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ROBUST EXPONENTIAL ATTRACTORS FOR A FAMILY OF NONCONSERVED PHASE-FIELD SYSTEMS WITH MEMORY

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Abstract. We consider a family of phase-field systems with memory effects in the temperature ϑ, depending on a parameter $\omega \geq 0$. Setting the problems in a suitable phase-space accounting for the past history of ϑ, we prove the existence of a family of exponential attractors $\mathcal{E}_\omega$ which is robust as $\omega \to 0$.

1. **Introduction.** A well-known and widely used mathematical model to describe phase transitions was proposed by Caginalp [1]. Suppose that, for any time $t \geq 0$, a two-phase material occupies a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$, and denote by ϑ its *relative temperature* with respect to some fixed critical temperature ϑ_c, and by χ the phase proportion (or *phase-field*). Taking some constants equal to 1, the Caginalp model reduces to the following system of partial differential equations

$$\partial_t (c\vartheta + \lambda\chi) - \kappa\Delta\vartheta = f,$$
$$\partial_t\chi - \Delta\chi + \phi(\chi) - \lambda\vartheta = 0,$$

in $\Omega \times \mathbb{R}^+$, where $\mathbb{R}^+ = (0, \infty)$. Here, $\lambda \in \mathbb{R}$ is a coupling constant related to the *latent heat*, c and κ are positive constants representing the *specific heat* and the *heat conductivity*, respectively, and f denotes an external heat source. The smooth function $\phi : \mathbb{R} \to \mathbb{R}$ accounts for the presence of two phases and, usually, it is the derivative of a double-well potential, i.e, $\phi(r) = r^3 - r$. This model is known as *nonconserved* phase-field system.

There are special materials like, for instance, viscous glass-forming liquids, for which c and κ show, in the frequency domain, a dependence on the frequency itself (see, e.g., [10, 11, 12] and references therein). This means that, in the time domain, the internal energy e and the heat flux $\boldsymbol{q}$ depend on the past history of ϑ through time convolution integrals, characterized by suitable memory kernels (see

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the pioneering papers [2, 14]). On account of this fact, in [7, 8] the constitutive laws used to derive the above system, namely,

$$e(t) = c\vartheta(t) + \lambda\chi(t), \qquad \boldsymbol{q}(t) = -\kappa\nabla\vartheta(t),$$

have been replaced by

$$e(t) = c_0\vartheta(t) + \int_0^\infty a(s)\vartheta(t-s)ds + \lambda\chi(t)$$

and

$$\boldsymbol{q}(t) = -\omega\nabla\vartheta(t) - \int_0^\infty b(s)\nabla\vartheta(t-s)ds,$$

with $c_0 > 0$ and $\omega \geq 0$. The smooth functions $a, b : \mathbb{R}^+ \to \mathbb{R}^+$ are the specific heat relaxation kernel and the heat conductivity relaxation kernel, respectively. Besides, for thermodynamic reasons, we also assume that b is nonincreasing and summable along with its first derivative, whereas a is bounded, nondecreasing, concave, and with summable first and second derivatives (see [7] for details). This choice entails that the evolution of (ϑ, χ) is ruled by the following integrodifferential system

$$\begin{aligned}
&\partial_t\vartheta + \vartheta + \int_0^\infty a'(s)\vartheta(t-s)ds + \partial_t\chi - \omega\Delta\vartheta + \int_0^\infty b(s)\Delta\vartheta(t-s)ds = f,\\
&\partial_t\chi - \Delta\chi + \phi(\chi) - \vartheta = 0,
\end{aligned}$$

in $\Omega \times \mathbb{R}^+$, where we set $c_0 = \lambda = a(0) = 1$ for the sake of simplicity.

In a series of papers (see [6, 7, 8, 9] and references therein), the above model has been analyzed within the theory of dissipative dynamical systems by introducing, following [3] (in the same spirit, see also [16, 17]), the additional (integrated) past history

$$\eta^t(s) = \int_{t-s}^t \vartheta(y)dy \qquad \text{in } \Omega,\ s \in \mathbb{R}^+.$$

supposing, in addition, that the past history of ϑ is given up to a given initial time (e.g., $t = 0$). This approach leads us to concentrate our attention on the equivalent system (see [13] for details)

$$\partial_t\vartheta + \vartheta + \partial_t\chi - \omega\Delta\vartheta + \int_0^\infty \nu(s)\eta(s)ds - \int_0^\infty \mu(s)\Delta\eta(s)ds = f, \tag{1.1}$$

$$\partial_t\chi - \Delta\chi + \phi(\chi) - \vartheta = 0, \tag{1.2}$$

$$\partial_t\eta + \partial_s\eta = \vartheta. \tag{1.3}$$

Here the memory kernels $\nu = -a''$ and $\mu = -b'$ are positive nonincreasing functions on $\mathbb{R}^+$ vanishing at infinity exponentially fast. Observe that, depending on the value of ω, equation (1.1) is of parabolic or of hyperbolic type. Hence, for short, we refer to the case $\omega > 0$ as *parabolic*, opposed to the case $\omega = 0$, that we call *hyperbolic*. Strictly speaking, this terminology is not completely correct, since some hyperbolicity is always present in the system due to equation (1.3).

