

NORMALIZED SOLUTIONS OF QUASILINEAR SCHRÖDINGER EQUATIONS WITH A GENERAL NONLINEARITY

TING DENG, MARCO SQUASSINA, JIANJUN ZHANG, AND XUEXIU ZHONG

ABSTRACT. We are concerned with solutions of the following quasilinear Schrödinger equations

$$-\operatorname{div}(\varphi^2(u)\nabla u) + \varphi(u)\varphi'(u)|\nabla u|^2 + \lambda u = f(u), \quad x \in \mathbb{R}^N$$

with prescribed mass

$$\int_{\mathbb{R}^N} u^2 dx = c,$$

where $N \geq 3, c > 0, \lambda \in \mathbb{R}$ appears as the Lagrange multiplier and $\varphi \in C^1(\mathbb{R}, \mathbb{R}^+)$. The nonlinearity $f \in C(\mathbb{R}, \mathbb{R})$ is allowed to be mass-subcritical, mass-critical and mass-supercritical at origin and infinity. Via a dual approach, the fixed point index and a global branch approach, we establish the existence of normalized solutions to the problem above. The results extend previous results by L. Jeanjean, J. J. Zhang and X.X. Zhong to the quasilinear case.

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1. INTRODUCTION

1.1. Background and motivation. This paper is concerned with the quasilinear Schrödinger equation of the form

$$(1.1) \quad -\operatorname{div}(\varphi^2(u)\nabla u) + \varphi(u)\varphi'(u)|\nabla u|^2 + \lambda u = f(u), \quad x \in \mathbb{R}^N,$$

which is relevant to solitary wave solutions of quasilinear Schrödinger equation

$$(1.2) \quad iz_t = -\Delta z + W(x)z - h(|z|^2)z - \Delta(l(|z|^2))l'(|z|^2)z, \quad x \in \mathbb{R}^N,$$

where $z : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{C}, W : \mathbb{R}^N \rightarrow \mathbb{R}$ is a given potential and l, h are real functions. Quasilinear equations of the form (1.2) appear naturally in mathematical physics and have been accepted

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as models of several physical phenomena corresponding to various types of nonlinear terms l . When $l(s) = s$, that is, $\varphi^2(u) = 1 + 2u^2$, Equation (1.2) arises in plasma physics for the superfluid film equation (see [14]), while when $l(s) = (1 + s)^{1/2}$, that is, $\varphi^2(u) = 1 + \frac{u^2}{2(1+u^2)}$, it models the self-channeling of a high-power ultrashort laser in matter (see [16]). Such equations were also showed in plasma physics and fluid mechanics [17], in dissipative quantum mechanics [9], in the theory of Heisenberg ferromagnetism and magnons [15, 25], and in condensed matter theory [23]. For more details, we refer to [18, 19].

In the last decades, quasilinear Schrödinger equations have received a considerable attentions from researchers on the existence of nontrivial solutions. In [4], M. Colin and L. Jeanjean considered the following quasilinear Schrödinger equations

$$(1.3) \quad -\Delta u - \Delta(u^2)u = g(x, u), \quad x \in \mathbb{R}^N.$$

Thanks to a dual approach, they transform equation (1.3) into a semilinear elliptic equation. By using the variational method, nontrivial solutions were obtained. In [8], E. Gloss adopted the dual approach in [4] to study the semiclassical states of quasilinear elliptic equations

$$-\varepsilon^2 \Delta u - \varepsilon^2 \Delta(u^2)u + V(x)u = h(u), \quad u > 0 \text{ in } \mathbb{R}^N.$$

Via the penalized argument as in [3], she established the existence and concentration of positive solutions without some growth conditions such as Ambrosetti-Rabinowitz. Different from [4], Y. Shen and Y. Wang [26] introduced a new variable replacement to study the existence of nontrivial solutions for generalized quasilinear Schrödinger equations. By using the dual approach in [26], Y. Deng, S. Peng and S. Yan [6] considered a generalized quasilinear Schrödinger equation with critical growth. By virtue of variational approaches, they obtained the existence of positive solutions. In [1], C. O. Alves, Y. Wang and Y. Shen considered the following quasilinear Schrödinger equation with one parameter

$$(1.4) \quad -\Delta u + V(x)u + \frac{\kappa}{2} \Delta(u^2)u = l(u), \quad x \in \mathbb{R}^N,$$

where κ is allowed to be positive. When $\kappa < 0$, one can use the dual approaches in [4, 26] to deal with the existence of nontrivial solutions for equation (1.4). The case of $\kappa > 0$ becomes more complicated. By adopting a truncation approach, the authors obtained the existence of nontrivial classical solution for $\kappa > 0$ small enough. For more related results on quasilinear elliptic equations, we refer to [10, 21, 22] and references therein. For a direct approach via nonsmooth critical point theory in cases where the change of variable is not usable, see [24].

In this paper, we focus on normalized solutions of equation (1.5) with prescribed mass, that is to find (λ, u) such that

$$(1.5) \quad \begin{cases} -\operatorname{div}(\varphi^2(u)\nabla u) + \varphi(u)\varphi'(u)|\nabla u|^2 + \lambda u = f(u), & x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = c, \end{cases}$$

where $N \geq 3$, $f \in C(\mathbb{R}, \mathbb{R})$, $\varphi : \mathbb{R} \rightarrow \mathbb{R}^+$ is a C^1 nondecreasing function with respect to $|s|$, $c > 0$ is a given mass, $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier. From the view of physics, prescribed mass represents the law of conservation of mass, so it is pretty meaningful to investigate the existence of normalized solutions. If $\varphi \equiv 1$, problem (1.5) is reduced to the

following scalar field equation with prescribed mass

$$(1.6) \quad \begin{cases} -\Delta u + \lambda u = f(u), & x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = c. \end{cases}$$

One classical argument to deal with such problems is to find critical points $u \in H^1(\mathbb{R}^N)$ of the energy functional

$$\mathcal{J}(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} F(u) dx, \quad \text{where } F(s) := \int_0^s f(t) dt,$$

subject to the constraint

$$S_c := \{u \in H^1(\mathbb{R}^N) : \|u\|_{L^2(\mathbb{R}^N)}^2 = c\}.$$

When $f(s) = |s|^{p-2}s$ with $p \in (2, 2N/(N-2))$, thanks to the well-known Gagliardo-Nirenberg inequality, it is known that the energy functional \mathcal{J} is bounded from below on S_a for any $a > 0$ if $p < 2 + 4/N$ and unbounded from below for any $a > 0$ if $p > 2 + 4/N$. $2 + 4/N$ is called the mass-critical exponent. In [27, 28], C. A. Stuart considered problem (1.6) with a mass-subcritical nonlinearity and obtained the existence of normalized solutions by seeking a global minimizer of the energy functional on S_c . In the mass supercritical case, since there exists no global minimizer of the associated energy functional restricted on the constraint S_c , it seems that the arguments in dealing with the mass subcritical case do not work. In [11], L. Jeanjean imposed the following global condition on f .

(H) $\exists(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$ satisfying

$$2 + \frac{4}{N} < \alpha \leq \beta < \frac{2N}{(N-2)_+}$$

such that

$$\alpha F(s) \leq f(s)s \leq \beta F(s).$$

The assumption (H) is used to guarantee the mountain pass geometry of $\tilde{\mathcal{J}}$ on $S_c \times \mathbb{R}$, where $\tilde{\mathcal{J}}(u, s) = \mathcal{J}(e^{sN/2}u(e^{\cdot}))$. Then a bounded Palais-Smale sequence of \mathcal{J} restricted on S_c is obtained. Via some compactness arguments, the author shows the existence of normalized solutions to problem (1.6). More recently, L. Jeanjean, J. Zhang and X. Zhong [13] introduced a new and non-variational approach to deal with problem (1.6) in the mass-subcritical, mass-critical and mass-supercritical case in one unified way. In particular, the assumption (H) can be removed. For more related results on problem (1.6), we refer to [13] and references therein.

