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# On fractional Choquard equations

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We investigate a class of nonlinear Schrödinger equations with a generalized Choquard nonlinearity and fractional diffusion. We obtain regularity, existence, nonexistence, symmetry as well as decays properties.

*Keywords*: Fractional Laplacian; Choquard equation; existence; nonexistence; multiplicity.

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## 1. Introduction

Given $\omega > 0$, $N \geq 3$, $\alpha \in (0, N)$, $p > 1$ and $s \in (0, 1)$, we consider the nonlocal problem

$$(-\Delta)^s u + \omega u = (\mathcal{K}_\alpha * |u|^p)|u|^{p-2}u, \quad u \in H^s(\mathbb{R}^N), \qquad (\mathcal{P}_\omega)$$

where $\mathcal{K}_\alpha(x) = |x|^{\alpha - N}$ and the Hilbert space $H^s(\mathbb{R}^N)$ is defined as

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : (-\Delta)^{s/2}u \in L^2(\mathbb{R}^N)\},$$

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with scalar product and norm given by

$$(u,v) = \int (-\Delta)^{s/2}u(-\Delta)^{s/2}v + \omega \int uv, \quad \|u\|^2 = \|(-\Delta)^{s/2}u\|_2^2 + \omega\|u\|_2^2.$$

The fractional Laplacian operator $(-\Delta)^s$ is defined by

$$(-\Delta)^s u(x) = -\frac{C(N,s)}{2} \int \frac{u(x+y) - u(x-y) - 2u(x)}{|y|^{N+2s}} dy, \quad x \in \mathbb{R}^N,$$

where $C(N,s)$ is a suitable normalization constant. Thus, problem $(\mathcal{P}_\omega)$ presents nonlocal characteristics in the nonlinearity as well as in the (fractional) diffusion.

We point out that when $s = 1$, $p = 2$ and $\alpha = 2$, then $(\mathcal{P}_\omega)$ boils down to the so-called Choquard or nonlinear Schrödinger–Newton equation

$$-\Delta u + \omega u = (\mathcal{K}_2 * u^2)u, \quad u \in H^1(\mathbb{R}^N). \tag{1.1}$$

This equation was elaborated by Pekar[30] in the framework of quantum mechanics. Subsequently, it was adopted as an approximation of the Hartree–Fock theory, see Ref. 6. More recently, Penrose[31] settled it as a model of self-gravitating matter. The first investigations for existence and symmetry of the solutions to (1.1) go back to the works of Lieb[23] and Lions.[25] On this basis, we will refer to $(\mathcal{P}_\omega)$ as to the generalized nonlinear Choquard equation. In the last few years, the study of equations involving pseudo-differential operators has steadily grown. In Refs. 26 and 27 the authors discuss recent developments in the description of anomalous diffusion via fractional dynamics and various fractional equations are derived asymptotically from Lévy random walk models, extending Brownian walk models in a natural way. In particular, in Ref. 20, a fractional Schrödinger equation with local power type nonlinearity was studied. This extends to a Lévy framework the classical statement that path integral over Brownian trajectories leads to the standard Schrödinger equation $-\Delta u + \omega u = f(u)$, see e.g. Ref. 8 and references therein. In the case $s = 1/2$, problem $(\mathcal{P}_\omega)$ has been used to model the dynamics of pseudo-relativistic boson stars. Indeed in Ref. 16 the following equation is studied:

$$\sqrt{-\Delta}u + u = (\mathcal{K}_2 * |u|^2)u, \quad u \in H^{1/2}(\mathbb{R}^3), \quad u > 0,$$

and in Ref. 13 it is shown that the dynamical evolution of boson stars is described by the nonlinear evolution equation

$$i\partial_t \psi = \sqrt{-\Delta + m^2}\psi - (\mathcal{K}_2 * |\psi|^2)\psi \quad (m \geq 0),$$

for a field $\psi : [0,T) \times \mathbb{R}^3 \to \mathbb{C}$ (see also Refs. 18, 19 and 21). The square root of the Laplacian also appears in the semi-relativistic Schrödinger–Poisson–Slater systems, see e.g. Ref. 4.

So motivated by the above-cited works, in this paper we have considered $(\mathcal{P}_\omega)$ as a generalization of (1.1) which takes into account more general convolution kernels and allows a distribution density of type $|u|^p$. Observe that mathematically Eq. $(\mathcal{P}_\omega)$ involves two fractional operators since it can be seen as a coupled system of two equations involving fractional Laplacians (see Sec. 5, in particular problem (5.2)).

We shall say that $u \in H^s(\mathbb{R}^N)$ is a weak solution of $(\mathcal{P}_\omega)$ if

$$\int (-\Delta)^{s/2}u(-\Delta)^{s/2}v + \omega \int uv = \int (\mathcal{K}_\alpha * |u|^p)|u|^{p-2}uv, \quad \text{for all } v \in H^s(\mathbb{R}^N).$$

Let

$$1 + \frac{\alpha}{N} < p < \frac{N+\alpha}{N-2s}, \tag{1.2}$$

and introduce the Nehari manifold

$$\mathcal{N}_\omega := \left\{ u \in H^s(\mathbb{R}^N)\backslash\{0\} : \|(-\Delta)^{s/2}u\|_2^2 + \omega\|u\|_2^2 - \int (\mathcal{K}_\alpha * |u|^p)|u|^p = 0 \right\},$$

and the $C^1$ functional $E_\omega : H^s(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$E_\omega(u) = \frac{1}{2}\int |(-\Delta)^{s/2}u|^2 + \frac{\omega}{2}\int u^2 - \frac{1}{2p}\int (\mathcal{K}_\alpha * |u|^p)|u|^p. \tag{1.3}$$

A *ground state* of $(\mathcal{P}_\omega)$ is a solution with minimal energy $E_\omega$ and can be characterized as

$$\min_{u \in \mathcal{N}_\omega} E_\omega(u).$$

The main result of the paper is the following.

**Theorem 1.1.** *Assume that $p$ satisfies* (1.2). *Then*

**Existence:** *There exists a ground state $u \in H^s(\mathbb{R}^N)$ to problem $(\mathcal{P}_\omega)$ which is positive, radially symmetric and decreasing*;

**Regularity:** $u \in L^1(\mathbb{R}^N)$ *and moreover if* $s \leq 1/2$, $u \in C^{0,\mu}(\mathbb{R}^N)$ *for some* $\mu \in (0, 2s)$, *if* $s > 1/2$, $u \in C^{1,\mu}(\mathbb{R}^N)$ *for some* $\mu \in (0, 2s-1)$;

**Asymptotics:** *If* $p \geq 2$, *there exists* $C > 0$ *such that*

$$u(x) = \frac{C}{|x|^{N+2s}} + o(|x|^{-N-2s}), \quad \text{as } |x| \to \infty;$$

**Morse index:** *If* $2 \leq p < 1 + (2s+\alpha)/N$ *and* $s > 1/2$, *the Morse index of $u$ is equal to one.*

Under some restrictions on the values of $p$, there exist different ways of obtaining ground state solutions, via minimization problems which turn out to be equivalent up to a suitable change of scale, as shown in Propositions 4.1 and 4.2. In particular, in the range

$$1 + \frac{\alpha}{N} < p < 1 + \frac{2s+\alpha}{N}, \tag{1.4}$$

the ground states can be found by minimizing the functional

$$E_0(u) = \frac{1}{2}\int |(-\Delta)^{s/2}u|^2 - \frac{1}{2p}\int (\mathcal{K}_\alpha * |u|^p)|u|^p, \tag{1.5}$$

on $L^2$-spheres, which allows to obtain the additional information about the Morse index of solutions. The information provided in Proposition 4.2 is also useful when studying the *orbital stability* property of the family of ground states for the equation

$$\mathrm{i}u_t = (-\Delta)^s u + \omega u - (\mathcal{K}_\alpha * |u|^p)|u|^{p-2}u \quad \mathbb{R}^N \times (0,\infty). \tag{1.6}$$

This topic was recently investigated in Ref. 36 in the case $p = 2$ and with $\alpha \in (N-2s, N)$, see the introduction therein for the physical motivations. We plan to investigate (1.6) — in presence of a parameter $\varepsilon$ of singular perturbation — from the point of view of *soliton dynamics* by following an approach used in Ref. 5 to study the local case $s = 1$ and motivated by the absence of general results about the nondegeneracy of ground states.

We point out that, contrary to the local case $s = 1$, the solutions can only decay at the polynomial rate $|x|^{-N-2s}$. We refer the reader to Ref. 29 for sharp results about the exponential decay of ground state solutions in the case $s = 1$.

Moreover, we have the following multiplicity result.

**Theorem 1.2.** *Assume that* (1.2) *holds. Then* $(\mathcal{P}_\omega)$ *admits infinitely many radial solutions with diverging norm and diverging energy levels. If in addition* $N = 4$ *or* $N \geq 6$, *then* $(\mathcal{P}_\omega)$ *admits infinitely many nonradial solutions with diverging norm and diverging energy levels.*

Next, we have the following nonexistence result.

**Theorem 1.3.** *Assume that either* $p \leq 1 + \alpha/N$ *or* $p \geq (N+\alpha)/(N-2s)$. *Then* $(\mathcal{P}_\omega)$ *does not admit nontrivial solutions* $u \in C^2(\mathbb{R}^N)$.

As a consequence, the range of $p$ detected in (1.2) is optimal for the existence of nontrivial solutions. The first complete study of Pohožaev identities and nonexistence results in star-shaped bounded domains for equations involving the fractional Laplacian and a local nonlinearity was done in Refs. 32 and 33. Then, more recently, for fractional equations set on the whole $\mathbb{R}^N$, in Ref. 9, the authors obtained a Pohožaev identity for power type nonlinearities. Theorem 1.3 is based upon Pohožaev identity (6.1) which is obtained, as in Ref. 9, by the localization procedure due to Caffarelli and Silvestre.[7]

Next, we denote by $\dot{H}^s(\mathbb{R}^N)$ the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the seminorm $\|(-\Delta)^{s/2}\cdot\|_2$, known as Gagliardo seminorm, and consider the problem

$$(-\Delta)^s u = (\mathcal{K}_\alpha * |u|^p)|u|^{p-2}u, \quad u \in \dot{H}^s(\mathbb{R}^N). \tag{$\mathcal{P}_0$}$$

We have the following result.

**Theorem 1.4.** *The following assertions hold*:

(1) *Let* $p \neq \frac{\alpha+N}{N-2s}$. *Then* $(\mathcal{P}_0)$ *does not admit nontrivial solutions* $u \in \dot{H}^s(\mathbb{R}^N) \cap L^{\frac{2pN}{N+\alpha}}(\mathbb{R}^N)$.

(2) *Let* $p = \frac{\alpha+N}{N-2s} = 2$. *Then the problem writes as*

$$(-\Delta)^s u = (|x|^{-4s} * |u|^2)u, \quad u \in \dot{H}^s(\mathbb{R}^N),\ \ N > 4s, \tag{1.7}$$

*and any of its solutions of fixed sign have the form*

$$C\left(\frac{t}{t^2+|x-x_0|^2}\right)^{\frac{N-2s}{2}}, \quad x\in\mathbb{R}^N, \tag{1.8}$$

*for some* $x_0 \in \mathbb{R}^N$, $C > 0$ *and* $t > 0$.

The classification of the solutions to problem (1.7) is reminiscent of that for the fixed-sign solutions to

$$(-\Delta)^s u = u^{\frac{N+2s}{N-2s}} \quad \text{in } \mathbb{R}^N.$$

In Ref. 10 (see also Ref. 22) the authors proved that any positive to this problem has the form of (1.8).