Assuming, for instance, Neumann boundary conditions for ϑ, χ and η, and taking f constant in time, it has been shown that (1.1)-(1.3) generates a strongly continuous semigroup $S_\omega(t)$ on a suitable phase-space. Besides, $S_\omega(t)$ possesses a global attractor $\mathcal{A}_\omega$ which is upper semicontinuous at $\omega = 0$, with respect to the standard Hausdorff semidistance (see [7, 8, 9]). In the parabolic case, the existence of exponential attractors has also been proved [6].

The aim of the present work is to complete the analysis carried out so far, showing that even in the hyperbolic case there exist exponential attractors for the semigroup. Indeed, we will prove a deeper fact. Namely, we will construct a *specific* family of exponential attractors $\mathcal{E}_\omega$ for $S_\omega(t)$ that are *robust* in the following sense: the symmetric Hausdorff distance between $\mathcal{E}_\omega$ and $\mathcal{E}_0$ goes to 0 as $\omega \to 0$ in an explicitly controlled way. The result is obtained as a nontrivial application of a recent abstract theorem due to Fabrie, Galusinski, Miranville and Zelik [5, Theorem 1.1] (reported below as Lemma 3.1), which provides sufficient conditions ensuring that certain (possibly singularly) perturbed dynamical systems associated with asymptotically compact semigroups possess robust exponential attractors.

Remark 1.1. We emphasize that we consider a linear coupling between ϑ and χ, since we want to capture also the limiting case $\omega = 0$. However, when $\omega > 0$, one can replace the terms $\partial_t\chi$ of equation (1.1) and $-\vartheta$ of equation (1.2) with $\lambda'(\chi)\partial_t\chi$ and $-\lambda'(\chi)\vartheta$, respectively, where λ is a function with quadratic growth (cf. [6]). Thus, from one side, it will be easier to obtain energy estimates, because of the linear coupling; nonetheless, the estimates will have to be independent of ω (cf. [9]).

Before getting into details, let us fix some notation first. On the Hilbert space $L^2(\Omega)$ (with inner product and norm denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively), we define the strictly positive operator

$$A = \mathbb{I} - \Delta \qquad \text{with domain} \qquad \mathcal{D}(A) = \big\{v \in H^2(\Omega) \;\big|\; \partial_{\boldsymbol{n}} v = 0 \;\text{ on } \partial\Omega\big\}.$$

For $r \in \mathbb{R}$, we introduce the Hilbert spaces $H_r = \mathcal{D}(A^{r/2})$, endowed with the inner products $\langle\cdot,\cdot\rangle_{H_r} = \langle A^{r/2}\cdot, A^{r/2}\cdot\rangle$, and the weighted Hilbert spaces

$$\mathcal{M}_r = L^2_\nu(\mathbb{R}^+; H_r) \cap L^2_\mu(\mathbb{R}^+; H_{r+1}),$$

whose inner products are given by

$$\langle\eta_1,\eta_2\rangle_{\mathcal{M}_r} = \int_0^\infty \nu(s)\langle\eta_1(s),\eta_2(s)\rangle_{H_r} ds + \int_0^\infty \mu(s)\langle\eta_1(s),\eta_2(s)\rangle_{H_{r+1}} ds.$$

Next, we consider the infinitesimal generator of the strongly continuous semigroup of right-translations on $\mathcal{M}_0$ (see [13]), namely, the linear operator on $\mathcal{M}_0$

$$T = -\partial_s \qquad \text{with domain} \qquad \mathcal{D}(T) = \big\{\eta \in \mathcal{M}_0 \;\big|\; \partial_s\eta \in \mathcal{M}_0,\ \eta(0) = 0\big\}.$$

Here $\partial_s\eta$ denotes the distributional derivative of η with respect to the internal variable s. Finally, we define the product Hilbert spaces

$$\mathcal{H}_r = H_r \times H_{r+1} \times \mathcal{M}_r.$$

It is worth noting that the embedding $\mathcal{H}_1 \hookrightarrow \mathcal{H}_0$ is continuous but not compact, due to the presence of the third component $\mathcal{M}_r$. Finding nice compact embeddings is actually crucial to prove our results. Hence, in order to remove this obstacle, we need to introduce a further space. Defining for every $\eta \in \mathcal{M}_0$ the *tail function*

$$\mathbb{T}_\eta(x) = \int_{(0,\frac{1}{x})\cup(x,\infty)} \Big[\nu(s)\|\eta(s)\|^2 + \mu(s)\|A^{1/2}\eta(s)\|^2\Big] ds, \qquad x \geq 1,$$

we set

$$\mathcal{Z} = \Big\{z = (\vartheta,\chi,\eta) \in \mathcal{H}_1 \;\Big|\; \eta \in \mathcal{D}(T) \quad \text{and} \quad \sup_{x\geq 1} x\mathbb{T}_\eta(x) < \infty\Big\}.$$

It is readily seen that $\mathcal{Z}$ is a Banach space with the norm

$$\|z\|_{\mathcal{Z}}^2 = \|z\|_{\mathcal{H}_1}^2 + \|T\eta\|_{\mathcal{M}_0}^2 + \sup_{x\geq 1} x\mathbb{T}_\eta(x).$$

Then, appealing to an immediate generalization of a compactness lemma from [15], it is possible to show that the embedding $\mathcal{Z} \hookrightarrow \mathcal{H}_0$ is indeed compact (cf. [6]).