When $\varphi^2(u) = 1 + 2u^2$, problem (1.1) reads as the following form

$$(1.7) \quad -\Delta u - u\Delta(u^2) + \lambda u = f(u) \quad \text{in } \mathbb{R}^N.$$

Different to the case $\varphi \equiv 1$, $4 + 4/N$ was proved by M. Colin, L. Jeanjean and M. Squassina [5] to be the mass-critical exponent for problem (1.7). In [12], L. Jeanjean, T. Luo and Z.-Q. Wang considered problem (1.7) with a mass-subcritical nonlinearity $f(u) = |u|^{p-2}u$, $p \in (2 + 4/N, 4 + 4/N)$ and prove the existence of two solutions if the prescribed L^2 -norm is large enough. In [30], H. Ye and Y. Yu focused on the existence of normalized solutions of problem (1.7) in the mass-critical case. For the mass-supercritical case, we refer to [20], where H. Li and W. Zou adopted one perturbation argument as in [21] to get the existence of the existence of ground state normalized solutions to problem (1.7). By applying the index theory,

infinitely many normalized solutions were also obtained. Moreover, they also established the concentration behavior of ground state solutions in the mass-critical case. In [31], via the dual approach in [4] and genus theory, L. Zhang, Y. Li and Z.-Q. Wang constructed multiple normalized solutions of problem (1.7) with $f(u) = |u|^{p-2}u$ in the mass-subcritical case.

When $\varphi^2(u) = 1 + \frac{u^2}{2(1+u^2)}$, equation (1.1) is reduced to the following quasilinear equation

$$(1.8) \quad -\Delta u + \lambda u - \left[\Delta (1 + u^2)^{\frac{1}{2}} \right] \frac{u}{2(1 + u^2)^{\frac{1}{2}}} = f(u) \text{ in } \mathbb{R}^N.$$

In [29], under the Berestycki-Lions condition, X. Yang, X. Tang and B. Cheng show that problem (1.8) admits multiple radial and nonradial normalized solutions in the mass-subcritical case. For more processes on normalized solutions of quasilinear Schrödinger equations, we refer to [12, 20, 30, 31] and the references therein.

1.2. Assumptions and main results. The main purpose of the present paper is to investigate the existence of normalized solutions to problem (1.5) with a relatively general nonlinearity. Throughout this paper, we impose the following assumptions on φ .

(φ_0) $\varphi \in C^1(\mathbb{R}, \mathbb{R}^+)$ is even, $\varphi'(t) \geq 0$ for all $t \geq 0$ and $\varphi(0) = 1$.

(φ_1) There exists $a^* > 0$ such that

$$\lim_{s \rightarrow +\infty} \varphi(s) = a^* > 0.$$

(φ_2) $\lim_{t \rightarrow \infty} t\varphi'(t) = 0$.

Remark 1.1. As a reference model, take $\varphi(t) = \left(1 + \frac{t^2}{2(1+t^2)}\right)^{\frac{1}{2}}$ satisfying (φ_0)-(φ_2).

As for the nonlinearity f , we assume that

(F_1) $f \in C^1[0, +\infty)$, $f(s) > 0$ for $s > 0$.

(F_2) There exist $\alpha, \beta, \mu_1, \mu_2 > 0$ satisfying

$$2 < \alpha, \beta < 2^* := \frac{2N}{N-2}$$

such that

$$\lim_{t \rightarrow 0^+} \frac{f'(t)}{t^{\alpha-2}} = \mu_1(\alpha - 1) > 0 \quad \text{and} \quad \lim_{t \rightarrow +\infty} \frac{f'(t)}{t^{\beta-2}} = \mu_2(\beta - 1) > 0.$$

(F_3) There exists no positive radial decreasing classical solution for $-\operatorname{div}(\varphi^2(u)\nabla u) + \varphi(u)\varphi'(u)|\nabla u|^2 = f(u)$ in \mathbb{R}^N .

Remark 1.2. By [13, Theorem 2.2-(ii)], (F_3) holds for $f(t) = |t|^{q-2}t$, $q \in (2, 2N/(N-2))$.

Before to state our result, denote by U the unique positive solution of

$$(1.9) \quad -\Delta U + U = \mu_1 U^{\alpha-1} \text{ in } \mathbb{R}^N, \quad U(0) = \max_{x \in \mathbb{R}^N} U(x)$$

and V the unique positive solution of

$$(1.10) \quad -\Delta V + V = \mu_2 V^{\beta-1} \text{ in } \mathbb{R}^N, \quad V(0) = \max_{x \in \mathbb{R}^N} V(x).$$

It is known that U, V are non-degenerated (See [13, Proposition 2.1]).

Due to the term φ , equation (1.1) is quasilinear. Similarly as in [4], one can investigate the following equivalent semilinear elliptic equation

$$(1.11) \quad -\Delta v + \lambda \frac{\phi^{-1}(v)}{\varphi(\phi^{-1}(v))} - \frac{f(\phi^{-1}(v))}{\varphi(\phi^{-1}(v))} = 0, \quad x \in \mathbb{R}^N$$

via the transformation

$$v = \phi(u) = \int_0^u \varphi(t) dt.$$

Moreover, the existence of solutions to equation (1.1) is reduced to the one of solutions to equation (1.11).

Now, our main result reads as follows.

Theorem 1.3. *Let $N \geq 3$ and assume that (φ_0) - (φ_2) , (F_1) - (F_3) hold, then we have the following conclusions.*

(i) **(mass subcritical case)** *If $2 < \alpha, \beta < 2 + \frac{4}{N}$, then for any given $c > 0$, (2.4) possesses a positive normalized solution $(\lambda, v_\lambda) \in (0, +\infty) \times H_{rad}^1(\mathbb{R}^N)$.*

(ii) **(exactly mass critical case)** *If $\alpha = \beta = 2 + \frac{4}{N}$, denote*

$$c_* := \min\{\|U\|_2^2, (a^*)^N \|V\|_2^2\}, c^* := \max\{\|U\|_2^2, (a^*)^N \|V\|_2^2\},$$

then (2.4) admits at least one positive normalized solution $(\lambda, v_\lambda) \in (0, +\infty) \times H_{rad}^1(\mathbb{R}^N)$ provided $c \in (c_, c^*)$ and no positive normalized solution if $c > 0$ small or large.*

(iii) **(at most mass critical case)**

(iii-1) *If $2 < \alpha < \beta = 2 + \frac{4}{N}$, (2.4) admits at least one positive normalized solution $(\lambda, v_\lambda) \in (0, +\infty) \times H_{rad}^1(\mathbb{R}^N)$ if $0 < c < (a^*)^N \|V\|_2^2$ and no positive normalized solution if $c > 0$ large.*

(iii-2) *If $2 < \beta < \alpha = 2 + \frac{4}{N}$, (2.4) admits at least one positive normalized solution $(\lambda, v_\lambda) \in (0, +\infty) \times H_{rad}^1(\mathbb{R}^N)$ if $c > \|U\|_2^2$ and no positive normalized solution if $c > 0$ small.*

(iv) **(mixed case)**

(iv-1) *If $2 < \alpha < 2 + \frac{4}{N} < \beta < 2^*$, (2.4) admits at least two distinct positive normalized solutions $(\lambda_i, v_{\lambda_i}) \in (0, +\infty) \times H_{rad}^1(\mathbb{R}^N)$ if $c > 0$ small and no positive normalized solution provided $c > 0$ large.*

