartheta F(s) \le s f(s) \text{ for all } s > 0,$$

$$F(s) = \int_0^s f(t)dt.$$

Conditions weaker than (AR) were used, first time, in [20, 25, 28, 27, 30]. In all the above cited papers, the nonlinearity f had subcritical growth, that is, in addition to $\lambda = 0$, the growth of f in comparable with s^p with $p < (N+2)/(N-2)$, for $N \ge 3$. In the case $s \in (0,2)$, nonlocal case, we say that f has a subcritical growth, if the growth of f in s is comparable with s^p for $p < (N+s)/(N-s)$, with $N > s$. In this situation, we would like to mention two works, one by Chang and Wang [19], where the authors recovered the Berestycki and Lions [11] results by improving Strauss compactness result [35], and a paper by Secchi [31] where the existence of ground state solutions is established. Motivated by the papers above, we are going to study the nonlocal case, with nonlinearities involving a critical growth and a subcritical perturbation f. Elliptic problems with critical growth, after the pioneering works by Brezis and Nirenberg [14] have had many progresses in several directions. We would like to mention [5, 29, 39] and the references therein, in local case. For nonlocal case, in bounded domain, we cite [9, 15, 23, 26, 32, 37] and references therein, while in whole space was studied recently in [34] for non-vanishing potential. Recently, Caffarelli and Silvestre [16] developed a local interpretation of the fractional Laplacian given in $\mathbb{R}^N$ by considering a Neumann type operator in the extended domain $\mathbb{R}^{N+1}_+$ defined by $\{(x,t) \in \mathbb{R}^{N+1} : t > 0\}$. A similar extension, for nonlocal problems on bounded domain with the zero Dirichlet boundary condition, was established, for instance, by Cabrè and Tan in [15], Tan [38], Capella *et al.* [17], Brändle *et al.* [13]. It is worth noticing that, in a bounded domain, the Fourier definition of the fractional Laplacian and its local Caffarelli–Silvestre interpretation do not agree, see the discussion developed [33] for more details. For $u \in H^{s/2}(\mathbb{R}^N)$, the solution $w \in X^s(\mathbb{R}^{N+1}_+)$ of

$$\begin{cases} -\mathrm{div}(y^{1-s}\nabla w) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ w = u & \text{on } \mathbb{R}^N \times \{0\} \end{cases} \tag{1.2}$$

is called s-harmonic extension $w = E_s(u)$ of u and it is proved in [16] (see also [13]) that

$$\lim_{y\to 0^+} y^{1-s}\frac{\partial w}{\partial y}(x,y) = -\frac{1}{k_s}(-\Delta)^{\frac{s}{2}}u(x),$$

where

$$k_s = \frac{2^{1-s}\Gamma\left(1-\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}.$$

Here the spaces $X^s(\mathbb{R}^{N+1}_+)$ and $H^{s/2}(\mathbb{R}^N)$ are defined as the completion of $C_0^\infty(\overline{\mathbb{R}^{N+1}_+})$ and $C_0^\infty(\mathbb{R}^N)$, under the norms (which actually do coincide, see [13, Lemma A.2])

$$\|w\|_{X^s} := \left(\int_{\mathbb{R}^{N+1}_+} k_s y^{1-s}|\nabla w|^2 \mathrm{d}x\mathrm{d}y\right)^{1/2},$$

$$\|u\|_{H^{\frac{s}{2}}} := \left(\int_{\mathbb{R}^N} |2\pi\xi|^s|\mathbb{F}(u(\xi))|^2 \mathrm{d}\xi\right)^{1/2} = \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x\right)^{1/2}.$$

Our problem (1.1) will be studied in the half-space, namely,

$$\begin{cases} -\mathrm{div}(y^{1-s}\nabla w) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ -k_s\dfrac{\partial w}{\partial \nu} = -V(x)u + K(x)f(u) + \lambda|u|^{2^*_s-2}u & \text{on } \mathbb{R}^N \times \{0\}, \end{cases} \tag{1.3}$$

where

$$\frac{\partial w}{\partial \nu} = \lim_{y\to 0^+} y^{1-s}\frac{\partial w}{\partial y}(x,y).$$

We are looking for a positive solution in the Hilbert space E defined by

$$E = \left\{w \in X^s(\mathbb{R}^{N+1}_+) : \int_{\mathbb{R}^N} V(x)w(x,0)^2 \mathrm{d}x < \infty\right\}$$

endowed with norm

$$\|w\| := \left(\int_{\mathbb{R}^{N+1}_+} k_s y^{1-s}|\nabla w|^2 \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(x)w(x,0)^2 \mathrm{d}x\right)^{1/2}.$$

Consider the Euler–Lagrange functional associated to (1.3) given by

$$J_\lambda(w) := \frac{1}{2}\|w\|^2 - \int_{\mathbb{R}^N} K(x)F(w(x,0))\mathrm{d}x - \frac{\lambda}{2^*_s}\int_{\mathbb{R}^N} w^+(x,0)^{2^*_s}\mathrm{d}x \tag{1.4}$$

which is C^1 with Gâteaux derivative

$$J'_\lambda(w)v = \int_{\mathbb{R}^{N+1}_+} k_s y^{1-s}\nabla w\cdot\nabla v\,\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(x)w(x,0)v(x,0)\mathrm{d}x$$
$$- \int_{\mathbb{R}^N} K(x)f(w(x,0))v(x,0)\mathrm{d}x - \lambda\int_{\mathbb{R}^N} w^+(x,0)^{2^*_s-1}v(x,0)\mathrm{d}x,$$
$$\text{for all } w, v\in E. \tag{1.5}$$

We now formulate assumptions for V, K, f in problem (1.1).

- Assumptions on V and K.
  - (I) **(sign of V and K):** V, K are continuous, $V, K > 0$ on $\mathbb{R}^N$ and $K \in L^\infty(\mathbb{R}^N)$;
  - (II) **(decay of K):** If $\{A_n\}$ is a sequence of Borel sets of $\mathbb{R}^N$ with $|A_n| \le R$ for some $R > 0$,
  $$\lim_{r\to\infty}\int_{A_n\cap B_r^c(0)} K(x)\mathrm{d}x = 0, \quad \text{uniformly with respect to } n\in\mathbb{N}; \tag{1.6}$$
  - (III) **(interrelation between V and K):** either
  $$\frac{K}{V}\in L^\infty(\mathbb{R}^N) \tag{1.7}$$
  or there exists $p\in(2, 2^*_s)$ such that
  $$\lim_{|x|\to\infty}\frac{K(x)}{V(x)^\gamma} = 0, \quad \gamma = \frac{ps - N(p-2)}{2s}\in(0,1). \tag{1.8}$$
- Assumptions on f.
  - (f1) **(behavior at zero):** $f:\mathbb{R}\to\mathbb{R}^+$ is continuous with $f = 0$ on $\mathbb{R}^-$. If (1.7) holds, then
  $$\limsup_{s\to 0^+}\frac{f(s)}{s} = 0.$$
  If condition (1.8) holds, we assume
  $$\limsup_{s\to 0^+}\frac{f(s)}{s^{p-1}} < +\infty.$$
  - (f2) **(quasi-critical growth):** If (1.7) holds, then
  $$\limsup_{s\to+\infty}\frac{f(s)}{s^{2^*_s-1}} = 0.$$
  If condition (1.8) holds, we assume
  $$\limsup_{s\to+\infty}\frac{F(s)}{s^p} < +\infty.$$
  - (f3) **(super-quadraticity):** $\frac{f(s)}{s}$ is non-decreasing in $\mathbb{R}^+$, and
  $$\limsup_{s\to+\infty}\frac{F(s)}{s^2} = +\infty.$$

(f3)$'$ **(super-quadraticity):** $\frac{f(s)}{s}$ is non-decreasing in $\mathbb{R}^+$ and there exist $C_0 > 0$ and $q \in (2, 2_s^*)$ with

$$F(s) \geq C_0 s^q, \quad \text{for all } s \in \mathbb{R}^+.$$

The following are the main results of the paper.