Conditions on the nonlinearity ϕ and the source term f.

$$\phi \in C^2(\mathbb{R}) \quad \text{with} \quad |\phi''(r)| \leq c(1+|r|), \quad \forall r \in \mathbb{R} \text{ and some } c \geq 0, \tag{H1}$$

$$\liminf_{|r|\to\infty} \frac{\phi(r)}{r} > 1 - \alpha_1, \ \text{ where } \alpha_1 \text{ is the first eigenvalue of } A, \tag{H2}$$

$$f \in H_1 \quad \text{constant in time.} \tag{H3}$$

Conditions on the memory kernels ν and μ.

$$\nu, \mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \tag{K1}$$

$$\nu(s) \geq 0, \quad \mu(s) \geq 0, \quad \forall s \in \mathbb{R}^+, \tag{K2}$$

$$\exists\, \delta > 0 : \ \nu'(s) + \delta\nu(s) \leq 0, \quad \mu'(s) + \delta\mu(s) \leq 0, \quad \forall s \in \mathbb{R}^+. \tag{K3}$$

Assuming Neumann boundary conditions for ϑ, χ and η, together with the additional constraint $\eta^t(0) = 0$, we interpret the differential operators Δ and ∂_s appearing in equations (1.1)-(1.3) as $\mathbb{I} - A$ and $-T$, respectively. Then, according to [7, 8], we have

Theorem 1.2. *For every $\omega \geq 0$, system* (1.1)-(1.3)*, endowed with Neumann boundary conditions, generates a strongly continuous semigroup $S_\omega(t)$ on the phase-space $\mathcal{H}_0$.*

The main result of this paper is

Theorem 1.3. *Let $\omega_0 > 0$ be fixed. Then, for every $\omega \in [0, \omega_0]$, the strongly continuous semigroup $S_\omega(t)$ has a compact invariant set $\mathcal{E}_\omega \subset \mathcal{H}_0$ of finite fractal dimension (called exponential attractor) that satisfies the following conditions:*

(i) *there exist $\kappa > 0$ and a positive increasing function J such that, for every bounded set $\mathcal{B} \subset \mathcal{H}_0$, there holds*

$$\mathrm{dist}_{\mathcal{H}_0}(S_\omega(t)\mathcal{B}, \mathcal{E}_\omega) \leq J(R)e^{-\kappa t}, \qquad \forall t \geq 0,$$

where $R = \sup_{z\in\mathcal{B}} \|z\|_{\mathcal{H}_0}$;

(ii) *the fractal dimension of $\mathcal{E}_\omega$ is uniformly bounded with respect to $\omega \in [0, \omega_0]$;*

(iii) *there exist $\tau \in (0,1)$ and $C > 0$ such that*

$$\mathrm{dist}^{\mathrm{sym}}_{\mathcal{H}_0}(\mathcal{E}_\omega, \mathcal{E}_0) \leq C\omega^\tau.$$

Here $\mathrm{dist}_{\mathcal{H}_0}$ and $\mathrm{dist}^{\mathrm{sym}}_{\mathcal{H}_0}$ denote the usual Hausdorff semidistance and symmetric Hausdorff distance in $\mathcal{H}_0$. All the quantities appearing here and in the sequel are independent of $\omega \in [0, \omega_0]$. It is worth pointing out that the basin of attraction of $\mathcal{E}_\omega$ is the *whole* phase-space, even in the hyperbolic case $\omega = 0$.

As a byproduct, we obtain the existence of a (unique) global attractor $\mathcal{A}_\omega$ for $S_\omega(t)$, recovering the results of [8, 9]. Indeed, $\mathcal{E}_\omega$ is by definition a compact attracting set (see, for instance, [18]). So, in particular, $\mathcal{A}_\omega \subset \mathcal{E}_\omega$.

We point out that the existence of $\mathcal{A}_\omega$ can be demonstrated under the weaker condition $f \in H_0$. However, if $f \in H_0$ and $\omega = 0$, we do not get enough regularity, which is basic in order to prove Theorem 1.3.

We conclude the section recalling some already known results (see [8]), that will be needed in the course of the investigation. The first regards the continuous dependence on the initial data.