(iv-2) *If $2 < \beta < 2 + \frac{4}{N} < \alpha < 2^*$, (2.4) admits at least two distinct positive normalized solutions $(\lambda_i, v_{\lambda_i}) \in (0, +\infty) \times H_{rad}^1(\mathbb{R}^N)$ if $c > 0$ large and no positive normalized solution provided $c > 0$ small.*

(v) **(at least mass critical case)**

(v-1) *If $2 + \frac{4}{N} = \alpha < \beta < 2^*$, (2.4) admits at least one positive normalized solution $(\lambda, v_\lambda) \in (0, +\infty) \times H_{rad}^1(\mathbb{R}^N)$ if $0 < c < \|U\|_2^2$ and no positive normalized solution provided $c > 0$ large.*

(v-2) *If $2 + \frac{4}{N} = \beta < \alpha < 2^*$, (2.4) admits at least one positive normalized solution $(\lambda, v_\lambda) \in (0, +\infty) \times H_{rad}^1(\mathbb{R}^N)$ if $c > (a^*)^N \|V\|_2^2$ and no positive normalized solution provided $c > 0$ small.*

(vi) **(mass supercritical case)** *If $2 + \frac{4}{N} < \alpha, \beta < 2^*$, then for any given $c > 0$, (2.4) admits a normalized positive solution $(\lambda, v_\lambda) \in (0, +\infty) \times H_{rad}^1(\mathbb{R}^N)$.*

1.3. Strategy of this paper. First, by virtue of a dual approach in [26], we transform equation (1.1) into a semilinear elliptic equation. Second, for any fixed $\lambda > 0$, we show the semilinear elliptic equation obtained admits at least one positive and radially symmetric solution v_λ . Similarly to [13], by using the blow-up argument and a Liouville theorem, as $\lambda \rightarrow 0^+$ or $\lambda \rightarrow +\infty$, the asymptotic behaviors of positive solutions are investigated, as well as the L^2 -norms of $\phi^{-1}(v_\lambda)$. Finally, via the fixed point index and the topological degree, we adopt the similar idea to [13] to establish a global branch of positive solutions for $\lambda \in (0, \infty)$. By applying a continuity argument, Theorem 1.3 is proved.

Notations

- $L^p(\mathbb{R}^N)$ denotes the Lebesgue space with norm $\|u\|_p := (\int_{\mathbb{R}^N} |u|^p dx)^{1/p}$.
- $H^1(\mathbb{R}^N)$ denotes the Sobolev space modeled in $L^2(\mathbb{R}^N)$ with its usual norm $\|u\| := (\|u\|_2^2 + \|\nabla u\|_2^2)^{1/2}$.
- $C_{r,0}(\mathbb{R}^N)$ denotes the space of radial continuous functions vanishing at infinity.

2. FUNCTIONAL SETTING AND PRELIMINARIES

To seek normalized solutions of (1.5), it suffices to find critical points of

$$(2.1) \quad I(u) = \frac{1}{2} \int_{\mathbb{R}^N} \varphi^2(u) |\nabla u|^2 dx - \int_{\mathbb{R}^N} F(u) dx$$

on the mass sphere S_c . Due to the presence of φ , some additional difficulties arise. In this paper, we adopt a dual approach in [26] to overcome them and make a change of variables as follows. Let (u, λ) be any solution of problem (1.5), that is, for any $\tilde{\phi} \in C_0^\infty(\mathbb{R}^N)$, there holds

$$(2.2) \quad \int_{\mathbb{R}^N} \left[\varphi^2(u) \nabla u \nabla \tilde{\phi} + \varphi(u) \varphi'(u) |\nabla u|^2 \tilde{\phi} + \lambda u \tilde{\phi} - f(u) \tilde{\phi} \right] dx = 0.$$

If ones take

$$v = \phi(u) = \int_0^u \varphi(t) dt,$$

and choose $\tilde{\phi} = \frac{1}{\varphi(u)} \psi$ in equation (2.2) for any $\psi \in C_0^\infty(\mathbb{R}^N)$, then it follows that

$$(2.3) \quad \int_{\mathbb{R}^N} \left[\nabla v \nabla \psi + \lambda \frac{\phi^{-1}(v)}{\varphi(\phi^{-1}(v))} \psi - \frac{f(\phi^{-1}(v))}{\varphi(\phi^{-1}(v))} \psi \right] dx = 0.$$

Then (v, λ) is a solution of the following semilinear elliptic problem

$$(2.4) \quad -\Delta v + \lambda \frac{\phi^{-1}(v)}{\varphi(\phi^{-1}(v))} - \frac{f(\phi^{-1}(v))}{\varphi(\phi^{-1}(v))} = 0, \quad x \in \mathbb{R}^N$$

with the prescribed mass

$$\int_{\mathbb{R}^N} |\phi^{-1}(v)|^2 dx = c.$$

And corresponding to the energy functional I , the energy functional associated with problem (2.4) is defined by

$$(2.5) \quad J(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx - \int_{\mathbb{R}^N} F(\phi^{-1}(v)) dx.$$

Since φ is a nondecreasing positive function, we get $|\phi^{-1}(s)| \leq |s|$ for any s . Moreover, it is clear that J is well defined in $H^1(\mathbb{R}^N)$ and of C^1 -class. Therefore, in order to find normalized solutions of equation (1.5), it is sufficient to turn to consider the existence of normalized solutions to problem (2.4).

In the following, we intend to borrow some ideas in [13] to find normalized solutions of problem (2.4).

Let

$$g_\lambda(v) = \frac{f(\phi^{-1}(v))}{\varphi(\phi^{-1}(v))} - \lambda \frac{\phi^{-1}(v)}{\varphi(\phi^{-1}(v))} + \lambda v,$$

then equation (2.4) turns into

$$(2.6) \quad -\Delta v + \lambda v = g_\lambda(v), \quad x \in \mathbb{R}^N.$$

Lemma 2.1. *Under the assumptions (F_1) - (F_3) and (φ_0) - (φ_1) , for any $\lambda > 0$, there hold that*

(i) *For some $1 < p < 2^* - 1$,*

$$\limsup_{s \rightarrow +\infty} \frac{g_\lambda(s)}{s^p} < \infty.$$

(ii) *$g_\lambda(s) = o(s)$, $s \rightarrow 0$.*

(iii) *There exists $T > 0$ such that $\int_0^T g_\lambda(\tau) d\tau > \frac{\lambda}{2} T^2$.*

(iv) *$\lim_{s \rightarrow +\infty} \frac{g_\lambda(s)}{s^{\beta-1}} = \frac{\mu_2}{(a^*)^\beta}$.*

(v) *$g_\lambda(s) \leq s g'_\lambda(s)$ for $s > 0$ small.*

Proof. (i) It is easy to check (i) holds for $p = \beta - 1$ by the assumption (F_2) .