Theorem 1.1. *Assume* (I)–(III), (f1)–(f3) *and* $\lambda = 0$. *Then* (1.1) *admits a positive solution* $u \in E$.

Theorem 1.2. *Assume* (I)–(III), (f1)–(f2)–(f3)$'$, (AR), $\lambda = 1$ *and that one of the following holds*:

(1) $N > 2s$,
(2) $N = 2s$,
(3) $s < N < 2s$ *and* $q > \frac{N}{N-s}$,
(4) $s < N < 2s$ *and* $q < \frac{N}{N-s}$, *with* C_0 *large enough.*

Then (1.1) *admits a positive solution* $u \in E$.

Throughout the paper, unless explicitly stated, the symbol C will always denote a generic positive constant, which may vary from line to line.

2. Preliminary Results

Consider the weighted Banach space:

$$L_K^p = \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ measurable and } \int_{\mathbb{R}^N} K(x)|u|^p \mathrm{d}x < \infty \right\},$$

$$\| \cdot \|_{L_K^p} = \left(\int_{\mathbb{R}^N} K(x)|u|^p \mathrm{d}x \right)^{1/p}.$$

The first result, on compact injections for E, follows by adapting the arguments in [3].

Proposition 2.1 (Compactness). *The following facts hold*:

(1) E *is compactly embedded into* L_K^q *for all* $q \in (2, 2_s^*)$, *provided that* (1.7) *holds*;
(2) E *is compactly embedded into* L_K^p *provided that* (1.8) *holds*;
(3) *If* $w_n \rightharpoonup w$ *in* E, *then up to a subsequence*,

$$\lim_n \int_{\mathbb{R}^N} K(x) F(w_n(x,0)) \mathrm{d}x = \int_{\mathbb{R}^N} K(x) F(w(x,0)) \mathrm{d}x;$$

(4) *If* $w_n \rightharpoonup w$ *in* E, *then up to a subsequence*,

$$\lim_n \int_{\mathbb{R}^N} K(x) w_n(x,0) f(w_n(x,0)) \mathrm{d}x = \int_{\mathbb{R}^N} K(x) w(x,0) f(w(x,0)) \mathrm{d}x;$$

(5) *If* $w_n \rightharpoonup w$ *in* E, *then, up to a subsequence, for any* $v \in E$,

$$\lim_n \int_{\mathbb{R}^N} w_n^+(x,0)^{2_s^*-1} v(x,0) \mathrm{d}x = \int_{\mathbb{R}^N} w^+(x,0)^{2_s^*-1} v(x,0) \mathrm{d}x.$$

(6) *If $w_n \rightharpoonup w$ in E, then up to a subsequence, for any $v \in E$,*

$$\lim_n \int_{\mathbb{R}^N} K(x) f(w_n(x,0)) v(x,0) \mathrm{d}x = \int_{\mathbb{R}^N} K(x) f(w(x,0)) v(x,0) \mathrm{d}x.$$

Proof. Assume that condition (1.7) holds, let $q \in (2, 2_s^*)$ and let us prove assertion (1). Let $\varepsilon > 0$. Then, there exist $0 < s_0(\varepsilon) < s_1(\varepsilon)$, a positive constant $C(\varepsilon)$ and C_0 depending only on V and K, such that

$$\begin{aligned} K(x)|s|^q &\le \varepsilon C_0 (V(x)|s|^2 + |s|^{2_s^*}) \\ &\quad + C(\varepsilon) K(x) \chi_{[s_0(\varepsilon), s_1(\varepsilon)]}(|s|)|s|^{2_s^*}, \quad \text{for all } s \in \mathbb{R}. \end{aligned} \tag{2.1}$$

Therefore we obtain, for every $w \in E$ and $r > 0$,

$$\begin{aligned} &\int_{B_r^{+c}(0) \cap \{y=0\}} K(x)|w(x,0)|^q \mathrm{d}x \\ &\quad \le \varepsilon Q(w) + C(\varepsilon) s_1(\varepsilon)^{2_s^*} \int_{A_\varepsilon \cap (B_r^{+c}(0) \cap \{y=0\})} K(x) \mathrm{d}x, \end{aligned} \tag{2.2}$$

where we have set

$$\begin{aligned} Q(w) &:= C_0 \int_{\mathbb{R}^N} (V(x)|w(x,0)|^2 + |w(x,0)|^{2_s^*}) \mathrm{d}x, \\ A_\varepsilon &:= \{x \in \mathbb{R}^N : s_0(\varepsilon) \le |w(x,0)| \le s_1(\varepsilon)\}. \end{aligned} \tag{2.3}$$

If $(w_n) \subset E$ is such that $w_n \rightharpoonup w$ weakly in E for some $w \in E$, there exists $M > 0$ with

$$\begin{aligned} \int_{\mathbb{R}_+^{N+1}} k_s y^{1-s} |\nabla w_n|^2 \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(x)|w_n(x,0)|^2 \mathrm{d}x \le M, &\quad \text{for all } n \in \mathbb{N}, \\ \int_{\mathbb{R}^N} |w_n(x,0)|^{2_s^*} \mathrm{d}x \le M, &\quad \text{for all } n \in \mathbb{N}, \end{aligned} \tag{2.4}$$

so that $Q(w_n)$ is bounded in $\mathbb{R}$. On the other hand, if $A_\varepsilon^n = \{s_0(\varepsilon) \le |w_n(x,0)| \le s_1(\varepsilon)\}$, we get

$$s_0(\varepsilon)^{2_s^*} |A_\varepsilon^n| \le \int_{A_\varepsilon^n} |w_n(x,0)|^{2_s^*} \mathrm{d}x \le \int_{\mathbb{R}^N} |w_n(x,0)|^{2_s^*} \mathrm{d}x \le M, \quad \text{for all } n \in \mathbb{N}.$$

which implies that $\sup_{n \in \mathbb{N}} |A_\varepsilon^n| < +\infty$. Then, in light of (1.6), there exists $r(\varepsilon) > 0$ such that

$$\int_{A_\varepsilon^n \cap (B_{r(\varepsilon)}^{+c}(0) \cap \{y=0\})} K(x) \mathrm{d}x < \frac{\varepsilon}{C(\varepsilon) s_1(\varepsilon)^{2_s^*}}, \quad \text{for all } n \in \mathbb{N}. \tag{2.5}$$

Whence, invoking (2.2), we get

$$\int_{B_{r(\varepsilon)}^{+c}(0) \cap \{y=0\}} K(x)|w_n(x,0)|^q \mathrm{d}x \le (2C_0 M + 1)\varepsilon. \tag{2.6}$$

By the fractional compact embedding [13], we have

$$\lim_{n\to\infty}\int_{B^+_{r(\varepsilon)}(0)\cap\{y=0\}} K(x)|w_n(x,0)|^q\mathrm{d}x = \int_{B^+_{r(\varepsilon)}(0)\cap\{y=0\}} K(x)|w(x,0)|^q\mathrm{d}x. \tag{2.7}$$

Combining (2.6) and (2.7), yields

$$\lim_{n}\int_{\mathbb{R}^N\cap\{y=0\}} K(x)|w_n(x,0)|^q\mathrm{d}x = \int_{\mathbb{R}^N\cap\{y=0\}} K(x)|w(x,0)|^q\mathrm{d}x,$$

which concludes the proof of (1).