Theorem 1.4. *For every $R > 0$ there exists a positive constant $K = K(R)$ such that, for any time $T > 0$ and any pair of initial data $z_1, z_1 \in \mathcal{H}_0$ with $\|z_i\|_{\mathcal{H}_0} \le R$, there hold*

$$\|S_\omega(t)z_1 - S_\omega(t)z_2\|_{\mathcal{H}_0} \le e^{Kt}\|z_1 - z_2\|_{\mathcal{H}_0}, \qquad \forall t \in [0,T], \tag{1.4}$$

and

$$\sqrt{\omega}\,\|\vartheta_1 - \vartheta_2\|_{L^2(0,T;H_1)} \le Ke^{KT}\|z_1 - z_2\|_{\mathcal{H}_0}, \tag{1.5}$$

where $\vartheta_i(t)$ is the first component of $S_\omega(t)z_i$.

The dissipative character of $S_\omega(t)$ follows from

Theorem 1.5. *There exists a bounded set $\mathcal{B}_0 \subset \mathcal{H}_0$ which is invariant and absorbing for $S_\omega(t)$, for every $\omega \in [0, \omega_0]$. That is, given any bounded set $\mathcal{B} \subset \mathcal{H}_0$, there exists $t_0 = t_0(\mathcal{B})$ such that $S_\omega(t)\mathcal{B} \subset \mathcal{B}_0$ for every $t \ge t_0$, and $t_0(\mathcal{B}_0) = 0$. Besides, there exists $K_0 = K_0(\mathcal{B}) \ge 0$ such that*

$$\sup_{z \in \mathcal{B}} \|S_\omega(t)z\|_{\mathcal{H}_0} \le K_0, \qquad \forall t \ge 0.$$

We will also make use of higher order integral estimates. Namely (see [6]),

Proposition 1.6. *For every $R \ge 0$, there exists $C_0 = C_0(R) \ge 0$ such that, for every $z \in \mathcal{H}_1$ with $\|z\|_{\mathcal{H}_1} \le R$, it follows that*

$$\omega \int_0^t \|A\vartheta(y)\|^2 dy \le C_0(1+t),$$

where $\vartheta(t)$ denotes the first component of $S_\omega(t)z$.

The independence of ω in the above theorems, which is not explicitly stated in [6, 8], comes from the fact that the diffusion term $-\omega\Delta\vartheta$ gives always a contribution of the right sign in the various estimates. In fact, this term provides a higher order control that is in most cases superfluous, due to the linearity of the coupling between ϑ and χ.

Remark 1.7. The term $\vartheta + \int_0^\infty \mu(s)\eta(s)ds$, coming from the constitutive assumption on the internal energy, can be neglected if, for instance, ϑ satisfies the homogeneous Dirichlet boundary condition (cf. [9]). On the contrary, this term plays a basic role in proving the dissipativity of the system when the material is thermally isolated, i.e., if ϑ satisfies the homogeneous Neumann boundary condition.

2. **Compact Attracting Sets in $\mathcal{H}_0$.** We first recall the so-called *transitivity property* of exponential attraction, recently devised in [5, Theorem 5.1].

Lemma 2.1. *Let $S(t)$ be a strongly continuous semigroup on a Banach space $\mathcal{H}$. Let $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2 \subset \mathcal{H}$ be such that*

$$\mathrm{dist}_{\mathcal{H}}(S(t)\mathcal{C}_0, \mathcal{C}_1) \le \Lambda_1 e^{-\beta_1 t}, \qquad \mathrm{dist}_{\mathcal{H}}(S(t)\mathcal{C}_1, \mathcal{C}_2) \le \Lambda_2 e^{-\beta_2 t},$$

for some $\beta_1, \beta_2 > 0$ and $\Lambda_1, \Lambda_2 \geq 0$. Assume also that, for all $z_1, z_2 \in \bigcup_{t\geq 0} S(t)\mathcal{C}_j$, there holds

$$\|S(t)z_1 - S(t)z_2\|_{\mathcal{H}} \leq \Lambda_0 e^{\beta_0 t}\|z_1 - z_2\|_{\mathcal{H}},$$

for some $\beta_0 \geq 0$ and some $\Lambda_0 \geq 0$. Then it follows that

$$\operatorname{dist}_{\mathcal{H}}(S(t)\mathcal{C}_0, \mathcal{C}_2) \leq \Lambda e^{-\beta t},$$

where $\beta = \frac{\beta_1\beta_2}{\beta_0+\beta_1+\beta_2}$ and $\Lambda = \Lambda_0\Lambda_1 + \Lambda_2$.

Then we have

Proposition 2.2. *There exists a bounded set $\mathcal{K} \subset \mathcal{Z}$, compact in $\mathcal{H}_0$, such that*

(i) *$S_\omega(t)\mathcal{K} \subset \mathcal{K}$;*

(ii) *there exist $\varepsilon > 0$ and $M > 1$ such that*

$$\operatorname{dist}_{\mathcal{H}_0}(S_\omega(t)\mathcal{B}_0, \mathcal{K}) \leq Me^{-\varepsilon t}, \qquad \forall t \geq 0.$$

Proof. Following [9, Proposition 4.6, Lemma 6.3], we learn that the solution $S_\omega(t)z$ with initial data $z \in \mathcal{B}_0$ can be decomposed into the sum $z_d(t) + z_c(t)$ satisfying the following properties:

