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Asymptotic behavior of a thermoviscoelastic plate with memory effects

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Abstract. We consider a coupled linear system describing a thermoviscoelastic plate with hereditary effects. The system consists of a hyperbolic integrodifferential equation, governing the temperature, which is linearly coupled with the partial differential equation ruling the evolution of the vertical deflection, presenting a convolution term accounting for memory effects. It is also assumed that the thermal power contains a memory term characterized by a relaxation kernel. We prove that the system is exponentially stable and we obtain a closeness estimate between the system with memory effects and the corresponding memory-free limiting system, as the kernels fade in a suitable sense.

Keywords: thermoviscoelastic plate, memory effects, singular limit, exponential decay

1. Introduction

Let Ω be a bounded planar domain with smooth boundary $\partial\Omega$. Suppose that Ω is occupied, for all time t, by a thin homogeneous isotropic viscoelastic plate. Denoting by u its *vertical deflection* and by ϑ the *temperature variation field*, we suppose that the evolution of the pair (u, ϑ) is governed by the following integrodifferential system

$$\begin{cases} u_{tt} + h(0)\Delta^2 u + \displaystyle\int_0^\infty h'(s)\Delta^2 u(t-s)\,\mathrm{d}s + \Delta\vartheta = 0, \\ \vartheta_t + a(0)\vartheta + \displaystyle\int_0^\infty a'(s)\vartheta(t-s)\,\mathrm{d}s - \int_0^\infty k(s)\Delta\vartheta(t-s)\,\mathrm{d}s - \Delta u_t = 0, \end{cases} \tag{1.1}$$

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in $\Omega \times \mathbb{R}^+$, where $\mathbb{R}^+ = (0, \infty)$. Here $k : [0, \infty) \to \mathbb{R}^+$ and $h : [0, \infty) \to \mathbb{R}^+$ are smooth decreasing convex functions which go to 0 and to $h(\infty) > 0$ at infinity, respectively. Instead, the memory kernel $a : [0, \infty) \to \mathbb{R}^+$ is a smooth increasing concave function with a' vanishing at infinity. Moreover, all the other physical constants have been set equal to 1. Observe that, if k and $h - h(\infty)$ coincide with the Dirac mass δ_0 at zero and $a \equiv 0$, then, supposing $h(\infty) = 1$, the above system formally collapses into the linear model of thermoviscoelastic plate

$$\begin{cases} u_{tt} + \Delta^2 u_t + \Delta(\Delta u + \vartheta) = 0, \\ \vartheta_t - \Delta\vartheta - \Delta u_t = 0. \end{cases} \tag{1.2}$$

We shall assume for simplicity that system (1.1) is endowed with Navier boundary conditions

$$\begin{aligned} &u(t) = \Delta u(t) = 0 \quad && \text{on } \partial\Omega,\ t \geqslant 0, \\ &\vartheta(t) = 0 && \text{on } \partial\Omega,\ t \in \mathbb{R}, \end{aligned}$$

and initial conditions

$$\begin{aligned} &\big(u(0), u_t(0), \vartheta(0)\big) = (u_0, u_1, \vartheta_0) \quad && \text{in } \Omega, \\ &\vartheta(-s) = \vartheta_0(s) && \text{in } \Omega \times \mathbb{R}^+, \\ &u(-s) = u_0(s) && \text{in } \Omega \times \mathbb{R}^+, \end{aligned}$$

where $u_0, u_1, \vartheta_0 : \Omega \to \mathbb{R}$ and $u_0, \vartheta_0 : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ are assigned functions. The choice of these boundary conditions (*edge-free plate*) simplifies the functional setup as well as some technical arguments with respect, e.g., to Neumann boundary data (*clamped plate*). The results will be obtained via the so-called past history approach (cf. [6] and references therein) which allows, under suitable assumptions, to express the solution by a strongly continuous semigroup acting on an appropriate (extended) phase space (cf. Theorem 2.1).