Assume now that condition (1.8) holds and let us prove assertion (2). By a direct calculation, for any $x\in\mathbb{R}^N$ and $s\ge 0$, if $\gamma\in(0,1)$ is the constant introduced in (1.8), we get

$$V(x)s^{2-p}+s^{2^*_s-p}\ge\omega(p,s)V(x)^\gamma,\quad \omega(p,s)=\left(\frac{2^*_s-2}{2^*_s-p}\right)\left(\frac{p-2}{2^*_s-p}\right)^{\frac{2-p}{2^*_s-2}}.$$

Let $\varepsilon>0$. Combining this inequality with (1.8), there exists $r(\varepsilon)>0$ such that

$$K(x)|s|^p\le\varepsilon(V(x)|s|^2+|s|^{2^*_s}),\quad \text{for all } s\in\mathbb{R} \text{ and } |x|\ge r(\varepsilon). \tag{2.8}$$

Then, for all $w\in E$, we conclude

$$\begin{aligned}&\int_{B^{+c}_{r(\varepsilon)}(0)\cap\{y=0\}} K(x)|w(x,0)|^p\mathrm{d}x\\ &\qquad\le\varepsilon\int_{B^{+c}_{r(\varepsilon)}(0)\cap\{y=0\}}(V(x)|w(x,0)|^2+|w(x,0)|^{2^*_s})\mathrm{d}x.\end{aligned}$$

If $(w_n)\subset E$ and $w_n\rightharpoonup w$ weakly in E, there exists $M>0$ such that (2.4) holds. Whence, for a suitable radius $r(\varepsilon)>0$ there holds

$$\int_{B^{+c}_{r(\varepsilon)}(0)\cap\{y=0\}} K(x)|w_n(x,0)|^p\mathrm{d}x\le 2\varepsilon M,\quad \text{for all } n\in\mathbb{N}. \tag{2.9}$$

Since $p\in(2,2^*_s)$, by the fractional compact embedding we have

$$\lim_{n}\int_{B^+_{r(\varepsilon)}(0)\cap\{y=0\}} K(x)|w_n(x,0)|^p\mathrm{d}x = \int_{B^+_{r(\varepsilon)}(0)\cap\{y=0\}} K(x)|w(x,0)|^p\mathrm{d}x. \tag{2.10}$$

Combining (2.9) and (2.10), we get

$$\lim_{n}\int_{\mathbb{R}^N\cap\{y=0\}} K(x)|w_n(x,0)|^p\mathrm{d}x = \int_{\mathbb{R}^N\cap\{y=0\}} K(x)|w(x,0)|^p\mathrm{d}x,$$

which concludes the proof of assertion (2).

Let us now turn to the proof of (3) and (4) under assumption (1.7). From (f_1)–(f_3), fixed $q\in(2,2^*_s)$ and given $\varepsilon>0$, there exist $0<s_0(\varepsilon)<s_1(\varepsilon)$, $C(\varepsilon)>0$ and

C_0 depending only upon V and K, with

$$|K(x)F(s)| \leq \varepsilon C_0(V(x)|s|^2 + |s|^{2_s^*}) + C(\varepsilon)K(x)\chi_{[s_0(\varepsilon),s_1(\varepsilon)]}(|s|)|s|^q, \quad \text{for all } s \in \mathbb{R}, \tag{2.11}$$

$$|K(x)f(s)s| \leq \varepsilon C_0(V(x)|s|^2 + |s|^{2_s^*}) + C(\varepsilon)K(x)\chi_{[s_0(\varepsilon),s_1(\varepsilon)]}(|s|)|s|^q, \quad \text{for all } s \in \mathbb{R}. \tag{2.12}$$

Notice that, by (1.6), arguing as for the proof of (1), there exists $r(\varepsilon) > 0$ such that

$$\int_{A_\varepsilon^n \cap B_{r(\varepsilon)}^{+c}(0)\cap\{y=0\}} K(x)\mathrm{d}x \leq \frac{\varepsilon}{C(\varepsilon)s_1(\varepsilon)^q}, \quad \text{for all } n \in \mathbb{N}. \tag{2.13}$$

Let $\{w_n\} \in E$ be bounded. Combining the above inequality with (2.4), (2.11) and (2.12), we have

$$\int_{B_{r(\varepsilon)}^{+c}(0)\cap\{y=0\}} K(x)F(w_n(x,0))\mathrm{d}x \leq (2C_0M+1)\varepsilon, \quad \text{for all } n \in \mathbb{N}, \tag{2.14}$$

$$\int_{B_{r(\varepsilon)}^{+c}(0)\cap\{y=0\}} K(x)f(w_n(x,0))w_n(x,0)\mathrm{d}x \leq (2C_0M+1)\varepsilon, \quad \text{for all } n \in \mathbb{N}. \tag{2.15}$$

Since $(w_n(x,0))$ is bounded in $L^{2_s^*}(\mathbb{R}^N)$, by Strauss lemma [11, Theorem A.I p. 338], we infer

$$\lim_n \int_{B_{r(\varepsilon)}^{+}(0)\cap\{y=0\}} K(x)F(w_n(x,0))\mathrm{d}x = \int_{B_{r(\varepsilon)}^{+}(0)\cap\{y=0\}} K(x)F(w(x,0))\mathrm{d}x,$$

$$\lim_n \int_{B_{r(\varepsilon)}^{+}(0)\cap\{y=0\}} K(x)f(w_n(x,0))w_n(x,0)\mathrm{d}x = \int_{B_{r(\varepsilon)}^{+}(0)\cap\{y=0\}} K(x)f(w(x,0)w(x,0))\mathrm{d}x.$$

Combining these limits with (2.14) and (2.15) we conclude the proof.