– there exist $\varepsilon > 0$ and $M > 1$ such that

$$\sup_{z\in\mathcal{B}_0} \|z_d(t)\|_{\mathcal{H}_0} \leq Me^{-\varepsilon t}, \qquad \forall t \geq 0;$$

– there exists a compact set $\mathcal{K}_0 \subset \mathcal{H}_0$ such that

$$\bigcup_{z\in\mathcal{B}_0} z_c(t) \subset \mathcal{K}_0, \qquad \forall t \geq 0.$$

Due to the two above results, we get at once the inequality

$$\operatorname{dist}_{\mathcal{H}_0}(S_\omega(t)\mathcal{B}_0, \mathcal{K}_0) \leq Me^{-\varepsilon t}, \qquad \forall t \geq 0. \tag{2.1}$$

In fact, a closer look to [9] shows that $\mathcal{K}_0$ is of the form

$$\mathcal{K}_0 = \left\{ z = (\vartheta, \chi, \eta) \;\Big|\; \|z\|^2_{\mathcal{H}_1} + \|T\eta\|^2_{\mathcal{M}_0} + \sup_{s\in\mathbb{R}^+} \frac{1}{1+s}\|A^{1/2}\eta(s)\|^2 \leq R_0 \right\},$$

for some $R_0 \geq 0$. So, in particular, $\mathcal{K}_0$ is a bounded subset of $\mathcal{Z}$. To reach the conclusion, we just have to refine a little bit the set $\mathcal{K}_0$. Hence, following [6], we introduce the set

$$\mathcal{K}_* = \left\{ z = (\vartheta, \chi, \eta) \;\Big|\; \|z\|^2_{\mathcal{H}_1} + \|T\eta\|^2_{\mathcal{M}_0} \leq R_1^2,\ \mathbb{T}_\eta(x) \leq \frac{R_1^2}{x},\ \forall x \geq x^* \right\}.$$

Here $x^* \geq 1$ is a fixed number depending only on the kernels ν and μ. Observe that $\mathcal{K}_*$ is a bounded subset of $\mathcal{Z}$ closed in $\mathcal{H}_0$, and thus compact in $\mathcal{H}_0$. Reasoning as in the proof of [6, Theorem 5.2], one can show that, provided that R_1 is big enough, the set $\mathcal{K}_*$ absorbs (uniformly in ω) bounded subsets of $\mathcal{Z}$. Indeed, in this case the proof is even easier (cf. Remark 1.1). So, there exists $t_* \geq 0$ such that

$$S_\omega(t)\mathcal{K}_0 \subset \mathcal{K}_* \qquad \text{and} \qquad S_\omega(t)\mathcal{K}_* \subset \mathcal{K}_*, \qquad \forall t \geq \frac{t_*}{2}.$$

Finally, we define the compact set

$$\mathcal{K} = \overline{\bigcup_{\omega\in[0,\omega_0]} \bigcup_{t\geq\frac{t_*}{2}} S_\omega(t)\mathcal{K}_*}^{\mathcal{H}_0} \subset \mathcal{K}_*.$$

The invariance of $\mathcal{K}$ easily follows from the continuity of $S_\omega(t)$. Besides, by construction,

$$S_\omega(t)\mathcal{K}_0 \subset \mathcal{K}, \qquad \forall t \geq t_*. \tag{2.2}$$

Notice that (2.2), thanks to Theorem 1.5, implies

$$\sup_{\omega\in[0,\omega_0]} \sup_{t\in[0,t_*]} \operatorname{dist}_{\mathcal{H}_0}\big(S_\omega(t)\mathcal{K}_0, \mathcal{K}\big) < \infty,$$

thus $\mathcal{K}$ attracts with an arbitrary exponential rate $S_\omega(t)\mathcal{K}_0$. At this point, we use Lemma 2.1. Thus, collecting (2.1)-(2.2) and the continuous dependence estimate (1.4), we get the desired conclusion. □

3. **Robust Exponential Attractors.** In order to prove the main Theorem 1.3, we first need to write the abstract result of [5] adapted to our situation.

Lemma 3.1. *Assume there exist* $\alpha \in (0, \frac{1}{4})$*, a time* $t_\alpha > 0$ *and constants* $C_j = C_j(\alpha) \geq 0$ *such that the following conditions hold.*

(L1) *The map* $S_\omega = S_\omega(t_\alpha)$ *admits the decomposition* $S_\omega = L_\omega + N_\omega$ *such that, for every* $z_1, z_2 \in \mathcal{K}$*,*

$$\|L_\omega z_1 - L_\omega z_2\|_{\mathcal{H}_0} \leq \alpha \|z_1 - z_2\|_{\mathcal{H}_0},$$

and

$$\|N_\omega z_1 - N_\omega z_2\|_{\mathcal{Z}} \leq C_1 \|z_1 - z_2\|_{\mathcal{H}_0}.$$