(ii) Since

$$g_\lambda(s) = \frac{f(\phi^{-1}(s))}{\varphi(\phi^{-1}(s))} - \lambda \frac{\phi^{-1}(s)}{\varphi(\phi^{-1}(s))} + \lambda s,$$

then

$$\begin{aligned} \lim_{s \rightarrow 0} \frac{g_\lambda(s)}{s} &= \lim_{s \rightarrow 0} \frac{f(\phi^{-1}(s))}{s\varphi(\phi^{-1}(s))} - \lambda \lim_{s \rightarrow 0} \frac{\phi^{-1}(s)}{s\varphi(\phi^{-1}(s))} + \lambda \\ &= \lim_{s \rightarrow 0} \frac{f(s)}{s} = 0. \end{aligned}$$

(iii) Since

$$\begin{aligned} \lim_{t \rightarrow +\infty} \frac{\int_0^t g_\lambda(\tau) d\tau}{t^2} &= \lim_{t \rightarrow +\infty} \frac{g_\lambda(t)}{2t} \\ &= \frac{1}{2} \left[\lim_{t \rightarrow +\infty} \frac{f(\phi^{-1}(t))}{t\varphi(\phi^{-1}(t))} - \lambda \lim_{t \rightarrow +\infty} \frac{\phi^{-1}(t)}{t\varphi(\phi^{-1}(t))} + \lambda \right]. \end{aligned}$$

Take $s = \phi^{-1}(t)$, then we get

$$t = \phi(s) = \int_0^s \varphi(\tau) d\tau$$

and

$$\lambda \lim_{t \rightarrow +\infty} \frac{\phi^{-1}(t)}{t\varphi(\phi^{-1}(t))} = \lambda \lim_{s \rightarrow +\infty} \frac{s}{\phi(s)\varphi(s)},$$

Noting that $\lim_{s \rightarrow +\infty} \varphi(s) = a^*$,

$$\begin{aligned} \lambda \lim_{s \rightarrow +\infty} \frac{s}{\phi(s) \varphi(s)} &= \frac{\lambda}{a^*} \lim_{s \rightarrow +\infty} \frac{s}{\int_0^s \varphi(\tau) d\tau} \\ &= \frac{\lambda}{a^*} \lim_{s \rightarrow +\infty} \frac{1}{\varphi(s)} \\ &= \frac{\lambda}{(a^*)^2}. \end{aligned}$$

Meanwhile, it follows from (F_2) that

$$\begin{aligned} \lim_{t \rightarrow +\infty} \frac{f(\phi^{-1}(t))}{t\varphi(\phi^{-1}(t))} &= \lim_{s \rightarrow +\infty} \frac{f(s)}{\phi(s) \varphi(s)} = \frac{1}{a^*} \lim_{s \rightarrow +\infty} \frac{f(s)}{\phi(s)} \\ &= \frac{1}{(a^*)^2} \lim_{s \rightarrow +\infty} \frac{f(s)}{s} = +\infty. \end{aligned}$$

Thus we obtain that

$$\lim_{t \rightarrow +\infty} \frac{\int_0^t g_\lambda(\tau) d\tau}{t^2} = \infty,$$

which implies that there exists $T > 0$ such that $\int_0^T g_\lambda(\tau) d\tau > \frac{\lambda}{2} T^2$.

(iv) Set $p = \beta - 1$ and notice that $\phi(s) \geq s$ for any $s > 0$, we have

$$\begin{aligned} \lim_{s \rightarrow +\infty} \frac{g_\lambda(s)}{s^p} &= \lim_{s \rightarrow +\infty} \frac{f(\phi^{-1}(s))}{s^p \varphi(\phi^{-1}(s))} + \lim_{s \rightarrow +\infty} \frac{\lambda s - \lambda \frac{\phi^{-1}(s)}{\varphi(\phi^{-1}(s))}}{s^p} \\ &= \lim_{t \rightarrow +\infty} \frac{f(t)}{(\phi(t))^p \varphi(t)} - \lambda \lim_{t \rightarrow +\infty} \frac{t}{(\phi(t))^p \varphi(t)} \\ &= \lim_{t \rightarrow +\infty} \frac{f(t)}{(\phi(t))^p \varphi(t)} = \lim_{t \rightarrow +\infty} \frac{f(t)}{(ta^*)^p a^*} \\ &= \lim_{t \rightarrow +\infty} \frac{f(t)}{t^p} \cdot \frac{1}{(a^*)^{p+1}} = \frac{\mu_2}{(a^*)^{p+1}}. \end{aligned}$$

(v) It suffices to show that $\varphi(t)g_\lambda(\phi(t)) \leq \phi(t) \frac{d}{dt}[g_\lambda(\phi(t))]$ for $t > 0$ small. Obviously,

$$\frac{d}{dt}[g_\lambda(\phi(t))] = \frac{f'(t)\varphi(t) - f(t)\varphi'(t)}{\varphi^2(t)} + \frac{\lambda\varphi^3(t) - \lambda\varphi(t) + \lambda t\varphi'(t)}{\varphi^2(t)}.$$

By (φ_0) , for any $t \geq 0$, $\phi(t) \leq t\varphi(t)$ and then

$$\phi(t)\varphi^{-2}(t) [\varphi^3(t) - \varphi(t) + t\varphi'(t)] \geq \phi(t)\varphi(t) - t.$$

Meanwhile, by (F_2) ,

$$\lim_{t \rightarrow 0^+} \frac{\phi(t)\varphi(t)f'(t)}{f(t)} = \alpha - 1 > 1,$$

which implies that for $t > 0$ small,

$$\phi(t) [f'(t)\varphi(t) - f(t)\varphi'(t)] \geq \varphi^2(t)f(t).$$

Thus, it follows that

$$\phi(t) \frac{d}{dt} [g_\lambda(\phi(t))] \geq f(t) - \lambda t + \lambda \phi(t) \varphi(t) = \varphi(t) g_\lambda(\phi(t)), \quad t > 0 \text{ small.}$$

□

By Lemma 2.1, for any fixed $\lambda > 0$, g_λ satisfies the Berestycki-Lions conditions. As a consequence of [2, Theorem 1], for any $\lambda > 0$, problem (2.6) admits a ground state solution in $H^1(\mathbb{R}^N)$, which is positive and radially symmetric. Denoting

$$\mathcal{S} = \left\{ (\lambda, v_\lambda) \in (0, +\infty) \times H_{rad}^1(\mathbb{R}^N) : v_\lambda > 0, (\lambda, v_\lambda) \text{ solves} \right. \\ \left. -\Delta v + \lambda \frac{\phi^{-1}(v)}{\varphi(\phi^{-1}(v))} - \frac{f(\phi^{-1}(v))}{\varphi(\phi^{-1}(v))} = 0 \text{ in } \mathbb{R}^N \right\}.$$

Remark 2.2. As a consequence of [7, Theorem 2], for any $\lambda > 0$, for any positive C^2 solution u of problem (2.6) with $u(x) \rightarrow 0$ as $|x| \rightarrow \infty$, one can show that u is radially symmetric about some point $x_0 \in \mathbb{R}^N$, that is, $u(x) = u_0(|x - x_0|)$, where $\frac{\partial u_0}{\partial r} < 0$ for $r = |x - x_0| > 0$. In what follows, we assume that $x_0 = 0$. Actually, thanks to the elliptic estimate, u and $|\nabla u|$ decays exponentially at infinity. Set $g_\lambda(t) = h_1(t) + h_2(t)$, $t \geq 0$, where

$$h_1(t) = \frac{f(\phi^{-1}(t))}{\varphi(\phi^{-1}(t))}, \quad h_2(t) = -\lambda \frac{\phi^{-1}(t)}{\varphi(\phi^{-1}(t))} + \lambda t,$$

then by (F_2) , $h_1(t) = O(|t|^{\alpha-1})$ as $t \rightarrow 0$ and by (φ_0) ,

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{h_2(t)}{t^2} &= \lambda \lim_{t \rightarrow 0} \frac{\phi(t)\varphi(t) - t}{t^2} \\ &= \lambda \lim_{t \rightarrow 0} \frac{\phi(t)[\varphi(t) - 1]}{t^2} + \lambda \lim_{t \rightarrow 0} \frac{\phi(t) - t}{t^2} \\ &= \frac{3\lambda}{2} \varphi'(0) = 0. \end{aligned}$$

It follows that $g_\lambda(t) = O(|t|^{\min\{\alpha-1, 2\}})$ as $t \rightarrow 0$. Moreover,

$$h_2'(t) = \lambda \varphi^{-3}(\phi^{-1}(t)) [\varphi^3(\phi^{-1}(t)) - \varphi(\phi^{-1}(t)) + \phi^{-1}(t)\varphi'(\phi^{-1}(t))] \geq 0, \quad t \geq 0,$$

which implies that h_2 is nondecreasing for $t \geq 0$. As for the condition on h_1 as follows, for some $C > 0, p > 1$,

$$|h_1(t_1) - h_1(t_2)| \leq C |t_1 - t_2| / |\log \min(t_1, t_2)|^p, \quad t_1, t_2 \in [0, \max_{x \in \mathbb{R}^N} u(x)],$$

to prove [7, Theorem 2], it is only used in the proof of [7, Lemma 6.3]. Since u decays exponentially, one can check that [7, Lemma 6.3] still holds in our case and as well as [7, Theorem 2].