The plan of the paper is as follows. In Sec. 2 we collect some preliminary notions and results. In Sec. 3 we investigate the Hölder regularity and the asymptotic behavior of weak solutions. In Sec. 4 we prove the existence of least energy solutions (ground states) determining equivalent ways of characterizing them. Here we also get their symmetry and monotonicity properties and we investigate the Morse index of ground states in the particular ranges $2 \le p < 1 + (2s+\alpha)/N$ and $s \ge 1/2$. In Sec. 5 we get the existence of infinitely many solutions, symmetric under the action of some group. In Sec. 6, we obtain a general Pohǒzaev identity and we prove Theorem 1.4. In the paper, $C$ will always denote a generic constant which may vary from line-to-line. Unless expressly specified, the integral are meant to be extended to $\mathbb{R}^N$.

## 2. Preliminaries

First of all, let us recall the following properties which follow from the fractional Sobolev embedding

$$H^s(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^N), \quad r \in [2, 2_s^*], \quad \text{where} \quad 2_s^* := \frac{2N}{N-2s},$$

the Hardy–Littlewood inequality and the fractional version of the Gagliardo–Niremberg inequality

$$\|u\|_q \le C\|(-\Delta)^{s/2}u\|_2^{\beta}\|u\|_2^{(1-\beta)}, \tag{2.1}$$

for $q \in [2, 2_s^*]$ and $\beta$ satisfying $\frac{1}{q} = \frac{\beta}{2_s^*} + \frac{1-\beta}{2}$. Notice that by Proposition 3.6 of Ref. 12),

$$\|(-\Delta)^{s/2}u\|_2^2 = \frac{C(N,s)}{2}\iint \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}. \tag{2.2}$$

**Lemma 2.1.** *Let* $p$ *satisfy* (1.2). *We have that*:

(i) $2Np/(N+\alpha) \in (2, 2_s^*)$ *and for every* $u \in H^s(\mathbb{R}^N)$:

$$\int (\mathcal{K}_\alpha * |u|^p)|u|^p \le C\|u\|_{2Np/(N+\alpha)}^{2p}. \tag{2.3}$$

(ii) *If*

$$\frac{N(2p-1)}{N+\alpha} \leq q < \frac{Np}{\alpha}, \tag{2.4}$$

*and* $u \in L^q(\mathbb{R}^N)$, *then*

$$(\mathcal{K}_\alpha * |u|^p)|u|^{p-2}u \in L^r(\mathbb{R}^N) \quad \textit{for} \quad \frac{1}{r} = \frac{2p-1}{q} - \frac{\alpha}{N}. \tag{2.5}$$

*In particular* (2.5) *defines a function* $r = r(q)$ *which is strictly increasing and maps* $[N(2p-1)/(N+\alpha), Np/\alpha)$ *onto* $[1, Np/(\alpha(p-1)))$.

(iii) *For every* $u \in H^s(\mathbb{R}^N)$:

$$\int (\mathcal{K}_\alpha * |u|^p)|u|^p \leq C\|(-\Delta)^{s/2}u\|_2^{2\beta p}\|u\|_2^{2(1-\beta)p}, \quad \beta = \frac{Np-N-\alpha}{2sp}. \tag{2.6}$$

**Proof.** Property (2.1) is trivial. In order to prove (2.5), let $q$ be as in (2.4) and $u \in L^q(\mathbb{R}^N)$. Using Hardy–Littlewood–Sobolev inequality we have that

$$\mathcal{K}_\alpha * |u|^p \in L^t(\mathbb{R}^N) \quad \text{with} \quad \frac{1}{t} = \frac{p}{q} - \frac{\alpha}{N}.$$

Since $q < Np/\alpha$, then $t > 0$. Moreover, since $p > 1$, then

$$\frac{Np}{N+\alpha} < \frac{N(2p-1)}{N+\alpha},$$

and so $t > 1$. Hence, since for $p > 1$,

$$\frac{Np}{\alpha} < \frac{N(2p-1)}{\alpha},$$

by using Hölder inequality we get (2.5). Finally (2.1) easily follows from (2.3) and (2.1). □

The next result is an adaptation of a classical lemma of Lions and it is crucial in the proofs of the existence theorems.

**Lemma 2.2.** *Let* $q \in [2, 2_s^*]$. *For every* $u \in H^s(\mathbb{R}^N)$ *we have that*

$$\|u\|_q^q \leq C\left(\sup_{x\in\mathbb{R}^N}\int_{B_1(x)}|u|^q\right)^{1-\frac{2}{q}}\|u\|^2.$$

**Proof.** If $q = 2$ it is obvious. Let now $q \in (2, 2_s^*]$. Since $r := N(q-2)/2s \leq q$, for a.e. $x \in \mathbb{R}^N$, by Theorem 6.7 of Ref. 12, we have

$$\int_{B_1(x)}|u|^q \leq \left(\int_{B_1(x)}|u|^r\right)^{\frac{q}{r}(1-\frac{2}{q})}\left(\int_{B_1(x)}|u|^{2_s^*}\right)^{\frac{2}{2_s^*}}$$

$$\leq C \left( \int_{B_1(x)} |u|^q \right)^{1-\frac{2}{q}} \|u\|^2_{H^s(B_1(x))}$$

$$\leq C \left( \sup_{x\in\mathbb{R}^N} \int_{B_1(x)} |u|^q \right)^{1-\frac{2}{q}} \|u\|^2_{H^s(B_1(x))},$$

where $\|u\|^2_{H^s(B_1(x))}$ is defined in Eq. (2.2) of Ref. 12. Hence, we cover $\mathbb{R}^N$ with balls of radius 1 in such a way that each point of $\mathbb{R}^N$ is contained in at most $N+1$ balls. This procedure works even if in the $H^s(B_1(x))$-norm there is a nonlocal term (the Gagliardo seminorm) and so we conclude. □

With the same procedure of Lemma 2.2, one proves that, for all $u \in H^s(\mathbb{R}^N)$, $2 \leq q < 2^*_s$, and $\sigma > 0$,

$$\|u\|_t^t \leq C \left( \sup_{x\in\mathbb{R}^N} \int_{B_\sigma(x)} |u|^q \right)^{\frac{\beta t}{q}} \|u\|^2,$$

where $t = q + 2(2^*_s - q)/2^*_s$ and $\beta = q(2^*_s - 2)/[q(2^*_s - 2) + 2 \cdot 2^*_s]$ and so one obtains the following lemma.

**Lemma 2.3.** *If $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$ and for some $\sigma > 0$ and $2 \leq q < 2^*_s$ we have*

$$\sup_{x\in\mathbb{R}^n} \int_{B_\sigma(x)} |u_n|^q \to 0 \quad \text{as } n \to \infty,$$

*then $u_n \to 0$ in $L^r(\mathbb{R}^N)$ for $2 < r < 2^*_s$.*

## 3. Regularity and Asymptotics

In this section we want to show that any $H^s(\mathbb{R}^N)$-solution of $(\mathcal{P}_\omega)$ is indeed regular as well as the asymptotic profile. Let us recall the definition of the fractional Sobolev spaces for $q \geq 1$ and $\beta \geq 0$:

$$\mathcal{W}^{\beta,q} = \{u \in L^q(\mathbb{R}^N) | \mathcal{F}^{-1}[(1 + |\xi|^\beta)\mathcal{F}u] \in L^q(\mathbb{R}^N)\} \tag{3.1}$$

(see Ref. 35 for more details) and the following results (see Theorem 3.2 in Ref. 15).

**Theorem 3.1.** *We have the following*:

(i) *If $\beta \geq 0$ and either $1 < r \leq q \leq r^*_\beta := Nr/(N - \beta r) < +\infty$ or $r = 1$ and $1 \leq q < N/(N - \beta)$, we have that $\mathcal{W}^{\beta,r}$ is continuously embedded in $L^q(\mathbb{R}^N)$.*

(ii) *Assume that $0 \leq \beta \leq 2$ and $\beta > N/r$. If $\beta - N/r < 1$ and $0 < \mu \leq \beta - N/r$ then $\mathcal{W}^{\beta,r}$ is continuously embedded in $C^{0,\mu}(\mathbb{R}^N)$. If $\beta - N/r > 1$ and $0 < \mu \leq \beta - N/r - 1$ then $\mathcal{W}^{\beta,r}$ is continuously embedded in $C^{1,\mu}(\mathbb{R}^N)$.*

We prove the following theorem.

**Theorem 3.2.** *Let $u$ be a solution of $(\mathcal{P}_\omega)$. If $s \le 1/2$, then $u \in L^1(\mathbb{R}^N) \cap C^{0,\mu}(\mathbb{R}^N)$ for $\mu \in (0, 2s)$. If $s > 1/2$, then $u \in L^1(\mathbb{R}^N) \cap C^{1,\mu}(\mathbb{R}^N)$ for $\mu \in (0, 2s-1)$.*

**Lemma 3.1.** *Let $u \in H^s(\mathbb{R}^N)$ be a solution of $(\mathcal{P}_\omega)$. Then for every $q \ge 1$ such that*

$$\frac{1}{q} > \frac{\alpha}{N}\left(1 - \frac{1}{p}\right) - \frac{2s}{N},$$

*we have that $u \in L^q(\mathbb{R}^N)$. Moreover, for every $r > 1$ such that*

$$\frac{1}{r} > \frac{\alpha}{N}\left(1 - \frac{1}{p}\right),$$

*we have that $u \in \mathcal{W}^{2s,r}$.*

**Proof.** Let us consider $q_0 = 2Np/(N+\alpha)$. Since $u \in H^s(\mathbb{R}^N)$, by Sobolev embeddings we have that $u \in L^{q_0}(\mathbb{R}^N)$. Moreover by (2.1) of Lemma 2.1 we have that $(\mathcal{K}_\alpha * |u|^p)|u|^{p-2}u \in L^{r_0}(\mathbb{R}^N)$ with $1/r_0 = (2p-1)/q_0 - \alpha/N$. Thus, since the Bessel operator preserves the Lebesgue spaces (see Ref. 35) and by (3.1) we have that $u \in \mathcal{W}^{2s,r_0}$. Then, by Sobolev embedding in (3.1) of Theorem 3.1, $u \in L^q(\mathbb{R}^N)$ for every $q \in [r_0, (r_0)^*_{2s}]$, i.e. for every $q$ such that

$$\left(\frac{\alpha}{N}\left(1 - \frac{1}{p}\right) - \frac{2s}{N} <\right) \frac{1}{r_0} - \frac{2s}{N} \le \frac{1}{q} \le \frac{1}{r_0} (< 1).$$

Hence let us define

$$q_1 := \max\left\{r_0, \frac{N(2p-1)}{N+\alpha}\right\} \quad \text{and} \quad q^1 := \min\left\{(r_0)^*_{2s}, \frac{Np}{\alpha}\right\}.$$

It is easy to see that $q_0 \in [q_1, q^1[$. Moreover, since for every $q \in [q_1, q^1[$ we have $u \in L^q(\mathbb{R}^N)$, then $(\mathcal{K}_\alpha * |u|^p)|u|^{p-2}u \in L^r(\mathbb{R}^N)$ and so $u \in \mathcal{W}^{2s,r}$ for every $r \in [r(q_1), r(q^1)[$, where the map $r = r(q)$ has been defined in (2.1) of Lemma 2.1. Hence by Sobolev embeddings and again by (2.1) of Lemma 2.1, $u \in L^q(\mathbb{R}^N)$ for every $q \in [r(q_1), (r(q^1))^*_{2s}[$. If $r(q_1) = 1$, namely $q_1 = N(2p-1)/(N+\alpha)$, we stop here from the left-hand side of the interval of $q$'s. Analogously, if $1/(r(q^1))^*_{2s} = \alpha/N(1-1/p) - 2s/N$, namely $q^1 = Np/\alpha$ we stop here from the right-hand side of the interval of $q$'s. Otherwise we iterate the procedure. We take

$$q_i := \max\left\{r(q_{i-1}), \frac{N(2p-1)}{N+\alpha}\right\} = \max\left\{r(r(q_{i-2})), \frac{N(2p-1)}{N+\alpha}\right\},$$

and

$$q^i := \min\left\{(r(q^{i-1}))^*_{2s}, \frac{Np}{\alpha}\right\} = \min\left\{(r((r(q^{i-2}))^*_{2s}))^*_{2s}, \frac{Np}{\alpha}\right\}.$$