System (1.1) with $h \equiv h(\infty)$ was considered and justified from the physical viewpoint in [4], while the viscoelastic case was treated in [13] (cf. also their references). More precisely, according to [4], the Laplace operator $-\Delta$ in the second equation of (1.1) should be replaced by $-\Delta + c\mathbb{I}$ with $c > 0$. However, this does not affect the mathematical analysis (see, e.g., [5,7]), so we decided to set $c = 0$ just for the sake of simplicity. In [4] the exponential stability was proved provided that $a(0) \neq 0$, namely not only the heat conduction law accounts for hereditary effects (see [8]), but also the constitutive assumption for the thermal power contains a memory term characterized by a nonzero relaxation kernel. Instead, if one assumes that $h \equiv h(\infty)$ and $a \equiv 0$, then, for nonzero initial histories, the system *fails* to be exponentially stable, no matter how fast the memory kernel k squeezes at infinity, provided that its growth around the origin is suitably controlled (cf. [5], Theorem 5.4). This confirms the conjecture formulated in [4], Remark 5.1, and also says that the presence of past history plays a discriminating role for the stability of the thermoelastic system. It must be noticed that the exponential stability was obtained in [4] and in [13] by exploiting some spectral analysis arguments, without detecting a precise decay rate. On the other hand, mainly in view of the asymptotic analysis that we wish to pursue with respect to the behavior of the memory kernels involved in (1.1), here we are interested in getting an explicit rate of decay. By exploiting a technique first introduced in [9,15], we detect the decay rate by building up an ad hoc perturbation of the energy functional which satisfy suitable differential inequalities (cf. Theorem 3.1). Concerning the case $h \not\equiv h(\infty)$ and $a \equiv 0$, in the Appendix we shall consider a quite general class of (abstract) thermoelastic systems with memory. We shall prove that every trajectory squeezes to zero asymptotically (nonuniformly with respect to initial data). Moreover, we shall exhibit some (weakly singular) memory kernels for which the corresponding system, not including (1.1), lacks of exponential stability.

The second main result of this paper is about the closeness between the solutions to system (1.1) and the solutions to the system (1.2). The set of boundary and initial conditions is the same but the ones for the past histories of ϑ and u. Concerning the memory kernels k and h, we proceed in the spirit of [1] (see also [2,3,7]) by replacing them with the rescalings k_ε and h_σ, defined by

$$k_\varepsilon(s) = \frac{1}{\varepsilon} k\left(\frac{s}{\varepsilon}\right), \qquad h_\sigma(s) = \frac{1}{\sigma} \tilde{h}\left(\frac{s}{\sigma}\right), \quad \forall s \in \mathbb{R}^+,$$

where $\tilde{h} = h - h(\infty)$, while $\varepsilon \in (0,1]$ and $\sigma \in (0,1]$ are time relaxation parameters. Notice that k_ε and h_σ approach the Dirac mass δ_0 as ε and σ go to zero, in the sense of distribution. Moreover, on the basis of physical motivations, concerning the parametrization of the memory kernel a, we think of the (model) situation

$$a_\tau(s) = \phi(\tau) + \psi(\tau)\big(1 - \mathrm{e}^{-\omega s}\big), \quad \forall s \in \mathbb{R}^+,$$

$\phi, \psi : [0,1] \to \mathbb{R}^+$ being continuous functions with $\phi(0) = \psi(0) = 0$ (see (2.9)–(2.11)). Therefore, while the kernels k and h undergo a singular perturbation procedure, a is parameterized just in order to be uniformly squeezing to zero as τ vanishes. A suitable reformulation of system (1.1), according to a well-established procedure, with k_ε in place of k, h_ε in place of h, and a_τ in place of a is shown to generate a semigroup of contractions $S_{\sigma,\tau,\varepsilon}(t)$ on a certain phase-space $\mathcal{H}^0_{\sigma,\tau,\varepsilon}$. Then, denoting by $S_{0,0,0}(t)$ the limiting semigroup generated by system (1.2), we establish an estimate of the difference between two different trajectories, in terms of σ, τ and ε, which holds on any bounded time interval. Basically, our estimate says that the solutions to system (1.1) are arbitrarily close, in the natural norm of $\mathcal{H}^0_{\sigma,\tau,\varepsilon}$, to the solutions of system (1.2), provided that σ, τ and ε are small enough and the initial data are chosen inside a suitable regular bounded subset of the phase-space. For the sake of generality we stress that, in addition to the singular limit estimate in the norm of the base phase-space $\mathcal{H}^0_{\sigma,\tau,\varepsilon}$, we shall actually provide the control with respect to the norms of the higher order phase-spaces $\mathcal{H}^m_{\sigma,\tau,\varepsilon}$, $m \geqslant 0$ for suitably regular initial data (cf. Theorem 4.5). Clearly, the limit process for σ and ε going to zero is *singular*, for the information on the past histories of the temperature field ϑ and of the vertical deflection u get lost in the limit. As we shall see, the closeness control has to be understood for time intervals which are bounded away from 0. In the particular case where we fix $\tau = 0$ and we only take care of the limit process with respect to σ and ε, the result strengthens since the estimate turns out to hold with constants which are independent of the time interval size, so that the differences between any two trajectories can be controlled for any time $t > 0$ (cf. Theorem 4.7).