3. ASYMPTOTIC BEHAVIORS

In this section, we consider the asymptotic behaviors of v_λ as $\lambda = \lambda_n \rightarrow 0^+$ or $\lambda = \lambda_n \rightarrow +\infty$. Similarly to [13], we have the following result in the case of $\lambda = \lambda_n \rightarrow 0^+$.

Lemma 3.1. Let $\{v_n\}_{n=1}^\infty \subset \mathcal{S}$ with $\lambda = \lambda_n \rightarrow 0^+$, then the following asserts hold.

(i)

$$\limsup_{n \rightarrow +\infty} \|v_n\|_\infty < +\infty.$$

(ii)

$$\liminf_{n \rightarrow +\infty} \frac{\|v_n\|_\infty^{\alpha-2}}{\lambda_n} > 0.$$

(iii)

$$\limsup_{n \rightarrow +\infty} \frac{\|v_n\|_\infty^{\alpha-2}}{\lambda_n} < +\infty.$$

(iv) Set

$$(3.1) \quad w_n(x) := \lambda_n^{\frac{1}{2-\alpha}} v_n \left(\frac{x}{\sqrt{\lambda_n}} \right),$$

then w_n satisfies

$$-\Delta w_n + w_n = \frac{g_{\lambda_n} \left(\lambda_n^{\frac{1}{\alpha-2}} w_n \right)}{\lambda_n^{\frac{\alpha-1}{\alpha-2}}} \text{ in } \mathbb{R}^N$$

and

$$(3.2) \quad w_n \rightarrow U \text{ in } H^1(\mathbb{R}^N) \text{ and } C_{r,0}(\mathbb{R}^N),$$

where $U \in C_{r,0}(\mathbb{R}^N)$ is the unique positive solution of (1.9)*Proof.* The proof is similar to [13] and we omit the details. □In the following, we consider the case of $\lambda = \lambda_n \rightarrow +\infty$.**Lemma 3.2.** Let $\{v_n\}_{n=1}^\infty \subset \mathcal{S}$ with $\lambda = \lambda_n \rightarrow +\infty$, then up to a subsequence,

$$\liminf_{n \rightarrow +\infty} \|v_n\|_\infty = +\infty.$$

Proof. By regularity, for any fixed n , $v_n \in C^2(\mathbb{R}^N)$ and we may suppose that $v_n(0) = \|v_n\|_\infty$. Setting

$$\bar{v}_n(x) := \frac{1}{v_n(0)} v_n \left(\frac{x}{\sqrt{\lambda_n}} \right),$$

we have

$$(3.3) \quad \begin{aligned} 1 = \bar{v}_n(0) &\leq -\Delta \bar{v}_n(0) + \bar{v}_n(0) \\ &= \frac{1}{\lambda_n v_n(0)} g_{\lambda_n}(v_n(0)) \\ &= \frac{f(\phi^{-1}(v_n(0)))}{\lambda_n v_n(0) \varphi(\phi^{-1}(v_n(0)))} - \frac{\phi^{-1}(v_n(0))}{v_n(0) \varphi(\phi^{-1}(v_n(0)))} + 1 \\ &\leq \frac{c \left(|\phi^{-1}(v_n(0))|^{\alpha-1} + |\phi^{-1}(v_n(0))|^{\beta-1} \right)}{\lambda_n v_n(0)} + 1 \\ &\quad - \frac{\phi^{-1}(v_n(0))}{v_n(0) \varphi(\phi^{-1}(v_n(0)))}, \end{aligned}$$

where we used the fact that for some $c > 0$, $|f(t)| \leq c(|t|^{\alpha-1} + |t|^{\beta-1})$, $t \geq 0$.

Let $\phi^{-1}(v_n(0)) = a_n$, then $v_n(0) = \phi(a_n) = \int_0^{a_n} \varphi(\tau) d\tau$. Next we argue by contradiction and suppose now that $\{a_n\}$ is bounded. Then (3.3) yields that

$$(3.4) \quad \begin{aligned} 1 &\leq \frac{c(a_n^{\alpha-1} + a_n^{\beta-1})}{\lambda_n \phi(a_n)} + 1 - \frac{a_n}{\phi(a_n)\varphi(a_n)} \\ &\leq \frac{c(a_n^{\alpha-2} + a_n^{\beta-2})}{\lambda_n} + 1 - \frac{a_n}{\phi(a_n)\varphi(a_n)}. \end{aligned}$$

If $a_n \rightarrow 0$, then $v_n(0) \rightarrow 0$. Recall that v_n satisfies

$$(3.5) \quad -\Delta v_n + \lambda_n \frac{\phi^{-1}(v_n)}{\varphi(\phi^{-1}(v_n))} = \frac{f(\phi^{-1}(v_n))}{\varphi(\phi^{-1}(v_n))} \text{ in } \mathbb{R}^N.$$

Thanks to (F₂), it follows that

$$-\Delta v_n + \frac{(\lambda_n - 1)\phi^{-1}(v_n)}{\varphi(\phi^{-1}(v_n))} \leq 0 \text{ in } \mathbb{R}^N.$$

Multiplying both sides by v_n and integrating in \mathbb{R}^N , we get

$$(3.6) \quad \int_{\mathbb{R}^N} |\nabla v_n|^2 dx + (\lambda_n - 1) \int_{\mathbb{R}^N} \frac{\phi^{-1}(v_n) v_n}{\varphi(\phi^{-1}(v_n))} dx \leq 0,$$

which is contradiction, due to $\phi^{-1}(v_n(x)) v_n(x) > 0, x \in \mathbb{R}^N$. So $\{a_n\}$ is positive and bounded away from zero. Passing to the limit of (3.4) as $n \rightarrow \infty$, we have

$$\limsup_{n \rightarrow \infty} \frac{a_n}{\phi(a_n)\varphi(a_n)} \leq 0,$$

which is a contraction. Thus, up to a subsequence, $v_n(0) \rightarrow \infty$. □

Lemma 3.3. *Let $\{v_n\}_{n=1}^\infty \subset \mathcal{S}$ with $\lambda = \lambda_n \rightarrow +\infty$, then*

$$(3.7) \quad \liminf_{n \rightarrow +\infty} \frac{\|v_n\|_\infty^{\beta-2}}{\lambda_n} > 0.$$

Proof. Recall that

$$-\Delta v_n = \frac{f(\phi^{-1}(v_n))}{\varphi(\phi^{-1}(v_n))} - \lambda_n \frac{\phi^{-1}(v_n)}{\varphi(\phi^{-1}(v_n))} \text{ in } \mathbb{R}^N.$$

Without loss of generality, we suppose that $v_n(0) = \|v_n\|_\infty, \forall n \in \mathbb{N}$. Since $-\Delta v_n(0) \geq 0$,

$$f(\phi^{-1}(\|v_n\|_\infty)) \geq \lambda_n \phi^{-1}(\|v_n\|_\infty),$$

which yields that

$$\liminf_{n \rightarrow +\infty} \frac{\|v_n\|_\infty^{\beta-2}}{\lambda_n} > 0,$$

so (3.7) is proved. □

Lemma 3.4. *Let $\{v_n\}_{n=1}^\infty \subset \mathcal{S}$ with $\lambda = \lambda_n \rightarrow +\infty$, then*

$$(3.8) \quad \limsup_{n \rightarrow +\infty} \frac{\|v_n\|_\infty^{\beta-2}}{\lambda_n} < +\infty.$$