We have that

$$q_{i+1} < q_i < \cdots < q_0 < \cdots < q^i < q^{i+1}.$$

Indeed, by induction, if we assume that $q_i < q_{i-1}$ then

$$\frac{1}{q_i} = \frac{1}{r(q_{i-1})} < \frac{1}{r(q_i)} = \frac{1}{q_{i+1}},$$

and, analogously, if $q^{i-1} < q^i$ then

$$\frac{1}{q^{i+1}} = \frac{1}{r(q^i)} - \frac{2s}{N} < \frac{1}{r(q^{i-1})} - \frac{2s}{N} = \frac{1}{q^i}.$$

We can conclude this procedure after a finite number of steps; indeed,

$$\frac{1}{q_i} = (2p-1)^i \left(\frac{1}{q_0} - \frac{\alpha}{2N(p-1)}\right) + \frac{\alpha}{2N(p-1)} \quad \text{with} \quad \frac{1}{q_0} - \frac{\alpha}{2N(p-1)} > 0,$$

and

$$\frac{1}{q^i} = (2p-1)^i \left(\frac{1}{q_0} - \frac{\alpha+2s}{2N(p-1)}\right) + \frac{\alpha+2s}{2N(p-1)} \quad \text{with} \quad \frac{1}{q_0} - \frac{\alpha}{2N(p-1)} < 0. \qquad \square$$

**Lemma 3.2.** *For every $r > 1$, the solution $u$ of $(\mathcal{P}_\omega)$ is in $\mathcal{W}^{2s,r}$.*

**Proof.** Let $r_0$ be such that $1/r_0 = \alpha(1 - 1/p)/N$. By Lemma 3.1 we have that $u \in \mathcal{W}^{2s,r}$ for every $r \in (1, r_0)$. Then by Sobolev embeddings, $u \in L^q(\mathbb{R}^N)$ for every $q \in [1, (r_0)^*_{2s})$. Hence, since $p < (N+\alpha)/(N-2s)$, then

$$\frac{1}{(r_0)^*_{2s}} = \frac{\alpha}{N}\left(1 - \frac{1}{p}\right) - \frac{2s}{N} < \frac{\alpha}{N}\left(1 - \frac{N-2s}{N+\alpha}\right) - \frac{2s}{N} < \frac{\alpha}{N}\frac{N-2s}{N+\alpha} < \frac{\alpha}{Np}.$$

Thus by $\mathcal{K}_\alpha * |u|^p \in L^\infty(\mathbb{R}^N)$ and so $(\mathcal{K}_\alpha * |u|^p)|u|^{p-2}u \in L^r(\mathbb{R}^N)$ for every $r \in (\max\{1/(p-1), 1\}, (r_0)^*_{2s}/(p-1))$. Thus $u \in \mathcal{W}^{2s,r}$ for every $r \in (\max\{1/(p-1), 1\}, (r_0)^*_{2s}/(p-1))$ and so for every $r \in (1, (r_0)^*_{2s}/(p-1))$. If $r_0 \geq N/(2s)$ we conclude. Otherwise we take $r_1 := (r_0)^*_{2s}/(p-1)$ and we iterate the procedure. If $p < 2$, then $r_1 > r_0$ and the procedure stops in a finite number of steps since

$$\begin{aligned}\frac{1}{r_i} = \frac{p-1}{(r_{i-1})^*_{2s}} &= \frac{(p-1)^i}{r_0} - \frac{2s(p-1)}{N}\sum_{j=0}^{i-1}(p-1)^j \\ &= \frac{(p-1)^i}{r_0} - \frac{2s(p-1)(1-(p-1)^i)}{N(2-p)}.\end{aligned}$$

If $p = 2$, then $r_1 > r_0$ and the procedure stops in a finite number of steps since

$$\frac{1}{r_i} = \frac{1}{(r_{i-1})^*_{2s}} = \frac{1}{r_0} - \frac{2si}{N}.$$

If $p > 2$, then, since

$$\frac{1}{r_i} < \frac{2s(p-1)}{N(p-2)},$$

we have that

$$\frac{1}{r_{i+1}} = (p-1)\left(\frac{1}{r_i} - \frac{2s}{N}\right) = \frac{1}{r_i} + \frac{p-2}{r_i} - \frac{2s(p-1)}{N} < \frac{1}{r_i},$$

and the procedure stops in a finite number of steps since

$$\frac{1}{r_i} = (p-1)^i\left(\frac{1}{r_0} - \frac{2s(p-1)}{N(p-2)}\right) + \frac{2s(p-1)}{N(p-2)}. \qquad \square$$

**Proof of Theorem 3.2.** The conclusions follow from Lemma 3.1 and combining Lemma 3.2 and (3.1) of Theorem 3.1. □

The proof of the regularity in Theorem 1.1 is thereby completed.

We note also the following result on the summability property of the fixed sign solutions which we will need in studying the Morse index. In this context we need the functional to be $C^2$, and this is achieved for $p \geq 2$.

**Proposition 3.1.** *Let $s > 1/2$ and $p \geq 2$. If $u \in H^s(\mathbb{R}^N)$ is a solution of $(\mathcal{P}_\omega)$ with $|u| > 0$, then $u \in H^{2s+1}(\mathbb{R}^N)$. In particular $\nabla u \in H^s(\mathbb{R}^N)$.*

Before to proceed with the proof, we show the following general fact.

**Lemma 3.3.** *Let $u$ be a function in $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then $\mathcal{K}_\alpha * |u|^p \in C_0(\mathbb{R}^N)$.*

**Proof.** Let $B_1 \subset \mathbb{R}^N$ be the unit ball centered in 0 and write $\mathcal{K}_\alpha = \mathbf{1}_{B_1}\mathcal{K}_\alpha + \mathbf{1}_{B_1^c}\mathcal{K}_\alpha$, with:

$$\mathbf{1}_{B_1}\mathcal{K}_\alpha \in L^r(\mathbb{R}^N) \quad \text{for every } r \in [1, N/(N-\alpha)),$$
$$\mathbf{1}_{B_1^c}\mathcal{K}_\alpha \in L^r(\mathbb{R}^N) \quad \text{for every } r \in (N/(N-\alpha), +\infty].$$

Since $u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, it is possible to choose a small positive $\varepsilon$ in such a way that $\mathbf{1}_{B_1}\mathcal{K}_\alpha \in L^{1+\varepsilon}(\mathbb{R}^N)$ and $|u|^p \in L^{1+1/\varepsilon}(\mathbb{R}^N)$ and we conclude that

$$(\mathbf{1}_{B_1}\mathcal{K}_\alpha) * |u|^p \in C_0(\mathbb{R}^N). \tag{3.2}$$

Here $C_0(\mathbb{R}^N)$ the space of continuous functions vanishing at infinity. Analogously, we can choose a small positive $\varepsilon$ such that $|u|^p \in L^{1+\varepsilon}(\mathbb{R}^N)$ and $\mathbf{1}_{B_1^c}\mathcal{K}_\alpha \in L^{1+1/\varepsilon}(\mathbb{R}^N)$ and we have

$$(\mathbf{1}_{B_1^c}\mathcal{K}_\alpha) * |u|^p \in C_0(\mathbb{R}^N). \tag{3.3}$$

By (3.2) and (3.3) we conclude. □

**Proof of Proposition 3.1.** Let us assume $u > 0$. By Theorem 3.2 and Lemma 3.2, it is $u \in L^1(\mathbb{R}^N) \cap C^{1,\mu}(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$. We will show that $\|(-\Delta)^{s+1/2}u\|_2 < \infty$. By Lemma 3.3 we know that $\mathcal{K}_\alpha * u^p \in C_0(\mathbb{R}^N)$. We observe now that $\mathcal{K}_\alpha * u^p \in C^1(\mathbb{R}^N)$. Indeed, consider $\eta \in C_c^\infty(\mathbb{R}^N)$ with $\mathrm{supp}(\eta) \subset B_1(0)$ and $\eta \equiv 1$ on $B_{1/2}(0)$. Then:

- $\eta\mathcal{K}_\alpha \in L^1(\mathbb{R}^N)$, $u^p \in C^1(\mathbb{R}^N)$ with bounded first-order derivatives;
- $(1-\eta)\mathcal{K}_\alpha \in C^\infty(\mathbb{R}^N)$ with bounded derivatives, $u^p \in L^1(\mathbb{R}^N)$.

Hence, by the usual properties of the convolution, $\mathcal{K}_\alpha * u^p$ is $C^1$ with derivatives given by

$$\partial_i(\mathcal{K}_\alpha * u^p) = \eta\mathcal{K}_\alpha * \partial_i u^p + ((1-\eta)\mathcal{K}_\alpha) * \partial_i u^p.$$

Now, since $u \in C^{1,\mu}(\mathbb{R}^N)$ and $p \geq 2$, we have:

$$\eta\mathcal{K}_\alpha * \partial_i u^p = \eta\mathcal{K}_\alpha * (pu^{p-1}\partial_i u) \in L^1(\mathbb{R}^N) * L^\infty(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N),$$

$$((1-\eta)\mathcal{K}_\alpha) * \partial_i u^p = ((1-\eta)\mathcal{K}_\alpha) * (pu^{p-1}\partial_i u) \in L^\infty(\mathbb{R}^N) * L^1(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N),$$

which prove that $\partial_i(\mathcal{K}_\alpha * u^p) \in L^\infty(\mathbb{R}^N)$. Set $v := (\mathcal{K}_\alpha * u^p)u^{p-1}$; since $p \geq 2$ we have

$$\partial_i v = u^{p-1}\partial_i(\mathcal{K}_\alpha * u^p) + (\mathcal{K}_\alpha * u^p)\partial_i u^{p-1} \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \tag{3.4}$$

and then

$$\|(-\Delta)^{s+1/2}u\|_2 = \|(-\Delta)^s[(-\Delta)^s + \omega I]^{-1}(-\Delta)^{1/2}v\|_2 \leq C\|\nabla v\|_2 < \infty.$$

The proof is thereby complete. □

**Remark 3.1.** Under the hypotheses of Proposition 3.1, we have $u \in C^2(\mathbb{R}^N)$. Indeed by Theorem 3.2 and (3.4) we know that $u \in C^{1,\mu}(\mathbb{R}^N)$ with

$$\partial_i(-\omega u + (\mathcal{K}_\alpha * u^p)u^{p-1}) \in L^\infty(\mathbb{R}^N).$$

Thus $\partial_i u$ satisfies

$$(-\Delta)^s\partial_i u = \partial_i(-\omega u + (\mathcal{K}_\alpha * u^p)u^{p-1}),$$

and, by Proposition 2.1.11 of Ref. 34, we conclude that $\partial_i u \in C^1(\mathbb{R}^N)$.

We conclude this section by showing the asymptotic profile of the solutions of $(\mathcal{P}_\omega)$. For the sake of simplicity we set

$$V := -(\mathcal{K}_\alpha * |u|^p)|u|^{p-2}.$$

We get the following theorem.