(L2) *For every* $n \in \mathbb{N}$ *and every* $z \in \mathcal{K}$

$$\|S_\omega^n z - S_0^n z\|_{\mathcal{H}_0} \leq C_2^n \sqrt{\omega}.$$

(L3) *For every* $z \in \mathcal{K}$ *and every* $t \in [t_\alpha, 2t_\alpha]$

$$\|S_\omega(t) z - S_0(t) z\|_{\mathcal{H}_0} \leq C_3 \sqrt{\omega}.$$

(L4) *The map*

$$(t, z) \mapsto S_\omega(t) z : [t_\alpha, 2t_\alpha] \times \mathcal{K} \to \mathcal{K}$$

is Lipschitz continuous. Here $\mathcal{K}$ *is endowed with the metric topology of* $\mathcal{H}_0$*.*

Then, for every $\omega \in [0, \omega_0]$*, there exists a family of sets* $\mathcal{E}_\omega \subset \mathcal{K}$ *satisfying* (ii)-(iii) *of Theorem 1.3 and the attraction property*

$$\operatorname{dist}_{\mathcal{H}_0}(S_\omega(t)\mathcal{K}, \mathcal{E}_\omega) \leq J_0 e^{-\kappa_0 t}, \qquad \forall t \geq 0,$$

for some $\kappa_0 > 0$ *and* $J_0 \geq 0$*.*

The check that conditions (L1)-(L4) of Lemma 3.1 hold true is postponed to the last section. Thus, to complete the proof of Theorem 1.3, we are left to show that the basin of attraction of $\mathcal{E}_\omega$ coincides with the whole phase-space $\mathcal{H}_0$ (that is, condition (i) of the theorem). To this aim, let $\mathcal{B} \subset \mathcal{H}_0$ be a bounded set and let $R = \sup_{z\in\mathcal{B}} \|z\|_{\mathcal{H}_0}$. Then, by means of Theorem 1.5, Proposition 2.2 and Lemma 3.1, we have the following chain of exponential attractions:

$$\operatorname{dist}_{\mathcal{H}_0}(S_\omega(t)\mathcal{B}, \mathcal{B}_0) \leq \Lambda(R) e^{-t},$$
$$\operatorname{dist}_{\mathcal{H}_0}(S_\omega(t)\mathcal{B}_0, \mathcal{K}) \leq M e^{-\varepsilon t},$$
$$\operatorname{dist}_{\mathcal{H}_0}(S_\omega(t)\mathcal{K}, \mathcal{E}_\omega) \leq J_0 e^{-\kappa_0 t},$$

for some increasing positive function Λ.

In light of (1.4), a further application of Lemma 2.1 allows us to connect the above chain, so getting the desired exponential attraction property (i) of Theorem 1.3.

4. **Verification of Conditions (L1)-(L4) of Lemma 3.1.** Throughout this section, let $c = c(\alpha) \geq 0$ be a generic constant. Again, c as well as all the other quantities appearing in the sequel are understood to be independent of $\omega \in [0, \omega_0]$.

4.1. *Proof of* (L1). First of all, we decompose the map $S_\omega(t) : \mathcal{K} \to \mathcal{K}$ as

$$S_\omega(t) = L_\omega(t) + N_\omega(t),$$

where, for $z \in \mathcal{K}$,

$$L_\omega(t)z = (\vartheta_L(t), \chi_L(t), \eta_L(t))$$

is the solution at time t to the linear problem

$$\begin{aligned}
&\partial_t\vartheta_L + \vartheta_L + \omega A\vartheta_L + \int_0^\infty \nu(s)\eta_L(s)ds + \int_0^\infty \mu(s)A\eta_L(s)ds = -\partial_t\chi_L,\\
&\partial_t\chi_L + A\chi_L = \vartheta_L,\\
&\partial_t\eta_L = T\eta_L + \vartheta_L,\\
&L_\omega(0)z = z,
\end{aligned}$$

and $N_\omega(t)$ is obtained by difference.

It is easy to prove (cf. [8, Lemma 6.2]) that $L_\omega(t)$ is an exponentially stable strongly continuous linear semigroup on $\mathcal{H}_0$. Thus, there exists $\sigma > 0$ such that

$$\|L_\omega(t)z_1 - L_\omega(t)z_2\|_{\mathcal{H}_0} \leq ce^{-\sigma t}\,\|z_1 - z_2\|_{\mathcal{H}_0}, \qquad \forall t \geq 0,$$

for every $z_1, z_2 \in \mathcal{H}_0$. It is then clear that we can fix (any) $\alpha \in (0, \frac{1}{4})$ and, accordingly, $t_\alpha > 0$ such that $L_\omega = L_\omega(t_\alpha)$ satisfies the first inequality of (L1).