Proof. We prove it by contradiction and suppose

$$\lim_{n \rightarrow +\infty} \frac{\|v_n\|_\infty^{\beta-2}}{\lambda_n} = +\infty.$$

Set $k = \frac{\beta-2}{2}$ and

$$\tilde{v}_n(x) = \frac{1}{\|v_n\|_\infty} v_n \left(\frac{x}{\|v_n\|_\infty^k} \right),$$

then $\tilde{v}_n(0) = \|\tilde{v}_n\|_\infty = 1$ and

$$-\Delta \tilde{v}_n(x) = -\frac{1}{\|v_n\|_\infty^{1+2k}} \Delta v_n \left(\frac{x}{\|v_n\|_\infty^k} \right),$$

then

$$(3.9) \quad -\Delta \tilde{v}_n(x) = \frac{1}{\|v_n\|_\infty^{1+2k}} \left[\frac{f(\phi^{-1}(\tilde{v}_n(x)\|v_n\|_\infty))}{\varphi(\phi^{-1}(\tilde{v}_n(x)\|v_n\|_\infty))} - \lambda_n \frac{\phi^{-1}(\tilde{v}_n(x)\|v_n\|_\infty)}{\varphi(\phi^{-1}(\tilde{v}_n(x)\|v_n\|_\infty))} \right],$$

By (F_2) , the right hand side of (3.9) is in $L^\infty(\mathbb{R}^N)$. Up to a subsequence, we may suppose that $\tilde{v}_n \rightarrow \tilde{v}$ in $C_{loc}^2(\mathbb{R}^N)$ and then \tilde{v} is a non-negative bounded solution to

$$-\Delta \tilde{v} = \mu_2(a^*)^{-\beta} \tilde{v}^{\beta-1} \text{ in } \mathbb{R}^N.$$

As a consequence of [13, Theorem 2.2-(ii)], $\tilde{v} \equiv 0$, which contradicts $\tilde{v}(0) = 1$. \square

Lemma 3.5. *Let $\{v_n\}_{n=1}^\infty \subset \mathcal{S}$ with $\lambda = \lambda_n \rightarrow +\infty$. Define*

$$(3.10) \quad w_n(x) := \lambda_n^{\frac{1}{2-\beta}} v_n \left(\frac{x}{\sqrt{\lambda_n}} \right),$$

then, up to a subsequence, $w_n \rightarrow V^*$ in $C_{r,0}(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$ as $n \rightarrow +\infty$, where $V^* = a^*V(\frac{\cdot}{a^*})$ and V is the unique positive solution of (1.10).

Proof. By Lemma 3.3 and Lemma 3.4, we have that

$$0 < \liminf_{n \rightarrow +\infty} \frac{v_n(0)^{\beta-2}}{\lambda_n} \leq \limsup_{n \rightarrow +\infty} \frac{v_n(0)^{\beta-2}}{\lambda_n} < +\infty,$$

which implies $\{w_n\}$ is uniformly bounded in $L^\infty(\mathbb{R}^N)$. Since w_n satisfies

$$(3.11) \quad -\Delta w_n + w_n = \frac{g_{\lambda_n} \left(\lambda_n^{\frac{1}{\beta-2}} w_n \right)}{\lambda_n^{\frac{\beta-1}{\beta-2}}} \text{ in } \mathbb{R}^N,$$

that is,

$$(3.12) \quad -\Delta w_n = \lambda_n^{\frac{1-\beta}{\beta-2}} \frac{f \left(\phi^{-1} \left(\lambda_n^{\frac{1}{\beta-2}} w_n \right) \right)}{\varphi \left(\phi^{-1} \left(\lambda_n^{\frac{1}{\beta-2}} w_n \right) \right)} - \lambda_n^{\frac{1}{2-\beta}} \frac{\phi^{-1} \left(\lambda_n^{\frac{1}{\beta-2}} w_n \right)}{\varphi \left(\phi^{-1} \left(\lambda_n^{\frac{1}{\beta-2}} w_n \right) \right)}.$$

Thanks to (F_2) , one can check that the right hand of (3.12) is in $L^\infty(\mathbb{R}^N)$. By the elliptic regularity, up to a subsequence, we assume that $w_n \rightarrow V^*$ in $C_{loc}^2(\mathbb{R}^N)$, where V^* satisfies

$$-\Delta V^* + (a^*)^{-2} V^* = \mu_2 (a^*)^{-\beta} (V^*)^{\beta-1} \text{ in } \mathbb{R}^N, \quad V^*(0) = \max_{x \in \mathbb{R}^N} V^*(x).$$

Similarly to [13], the result desired can be obtained. \square

Next, we focus on the asymptotic behaviour of $\|\phi^{-1}(v_n)\|_2$ as $n \rightarrow \infty$.

Theorem 3.6. (i) Let $\{v_n\}_{n=1}^\infty \subset H^1(\mathbb{R}^N)$ be positive solutions to (2.4) with $\lambda = \lambda_n \rightarrow 0^+$.

Then

$$\lim_{n \rightarrow +\infty} \|v_n\|_\infty = 0, \quad \lim_{n \rightarrow +\infty} \|\nabla v_n\|_2 = 0$$

and

$$\lim_{n \rightarrow +\infty} \|\phi^{-1}(v_n)\|_2 = \begin{cases} 0 & \alpha < 2 + \frac{4}{N}, \\ \|U\|_2 & \alpha = 2 + \frac{4}{N}, \\ +\infty & \alpha > 2 + \frac{4}{N}. \end{cases}$$

(ii) Let $\{v_n\}_{n=1}^\infty \subset H^1(\mathbb{R}^N)$ be positive solutions to (2.4) with $\lambda = \lambda_n \rightarrow +\infty$. Then

$$\lim_{n \rightarrow +\infty} \|v_n\|_\infty = +\infty, \quad \lim_{n \rightarrow +\infty} \|\nabla v_n\|_2 = +\infty$$

and

$$\lim_{n \rightarrow +\infty} \|\phi^{-1}(v_n)\|_2 = \begin{cases} +\infty & \beta < 2 + \frac{4}{N}, \\ (a^*)^{N/2} \|V\|_2 & \beta = 2 + \frac{4}{N}, \\ 0 & \beta > 2 + \frac{4}{N}. \end{cases}$$

Proof. Similarly to [13], one can show the asymptotic behaviour of $\|v_n\|_\infty$ and $\|\nabla v_n\|_2$. In the following, we only consider $\|\phi^{-1}(v_n)\|_2^2$.