**Theorem 3.3.** *Let $p \geq 2$ and $u$ be a solution of $(\mathcal{P}_\omega)$. Then there exist two positive constants $C_1, C_2$ such that, for any $x \in \mathbb{R}^N$,*

$$|u(x)| \leq C_1\langle x\rangle^{-N-2s}, \quad \text{where } \langle x\rangle = (1+|x|^2)^{1/2},$$

*and*

$$u(x) = -C_2\left(\int Vu\right)\frac{1}{|x|^{N+2s}} + o(|x|^{-N-2s}) \quad \text{for } |x| \to +\infty.$$

**Proof.** For a solution $u$ of $(\mathcal{P}_\omega)$, we have

$$(-\Delta)^s u + Vu = -\omega u.$$

By Lemma 3.3 we have $V \in L^\infty(\mathbb{R}^N)$ and $V(x) \to 0$ for $|x| \to \infty$. Then, for every $\tau \in (0,1)$ there exists $R > 0$ such that $V(x) \geq -\tau$, whenever $|x| \geq R$. Then, we are in a position to apply Lemma C.2 in Ref. 17 to obtain the conclusion. □

As it can be seen in Lemma C.2 of Ref. 17, the constants $C_1, C_2$ depend on the solutions by their $L^2$-norm.

The decay estimate in Theorem 1.1 is proved.

## 4. Ground States

Ground states solutions for $(\mathcal{P}_\omega)$ can be found minimizing $E_0$, defined in (1.5), on the sphere $\Sigma_\rho = \{u \in H^s(\mathbb{R}^N) : \|u\|_2 = \rho\}$ with $\rho > 0$, or

$$S(u) := \frac{\|u\|^2}{(\int (\mathcal{K}_\alpha * |u|^p)|u|^p)^{1/p}},$$

on $(H^s(\mathbb{R}^N) \cap L^{2Np/(N+\alpha)}(\mathbb{R}^N))\backslash\{0\}$, or considering

$$W(u) := \frac{\|(-\Delta)^{s/2}u\|_2^{\frac{N(p-1)-\alpha}{sp}} (\omega\|u\|_2^2)^{\frac{N+\alpha-(N-2s)p}{2sp}}}{(\int (\mathcal{K}_\alpha * |u|^p)|u|^p)^{1/p}}.$$

Indeed, straightforward calculations show the following relationships between these three functionals.

**Proposition 4.1.** *For every $p > 1$ and $u \in (H^s(\mathbb{R}^N) \cap L^{2Np/(N+\alpha)}(\mathbb{R}^N))\backslash\{0\}$,*

$$\max_{\tau>0} E_\omega(\tau u) = \left(\frac{1}{2} - \frac{1}{2p}\right) S(u)^{p/(p-1)}.$$

*Moreover let $u_\tau(\cdot) = u(\tau\cdot)$. We have that*:

(i) *if $p$ satisfies* (1.2) *then*

$$\min_{\tau>0} S(u_\tau) = \frac{2sp}{N+\alpha-(N-2s)p}\left(\frac{N+\alpha-(N-2s)p}{Np-(N+\alpha)}\right)^{\frac{Np-(N+\alpha)}{2sp}} W(u);$$

(ii) *if $p$ satisfies* (1.4) *then*

$$\min_{\tau>0} E_0(\tau^{N/2}u_\tau) = -\mathfrak{a}\left(\frac{(\omega\|u\|_2^2)^{N+\alpha-(N-2s)p}}{W(u)^{2sp}}\right)^{\frac{1}{(2s+\alpha)-N(p-1)}},$$

*where*

$$\mathfrak{a} = \frac{(\alpha+2s)-N(p-1)}{4sp}\left(\frac{N(p-1)-\alpha}{2sp}\right)^{\frac{N(p-1)-\alpha}{(\alpha+2s)-N(p-1)}};$$

(iii) *if $p = 1 + (2s+\alpha)/N$ then $E_0(\tau^{N/2}u_\tau) = \tau^{2s}E_0(u)$;*

(iv) *if $p > 1 + (2s+\alpha)/N$ then*

$$\lim_{\tau\to+\infty} E_0(\tau^{N/2}u_\tau) = -\infty,$$

*and*

$$\max_{\tau>0} E_0(\tau^{N/2}u_\tau) = \mathfrak{b}\left(\frac{W(u)^{2sp}}{(\omega\|u\|_2^2)^{N+\alpha-(N-2s)p}}\right)^{\frac{1}{N(p-1)-(2s+\alpha)}},$$

*where*

$$\mathfrak{b} = \frac{N(p-1)-(\alpha+2s)}{2[N(p-1)-\alpha]}\left(\frac{2sp}{N(p-1)-\alpha}\right)^{\frac{2s}{N(p-1)-(\alpha+2s)}}.$$

Hence, arguing as in the proof of Proposition 2.2 of Ref. 29 and applying Lemma 2.2, we obtain the following theorem.

**Theorem 4.1.** *If $p$ satisfies* (1.2), *then $S$ achieves the minimum on $H^s(\mathbb{R}^N)\backslash\{0\}$.*

This proves the existence part of Theorem 1.1. Concerning the symmetry of these ground states, we have the following result.

**Theorem 4.2.** *Let $u \in H^s(\mathbb{R}^N)$ be a ground state of $(\mathcal{P}_\omega)$. Then $u$ has fixed sign and there exist $x_0 \in \mathbb{R}^N$ and a monotone function $v : \mathbb{R} \to \mathbb{R}$ with fixed sign such that $u(x) = v(|x - x_0|)$.*

**Proof.** Given a ground state $u$ of $(\mathcal{P}_\omega)$, $u \neq 0$ and $u$ is a solution of

$$S(u) = \inf_{\varphi \in H^s(\mathbb{R}^N)\backslash\{0\}} S(\varphi).$$

Taking into account $\|(-\Delta)^{s/2}|u|\|_2 \leq \|(-\Delta)^{s/2}u\|_2$ also $|u|$ is a ground state. Then

$$(-\Delta)^s|u| + \omega|u| = (\mathcal{K}_\alpha * |u|^p)|u|^{p-1}.$$

By arguing as in the end of Sec. 3 of Ref. 15, if $u(x_0) = 0$ for some $x_0 \in \mathbb{R}^N$, then one obtains

$$\int \frac{|u(x_0 + y)| + |u(x_0 - y)|}{|x_0 - y|^{N+2s}} = 0,$$

yielding $u = 0$, a contradiction. Whence $|u| > 0$ and $u$ does not change sign. We shall assume $u > 0$. Given $v \in H^s(\mathbb{R}^N)$ with $v \geq 0$ and any half-space $H \subset \mathbb{R}^N$, the polarization $v^H$ is defined as

$$v^H(x) = \begin{cases} \max\{v(x), v(\sigma_H(x))\} & \text{if } x \in H, \\ \min\{v(x), v(\sigma_H(x))\} & \text{if } x \in \mathbb{R}^N\backslash H, \end{cases}$$

where $\sigma^H(x)$ is the reflected of $x$ with respect to $\partial H$. Then, $\|v^H\|_2^2 = \|v\|_2^2$ and, by (2.2) and Theorem 2 of Ref. 1, $\|(-\Delta)^{s/2}v^H\|_2^2 \leq \|(-\Delta)^{s/2}v\|_2^2$. In turn, since $S(u) \leq S(u^H)$, we conclude that

$$\int (\mathcal{K}_\alpha * |u|^p)|u|^p = \frac{\|u\|^{2p}}{[S(u)]^p} \geq \frac{\|u^H\|^{2p}}{[S(u^H)]^p} = \int (\mathcal{K}_\alpha * |u^H|^p)|u^H|^p.$$

Then, by combining Lemma 5.3 and Lemma 5.4 of Ref. 29, we conclude the proof. □

As we said in the Introduction, here we are particularly interested into the precompactness properties of the minimizing sequences of $E_0$ on $\Sigma_\rho$. In this case we have to assume that $p$ satisfies (1.4). Indeed, if $p \geq 1 + \frac{2s+\alpha}{N}$, using the same rescaling $\tau^{N/2}u_\tau$ as in Proposition 4.1, we deduce that $E_0$ is unbounded from below. In the following lemma we collect some basic facts.

**Lemma 4.1.** *Let $\rho > 0$ be fixed. Then*:

(i) *$E_0$ is coercive and bounded from below on $\Sigma_\rho$;*

(ii) $m_{\rho^2} := \inf_{u\in\Sigma_\rho} E_0(u) < 0$;
(iii) *every minimizing sequence for $E_0$ in $\Sigma_\rho$ is bounded and can be assumed non-negative, radially symmetric and decreasing.*

**Proof.** Let $u \in \Sigma_\rho$. By (2.6) we have

$$E_0(u) \geq \frac{1}{2}\|(-\Delta)^{s/2}u\|_2^2 - C\|(-\Delta)^{s/2}u\|_2^{2\beta p}\rho^{2(1-\beta)p}.$$

Since $p$ satisfies (1.4), then $0 < \beta p < 1$ we get (4.1). To show (4.1), fix $u \in \Sigma_\rho$ and observe that the rescaling $\tau^{N/2}u_\tau$ preserves $L^2$-norm. We have that $E_0(\tau^{N/2}u_\tau)$ becomes negative for small $\tau$. Finally, the statements in (4.1) easily follow from the coercivity of $E_0$ and from the fact that $\|(-\Delta)^{s/2}u^*\|_2 \leq \|(-\Delta)^{s/2}u\|_2$, where $u^*$ is the symmetric radial decreasing rearrangement of $u$ (see Theorem 3 in Ref. 1). □

Hence we have the following compactness result.

**Theorem 4.3.** *For every $\rho > 0$, every minimizing sequence for $E_0$ in $\Sigma_\rho$ is relatively compact in $H^s(\mathbb{R}^N)$ up to a translation. In particular $E_0$ has a minimum point on $\Sigma_\rho$, that can be assumed non-negative, radially symmetric and decreasing.*

**Proof.** Let $\{u_n\}$ be a minimizing sequence for $E_0$ on $\Sigma_\rho$ satisfying $E_0'(u_n) \to 0$ as $n \to +\infty$. In view of Lemma 4.1 it is bounded in $H^s(\mathbb{R}^N)$ and then there exists $u \in H^s(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$. Now let $R > 0$. If it were

$$\lim_n \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} u_n^2 = 0,$$

then, by Lemma 2.3, we would have $u_n \to 0$ in $L^q(\mathbb{R}^N)$ and then, by (i) of Lemma 2.1,

$$\int (\mathcal{K}_\alpha * |u_n|^p)|u_n|^p \to 0,$$

implying that $\lim_n E_0(u_n) \geq 0$, which is a contradiction with (ii) in Lemma 4.1. Then, possibly passing to a subsequence, there exists a $\delta > 0$ such that

$$\sup_n \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^2 \geq \delta.$$

We infer that there exists $\{y_n\} \subset \mathbb{R}^N$ such that

$$\int_{B_R(y_n)} |u_n|^2 \geq \delta.$$

Hence, defining $v_n = u_n(\cdot + y_n)$ and by the compact embedding of $H^s_{\mathrm{loc}}(\mathbb{R}^N)$ into $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ (see e.g. Corollary 7.2 in Ref. 12) we get a bounded minimizing sequence whose weak limit $v$ is nontrivial, $\|v\|_2 \leq \rho$,

$$\|v_n - v\|_2^2 + \|v\|_2^2 = \|v_n\|_2^2 + o_n(1), \tag{4.1}$$

$$\|(-\Delta)^{s/2}(v_n - v)\|_2^2 + \|(-\Delta)^{s/2}v\|_2^2 = \|(-\Delta)^{s/2}v_n\|_2^2 + o_n(1), \tag{4.2}$$

and, by Lemma 2.4 of Ref. 29,

$$\begin{aligned}\int(\mathcal{K}_\alpha * |v_n - v|^p)|v_n - v|^p + \int(\mathcal{K}_\alpha * |v|^p)|v|^p \\ = \int(\mathcal{K}_\alpha * |v_n|^p)|v_n|^p + o_n(1).\end{aligned} \tag{4.3}$$