Let us now turn to the proof of (3) and (4) under assumption (1.8). Let $\varepsilon > 0$. We learned that there exists $r(\varepsilon) > 0$ such that (2.8) holds, yielding

$$K(x)|F(s)| \leq \varepsilon(V(x)|F(s)||s|^{2-p} + |F(s)||s|^{2_s^*-p}), \quad \text{for all } s \in \mathbb{R} \text{ and } |x| \geq r(\varepsilon),$$

$$K(x)f(s)s \leq \varepsilon(V(x)f(s)s|s|^{2-p} + f(s)s|s|^{2_s^*-p}), \quad \text{for all } s \in \mathbb{R}^+ \text{ and } |x| \geq r(\varepsilon).$$

From (f_1) and (f_2), there exist $0 < s_0(\varepsilon) < s_1(\varepsilon)$ satisfying

$$K(x)|F(s)| \leq \varepsilon(V(x)|s|^2 + |s|^{2_s^*}), \quad \text{for all } s \in I_\varepsilon \text{ and } |x| \geq r(\varepsilon),$$

$$K(x)f(s)s \leq \varepsilon(V(x)|s|^2 + |s|^{2_s^*}), \quad \text{for all } s \in I_\varepsilon \cap \mathbb{R}^+ \text{ and } |x| \geq r(\varepsilon),$$

where $I_\varepsilon = \{|s| < s_0(\varepsilon) \text{ or } |s| > s_1(\varepsilon)\}$. Then, we have

$$\int_{B^{+c}_{r(\varepsilon)}(0)\cap\{y=0\}} K(x)F(w_n(x,0)\mathrm{d}x$$
$$\leq \varepsilon Q(w_n) + C(\varepsilon)\int_{A^n_\varepsilon\cap(B^{+c}_{r(\varepsilon)}(0)\cap\{y=0\})} K(x)\mathrm{d}x, \tag{2.16}$$

$$\int_{B^{+c}_{r(\varepsilon)}(0)\cap\{y=0\}} K(x)f(w_n(x,0)w_n(x,0)\mathrm{d}x$$
$$\leq \varepsilon Q(w_n) + C(\varepsilon)\int_{A^n_\varepsilon\cap(B^{+c}_{r(\varepsilon)}(0)\cap\{y=0\})} K(x)\mathrm{d}x, \tag{2.17}$$

where

$$C(\varepsilon) = \max\left\{\max_{[s_0(\varepsilon),s_1(\varepsilon)]}|F(s)|, \max_{[s_0(\varepsilon),s_1(\varepsilon)]}|f(s)s|\right\}.$$

Arguing as for the proof of (1), we have

$$\left|\int_{(B^{+c}_{r(\varepsilon)}(0)\cap\{y=0\})} K(x)F(w_n(x,0))\mathrm{d}x\right| \leq (2M+1)\varepsilon, \quad \text{for all } n\in\mathbb{N},$$

$$\left|\int_{(B^{+c}_{r(\varepsilon)}(0)\cap\{y=0\})} K(x)f(w_n(x,0))w_n(x,0)\mathrm{d}x\right| \leq (2M+1)\varepsilon, \quad \text{for all } n\in\mathbb{N}.$$

Invoking again Strauss lemma, by the above inequalities, conclusions (3) and (4) follow. To prove (5), it is enough to observe that $w_n^+(x,0)^{2^*_s-1} \rightharpoonup w^+(x,0)^{2^*_s-1}$ weakly in $(L^{2^*_s})'$. Finally, let us prove (6). If (1.7) holds, then the sequence $(\sqrt{K(x)}f(w_n(x,0))\chi_{\{|w_n(x,0)|\leq 1\}})$ is bounded in $L^2(\mathbb{R}^N)$ being

$$|\sqrt{K(x)}f(w_n(x,0))\chi_{\{|w_n(x,0)|\leq 1\}}|^2 \leq CV(x)|w_n(x,0)|^2.$$

This, by pointwise convergence, yields for every $\varphi\in L^2(\mathbb{R}^N)$

$$\lim_k \int_{\mathbb{R}^N} \sqrt{K(x)}f(w_n(x,0))\chi_{\{|w_n(x,0)|\leq 1\}}\varphi(x)\mathrm{d}x$$
$$= \int_{\mathbb{R}^N} \sqrt{K(x)}f(w(x,0))\chi_{\{|w(x,0)|\leq 1\}}\varphi(x)\mathrm{d}x.$$

Given $v\in E$, since $\sqrt{K(x)} \leq C\sqrt{V(x)}$, it follows $\sqrt{K(x)}v(x,0)\in L^2(\mathbb{R}^N)$, yielding

$$\lim_k \int_{\mathbb{R}^N} K(x)f(w_n(x,0))\chi_{\{|w_n(x,0)|\leq 1\}}v(x,0)\mathrm{d}x$$
$$= \int_{\mathbb{R}^N} K(x)f(w(x,0))\chi_{\{|w(x,0)|\leq 1\}}v(x,0)\mathrm{d}x. \tag{2.18}$$

In a similar fashion, the sequence $(K(x)f(w_n(x,0))\chi_{\{|w_n(x,0)|\geq 1\}})$ is bounded in $L^{\frac{2_s^*}{2_s^*-1}}(\mathbb{R}^N)$ being

$$|K(x)f(w_n(x,0))\chi_{\{|w_n(x,0)|\geq 1\}}|^{\frac{2_s^*}{2_s^*-1}} \leq |w_n(x,0)|^{2_s^*}.$$

This, by pointwise convergence, and since $v \in E$, yields

$$\lim_k \int_{\mathbb{R}^N} K(x)f(w_n(x,0))\chi_{\{|w_n(x,0)|\geq 1\}}v(x,0)\mathrm{d}x$$
$$= \int_{\mathbb{R}^N} K(x)f(w(x,0))\chi_{\{|w(x,0)|\geq 1\}}v(x,0)\mathrm{d}x. \tag{2.19}$$

Combining (2.18) and (2.19) yields the assertion. In a similar fashion one can treat the case when (1.8) holds since, by means of (2), $K^{1/p}v(x,0) \in L^p(\mathbb{R}^N)$ for all $v \in E$ and, up to a subsequence,

$$|K(x)^{\frac{p-1}{p}} f(w_n(x,0))\chi_{\{|w_n(x,0)|\leq 1\}}|^{p'} \leq K(x)|w_n(x,0)|^p \leq z(x) \in L^1(\mathbb{R}^N).$$

This concludes the proof. □

From (f_1) and (f_2) one can prove that J_λ satisfies the Mountain–Pass geometry (cf. [6]).

Lemma 2.2 (Geometry). *The functional J_λ satisfies*

(1) *There exists $\beta, \rho > 0$ such that $J_\lambda(u) \geq \beta$ if $\|u\| = \rho$;*
(2) *There exists $e \in E\backslash\{0\}$ with $\|u\| > \rho$ such that $J_\lambda(e) \leq 0$.*

Proof. (2) is obvious. Concerning (1), observe that in light of condition (1.8) on V and K, the space E is *continuously* embedded into $L^p_K(\mathbb{R})$ where $p \in (2, 2_s^*)$ is the precisely the value which appears in condition (1.8). This can be readily obtained by arguing as in the proof of [12, part (i) of Theorem 4] (see formula (8) therein obtained by Hölder inequality) and by using the fractional Sobolev inequality. This is possible since in any of the two assumptions between V and K, we have that

$$\frac{K}{V^\gamma} \in L^\infty(\mathbb{R}), \quad \gamma = \frac{ps - N(p-2)}{2s}.$$

This is the fractional counterpart of the assumption on $\mathcal{W}$ in [12]. Once this embedding is available, recall that we can write the inequality, for ε_0 to be fixed small

$$K(x)F(s) \leq \varepsilon_0 V(x)s^2 + Cs^{2_s^*} + CK(x)s^p, \quad x \in \mathbb{R}, \ \ s \in \mathbb{R}^+,$$

and the Mountain–Pass geometry can be proved. □

Therefore, there exists a sequence $\{w_n\} \subset E$, so-called *Cerami sequence* [18], such that

$$J_\lambda(w_n) \to c, \quad (1 + \|w_n\|)\|J'_\lambda(w_n)\| \to 0, \tag{2.20}$$

where c is given by

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\lambda(\gamma(t)),$$

with

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0 \text{ and } J_\lambda(\gamma(1)) \le 0\}.$$

Next we turn to the boundedness of (w_n) in E.