Concerning the second one, we leave to the reader the easy check that $N_\omega(t)$ maps $\mathcal{K}$ into $\mathcal{Z}$. Then, for $z_1, z_2 \in \mathcal{K}$, we set

$$\begin{aligned}
(\vartheta(t), \chi(t), \eta^t) &= S_\omega(t)z_1 - S_\omega(t)z_2,\\
\bar{z}(t) = (\bar{\vartheta}(t), \bar{\chi}(t), \bar{\eta}^t) &= N_\omega(t)z_1 - N_\omega(t)z_2.
\end{aligned}$$

By straightforward computations, we find the system

$$\begin{aligned}
&\partial_t\bar{\vartheta} + \bar{\vartheta} + \omega A\bar{\vartheta} + \int_0^\infty \nu(s)\bar{\eta}(s)ds + \int_0^\infty \mu(s)A\bar{\eta}(s)ds = \omega\vartheta - \partial_t\bar{\chi} + \int_0^\infty \mu(s)\eta(s)ds,\\
&\partial_t\bar{\chi} + A\bar{\chi} = \bar{\vartheta} - \phi(\chi_1) + \phi(\chi_2) + \chi,\\
&\partial_t\bar{\eta} = T\bar{\eta} + \bar{\vartheta},\\
&\bar{z}(0) = 0,
\end{aligned}$$

where χ_i is the second component of $S_\omega(t)z_i$. Next, multiply the first equation by $A\bar{\vartheta}$ in H_0, the second by $A\partial_t\bar{\chi}$ in H_0, and the third by $\bar{\eta}$ in $\mathcal{M}_1$, and add the resulting equations. Integrating by parts, in view of (K3), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\bar z\|^2_{\mathcal{H}_1} + \|A^{1/2}\bar\vartheta\|^2 + \omega\|A\bar\vartheta\|^2 + \|A^{1/2}\partial_t\bar\chi\|^2\\ &\leq \omega\langle A^{1/2}\vartheta, A^{1/2}\bar\vartheta\rangle + \int_0^\infty \mu(s)\langle A^{1/2}\eta(s), A^{1/2}\bar\vartheta\rangle ds + \langle A^{1/2}\chi, A^{1/2}\partial_t\bar\chi\rangle\\ &\quad - \langle A^{1/2}\phi(\chi_1) - A^{1/2}\phi(\chi_2), A^{1/2}\partial_t\bar\chi\rangle.\end{aligned}$$

On account of (H1) and the boundedness (and the invariance) of $\mathcal{K}$, we deduce that

$$\|A^{1/2}\phi(\chi_1) - A^{1/2}\phi(\chi_2)\| \leq c\|A^{1/2}\chi\|.$$

Hence, using the Hölder and the Young inequalities together with (K1), we easily end up with

$$\frac{d}{dt}\|\bar z\|^2_{\mathcal{H}_1} \leq c\big(\omega\|A^{1/2}\vartheta\|^2 + \|\eta\|^2_{\mathcal{M}_0} + \|A^{1/2}\chi\|^2\big).$$

Finally, integrating on $(0, t_\alpha)$, from (1.4)-(1.5), we get

$$\|N_\omega z_1 - N_\omega z_2\|_{\mathcal{H}_1} \leq c\|z_1 - z_2\|_{\mathcal{H}_0},$$

having set $N_\omega = N_\omega(t_\alpha)$. Actually, to conclude that the second inequality of (L1) holds, we need to control the left-hand side in the norm of $\mathcal{Z}$. This, on account of the above estimate, amounts to showing that

$$\|T\bar\eta^{t_\alpha}\|^2_{\mathcal{M}_0} + \sup_{x\geq 1} x\mathbb{T}_{\bar\eta^{t_\alpha}}(x) \leq c\|z_1 - z_2\|^2_{\mathcal{H}_0}.$$

The proof, similar to the one of [6, Lemma 7.4], is left to the reader.

4.2. *Proofs of* (L2)-(L3)**.** Both conditions follow directly from

Lemma 4.1. *There holds*

$$\|S_\omega(t)z - S_0(t)z\|_{\mathcal{H}_0} \leq c\sqrt{\omega}\, e^{ct}, \qquad \forall t \geq 0,$$

for every $z \in \mathcal{K}$.

Proof. For $z \in \mathcal{K}$, we set

$$\begin{aligned}(\vartheta_\omega(t), \chi_\omega(t), \eta^t_\omega) &= S_\omega(t)z,\\ \bar z(t) = (\bar\vartheta(t), \bar\chi(t), \bar\eta^t) &= S_\omega(t)z - S_0(t)z.\end{aligned}$$

Then $\bar z$ solves the system

$$\begin{aligned}&\partial_t\bar\vartheta + \bar\vartheta + \partial_t\bar\chi + \int_0^\infty \nu(s)\bar\eta(s)ds + \int_0^\infty \mu(s)A\bar\eta(s)ds = -\omega A\vartheta_\omega + \omega\vartheta_\omega + \int_0^\infty \mu(s)\bar\eta(s)ds,\\ &\partial_t\bar\chi + A\bar\chi + \phi(\chi_\omega) - \phi(\chi_0) = \bar\vartheta + \bar\chi,\\ &\partial_t\bar\eta = T\bar\eta + \bar\vartheta,\\ &\bar z(0) = 0.\end{aligned}$$