Assume by contradiction that $\|v\|_2 = \mu < \rho$. Since, by (4.1),

$$a_n = \frac{\sqrt{\rho^2 - \mu^2}}{\|v_n - v\|_2} \to 1,$$

and, by (4.2) and (4.3),

$$E_0(v_n - v) + E_0(v) = m_{\rho^2} + o_n(1),$$

and then

$$E_0(a_n(v_n - v)) + E_0(v) = E_0(v_n - v) + E_0(v) + o_n(1) = m_{\rho^2} + o_n(1).$$

Then, since $\|a_n(v_n - v)\|_2^2 = \rho^2 - \mu^2$, we get

$$m_{\rho^2-\mu^2} + m_{\mu^2} \leq m_\rho + o_n(1). \tag{4.4}$$

Now let us define for $\nu > 0$, $\Sigma_\rho^\nu = \{w \in \Sigma_\rho : \int(\mathcal{K}_\alpha * |w|^p)|w|^p \geq \nu\}$. We show that there exists $\nu > 0$ such that

$$m_{\rho^2} = \inf_{w\in\Sigma_\rho^\nu} E_0(w). \tag{4.5}$$

Of course $m_{\rho^2} \leq \inf_{u\in\Sigma_\rho^\nu} E_0(u)$. Assuming by contradiction that, for every $\nu > 0$, $m_{\rho^2} < \inf_{w\in\Sigma_\rho^\nu} E_0(w)$, we can find a minimizing sequence $\{w_n\}$ such that

$$E_0(w_n) \to m_{\rho^2} \quad \text{and} \quad \int(\mathcal{K}_\alpha * |w_n|^p)|w_n|^p \to 0.$$

Thus

$$0 \leq \frac{1}{2}\|(-\Delta)^{s/2}w_n\|_2^2 = E_0(w_n) + \frac{1}{p}\int(\mathcal{K}_\alpha * |w_n|^p)|w_n|^p \to m_{\rho^2} < 0.$$

Then, by (4.5), we easily get $m_{\tau^2\rho^2} < \tau^2 m_{\rho^2}$ for every $\tau > 1$. Thus, for all $\mu \in (0, \rho)$:

$$m_{\rho^2} < m_{\mu^2} + m_{\rho^2-\mu^2},$$

which is in contradiction with (4.4). Hence $v \in \Sigma_\rho$, $\|v_n - v\|_2 = o_n(1)$ and, by applying (2.1),

$$\|v_n - v\|_{2Np/(N+\alpha)} = o_n(1). \tag{4.6}$$

It remains to show that $\|(-\Delta)^{s/2}(v_n - v)\|_2 = o_n(1)$. Since $\{v_n\}$ is a bounded Palais–Smale sequence, there exists $\{\lambda_n\} \subset \mathbb{R}$ such that for every $w \in H^s(\mathbb{R}^N)$:

$$E_0'(v_n)[w] - \lambda_n \int v_n w = o_n(1) \quad \text{and} \quad E_0'(v_n)[v_n] - \lambda_n \|v_n\|_2^2 = o_n(1).$$

Then we obtain that $\{\lambda_n\}$ is bounded and

$$(E_0'(v_n) - E_0'(v_m))[v_n - v_m] - \lambda_n \int v_n(v_n - v_m) + \lambda_m \int v_m(v_n - v_m) \to 0 \quad \text{as } m, n \to \infty.$$

Since, by Hardy–Littlewood–Sobolev inequality and (4.6):

$$\left| \int (\mathcal{K}_\alpha * |v_n|^p)|v_n|^{p-2} v_n(v_n - v_m) \right| \le C\|v_n\|_{2Np/(N+\alpha)}^{2p-1} \|v_n - v_m\|_{2Np/(N+\alpha)} \to 0,$$

and

$$\lambda_n \int v_n(v_n - v_m) \to 0,$$

as $m, n \to \infty$, we have that $\{v_n\}$ is a Cauchy sequence in $H^s(\mathbb{R}^N)$ and we get that $\{v_n\}$ is relatively compact. The last statement of the theorem is achieved by taking into account (4.1) of Lemma 4.1. □

Finally, following step-by-step (see proof of Lemma 2.6 in Ref. 11), we get the following relations between the ground states (as minima of $E_\omega$ on $\mathcal{N}_\omega$) and the minima of $E_0$ on $\Sigma_\rho$.

**Proposition 4.2.** *For every $\rho > 0$, the minimization problems*

$$\min_{u \in \Sigma_\rho} E_0(u) \quad \textit{and} \quad \min_{u \in \mathcal{N}_\omega} E_\omega(u)$$

*are equivalent. Moreover the $L^2$-norm $\rho$ of any ground state $u$ of $(\mathcal{P}_\omega)$ satisfies*

$$\rho^2 = \frac{N + \alpha - (N - 2s)p}{\omega s(p-1)} \min_{u \in \mathcal{N}_\omega} E_\omega(u)$$

*and*

$$\min_{u \in \Sigma_\rho} E_\omega(u) = \min_{u \in \mathcal{N}_\omega} E_\omega(u).$$

**Proof.** Let $\rho, \omega > 0$,

$$\mathfrak{K}_{\Sigma_\rho} = \{m \in \mathbb{R}_- : \exists\, u \in \Sigma_\rho \text{ s.t. } E_0'|_{\Sigma_\rho}(u) = 0 \text{ and } E_0(u) = m\}$$

and

$$\mathfrak{K}_{\mathcal{N}_\omega} = \{c \in \mathbb{R} : \exists\, u \in \mathcal{N}_\omega \text{ s.t. } E_\omega'(u) = 0 \text{ and } E_\omega(u) = c\},$$

where, for all $u, v \in H^s(\mathbb{R}^N)$,

$$E'_\omega(u)[v] = \int (-\Delta)^{s/2}u\ (-\Delta)^{s/2}v + \omega \int uv - \int (\mathcal{K}_\alpha * |u|^p)|u|^{p-2}uv.$$

First of all we observe that, by Lemma 4.1 and Theorem 4.3, $\mathfrak{K}_{\Sigma_\nu}$ is well defined.

Let now $u \in \Sigma_\rho$ such that $E'_0|_{\Sigma_\rho}(u) = 0$ and $E_0(u) = m$ with $m < 0$. Then there exists $\lambda \in \mathbb{R}$ such that

$$(-\Delta)^s u - (\mathcal{K}_\alpha * |u|^p)|u|^{p-2}u = -\lambda u, \tag{4.7}$$

and so

$$\|(-\Delta)^{s/2}u\|_2^2 - \int (\mathcal{K}_\alpha * |u|^p)|u|^p = -\lambda\rho^2. \tag{4.8}$$

Then, since $E_0(u) = m$, by (4.8) we get

$$\frac{p-1}{2p}\|(-\Delta)^{s/2}u\|_2^2 = m + \frac{\lambda\rho^2}{2p}, \tag{4.9}$$

and so $\lambda > 0$. Now let

$$w(x) := \tau^{\frac{\alpha+2s}{2(p-1)}}u(\tau x) \quad \text{with} \quad \tau = \left(\frac{\omega}{\lambda}\right)^{1/(2s)}.$$

We have that $w$ solves

$$(-\Delta)^s w + \omega w - (\mathcal{K}_\alpha * |w|^p)|w|^{p-2}w = 0,$$

and so $w \in \mathcal{N}_\omega$, $E'_\omega(w) = 0$ and $c = E_\omega(w) \in \mathfrak{K}_{\mathcal{N}_\omega}$.

Vice versa, if $w \in \mathcal{N}_\omega$ such that $E'_\omega(w) = 0$ and $c = E_\omega(w)$, we consider

$$u(x) := \tau^{\frac{\alpha+2s}{2(p-1)}}w(\tau x) \quad \text{with} \quad \tau = \left(\frac{\rho}{\|w\|_2}\right)^{\frac{2(p-1)}{\alpha+2s-N(p-1)}}.$$

We have that $u \in \Sigma_\rho$, (4.7) holds for

$$\lambda = \omega\tau^{2s} = \omega\left(\frac{\rho}{\|w\|_2}\right)^{\frac{4s(p-1)}{\alpha+2s-N(p-1)}},$$

and

$$\begin{aligned} m &= \tau^{\frac{\alpha+2sp-N(p-1)}{p-1}}\left(c - \frac{\omega}{2}\|w\|_2^2\right) \\ &= \left(\frac{\rho}{\|w\|_2}\right)^{\frac{2(\alpha+2sp-N(p-1))}{\alpha+2s-N(p-1)}}\left(c - \frac{\omega}{2}\|w\|_2^2\right). \end{aligned} \tag{4.10}$$

By Pohožaev identity (6.1) and since $w \in \mathcal{N}_\omega$ and $E_\omega(w) = c$ we get the system:

$$\begin{cases} (N-2s)\|(-\Delta)^{s/2}w\|_2^2 + \omega N\|w\|_2^2 - \dfrac{\alpha+N}{p}\displaystyle\int (\mathcal{K}_\alpha * |w|^p)|w|^p = 0, \\ \|(-\Delta)^{s/2}w\|_2^2 + \omega\|w\|_2^2 - \displaystyle\int (\mathcal{K}_\alpha * |w|^p)|w|^p = 0, \\ \dfrac{1}{2}\|(-\Delta)^{s/2}w\|_2^2 + \dfrac{\omega}{2}\|w\|_2^2 - \dfrac{1}{2p}\displaystyle\int (\mathcal{K}_\alpha * |w|^p)|w|^p = c, \end{cases}$$

from which

$$\|w\|_2^2 = \frac{N+\alpha-(N-2s)p}{\omega s(p-1)}c.$$

Thus (4.10) becomes:

$$m = -\frac{\alpha+2s-N(p-1)}{2}\left(\frac{\omega\rho^2}{N+\alpha-(N-2s)p}\right)^{\frac{\alpha+2sp-N(p-1)}{\alpha+2s-N(p-1)}} \times \left(\frac{s(p-1)}{c}\right)^{\frac{2s(p-1)}{\alpha+2s-N(p-1)}}, \tag{4.11}$$

and, for

$$\rho^2 = \frac{N+\alpha-(N-2s)p}{\omega s(p-1)}c,$$

(4.11) implies

$$m + \frac{\omega}{2}\rho^2 = c.$$

Hence the conclusions easily follow. □

Finally we study the Morse index of the ground state. In the last part of this section we assume $2 \leq p < 1 + (2s+\alpha)/N$ to have that the functional $E_\omega$ is $C^2$ and $s > 1/2$. If $u$ is the minimum of $E_0$ on $\Sigma_\rho$ we have

$$\int |(-\Delta)^{s/2}u|^2 - \int (\mathcal{K}_\alpha * |u|^p)|u|^p = -\lambda\rho^2, \tag{4.12}$$

with $\lambda > 0$ (by (4.9)). Now consider

$$E''_\lambda(u)[\xi,\eta] = \int (-\Delta)^{s/2}\xi(-\Delta)^{s/2}\eta + \lambda\int \xi\eta - p\int (\mathcal{K}_\alpha * |u|^{p-2}u\eta)|u|^{p-2}u\xi - (p-1)\int (\mathcal{K}_\alpha * |u|^p)|u|^{p-2}\xi\eta. \tag{4.13}$$

To obtain information on the Morse index, we need to study $\ker E''_\lambda(u)$.