Lemma 2.3 (Boundedness). *Let $\lambda \in \{0,1\}$. Then the Cerami sequence $(w_n) \subset E$ is bounded.*

Proof. First of all, we observe that $w_n^- \in E$ and, by the definition of J_λ,

$$\begin{aligned} J'_\lambda(w_n)(-w_n^-) &= -\int_{\mathbb{R}^{N+1}_+} k_s y^{1-s} \nabla w_n \cdot \nabla w_n^- \mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^N} V(x) w_n(x,0) w_n^-(x,0) \\ &= \int_{\mathbb{R}^{N+1}_+} k_s y^{1-s} |\nabla w_n^-|^2 \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(x) w_n^-(x,0)^2 = \|w_n^-\|^2. \end{aligned}$$

Since $(1 + \|w_n\|) J'_\lambda(w_n)(-\|w_n^-\|^{-1} w_n^-) = o_n(1)$ as $n \to \infty$, it follows that $J'_\lambda(w_n)(-w_n^-) = o_n(1)$ as $n \to \infty$, which in turn implies that $\|w_n^-\| = o_n(1)$, as $n \to \infty$.

Case $\boldsymbol{\lambda = 0}$. Denote $J_0 = J$. Let $t_n \in [0,1]$ be such that

$$J(t_n w_n) = \max_{t \in [0,1]} J(t w_n).$$

We *claim* that $J(t_n w_n)$ is bounded from above. Without loss of generality, we may assume that $t_n \in (0,1)$ for all n. Then, we have $J'(t_n w_n)(w_n) = 0$ and

$$\begin{aligned} 2J(t_n w_n) &= 2J(t_n w_n) - J'(t_n w_n)(t_n w_n) \\ &= \int_{\mathbb{R}^N} K(x) \mathcal{H}(t_n w_n(x,0)) \mathrm{d}x \\ &= \int_{\mathbb{R}^N} K(x) \mathcal{H}(t_n w_n^+(x,0)) \mathrm{d}x, \end{aligned} \tag{2.21}$$

where $\mathcal{H}(s) = sf(s) - 2F(s)$ is non-decreasing and $\mathcal{H} = 0$ on $\mathbb{R}^-$. Thus, since $t_n \in (0,1)$ and $w_n^+ \ge 0$, from formula (2.21) we obtain that

$$\begin{aligned} 2J(t_n w_n) &\le \int_{\mathbb{R}^N} K(x) \mathcal{H}(w_n^+(x,0)) \mathrm{d}x = \int_{\mathbb{R}^N} K(x) \mathcal{H}(w_n(x,0)) \mathrm{d}x \\ &= 2J(w_n) - J'(w_n)(w_n) = 2J(w_n) + o_n(1), \end{aligned}$$

which proves the claim. Now, we prove that $(w_n) \subset E$ is bounded. Assume by contradiction that, up to subsequence, $\|w_n\| \to +\infty$ as $n \to \infty$. Set $z_n := w_n / \|w_n\|$ and suppose that $z_n \rightharpoonup z$, as $n \to \infty$, in E. We now *claim* that $z(x,0) = 0$ almost everywhere in $\mathbb{R}^N$. In fact,

$$o_n(1) + \frac{1}{2} = \int_{\mathbb{R}^N} \frac{K(x) F(w_n(x,0))}{\|w_n\|^2} \mathrm{d}x = \int_{\mathbb{R}^N} \frac{K(x) F(w_n(x,0))}{|w_n(x,0)|^2} z_n^2(x,0) \mathrm{d}x.$$

By (f_3), given $\tau > 0$ there exists $\xi_\tau > 0$ such that $F(s) \geq \tau s^2$ for all $|s| \geq \xi_\tau$. Thus,

$$o_n(1) + \frac{1}{2} \geq \int_{\{|w_n(x,0)| \geq \xi_\tau\}} \frac{K(x)F(w_n(x,0))}{|w_n(x,0)|^2} z_n^2(x,0)\mathrm{d}x$$
$$\geq \tau \int_{\mathbb{R}^N} K(x) z_n^2(x,0) \chi_{\{|z_n(x,0)| \geq \frac{\xi_\tau}{\|w_n\|}\}} \mathrm{d}x.$$

Thus, by Fatou lemma, since $z_n^2(x,0)\chi_{\{|z_n(x,0)| \geq \frac{\xi_\tau}{\|w_n\|}\}} \to z(x,0)$ a.e., for any $\tau > 0$, we conclude

$$\frac{1}{2} \geq \tau \int_{\mathbb{R}^N} K(x) z^2(x,0)\mathrm{d}x.$$

Since $K > 0$, it follows $z(x,0) = 0$, by the arbitrariness of $\tau > 0$ and the claim follows. Now, let $B > 0$. Of course $B\|w_n\|^{-1} \in [0,1]$ eventually for $n \geq n_B$, for some $n_B \in \mathbb{N}$. Thus,

$$J(t_n w_n) \geq J(B z_n) = \frac{B^2}{2} - \int_{\mathbb{R}^N} K(x) F(B z_n(x,0))\mathrm{d}x,$$

since t_n is a maximum point. By Proposition 2.1, it follows

$$\int_{\mathbb{R}^N} K(x)F(Bz_n(x,0))\mathrm{d}x \to \int_{\mathbb{R}^N} K(x)F(Bz(x,0)) = 0,$$

and we have $J(t_n w_n) + o_n(1) \geq B^2/2$, which yields $\sup\{J(t_n w_n) : n \in \mathbb{N}\} \geq B^2/2$, a contradiction if

$$B = 2\sqrt{\sup\{J(t_n w_n) : n \in \mathbb{N}\}} \in (0, \infty).$$

This concludes the proof.