Multiplying the above equations by $\bar\vartheta$ in H_0, $\partial_t\bar\chi$ in H_0 and $\bar\eta$ in $\mathcal{M}_0$ respectively, and adding the resulting equations, in view of (K3) we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\bar z\|^2_{\mathcal{H}_0} + \|\bar\vartheta\|^2 + \|\partial_t\bar\chi\|^2\\ &\leq -\omega\langle A\vartheta_\omega, \bar\vartheta\rangle + \omega\langle\vartheta_\omega, \bar\vartheta\rangle + \int_0^\infty \mu(s)\langle\bar\eta(s), \bar\vartheta\rangle ds - \langle\phi(\chi_\omega) - \phi(\chi_0), \partial_t\bar\chi\rangle + \langle\partial_t\bar\chi, \bar\chi\rangle.\end{aligned}$$

By virtue of (H1),

$$\|\phi(\chi_0) - \phi(\chi_\omega)\|^2 \le c\|A^{1/2}\bar{\chi}\|^2,$$

so the Hölder and the Young inequalities, together with (K1), yield

$$\frac{d}{dt}\|\bar{z}\|^2_{\mathcal{H}_0} \le c\|\bar{z}\|^2_{\mathcal{H}_0} + c\omega^2\|A\vartheta_\omega\|^2.$$

Applying the standard Gronwall Lemma on $[0,t]$, in view of Proposition 1.6, we obtain the desired estimate. □

4.3. *Proof of* (L4). Throughout this proof, the positive constant c may depend on T. On account of estimate (1.4), we have

$$\|S_\omega(t_1)z_1 - S_\omega(t_2)z_2\|_{\mathcal{H}_0} \le \|S_\omega(t_1)z_1 - S_\omega(t_2)z_1\|_{\mathcal{H}_0} + c\|z_1 - z_2\|_{\mathcal{H}_0},$$

for every $t_1, t_2 \in [0,T]$ and $z_1, z_2 \in \mathcal{K}$. In order to prove the result it then suffices to show that

$$\sup_{z\in\mathcal{K}}\sup_{t\in[0,T]}\|\partial_t S_\omega(t)z\|_{\mathcal{H}_0} \le c. \tag{4.1}$$

For $z = (\vartheta_0, \chi_0, \eta_0) \in \mathcal{K}$ we consider the system, obtained via formal time differentiation from the original one,

$$\begin{aligned}
&\partial_t\varphi + \varphi + \omega A\varphi + \int_0^\infty \nu(s)\xi(s)ds + \int_0^\infty \mu(s)A\xi(s)ds = \omega\varphi + \int_0^\infty \mu(s)\xi(s)ds - \partial_t\psi,\\
&\partial_t\psi + A\psi = -\phi'(\chi)\partial_t\chi + \varphi + \psi,\\
&\partial_t\xi = T\xi + \varphi,\\
&\varphi(0) = (\omega-1)\vartheta_0 - \omega A\vartheta_0 - \int_0^\infty \nu(s)\eta_0(s)ds - \int_0^\infty \mu(s)A\eta_0(s)ds + \int_0^\infty \mu(s)\eta_0(s)ds\\
&\qquad + \big(A\chi_0 + \phi(\chi_0) - \vartheta_0 - \chi_0\big) + f,\\
&\psi(0) = -A\chi_0 - \phi(\chi_0) + \vartheta_0 + \chi_0,\\
&\xi^0 = T\eta_0 + \vartheta_0,
\end{aligned}$$

which, by standard arguments, admits a unique solution (φ, ψ, ξ) with

$$\begin{aligned}
&\varphi \in C^0([0,T], H_{-1}) \cap L^2(0,T; H_0),\\
&\psi \in C^0([0,T], H_0) \cap L^2(0,T; H_1) \cap H^1(0,T; H_0),\\
&\xi \in C^0([0,T], \mathcal{M}_{-1}).
\end{aligned}$$

Hence, for all $t \in [0,T]$,

$$\varphi(t) = \partial_t\vartheta(t), \qquad \psi(t) = \partial_t\chi(t), \qquad \xi^t = \partial_t\eta^t. \tag{4.2}$$

Let us introduce the functional

$$\Psi(t) = \|\varphi(t)\|^2 + \|A^{1/2}\psi(t)\|^2 + \|\xi(t)\|^2_{\mathcal{M}_0}.$$

Next, multiply the first equation by φ in H_0, the second by $\partial_t\psi$ in H_0, and the third by ξ in $\mathcal{M}_0$. Adding the resulting identities and integrating by parts with respect to s the term $\langle T\xi, \xi\rangle_{\mathcal{M}_0}$ with the aid of (K3), we find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Psi + \|\varphi\|^2 + \|\partial_t\psi\|^2 + \omega\|A^{1/2}\varphi\|^2\\
&\le \omega_0\|\varphi\|^2 + \int_0^\infty \mu(s)\langle\xi(s), \varphi\rangle ds - \langle\phi'(\chi)\partial_t\chi, \partial_t\psi\rangle + \langle\psi, \partial_t\psi\rangle.
\end{aligned}$$

By (K1)-(K2), the Hölder and the Young inequalities and (4.2), we deduce

$$\frac{d}{dt}\Psi \leq c\Psi.$$

Applying the Gronwall's lemma on $[0, T]$ and taking the initial data into account, we finally obtain (4.1).

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