Since the problem is invariant for the group of translations, the solutions of $(\mathcal{P}_\omega)$ will never be isolated: in other words $\ker E''_\lambda(u) \neq \{0\}$ and in particular

$$\operatorname{span}\{\nabla u\} \subset \ker E''_\lambda(u). \tag{4.14}$$

Indeed, for every $a \in \mathbb{R}^N$, consider the action of the group of the translations in $\mathbb{R}^N$ induced on $H^s(\mathbb{R}^N)$, that is

$$\mathfrak{t}_a : u \in H^s(\mathbb{R}^N) \longmapsto u(\cdot + a) \in H^s(\mathbb{R}^N),$$

which is linear and isometric. Since $E_\lambda \circ \mathfrak{t}_a = E_\lambda$, we have $E'_\lambda(\mathfrak{t}_a u)[v] = E'_\lambda(u)[\mathfrak{t}_{-a}v]$, for every $u, v \in H^s(\mathbb{R}^N)$. For every $u \in H^s(\mathbb{R}^N)$ it is also convenient to introduce the following map

$$\mathfrak{s}_u : a \in \mathbb{R}^N \longmapsto u(\cdot + a) \in H^s(\mathbb{R}^N).$$

Of course, for a generic fixed $u \in H^s(\mathbb{R}^N)$, the map $\mathfrak{s}_u$ does not need to be differentiable but (for example) whenever $u \in H^s(\mathbb{R}^N)$ is a solution of $(\mathcal{P}_\omega)$ as in Proposition 3.1 it does, and the differential in 0 given by

$$\mathfrak{s}_u'(0)[b] = \nabla u \cdot b \in H^s(\mathbb{R}^N), \quad \text{for all } b \in \mathbb{R}^N.$$

Hence, in this case, by differentiating in 0 the map

$$a \in \mathbb{R}^N \longmapsto E_\lambda'(\mathfrak{s}_u(a)) \in H^{-s}(\mathbb{R}^N),$$

we get $E_\lambda''(\mathfrak{s}_u(0))[\mathfrak{s}_u'(0)[b], \cdot] = 0$ for all $b \in \mathbb{R}^N$ and this gives (4.14).

It would be interesting to understand if the ground state is nondegenerate in the sense that

$$\operatorname{span}\{\nabla u\} = \ker E_\lambda''(u).$$

We define the Morse index $\mathfrak{i}_{\text{Morse}}(u)$ as the maximal dimension of subspaces of $H^s(\mathbb{R}^N)$ on which $E_\lambda''(u)$ is negative definite. We have the following result which completes the proof of Theorem 1.1.

**Proposition 4.3.** *Let $u \in \Sigma_\rho$ be a ground state and $T_u\Sigma_\rho = \{w \in H^s(\mathbb{R}^N) : \int uw = 0\}$. Then*:

(i) *$E_\lambda''(u)$ is positive semidefinite on $T_u\Sigma_\rho$;*
(ii) $\inf_{w \in T_u\Sigma_\rho} E_\lambda''(u)[w,w] = 0$;
(iii) $\mathfrak{i}_{\text{Morse}}(u) = 1$.

**Proof.** Let $v$ be any element of $T_u\Sigma_\rho$ and $\gamma : (-\varepsilon, \varepsilon) \to \Sigma_\rho$ a smooth curve such that $\gamma(0) = u$ and $\gamma'(0) = v$. Since $u$ is the minimum of $E_0$ on $\Sigma_\rho$, it is

$$\frac{d^2}{d\tau^2} E_0(\gamma(\tau))|_{\tau=0} \geq 0,$$

which explicitly reads as

$$0 \leq E_0''(u)[v,v] + E_0'(u)[\gamma''(0)] = E_0''(u)[v,v] - \lambda \int u\gamma''(0). \tag{4.15}$$

Of course, $0 = \frac{d}{d\tau} \int |\gamma(\tau)|^2 = 2\int \gamma(\tau)\gamma'(\tau)$ implies

$$\int |v|^2 + \int u\gamma''(0) = 0,$$

which, plugged into (4.15) gives (i). Property (ii) follows by Proposition 3.1 and the translation invariance of $\Sigma_\rho$: indeed $\partial_{x_i} u \in T_u\Sigma_\rho$ and we know $E_\lambda''(u)[\partial_{x_i} u, \partial_{x_i} u] = 0$.

Finally, to prove (4.3), note that by (4.13) and (4.12):

$$\begin{aligned} E_\lambda''(u)[u,u] &= \int |(-\Delta)^{s/2} u| + \lambda\rho^2 + (1-2p) \int (\mathcal{K}_\alpha * |u|^p)|u|^p \\ &= 2(1-p) \int (\mathcal{K}_\alpha * |u|^p)|u|^p < 0. \end{aligned}$$

The result then follows from (i) and the direct sum decomposition (see Ref. 2 for the general setting): $H^s(\mathbb{R}^N) = T_u\Sigma_\rho \oplus \operatorname{span}\{u\}$. □

## 5. Multiplicity

We begin with some geometric properties of the functional $E_\omega$ in (1.3). The assumption (1.2) will be tacitly assumed in the whole section.

**Proposition 5.1.** *The functional $E_\omega$ satisfies the following geometric assumptions of the symmetric mountain pass theorem*:

(i) *it is even, that is $E_\omega(u) = E_\omega(-u)$;*
(ii) *it has a strict local minimum in* 0 *with $E_\omega(0) = 0$;*
(iii) *there exist a nested sequence $\{V_k\}$ of finite-dimensional subspaces of $H^s(\mathbb{R}^N)$ and $\{R_k\} \subset \mathbb{R}^+$ such that $E_\omega(u) \le 0$ for every $u \in V_k$ with $\|u\| \ge R_k$.*

**Proof.** Property (i) is immediate. By (2.3) it holds

$$E_\omega(u) \ge \frac{1}{2}\|u\|^2 - C\|u\|^{2p},$$

getting (ii). Finally, if $\{e_i\}_{i=1,\dots,k}$ is an orthogonal basis of a $k$-dimensional subspace $V_k$ of $H^s(\mathbb{R}^N)$, then, writing $u = \sum_{i=1}^k t_i e_i$, it is $E_\omega(u) \to -\infty$ for $\|u\| \to \infty$, proving (iii). □

To ensure existence of critical points of $E_\omega$, a compactness condition is necessary. To this aim some preliminaries are in order.

Firstly, let $\ell > 1$, $N_i \ge 2$, $i = 1, \dots, \ell$, or $\ell = 1$ and $N \ge 3$, and $N = \sum_{i=1}^{\ell} N_i$. A point in $\mathbb{R}^N$ is now denoted with $x = (x_1, \dots, x_\ell)$, $x_i \in \mathbb{R}^{N_i}$. Let $\mathscr{O}(N_i)$ be the orthogonal group on $\mathbb{R}^{N_i}$ and consider the product group

$$G := \mathscr{O}(N_1) \times \cdots \times \mathscr{O}(N_\ell),$$

acting on $\mathbb{R}^N$ by

$$g \cdot x = (g_1 x_1, \dots, g_\ell x_\ell), \quad g = (g_1, \dots, g_\ell) \in G,$$

and whose representation in $H^s(\mathbb{R}^N)$ is given by the linear and isometric action

$$(T_g u)(x) = u(g^{-1} \cdot x). \tag{5.1}$$

Set

$$X := \{u \in H^s(\mathbb{R}^N) : T_g u = u \text{ for all } g \in G\}.$$

In particular for $\ell = 1$ we have the radial functions, $u(x) = u(|x|)$. In this we say that the functions in $X$ are "symmetric". Then $X$ is exactly the closed and infinite-dimensional subspace of fixed points for the action (5.1). The importance of this setting is twofold. Indeed the functional $E_\omega$ is $G$-invariant, i.e. for every $g \in G$, $E_\omega \circ T_g = E_\omega$ and the space $X$ has compact embedding into $L^q(\mathbb{R}^N)$, $q \in (2, 2_s^*)$, see Ref. 24.

Secondly, for every fixed $u \in H^s(\mathbb{R}^N)$, consider the problem:

$$\begin{cases} (-\Delta)^{\alpha/2}\varphi = \gamma(\alpha)|u|^p, \quad \text{where } \gamma(\alpha) := \dfrac{\pi^{N/2}2^\alpha\Gamma(\alpha/2)}{\Gamma(N/2-\alpha/2)}, \\ \varphi \in \dot{H}^{\alpha/2}(\mathbb{R}^N), \end{cases} \tag{5.2}$$

(where $\Gamma$ is the gamma function) whose weak formulation is the following one: we say that $\varphi \in \dot{H}^{\alpha/2}(\mathbb{R}^N)$ is a weak solution if for every $\xi \in \dot{H}^{\alpha/2}(\mathbb{R}^N)$:

$$\int (-\Delta)^{\alpha/4}\varphi(-\Delta)^{\alpha/4}\xi = \gamma(\alpha)\int \xi|u|^p. \tag{5.3}$$

Recall that for every $\alpha \in (0,N)$, $(-\Delta)^{\alpha/2}u$ is defined via the Fourier transform and $\dot{H}^{\alpha/2}(\mathbb{R}^N)$ is defined as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the associated Gagliardo seminorm (these notions coincide with that given in the Introduction for $\alpha \in (0,2)$). Observe now that, under the assumption on $p$, the right-hand side in (5.3) defines the map

$$L : v \in \dot{H}^{\alpha/2}(\mathbb{R}^N) \mapsto \int v|u|^p \in \mathbb{R},$$

which is linear and continuous; indeed

$$|Lv| \le C\|u\|^p_{2Np/(N+\alpha)}\|v\|_{\dot{H}^{\alpha/2}} \le C\|u\|^p\|v\|_{\dot{H}^{\alpha/2}}.$$

By the Riesz representation theorem there exists a unique weak solution $\varphi$ of (5.2), represented as a convolution with the kernel $\mathcal{K}_\alpha/\gamma(\alpha)$, i.e. $\varphi = \mathcal{K}_\alpha * |u|^p$ and

$$\|\mathcal{K}_\alpha * |u|^p\|_{\dot{H}^{\alpha/2}} = \|L\| \le C\|u\|^p.$$

As a consequence of the above setting we can prove the following result, which will help us to recover compactness.

**Lemma 5.1.** *Let $\{u_n\}, u \in X$ be such that $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^N)$. Then:*

(i) $\mathcal{K}_\alpha * |u_n|^p \to \mathcal{K}_\alpha * |u|^p$ *in* $\dot{H}^{\alpha/2}(\mathbb{R}^N)$;
(ii) $\int(\mathcal{K}_\alpha * |u_n|^p)|u_n|^p \to \int(\mathcal{K}_\alpha * |u|^p)|u|^p$;
(iii) $\int(\mathcal{K}_\alpha * |u_n|^p)|u_n|^{p-2}u_n u \to \int(\mathcal{K}_\alpha * |u|^p)|u|^p$.