(i) Let w_n be defined by (3.1) and by Lemma 3.1, we have

$$\begin{aligned} \|\phi^{-1}(v_n)\|_2^2 &= \int_{\mathbb{R}^N} |\phi^{-1}(v_n(x))|^2 dx \\ &= \lambda_n^{-\frac{N}{2}} \int_{\mathbb{R}^N} \left| \phi^{-1} \left(v_n \left(\frac{y}{\sqrt{\lambda_n}} \right) \right) \right|^2 dy \\ &= \lambda_n^{-\frac{N}{2}} \int_{\mathbb{R}^N} \left(\frac{\phi^{-1} \left(v_n \left(\frac{y}{\sqrt{\lambda_n}} \right) \right)}{v_n \left(\frac{y}{\sqrt{\lambda_n}} \right)} \right)^2 \left(v_n \left(\frac{y}{\sqrt{\lambda_n}} \right) \right)^2 dy \\ &= \lambda_n^{-\frac{N}{2}} \int_{\mathbb{R}^N} \left(\frac{\phi^{-1} \left(v_n \left(\frac{y}{\sqrt{\lambda_n}} \right) \right)}{v_n \left(\frac{y}{\sqrt{\lambda_n}} \right)} \right)^2 \left(\frac{w_n(y)}{\lambda_n^{\frac{1}{2-\alpha}}} \right)^2 dy \\ &= \lambda_n^{-\frac{N}{2} + \frac{2}{\alpha-2}} \int_{\mathbb{R}^N} \left(\frac{\phi^{-1} \left(v_n \left(\frac{y}{\sqrt{\lambda_n}} \right) \right)}{v_n \left(\frac{y}{\sqrt{\lambda_n}} \right)} \right)^2 w_n^2(y) dy \\ &= \lambda_n^{-\frac{N}{2} + \frac{2}{\alpha-2}} (\|U\|_2^2 + o_n(1)), \end{aligned}$$

which yields the result desired.

(ii) For the case of $\lambda = \lambda_n \rightarrow +\infty$, let w_n be defined by (3.10). Since

$$\lim_{s \rightarrow +\infty} \frac{\phi^{-1}(s)}{s} = \lim_{t \rightarrow +\infty} \frac{t}{\phi(t)} = \frac{1}{a^*},$$

by Lemma 3.5,

$$\begin{aligned} \|\phi^{-1}(v_n)\|_2^2 &= \lambda_n^{-\frac{N}{2}} \int_{\mathbb{R}^N} \left(\frac{\phi^{-1}\left(v_n\left(\frac{y}{\sqrt{\lambda_n}}\right)\right)}{v_n\left(\frac{y}{\sqrt{\lambda_n}}\right)} \right)^2 \left(v_n\left(\frac{y}{\sqrt{\lambda_n}}\right)\right)^2 dy \\ &= \lambda_n^{-\frac{N}{2} + \frac{2}{\beta-2}} \int_{\mathbb{R}^N} \left(\frac{\phi^{-1}\left(v_n\left(\frac{y}{\sqrt{\lambda_n}}\right)\right)}{v_n\left(\frac{y}{\sqrt{\lambda_n}}\right)} \right)^2 w_n^2(y) dy \\ &= \lambda_n^{-\frac{N}{2} + \frac{2}{\beta-2}} ((a^*)^{-2} \|V^*\|_2^2 + o_n(1)), \end{aligned}$$

which yields the result desired. □

4. UNIQUENESS FOR λ SMALL OR LARGE

In this section, we prove the uniqueness of positive solutions to (2.4) provided $\lambda > 0$ small or large enough.

Theorem 4.1. *Let $N \geq 3$ and assume that (φ_0) - (φ_1) , (F_1) - (F_2) hold. Then (2.4) has at most one positive solution if $\lambda > 0$ large and in addition (φ_2) holds or if $\lambda > 0$ small and in addition (F_3) holds.*

Proof. We adopt the idea in [13] to give the proof.

If $\lambda > 0$ small and (F_1) - (F_3) hold, assume that problem (2.4) admits two families of positive solutions $u_\lambda^{(1)}$ and $u_\lambda^{(2)}$ with $\lambda \rightarrow 0^+$. Let

$$v_\lambda^{(i)}(\cdot) := \lambda^{-\frac{1}{\alpha-2}} u_\lambda^{(i)}\left(\cdot/\sqrt{\lambda}\right), \quad i = 1, 2.$$

Then by Lemma 3.1, $v_\lambda^{(1)}, v_\lambda^{(2)} \in H_{rad}^1(\mathbb{R}^N)$ satisfy

$$-\Delta v + v = \frac{g_\lambda(\lambda^{\frac{1}{\alpha-2}} v)}{\lambda^{\frac{\alpha-1}{\alpha-2}}} \text{ in } \mathbb{R}^N$$

and as $\lambda \rightarrow 0^+$, $v_\lambda^{(i)} \rightarrow U$ both in $C_{r,0}(\mathbb{R}^N)$ and in $H^1(\mathbb{R}^N)$, $i = 1, 2$. Setting

$$\xi_\lambda := \frac{v_\lambda^{(1)} - v_\lambda^{(2)}}{\|v_\lambda^{(1)} - v_\lambda^{(2)}\|_\infty},$$

for any $x \in \mathbb{R}^N$, there exists some $\theta(x) \in [0, 1]$ such that

$$-\Delta \xi_\lambda + \xi_\lambda = \lambda^{-1} g'_\lambda \left(\lambda^{\frac{1}{\alpha-2}} [\theta(x) v_\lambda^{(1)}(x) + (1 - \theta(x)) v_\lambda^{(2)}(x)] \right) \xi_\lambda \text{ in } \mathbb{R}^N,$$

where for any $t \in \mathbb{R}$,

$$\begin{aligned} \lambda^{-1}g'_\lambda(t) &= \frac{f'(\phi^{-1}(t))\varphi(\phi^{-1}(t)) - f(\phi^{-1}(t))\varphi'(\phi^{-1}(t))}{\lambda[\varphi(\phi^{-1}(t))]^3} \\ &\quad - \frac{\varphi(\phi^{-1}(t)) - \phi^{-1}(t)\varphi'(\phi^{-1}(t))}{[\varphi(\phi^{-1}(t))]^3} + 1. \end{aligned}$$

Thanks to $v_\lambda^{(i)} \rightarrow U$ in $C_{r,0}(\mathbb{R}^N)$, one can see that

$$\lim_{\lambda \rightarrow 0^+} \lambda^{\frac{1}{\alpha-2}} [\theta(x)v_\lambda^{(1)}(x) + (1-\theta(x))v_\lambda^{(2)}(x)] = 0 \quad \text{uniformly for } x \in \mathbb{R}^N,$$

which implies that for any $x \in \mathbb{R}^N$,

$$\lim_{\lambda \rightarrow 0^+} \lambda^{-1}g'_\lambda \left(\lambda^{\frac{1}{\alpha-2}} [\theta(x)v_\lambda^{(1)}(x) + (1-\theta(x))v_\lambda^{(2)}(x)] \right) = \mu_1(\alpha-1)U^{\alpha-2}.$$

Hence, up to a subsequence, $\xi_\lambda \rightarrow \xi$ in $C_{loc}^2(\mathbb{R}^N)$ and ξ is a radial bounded solution of

$$-\Delta\xi + \xi = (\alpha-1)\mu_1U^{\alpha-2}\xi.$$

Similarly to [13], we can get one contradiction.

Now, we consider the case of $\lambda > 0$ large and assume (φ_2) holds. Assume problem (2.4) admits two families of positive solutions $u_\lambda^{(1)}$ and $u_\lambda^{(2)}$ with $\lambda \rightarrow +\infty$. Let

$$v_\lambda^{(i)}(\cdot) := \lambda^{-\frac{1}{\beta-2}} u_\lambda^{(i)} \left(\cdot / \sqrt{\lambda} \right), \quad i = 1, 2,$$

and

$$\xi_\lambda := \frac{v_\lambda^{(1)} - v_\lambda^{(2)}}{\|v_\lambda^{(1)} - v_\lambda^{(2)}\|_\infty}.$$

Similarly as above, there exists some $\theta(x) \in [0, 1]$ such that

$$-\Delta\xi_\lambda + \xi_\lambda = \lambda^{-1}g' \left(\lambda^{\frac{1}{\beta-2}} [\theta(x)v_\lambda^{(1)}(x) + (1-\theta(x))v_\lambda^{(2)}(x)] \right) \xi_\lambda \quad \text{in } \mathbb{R}^N.$$