Case $\boldsymbol{\lambda = 1}$. Denote $J_\lambda = J$. The boundedness of the $\{w_n\}$ in E follows easily from (AR), since

$$o_n(1) + c \geq J(w_n) - \frac{1}{\vartheta} J'(w_n)(w_n) \geq \left(\frac{1}{2} - \frac{1}{\vartheta}\right) \|w_n\|^2.$$

This concludes the proof. □

The following Sobolev inequality can be found in [13],

$$\int_{\mathbb{R}_+^{N+1}} y^{1-s} |\nabla w|^2 \mathrm{d}x\mathrm{d}y \geq \mathbb{S}(s,N) \left(\int_{\mathbb{R}^N} |w(x,0)|^{2_s^*} \mathrm{d}x \right)^{\frac{2}{2_s^*}},$$
$$\text{for all } w \in X^s(\mathbb{R}_+^{N+1}), \tag{2.22}$$

where

$$\mathbb{S}(s,N) = \frac{\Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{1}{2}(N-s)\right) (\Gamma(N))^{\frac{s}{N}}}{2\pi^{\frac{s}{N}} \Gamma\left(\frac{1}{2}(2-s)\right) \Gamma\left(\frac{1}{2}(N+s)\right) \left(\Gamma\left(\frac{1}{2}N\right)\right)^{\frac{s}{N}}}.$$

This constant is achieved on the family of functions [13, 21, 32] $w_\varepsilon = E_s(u_\varepsilon)$ (by [36] for $s = 2$), where

$$u_\varepsilon(x) = \frac{\varepsilon^{\frac{N-s}{2}}}{(|x|^2 + \varepsilon^2)^{\frac{N-s}{2}}}, \quad \varepsilon > 0.$$

Furthermore, take $\phi(x, y) = \phi_0(|(x, y)|)$, where $\phi_0 \in C^\infty(0, \infty)$ is a non-increasing cut-off such that

$$\phi_0(s) = 1 \text{ if } s \in [0, 1/2], \quad \phi_0(s) = 0 \text{ if } s \geq 1.$$

Let ϕw_ε which belongs to $X^s(\mathbb{R}^{N+1}_+)$. By [9, Lemma 3.8] (which is formulated on a bounded domain Ω, but which holds with the very same proof when taking $\Omega = \mathbb{R}^N$), we have the following.

Lemma 2.4 (Concentration). *The family $\{\phi w_\varepsilon\}$, and its trace on $\{y = 0\}$, namely, ϕu_ε, satisfy*

$$\|\phi w_\varepsilon\|^2_{X^s} \leq \|w_\varepsilon\|^2_{X^s} + C\varepsilon^{N-s}, \tag{2.23}$$

$$\|\phi u_\varepsilon\|^2_{L^2} = \begin{cases} \mathcal{O}(\varepsilon^s) & \text{if } N > 2s, \\ \mathcal{O}(\varepsilon^s \log(1/\varepsilon)) & \text{if } N = 2s, \\ \mathcal{O}(\varepsilon^{N-s}) & \text{if } N < 2s, \end{cases} \tag{2.24}$$

for $\varepsilon > 0$ small enough. Define $\eta_\varepsilon = \phi w_\varepsilon / \|\phi u_\varepsilon\|_{L^{2^*_s}}$, then

$$\|\eta_\varepsilon\|^2_{X^s} \leq k_s \mathbb{S}(s, N) + C\varepsilon^{N-s}, \tag{2.25}$$

$$\|\eta_\varepsilon(x, 0)\|^2_{L^2} = \begin{cases} \mathcal{O}(\varepsilon^s) & \text{if } N > 2s, \\ \mathcal{O}(\varepsilon^s \log(1/\varepsilon)) & \text{if } N = 2s, \\ \mathcal{O}(\varepsilon^{N-s}) & \text{if } N < 2s \end{cases} \tag{2.26}$$

and

$$\|\eta_\varepsilon(x, 0)\|^q_{L^q} = \begin{cases} \mathcal{O}(\varepsilon^{\frac{2N-(N-s)q}{2}}) & \text{if } q > \dfrac{N}{N-s} \text{ (or } N \geq 2s), \\ \\ \mathcal{O}(\varepsilon^{\frac{(N-s)q}{2}}) & \text{if } q < \dfrac{N}{N-s}. \end{cases} \tag{2.27}$$

Here $a_\varepsilon = \mathcal{O}(b_\varepsilon)$ means that $C_1 \leq a_\varepsilon / b_\varepsilon \leq C_2$ for some $C_1, C_2 > 0$, independent of ε.

Remark 2.5. We remark that, actually, except (2.23) and (2.25), the other estimates follow exactly as in local case (see [14]), because in these cases, we know the explicit expression for u_ε. While, for w_ε, except for $s = 1$ (see [37]) and $s = 2$ (local case), the explicit expressions are not available. But, in [9], the authors were clever to overcome this difficulty, by exploring some properties of the Poisson kernel.

The s-harmonic extension of the u_ε has the following explicit expression

$$w_\varepsilon(x,y) = P_y^s * u_\varepsilon(x) = C_{N,s} y^s \int_{\mathbb{R}^N} \frac{u_\varepsilon(\xi)}{(|x-\xi|^2+y^2)^{\frac{N+s}{2}}} \mathrm{d}\xi, \quad \text{for some } C_{N,s} > 0.$$

Noticing that as u_ε and P_y^s are self-similar functions, namely

$$u_\varepsilon(x) = \varepsilon^{\frac{s-N}{2}} u_1\left(\frac{x}{\varepsilon}\right), \quad P_y^s(x) = \frac{1}{y^N} P_1^s\left(\frac{x}{|y|}\right) = \frac{y^s}{(|x|^2+y^2)^{\frac{N+s}{2}}},$$

then

$$w_\varepsilon(x,y) = \varepsilon^{\frac{s-N}{2}} w_1\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right).$$

Exploiting this fact, they estimate as follows

$$\int_{\mathbb{R}^{N+1}_+} y^{1-s} w_\varepsilon \phi \nabla\phi \cdot \nabla w_\varepsilon \mathrm{d}x\mathrm{d}y \le C\varepsilon^{N-s},$$

$$\int_{\mathbb{R}^{N+1}_+} y^{1-s} |w_\varepsilon \nabla\phi|^2 \mathrm{d}x\mathrm{d}y \le C\varepsilon^{N-s}.$$

Combining these inequalities, (2.23) holds. The inequality (2.25) comes as a consequence. Concerning (2.27), we justify it in the case $q < N/(N-s)$, the opposite case being similar. We have

$$\begin{aligned}
\|\phi u_\varepsilon\|_{L^q}^q &= \int_{\mathbb{R}^N} |\phi|^q |u_\varepsilon|^q \mathrm{d}x \ge \int_{B(0,\frac{1}{2})} |u_\varepsilon|^q \mathrm{d}x \\
&= \varepsilon^{\frac{(N-s)q}{2}} \int_{B(0,\frac{1}{2})} \frac{1}{(\varepsilon^2+|x|^2)^{\frac{(N-s)q}{2}}} \mathrm{d}x \\
&= \varepsilon^{\frac{(N-s)q}{2}} \int_{B(0,\frac{1}{2\varepsilon})} \frac{1}{\varepsilon^{(N-s)q}(1+|y|^2)^{\frac{(N-s)q}{2}}} \varepsilon^N \mathrm{d}y \\
&= \varepsilon^{N-\frac{(N-s)q}{2}} \int_{B(0,\frac{1}{2\varepsilon})} \frac{1}{(1+|y|^2)^{\frac{(N-s)q}{2}}} \mathrm{d}y \\
&= \varepsilon^{N-\frac{(N-s)q}{2}} C \int_0^{1/(2\varepsilon)} \frac{1}{(1+\varrho^2)^{\frac{(N-s)q}{2}}} \varrho^{N-1} \mathrm{d}\varrho \\
&= \varepsilon^{N-\frac{(N-s)q}{2}} C \left(\int_0^1 \frac{1}{(1+\varrho^2)^{\frac{(N-s)q}{2}}} \varrho^{N-1} \mathrm{d}\varrho \right. \\
&\quad \left. + \int_1^{1/(2\varepsilon)} \frac{1}{(1+\varrho^2)^{\frac{(N-s)q}{2}}} \varrho^{N-1} \mathrm{d}\varrho \right) \\
&\ge \varepsilon^{N-\frac{(N-s)q}{2}} \left(C + C \int_1^{1/(2\varepsilon)} \frac{1}{\varrho^{(N-s)q-N+1}} \mathrm{d}\varrho \right) \\
&\ge \varepsilon^{N-\frac{(N-s)q}{2}} (C + C\varepsilon^{(N-s)q-N}) \ge C\varepsilon^{\frac{(N-s)q}{2}},
\end{aligned}$$