**Proof.** Define the linear and continuous maps $L_n, L : \dot{H}^{\alpha/2}(\mathbb{R}^N) \to \mathbb{R}$ such that

$$L_n v = \int v|u_n|^p, \quad Lv = \int v|u|^p, \quad v \in \dot{H}^{\alpha/2}(\mathbb{R}^N).$$

By the compact embedding we may assume

$$\|u_n - u\|_q \to 0, \quad \text{for all } q \in (2, 2_s^*).$$

Then, denoting with $(2^*_{\alpha/2})'$ the conjugate exponent of $2^*_{\alpha/2}$,

$$\begin{aligned}|L_n v - Lv| &\le \int |v|||u_n|^p - |u|^p| \le \|v\|_{2^*_{\alpha/2}}\||u_n|^p - |u|^p\|_{(2^*_{\alpha/2})'} \\ &\le \|v\|_{\dot{H}^{\alpha/2}}\varepsilon_n \to 0,\end{aligned}$$

which proves the convergence of $L_n$ to $L$ in the operator norm, yielding (i). We now observe that:

$$|\mathcal{K}_\alpha * |u_n|^p| \leq \xi \in L^{2^*_{\alpha/2}}(\mathbb{R}^N), \quad |u_n|^p \leq \mu \in L^{(2^*_{\alpha/2})'}(\mathbb{R}^N),$$
$$|u_n|^{p-1} \leq \eta \in L^{\frac{2Np}{(N+\alpha)(p-1)}}(\mathbb{R}^N).$$

Hence by the Young inequality we have

$$(\mathcal{K}_\alpha * |u_n|^p)|u_n|^p \leq \frac{1}{2^*_{\alpha/2}}\xi^{2^*_{\alpha/2}} + \frac{1}{(2^*_{\alpha/2})'}\mu^{(2^*_{\alpha/2})'} \in L^1(\mathbb{R}^N),$$

as well as

$$(\mathcal{K}_\alpha * |u_n|^p)|u_n|^{p-1}|u|$$
$$\leq \frac{1}{2^*_{\alpha/2}}\xi^{2^*_{\alpha/2}} + \frac{(N+\alpha)(p-1)}{2Np}\eta^{\frac{2Np}{(N+\alpha)(p-1)}} + \frac{N+\alpha}{2Np}|u|^{\frac{2Np}{N+\alpha}} \in L^1(\mathbb{R}^N).$$

The dominated convergence theorem allows to obtain (ii) and (iii). □

**Theorem 5.1.** *The functional $E_\omega$ satisfies the Palais–Smale condition in $X$.*

**Proof.** Let $\{u_n\} \subset X$ be a Palais–Smale sequence, that is,

$$|E_\omega(u_n)| \leq M, \quad E'_\omega(u_n) \to 0 \quad \text{in } H^{-s}(\mathbb{R}^N).$$

Then we deduce in a standard way the boundedness of $\{u_n\}$ in $H^s(\mathbb{R}^N)$. Hence, there exists $u \in X$ such that, up to subsequences, $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^N)$. By Lemma 5.1 we have the convergences:

$$0 \leftarrow E'_\omega(u_n)[u] = (u_n, u) - \int (\mathcal{K}_\alpha * |u_n|^p)|u_n|^{p-2}u_n u \to \|u\|^2 - \int (\mathcal{K}_\alpha * |u|^p)|u|^p,$$

$$E'_\omega(u_n)[u_n] = \|u_n\|^2 - \int (\mathcal{K}_\alpha * |u_n|^p)|u_n|^p \to 0,$$

$$\int (\mathcal{K}_\alpha * |u_n|^p)|u_n|^p \to \int (\mathcal{K}_\alpha * |u|^p)|u|^p,$$

from which we deduce that $\|u_n\| \to \|u\|$. This gives the desired conclusion. □

**Theorem 5.2.** *The functional $E_\omega$ possesses infinitely many critical points $\{u_n\} \subset X$ such that $E_\omega(u_n) \to \infty$, and $\|u_n\| \to \infty$. In particular, problem $(\mathcal{P}_\omega)$ has infinitely many solutions in $X$.*

**Proof.** All the hypotheses (geometry and compactness) of the symmetric mountain pass theorem on the space $X$ are satisfied, so that the existence of infinitely many critical points $\{u_n\} \subset X$ with $E_\omega(u_n) \to \infty$ is guaranteed. Then, since $\int(\mathcal{K}_\alpha * |u_n|)|u_n|^p \leq C\|u_n\|^{2p}$, it has to be $\|u_n\| \to \infty$. By the Palais Principle of Symmetric Criticality, the constrained critical points $\{u_n\} \subset X$ for $E_\omega$ are indeed "true" critical points and hence solutions of $(\mathcal{P}_\omega)$. □

Observe that Proposition 5.1 holds also in the limit cases $p = 1 + \alpha/N$ and $p = (N + \alpha)/(N - 2s)$. Due to the nonexistence result (see Sec. 6), we see that the Palais–Smale condition cannot be satisfied for these values.

To obtain nonradial solutions we need a slight modification in the above setting, as introduced in Ref. 3. Let $N = 4$ or $N \geq 6$ and choose an integer $m \neq (N-1)/2$ such that $2 \leq m \leq N/2$. Let us define

$$G := \mathscr{O}(m) \times \mathscr{O}(m) \times \mathscr{O}(N - 2m),$$

whose induced action on $H^s(\mathbb{R}^N)$ is as usual,

$$(T_g u)(x) = u(g_1^{-1}x_1, g_2^{-1}x_2, g_3^{-1}x_3), \quad g = (g_1, g_2, g_3) \in G, \tag{5.4}$$

where, now $x = (x_1, x_2, x_3) \in \mathbb{R}^m \oplus \mathbb{R}^m \oplus \mathbb{R}^{N-2m}$. We know that $X$, associated to the action (5.4), has compact embedding into $L^q(\mathbb{R}^N), q \in (2, 2_s^*)$. Consider the involution in $\mathbb{R}^N$:

$$\tau \cdot x = (x_2, x_1, x_3),$$

and the action

$$(\mathcal{I}u)(x) = u(x), \quad (\mathcal{T}u)(x) = -u(\tau^{-1} \cdot x),$$

induced by $H = \{\iota_H, \tau\}$ on $H^s(\mathbb{R}^N)$. Define also the group

$$K := G \rtimes_\psi H \subset \mathscr{O}(N),$$

via the group homomorphism $\psi : H \to \mathrm{Aut}(G)$ given by

$$\psi(\iota_H)g = g, \quad \psi(\tau)g = g^{-1}, \quad g \in G.$$

Moreover, if

$$\pi : K \to \{+1, -1\} \quad \text{such that } \pi(g, \iota_H) = 1, \quad \pi(g, \tau) = -1,$$

denotes the canonical epimorphism, we define the action of $K$ on $H^s(\mathbb{R}^N)$ by

$$(T_k u)(x) := \pi(k)u(k^{-1} \cdot x), \quad k \in K.$$

Of course, this action is linear and isometric and in particular if $k = (g, \iota_H)$, then $(T_k u)(x) = u(g^{-1} \cdot x)$, if $k = (\iota_G, \tau)$, then $(T_k u)(x) = -u(\tau^{-1} \cdot x)$. Set

$$Y := \{u \in H^s(\mathbb{R}^N) : T_k u = u \text{ for all } k \in K\},$$

and note that the unique radial function in $Y$ is $u \equiv 0$. Since $E_\omega$ is $K$-invariant and $Y \subset X$ is closed and infinite-dimensional, we can argue as before obtaining the following multiplicity result.

**Theorem 5.3.** *Assume $N = 4$ or $N \geq 6$. The functional $E_\omega$ possesses infinitely many critical points $\{u_n\} \subset Y$ such that $E_\omega(u_n) \to \infty$ and $\|u_n\| \to \infty$. In particular, problem $(\mathcal{P}_\omega)$ has infinitely many solutions in $Y$.*

Hence the proof of Theorem 1.2 is completed.

## 6. Nonexistence

As known, in order to formally deduce a Pohožaev identity, one can compute

$$\frac{d}{d\vartheta} J(\gamma_u(\vartheta))|_{\vartheta=1} = 0,$$

where $\gamma_u(\vartheta) := u(\vartheta x)$ and $u$ is a solution to problem $(\mathcal{P}_\omega)$. We find

$$(N-2s)\int |(-\Delta)^{s/2}u|^2 + \omega N \int |u|^2 = \frac{\alpha+N}{p}\int (\mathcal{K}_\alpha * |u|^p)|u|^p. \qquad (6.1)$$

We shall rigorously justify this identity. We follow the localization argument developed in Ref. 9 by defining the space $X^s(\mathbb{R}^{N+1}_+)$ as the completion of $C_0^\infty(\mathbb{R}^{N+1}_+)$ for the norm

$$\|w\|_{X^s(\mathbb{R}^{N+1}_+)} := \left(\varpi_s^{-1}\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla w|^2 dxdy\right)^{1/2}, \quad \varpi_s := 2^{1-2s}\frac{\Gamma(1-s)}{\Gamma(s)}.$$

For a given $u \in H^s(\mathbb{R}^N)$, the solution $w \in X^s(\mathbb{R}^{N+1}_+)$ of the minimization problem

$$\min\left\{\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla w|^2 dxdy : w(x,0) = u(x) \text{ on } \mathbb{R}^N\right\}$$

is the solution to the boundary value problem:

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla w) = 0 & \text{on } \mathbb{R}^{N+1}_+, \\ w(x,0) = u(x) & \text{on } \mathbb{R}^N, \end{cases}$$

and it is usually called the $s$-harmonic extension of $u$, and

$$\|w\|_{X^s(\mathbb{R}^{N+1}_+)} = \|(-\Delta)^{s/2}u\|_2.$$

As known, the fractional Laplacian can be defined as the Dirichlet-to-Neumann map

$$(-\Delta)^s u(x) = -\frac{1}{\varpi_s}\lim_{y\to 0^+} y^{1-2s}\frac{\partial w}{\partial y}(x,y), \quad u \in H^s(\mathbb{R}^N).$$

Therefore, our nonlocal equation $(\mathcal{P}_\omega)$ can be restated into a local form as:

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla w) = 0 & \text{on } \mathbb{R}^{N+1}_+, \\ \partial^s_\nu w(x,0) = -\omega u + (\mathcal{K}_\alpha * |u|^p)|u|^{p-2}u & \text{on } \mathbb{R}^N, \end{cases}$$

where we have set

$$\partial^s_\nu w(x,0) := -\frac{1}{\varpi_s}\lim_{y\to 0^+} y^{1-2s}\partial_y w(x,y).$$

Without loss of generality, we shall set $\varpi_s = 1$. Of course, if $w$ is a weak solution to this problem, then $u(x) = w(x,0)$ is a weak solution to $(\mathcal{P}_\omega)$.

We have the following result.

**Theorem 6.1.** *Let $u \in C^2(\mathbb{R}^N)\cap H^s(\mathbb{R}^N)\cap L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$ be a weak solution to $(\mathcal{P}_\omega)$. Then* (6.1) *holds.*

Taking into account Remark 3.1, it is not restrictive to assume $u \in C^2(\mathbb{R}^N)$.

**Proof.** Since $u \in C^2(\mathbb{R}^N)$ we have $w \in C^2(\mathbb{R}^{N+1}_+)$. Set $\mathbb{D} = \{z = (x,y) \in \mathbb{R}^N \times [0,+\infty) : |z| \le 1\}$ and consider a cut-off function $\varphi \in C^1_c(\mathbb{R}^N \times [0,+\infty))$ such that $\varphi = 1$ on $\mathbb{D}$, and $\varphi_R(x,y) := \varphi(x/R, y/R)$. A direct computation yields:

$$\begin{aligned}
&\operatorname{div}(y^{1-2s}\nabla w)[\varphi_R(z\cdot\nabla w)] \\
&\quad = \operatorname{div}[(y^{1-2s}\nabla w)\varphi_R(z\cdot\nabla w)] - y^{1-2s}\nabla w\cdot\nabla[\varphi_R(z\cdot\nabla w)] \\
&\quad = \operatorname{div}[(y^{1-2s}\nabla w)\varphi_R(z\cdot\nabla w)] - y^{1-2s}(\nabla\varphi_R\cdot\nabla w)(z\cdot\nabla w) \\
&\qquad - y^{1-2s}\varphi_R|\nabla w|^2 - \frac{1}{2}y^{1-2s}\varphi_R(z\cdot\nabla(|\nabla w|^2)) \\
&\quad = \operatorname{div}\left[(y^{1-2s}\nabla w)\varphi_R(z\cdot\nabla w) - \frac{1}{2}y^{1-2s}\varphi_R z|\nabla w|^2\right] \\
&\qquad - y^{1-2s}(\nabla\varphi_R\cdot\nabla w)(z\cdot\nabla w) + \frac{N-2s}{2}y^{1-2s}\varphi_R|\nabla w|^2 \\
&\qquad + \frac{1}{2}y^{1-2s}(z\cdot\nabla\varphi_R)|\nabla w|^2,
\end{aligned}$$

and, integrating on $\mathbb{R}^{N+1}_+$, we get:

$$\begin{aligned}
\int_{\mathbb{R}^{N+1}_+} \operatorname{div}[(y^{1-2s}\nabla w)\varphi_R(z\cdot\nabla w)] &= \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N\times[\varepsilon,+\infty[} \operatorname{div}[(y^{1-2s}\nabla w)\varphi_R(z\cdot\nabla w)] \\
&= \lim_{\varepsilon\to 0^+}\int_{\partial(\mathbb{R}^N\times[\varepsilon,+\infty[)} (y^{1-2s}\nabla w)\cdot\nu\varphi_R(z\cdot\nabla w) \\
&= -\lim_{\varepsilon\to 0^+}\int_{\partial(\mathbb{R}^N\times[\varepsilon,+\infty[)} (y^{1-2s}\partial_y w)\varphi_R(z\cdot\nabla w) \\
&= \int(-\omega u + (\mathcal{K}_\alpha * |u|^p)|u|^{p-2}u)\varphi_R(x,0)(x\cdot\nabla u),
\end{aligned}$$

where $\nu(x) = (0,\ldots,0,-1)$. Now following the proof of Proposition 3.1 in Ref. 29 we get

$$\omega\int u\varphi_R(x,0)(x\cdot\nabla u) \to -\frac{\omega N}{2}\|u\|_2^2 \quad \text{as } R\to+\infty,$$

and

$$\begin{aligned}
&\int((\mathcal{K}_\alpha * |u|^p)|u|^{p-2}u)\varphi_R(x,0)(x\cdot\nabla u) \\
&\qquad \to -\frac{N+\alpha}{2p}\int(\mathcal{K}_\alpha * |u|^p)|u|^p \quad \text{as } R\to+\infty.
\end{aligned}$$