By Lemma 3.5, $v_\lambda^{(i)} \rightarrow V$ in $C_{r,0}(\mathbb{R}^N)$ as $\lambda \rightarrow +\infty$ for $i = 1, 2$. It is easy to know that, for any $x \in \mathbb{R}^N$,

$$\lim_{\lambda \rightarrow +\infty} \lambda^{\frac{1}{\beta-2}} [\theta(x)v_\lambda^{(1)}(x) + (1-\theta(x))v_\lambda^{(2)}(x)] = +\infty.$$

By (φ_2) , for any $x \in \mathbb{R}^N$,

$$\lim_{\lambda \rightarrow +\infty} \frac{\varphi(\phi^{-1}(t)) - \phi^{-1}(t)\varphi'(\phi^{-1}(t))}{[\varphi(\phi^{-1}(t))]^3} \Bigg|_{t=\lambda^{\frac{1}{\beta-2}} [\theta(x)v_\lambda^{(1)}(x) + (1-\theta(x))v_\lambda^{(2)}(x)]} = (a^*)^{-2}$$

and

$$\lim_{\lambda \rightarrow +\infty} \frac{f'(\phi^{-1}(t))\varphi(\phi^{-1}(t)) - f(\phi^{-1}(t))\varphi'(\phi^{-1}(t))}{\lambda[\varphi(\phi^{-1}(t))]^3} \Bigg|_{t=\lambda^{\frac{1}{\beta-2}} [\theta(x)v_\lambda^{(1)}(x) + (1-\theta(x))v_\lambda^{(2)}(x)]} = \mu_2(a^*)^{-\beta}V^{\beta-2}.$$

Hence, up to a subsequence, $\xi_\lambda \rightarrow \xi$ in $C_{loc}^2(\mathbb{R}^N)$ and ξ is a radial bounded solution of

$$-\Delta\xi + (a^*)^{-2}\xi = \mu_2(a^*)^{-\beta}V^{\beta-2}\xi.$$

Similarly, we get a contradiction. \square

5. PROOF OF THEOREM 1.3

We give the proof of Theorem 1.3. Being similar to that of [13], we only sketch it.

• **Step 1.** For any $\lambda > 0$, problem (2.4) admits a positive and radially symmetric solution u_λ . By Theorem 4.1, there exists $\lambda_0 > 0$ such that, up to translation, u_λ is the unique positive solution of problem (2.4) and then is of mountain-pass-type (see [13, Definition 2.1]). Furthermore, the map $\lambda \mapsto u_\lambda, \lambda \in (0, \lambda_0)$ is continuous. That is, $\{(\lambda, u_\lambda) : \lambda \in (0, \lambda_0)\}$ is a curve in $\mathbb{R} \times H_{rad}^1(\mathbb{R}^N)$.

• **Step 2.** Let $\tilde{\mathcal{S}} \subset \mathcal{S}$ be the connected component of \mathcal{S} containing the curve $\{(\lambda, u_\lambda) : \lambda \in (0, \lambda_0)\}$, where \mathcal{S} is given in Section 2. Denoting by $P_1 : (0, +\infty) \times H_{rad}^1(\mathbb{R}^N) \rightarrow (0, +\infty)$ the projection onto the λ -component, we show that $P_1(\tilde{\mathcal{S}}) = (0, +\infty)$.

For any $v \in H_{rad}^1(\mathbb{R}^N)$, define

$$\mathbb{T}_\lambda(v) =: (-\Delta + \lambda)^{-1}g_\lambda(v).$$

Then $\mathbb{T}_\lambda : H_{rad}^1(\mathbb{R}^N) \rightarrow H_{rad}^1(\mathbb{R}^N)$ is completely continuous and u is a radial solution of problem (2.4) if and only if u is a fixed point of \mathbb{T}_λ in $H_{rad}^1(\mathbb{R}^N)$. For $\lambda \in (0, \lambda_0)$, thanks to Lemma 2.1-(v), similarly to [13, Lemma 7.2], one can get the local fixed point index

$$\text{ind}(\mathbb{T}_\lambda, u_\lambda) = \text{deg}_{LS}(\text{Id} - \mathbb{T}_\lambda, N_\varepsilon(u_\lambda), 0) = -1,$$

where $\varepsilon > 0$ is small, N_ε denotes the ε -neighborhood in $H_{rad}^1(\mathbb{R}^N)$.

• **Step 3.** For any fixed $0 < a < b$, define

$$\mathcal{S}(a, b) = \left\{ (\lambda, v_\lambda) \in [a, b] \times H_{rad}^1(\mathbb{R}^N) : v_\lambda > 0, (\lambda, v_\lambda) \text{ solves} \right. \\ \left. -\Delta v + \lambda \frac{\phi^{-1}(v)}{\varphi(\phi^{-1}(v))} - \frac{f(\phi^{-1}(v))}{\varphi(\phi^{-1}(v))} = 0 \text{ in } \mathbb{R}^N \right\}.$$

By using the similar blow-up technique and ODE approach, one can show that the set $\mathcal{S}(a, b)$ is compact in $C_{r,0}(\mathbb{R}^N)$ and $H_{rad}^1(\mathbb{R}^N)$. Via the topological degree theory, thanks to the compactness of $\mathcal{S}(a, b)$ and $\text{ind}(\mathbb{T}_\lambda, u_\lambda) = -1$ for $\lambda \in (0, \lambda_0)$, there holds that $P_1(\tilde{\mathcal{S}}) = (0, \infty)$.

• **Step 4.** Similarly to [13], define

$$\tilde{\rho} : \tilde{\mathcal{S}} \rightarrow \mathbb{R}^+, \quad (\lambda, v) \mapsto \|\phi^{-1}(v)\|_2^2.$$

We just consider the case (i). By $P_1(\tilde{\mathcal{S}}) = (0, +\infty)$ and Theorem 3.6, if $2 < \alpha < 2 + \frac{4}{N}$, then there exists $(\lambda_n, v_n) \in \tilde{\mathcal{S}}$ with $\lambda_n \rightarrow 0^+$ and $\|\phi^{-1}(v_n)\|_2^2 \rightarrow 0$. Similarly, if $2 < \beta < 2 + \frac{4}{N}$, there exists $(\lambda'_n, v'_n) \in \tilde{\mathcal{S}}$ with $\lambda'_n \rightarrow +\infty$ and $\|\phi^{-1}(v'_n)\|_2^2 \rightarrow +\infty$. Since $\tilde{\mathcal{S}}$ is connected, for any given $c > 0$, there exists $(\lambda_c, u_c) \in \tilde{\mathcal{S}}$ such that $\tilde{\rho}(u_c) = c$, that is, (2.4) possesses a positive normalized solution. Since the proof of the other cases can be done similarly as in [13], we omit the details.

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(T. Deng)

COLLEGE OF MATHEMATICS AND STATISTICS
CHONGQING JIAOTONG UNIVERSITY
CHONGQING 400074, PR CHINA
Email address: 1666106471@qq.com

(M. Squassina)

DIPARTIMENTO DI MATEMATICA E FISICA
UNIVERSITÀ CATTOLICA DEL SACRO CUORE
VIA DEI MUSEI 41, BRESCIA, ITALY
Email address: marco.squassina@unicatt.it

(J. J. Zhang)

COLLEGE OF MATHEMATICS AND STATISTICS
CHONGQING JIAOTONG UNIVERSITY
CHONGQING 400074, PR CHINA
Email address: zhangjianjun09@tsinghua.org.cn

(X.X. Zhong)

SOUTH CHINA RESEARCH CENTER FOR APPLIED MATHEMATICS AND INTERDISCIPLINARY STUDIES
SOUTH CHINA NORMAL UNIVERSITY
GUANGZHOU 510631, P. R. CHINA
Email address: zhongxuexiu1989@163.com