for $\varepsilon > 0$ small enough. Analogously, we get

$$\begin{aligned}
\|\phi u_\varepsilon\|_{L^q}^q &= \int_{\mathbb{R}^N} |\phi|^q |u_\varepsilon|^q \mathrm{d}x \le \int_{B(0,1)} |u_\varepsilon|^q \mathrm{d}x \\
&= \varepsilon^{\frac{(N-s)q}{2}} \int_{B(0,1)} \frac{1}{(\varepsilon^2 + |x|^2)^{\frac{(N-s)q}{2}}} \mathrm{d}x \\
&= \varepsilon^{\frac{(N-s)q}{2}} \int_{B(0,\frac{1}{\varepsilon})} \frac{1}{\varepsilon^{(N-s)q}(1+|y|^2)^{\frac{(N-s)q}{2}}} \varepsilon^N \mathrm{d}y \\
&= \varepsilon^{N-\frac{(N-s)q}{2}} \int_{B(0,\frac{1}{\varepsilon})} \frac{1}{(1+|y|^2)^{\frac{(N-s)q}{2}}} \mathrm{d}y \\
&= \varepsilon^{N-\frac{(N-s)q}{2}} C \int_0^{1/\varepsilon} \frac{1}{(1+\varrho^2)^{\frac{(N-s)q}{2}}} \varrho^{N-1} \mathrm{d}\varrho \\
&= \varepsilon^{N-\frac{(N-s)q}{2}} C \left(\int_0^1 \frac{1}{(1+\varrho^2)^{\frac{(N-s)q}{2}}} \varrho^{N-1} \mathrm{d}\varrho \right. \\
&\qquad \left. + \int_1^{1/\varepsilon} \frac{1}{(1+\varrho^2)^{\frac{(N-s)q}{2}}} \varrho^{N-1} \mathrm{d}\varrho \right) \\
&\le \varepsilon^{N-\frac{(N-s)q}{2}} \left(C + C \int_1^{1/\varepsilon} \frac{1}{\varrho^{(N-s)q-N+1}} \mathrm{d}\varrho \right) \\
&\le \varepsilon^{N-\frac{(N-s)q}{2}} (C + C\varepsilon^{(N-s)q-N}) \le C\varepsilon^{\frac{(N-s)q}{2}},
\end{aligned}$$

for $\varepsilon > 0$ small enough. Since $\|\phi u_\varepsilon\|_{L^{2_s^*}}$ converges to a positive constant, the assertion follows. The remaining assertions can be obtained analogously.

The following result will be crucial for the proof of our main result.

Lemma 2.6 (MP energy bound). *Let* $\lambda = 1$ *and let* (f1)–(f2)–(f3)$'$ *hold. Then* $c < \frac{s}{2N}(k_s \mathbb{S}(s,N))^{N/s}$.

Proof. By definition of c, it is sufficient to prove that there exists $\varepsilon > 0$ small enough that

$$\sup_{t\ge 0} J(t\eta_\varepsilon) < \frac{s}{2N}(k_s \mathbb{S}(s,N))^{N/s}, \quad J = J_1.$$

By definition of J, we have

$$J(t\eta_\varepsilon) = \frac{t^2}{2}\|\eta_\varepsilon\|^2 - \int_{\mathbb{R}^N} K(x) F(t\eta_\varepsilon(x,0)) \mathrm{d}x - \frac{t^{2_s^*}}{2_s^*}.$$

By the assumptions of f, there exist $q \in (2, 2_s^*)$ and $C_0 > 0$ with $F(s) \ge C_0 s^q$ for any $s \in \mathbb{R}^+$. Then

$$J(t\eta_\varepsilon) \le \psi(t), \quad \psi(t) = \frac{t^2}{2}\|\eta_\varepsilon\|^2 - C_0 t^q \int_{\mathbb{R}^N} |\eta_\varepsilon(x,0)|^q \mathrm{d}x - \frac{t^{2_s^*}}{2_s^*}.$$

Since $\psi(t) \to -\infty$ as $t \to +\infty$, we have $\sup\{\psi(t) : t \geq 0\} = \psi(t_\varepsilon)$ for some $t_\varepsilon > 0$, so that

$$\|\eta_\varepsilon\|^2 - C_0 q t_\varepsilon^{q-2} \int_{\mathbb{R}^N} |\eta_\varepsilon(x,0))|^q \mathrm{d}x = t_\varepsilon^{2_s^*-2},$$

which yields $\sigma_0 \leq t_\varepsilon \leq \|\eta_\varepsilon\|^{\frac{2}{2_s^*-2}} \leq K_0$ for some $\sigma_0, K_0 > 0$ independent of ε, in view of Lemma 2.4 and the above equality. Since the map

$$[0, \|\eta_\varepsilon\|^{\frac{2}{2_s^*-2}}] \ni t \mapsto \frac{t^2}{2}\|\eta_\varepsilon\|^2 - \frac{t^{2_s^*}}{2_s^*},$$

increases, we get for some universal constant $C > 0$,

$$\begin{aligned}
\sup_{\mathbb{R}^+} \psi &\leq \frac{s}{2N}\left(\|\eta_\varepsilon\|_{X^s}^2 + \int_{\mathbb{R}^N} V(x)\eta_\varepsilon(x,0)^2 \mathrm{d}x\right)^{N/s} - C_0 C \|\eta_\varepsilon(x,0))\|_{L^q}^q \\
&\leq \frac{s}{2N}\left(k_s \mathbb{S}(s,N) + C\varepsilon^{N-s} + \int_{\mathbb{R}^N} V(x)\eta_\varepsilon(x,0)^2 \mathrm{d}x\right)^{N/s} \\
&\quad - C_0 C \|\eta_\varepsilon(x,0))\|_{L^q}^q.
\end{aligned}$$