Moreover by the dominated convergence theorem we have:

$$\begin{aligned}
\int_{\mathbb{R}^{N+1}_+} \operatorname{div}(y^{1-2s}\varphi_R z|\nabla w|^2) &= \lim_{\varepsilon\to 0+}\int_{\mathbb{R}^N\times[\varepsilon,+\infty[} \operatorname{div}(y^{1-2s}\varphi_R z|\nabla w|^2) \\
&= -\lim_{\varepsilon\to 0^+}\varepsilon^{2-2s}\int_{\partial(\mathbb{R}^N\times[\varepsilon,+\infty[)} \varphi_R|\nabla w|^2 = 0,
\end{aligned}$$

$$\int_{\mathbb{R}^{N+1}_+} y^{1-2s}(\nabla\varphi_R\cdot\nabla w)(z\cdot\nabla w)\to 0 \quad \text{as } R\to+\infty,$$

$$\int_{\mathbb{R}^{N+1}_+} y^{1-2s}(z\cdot\nabla\varphi_R)|\nabla w|^2\to 0 \quad \text{as } R\to+\infty,$$

$$\int_{\mathbb{R}^{N+1}_+} y^{1-2s}\varphi_R|\nabla w|^2\to \int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla w|^2 \quad \text{as } R\to+\infty,$$

which conclude the proof. □

By combining the Pohožaev identity (6.1) with

$$\int |(-\Delta)^{s/2}u|^2+\omega\int |u|^2=\int (\mathcal{K}_\alpha * |u|^p)|u|^p,$$

we get

$$\left(N-2s-\frac{\alpha+N}{p}\right)\int |(-\Delta)^{s/2}u|^2+\omega\left(N-\frac{\alpha+N}{p}\right)\int |u|^2=0.$$

Now, since $\omega>0$, if both the coefficients are positive, that is $p\geq \alpha+N/(N-2s)$, the unique solution is the trivial one. Analogously, if they are negative, that is $p\leq 1+\alpha/N$, nontrivial solutions cannot exist. Thus we conclude the proof of Theorem 6.1. Now, the first statement of Theorem 1.4 follows by Pohožaev identity (6.1).

In case $s=1$, the assertion that any solution to problem (1.7) of fixed sign has the form given in formula (1.8), was stated in Proposition A.1 of Ref. 28 and the authors claim in Remark A.2-(3) that the same holds for the fractional Laplacian. In order to justify this conclusion and make the paper self-contained, we provide the following analysis on how to rigorously prove the statement.

- *Invariance under Kelvin transform.* Consider the equation

$$(-\Delta)^s u=v_u u, \quad \text{with} \quad v_u=|\cdot|^{-4s}*u^2,$$

and define the following operators, on functions $g$ defined a.e.,

$$(Kg)(x):=|x|^{2s-N}g(x/|x|^2) \quad \text{and} \quad (Hg)(x):=|x|^{-N-2s}g(x/|x|^2).$$

$K$ is a Kelvin transform type operator, which is an isometry in $\dot H^s(\mathbb{R}^N)$, see Lemma 2.2 in Ref. 14. Observe that $K,H$ are involutions, namely $K^2=I=H^2$. Let us see now the behavior of $(-\Delta)^s$ and $v_u u$ under the operators $H,K$. Fall and Weth (see Corollary 2.3 in Ref. 14) prove that

$$H(-\Delta)^s=(-\Delta)^s K.$$

The behavior of the convolution term $v_u$ is proved in Lemma A.3 of Ref. 28, with $s=1$, where the authors use the identity $|y|^4|x-y/|y|^2|^4=|x|^4|x/|x|^2-y|^4$. In our case, by replacing the exponent 4 with $4s$, exactly the same computation gives

$$v_u(x)=|x|^{-4s}v_{K^{-1}u}(x/|x|^2).$$

Notice that, by the definition,

$$u(x) = |x|^{-(N-2s)} Ku(x/|x|^2),$$

and then

$$\begin{aligned} v_u(x)u(x) &= |x|^{-4s} v_{Ku}(x/|x|^2)|x|^{-(N-2s)} Ku(x/|x|^2) \\ &= |x|^{-(N+2s)} v_{Ku}(x/|x|^2) Ku(x/|x|^2) \\ &= H(v_{Ku}Ku)(x), \end{aligned}$$

namely, $H[v_u u] = v_{Ku}Ku$. If $u$ is a solution of (1.7), by applying $H$ to both sides we have

$$(-\Delta)^s Ku = v_{Ku} Ku,$$

and so $Ku \in \dot{H}^s(\mathbb{R}^N)$ is a solution of (1.7) too.

• *Radial symmetry and monotonicity.* We want to prove that each positive solution $u$ of (1.7) is radially symmetric and monotone decreasing about some point $x_0 \in \mathbb{R}^N$. Let $u \in \dot{H}^s(\mathbb{R}^N)$, $u > 0$, be a solution of (1.7) and, for simplicity, let $v := v_u$. By Sobolev embedding we have that $u \in L^{2_s^*}(\mathbb{R}^N)$ and by Hardy–Littlewood–Sobolev inequality, it follows $v \in L^{N/(2s)}(\mathbb{R}^N)$. Moreover, by arguing as in Theorem 4.5 of Ref. 10, we have that Eq. (1.7) is equivalent to the system

$$u(x) = \int \frac{v(y)u(y)}{|x-y|^{N-2s}} dy, \quad v = |x|^{-4s} * u^2. \tag{6.2}$$

We use classical notations for the moving plane, namely $\Sigma_\lambda = \{x_1 \geq \lambda\}$ and $u_\lambda(x) = u(x^\lambda) = u(2\lambda - x_1, x_2, \ldots, x_N)$. Simple calculations show that:

$$u_\lambda(x) - u(x) = \int_{\Sigma_\lambda} \left( \frac{1}{|x-y|^{N-2s}} - \frac{1}{|x^\lambda - y|^{N-2s}} \right) (u_\lambda(y)v_\lambda(y) - u(y)v(y))dy,$$

$$v_\lambda(x) - v(x) = \int_{\Sigma_\lambda} \left( \frac{1}{|x-y|^{4s}} - \frac{1}{|x^\lambda - y|^{4s}} \right) (u_\lambda^2(y) - u^2(y))dy.$$

Then, for any $x \in \Sigma_\lambda$, we have:

$$\begin{aligned} u_\lambda(x) - u(x) &\leq \int_{\{y \in \Sigma_\lambda : uv \leq u_\lambda v_\lambda\}} \frac{u_\lambda(y)v_\lambda(y) - u(y)v(y)}{|x-y|^{N-2s}} dy \\ &= \int_{\{y \in \Sigma_\lambda : uv \leq u_\lambda v_\lambda\}} \frac{u(y)[v_\lambda(y) - v(y)] + v_\lambda(y)[u_\lambda(y) - u(y)]}{|x-y|^{N-2s}} dy \\ &\leq \int_{\Sigma_\lambda^u} \frac{v_\lambda(y)[u_\lambda(y) - u(y)]}{|x-y|^{N-2s}} dy + \int_{\Sigma_\lambda^v} \frac{u(y)[v_\lambda(y) - v(y)]}{|x-y|^{N-2s}}, \end{aligned}$$

where we have set $\Sigma_\lambda^u = \Sigma_\lambda \cap \{u_\lambda > u\}$ and $\Sigma_\lambda^v = \Sigma_\lambda \cap \{v_\lambda > v\}$. Then, by Hardy–Littlewood–Sobolev and Hölder inequalities we have

$$\begin{aligned} \|u_\lambda - u\|_{L^{2_s^*}(\Sigma_\lambda^u)} \leq C(&\|v\|_{L^{N/(2s)}(\Sigma_\lambda^c)} \|u_\lambda - u\|_{L^{2_s^*}(\Sigma_\lambda^u)} \\ &+ \|u\|_{L^{2_s^*}(\Sigma_\lambda^v)} \|v_\lambda - v\|_{L^{N/(2s)}(\Sigma_\lambda^v)}). \end{aligned} \tag{6.3}$$

Analogously we get, for all $x \in \Sigma_\lambda$,

$$v_\lambda(x) - v(x) \le 2 \int_{\Sigma_\lambda^u} \frac{u_\lambda(y)(u_\lambda(y) - u(y))}{|x-y|^{4s}} dy,$$

and

$$\|v_\lambda - v\|_{L^{N/(2s)}(\Sigma_\lambda^v)} \le C \|u\|_{L^{2_s^*}(\Sigma_\lambda^c)} \|u_\lambda - u\|_{L^{2_s^*}(\Sigma_\lambda^u)}. \tag{6.4}$$

Since $\|v\|_{L^{N/(2s)}(\Sigma_\lambda^c)}, \|u\|_{L^{2_s^*}(\Sigma_\lambda^c)} \to 0$ as $\lambda \to -\infty$, combining (6.3) and (6.4), we obtain $\|u_\lambda - u\|_{L^{2_s^*}(\Sigma_\lambda^u)} = 0$ and hence $|\Sigma_\lambda^u| = 0$ and $|\Sigma_\lambda^v| = 0$. The proof of radial symmetry and monotonicity of $u$ and $v$ can be obtained in the same way of Step 2 and Step 3 in Ref. 28 using the analogous inequalities given above.

• *Classification result.* The same geometrical argument as in the proof of Step 3, on p. 335 of Ref. 10, which exploits the invariance of the problem under the Kelvin transform, shows that there exists a positive constant $u_\infty$ such that

$$\lim_{|x|\to\infty} |x|^{N-2s} u(x) = u_\infty. \tag{6.5}$$

With the above tools available, namely *Kelvin invariance*, *radial symmetry*, the *scaling invariance*

$$u_\lambda(x) = \lambda^{\frac{N-2s}{2}} u(\lambda x), \quad \lambda > 0,$$

and the *asymptotics* as in (6.5), then the desired classification follows as in Sec. 3.1 of Ref. 10, where the authors deal with the problem $(-\Delta)^s u = u^{2_s^*-1}$ in $\mathbb{R}^N$. More precisely, having formula (6.5) available, the arguments of Sec. 3.1 in Ref. 10, which rely on the validity of Lemmas 3.1 and 3.2 in Ref. 10, carry on with no variations since they contain calculations independent of the particular structure of the nonlinear term.

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