Now, by the elementary inequality $(a+b)^\alpha \leq a^\alpha + \alpha(a+b)^{\alpha-1}b$, $\alpha \geq 1$ and $a, b > 0$, we get by (2.25)

$$\begin{aligned}
\sup_{\mathbb{R}^+} \psi &\leq \frac{s}{2N}(k_s \mathbb{S}(s,N))^{N/s} + C\varepsilon^{N-s} \\
&\quad + C\int_{\mathbb{R}^N} V(x)\eta_\varepsilon(x,0)^2 \mathrm{d}x - C_0 C \|\eta_\varepsilon(x,0))\|_{L^q}^q \\
&\leq \frac{s}{2N}(k_s \mathbb{S}(s,N))^{N/s} + C\varepsilon^{N-s} + C\|\eta_\varepsilon(x,0)\|_{L^2}^2 - C_0 C \|\eta_\varepsilon(x,0))\|_{L^q}^q.
\end{aligned}$$

- In the case $N > 2s$, by means of (2.26) and (2.27), we get

$$\sup_{\mathbb{R}^+} \psi \leq \frac{s}{2N}(k_s \mathbb{S}(s,N))^{N/s} + C\varepsilon^{N-s} + C\varepsilon^s - C_0 C \varepsilon^{\frac{2N-(N-s)q}{2}}.$$

  Since $\frac{2N-(N-s)q}{2} < s < N-s$, we get the conclusion for ε sufficiently small.
- If $N = 2s$ and $2 < q < 2_s^* = 4$, by (2.26) and (2.27), we get

$$\sup_{\mathbb{R}^+} \psi \leq \frac{s}{2N}(k_s \mathbb{S}(s,N))^{N/s} + C\varepsilon^s(1 + \log(\varepsilon^{-1})) - C_0 C \varepsilon^{\frac{2N-sq}{2}}.$$

  Since it holds

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^{\frac{2N-sq}{2}}}{\varepsilon^s \log(\varepsilon^{-1})} = +\infty,$$

  again we can get the conclusion, for ε sufficiently small.
- If $s < N < 2s$ and $\frac{N}{N-s} < q < 2_s^*$, by (2.26) and (2.27), we get

$$\sup_{\mathbb{R}^+} \psi \leq \frac{s}{2N}(k_s \mathbb{S}(s,N))^{N/s} + C\varepsilon^{N-s} - C_0 C \varepsilon^{\frac{2N-(N-s)q}{2}}.$$

Since $\frac{2N-(N-s)q}{2} < N-s$ means $q > \frac{2s}{N-s}(> 2)$, we get the conclusion for ε sufficiently small.

- If $s < N < 2s$ and $2 < q < \frac{N}{N-s}$, by (2.26) and (2.27), we get

$$\sup_{\mathbb{R}^+} \psi \leq \frac{s}{2N}(k_s \mathbb{S}(s,N))^{N/s} + C\varepsilon^{N-s} - C_0 C \varepsilon^{\frac{(N-s)q}{2}},$$

and for $C_0 = \varepsilon^{-\vartheta}$ with $\vartheta > \frac{(N-s)(q-2)}{2}$, we get the conclusion. This concludes the proof. □

3. Proof of Theorem 1.1

Proof of Theorem 1.1. In light of Lemma 2.2, there exists a *Cerami* sequence $\{w_n\} \subset E$ for $J = J_0$. From Lemma 2.3, it follows that $w_n^- \to 0$ in E as $n \to \infty$ and that $\{w_n\}$ is bounded and has a non-negative weak limit $w \in E$. By Proposition 2.1, it follows that w is a weak non-negative solution, to which a weak solution $u \in H^{s/2}(\mathbb{R}^N)$ to (1.1) corresponds. We have $u > 0$ if $u \neq 0$. In fact, if $u(x_0) = 0$ for some $x_0 \in \mathbb{R}^N$, then $(-\Delta)^{s/2}u(x_0) = 0$ and by the representation formula [22]

$$(-\Delta)^{s/2}u(x) = -\frac{c(N,s/2)}{2}\int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|x-y|^{N+s}}\mathrm{d}y,$$

one obtains, at x_0, that

$$\int_{\mathbb{R}^N} \frac{u(x_0+y)+u(x_0-y)}{|x_0-y|^{N+s}}\mathrm{d}y = 0,$$

yielding $u = 0$, a contradiction. Let us prove that, indeed, $u \neq 0$. We prove that $w = E_s(u) \not\equiv 0$. In fact, (w_n) converges to w strongly in E. Indeed, up to a subsequence, $w_n \rightharpoonup w$ in E as $n \to \infty$, and since $J'(w_n)(w_n) = o_n(1)$, we have, again by virtue of Proposition 2.1,

$$\begin{aligned}\lim_{n\to\infty} \|w_n\|^2 &= \lim_{n\to\infty} \int_{\mathbb{R}^N} K(x)f(w_n(x,0))w_n(x,0)dx \\ &= \int_{\mathbb{R}^N} K(x)f(w(x,0))w(x,0)dx = \|w\|^2,\end{aligned}$$

that is, $w_n \to w$ in E. Hence $J(w) = c$ and $J'(w) = 0$, this implies that $w \not\equiv 0$.

The proof is completed. □

4. Proof of Theorem 1.2

Proof of Theorem 1.2. In light of Lemma 2.2, there exists a *Cerami* sequence $\{w_n\} \subset E$ for $J = J_1$. From Lemma 2.3 it follows that $w_n^- \to 0$ in E as $n \to \infty$ and that $\{w_n\}$ is bounded and has a non-negative weak limit $w \in E$. By Proposition 2.1, it follows that w is a weak non-negative solution, to which a weak solution $u \in H^{s/2}(\mathbb{R}^N)$ to (1.1) corresponds. We have $u > 0$ if $u \neq 0$, arguing as in Sec. 3. Let us prove that, indeed, $u \neq 0$. We prove that $w = E_s(u) \not\equiv 0$. By virtue of (2.20), we have

$$\frac{1}{2}\|w_n\|^2 - \int_{\mathbb{R}^N} K(x)F(w(x,0))\mathrm{d}x - \frac{1}{2_s^*}\int_{\mathbb{R}^N} w_n^+(x,0)^{2_s^*}\mathrm{d}x = c + o_n(1),$$

$$\|w_n\|^2 - \int_{\mathbb{R}^N} K(x)f(w(x,0))w(x,0)\mathrm{d}x - \int_{\mathbb{R}^N} w_n^+(x,0)^{2_s^*}\mathrm{d}x = o_n(1).$$

Suppose, by contradiction, that $w \equiv 0$. Then, we entail

$$\left(\frac{1}{2} - \frac{1}{2_s^*}\right)\|w_n\|^2 = c + o_n(1),$$

which combined with $\|w_n\|^2 = \|w_n(x,0)\|_{2_s^*}^{2_s^*} + o_n(1)$ as $n \to \infty$ and the Sobolev inequality

$$\|w_n\|^2 \geq \int_{\mathbb{R}_+^{N+1}} k_s y^{1-s}|\nabla w|^2\mathrm{d}x\mathrm{d}y \geq k_s\mathbb{S}(s,N)\|w_n(x,0)\|_{2_s^*}^2$$

implying

$$c = \lim_n J(w_n) = \left(\frac{1}{2} - \frac{1}{2_s^*}\right)\lim_n \|w_n\|^2 \geq \frac{s}{2N}(k_s\mathbb{S}(s,N))^{N/s}.$$

This contradicts Lemma 2.6. Hence $w \not\equiv 0$ and the proof is complete. □

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