An interesting open problem is the analysis of the present model when $\tilde{h}$ is approximated as a, that is, by a vanishing sequence of kernels. In fact, recalling that there is no exponential decay when a and h'

vanish (see [5]), there should be a relation between the relaxation times ε, σ and τ in order to preserve the exponential stability when they approach zero.

The content of the paper is organized as follows.

In Section 2 we introduce the notation and the basic tools, and we formulate the problems in the proper functional setting. In Section 3 we prove that, for every $\tau \neq 0$, the solutions to (1.1) are exponentially decaying with a rate of decay proportional to $\phi(\tau)$. In Section 4 we demonstrate the closeness estimate between the strongly continuous semigroups associated with systems (1.1) and (1.2) when the time rescaling parameters σ, τ and ε tend to zero. Finally, in the Appendix, we deal with the pointwise decay and the lack of exponential stability for an abstract class of thermoelastic systems with memory.

2. Preliminaries and well-posedness

In this section we provide the proper functional framework and the well-posedness result for problem (1.1).

2.1. The kernels parametrization

We assume that $k:\mathbb{R}^+ \to \mathbb{R}^+$ and $h:\mathbb{R}^+ \to \mathbb{R}^+$ are smooth, decreasing and summable functions satisfying, for the sake of simplicity, the normalization conditions

$$\int_0^\infty k(s)\,\mathrm{d}s = \int_0^\infty \tilde{h}(s)\,\mathrm{d}s = 1, \quad h(0) = 2,\ k(0) = h(\infty) = 1.$$

Then, we set

$$\mu(s) = -k'(s), \qquad \beta(s) = -h'(s), \quad \forall s \in \mathbb{R}^+,$$

where μ and β are supposed to satisfy

$$\mu, \beta \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \tag{2.1}$$

$$\mu(s) \geqslant 0, \qquad \beta(s) \geqslant 0, \quad \forall s \in \mathbb{R}^+, \tag{2.2}$$

$$\mu'(s) \leqslant 0, \qquad \beta'(s) \leqslant 0, \quad \forall s \in \mathbb{R}^+, \tag{2.3}$$

$$\mu'(s) + \delta_1 \mu(s) \leqslant 0, \qquad \forall s \in \mathbb{R}^+, \tag{2.4}$$

$$\beta'(s) + \delta_2 \beta(s) \leqslant 0, \qquad \forall s \in \mathbb{R}^+, \tag{2.5}$$

for some $\delta_1 > 0$ and $\delta_2 > 0$. For any $\varepsilon \in (0,1]$ and $\sigma \in (0,1]$ we define the rescalings

$$\mu_\varepsilon(s) = \frac{1}{\varepsilon^2}\mu\left(\frac{s}{\varepsilon}\right) = -k'_\varepsilon(s), \qquad \beta_\sigma(s) = \frac{1}{\sigma^2}\beta\left(\frac{s}{\sigma}\right) = -h'_\sigma(s). \tag{2.6}$$

Without loss of generality, we suppose that, for $\varepsilon \in (0,1]$ and $\sigma \in (0,1]$, there holds

$$\int_0^\infty \mu_\varepsilon(s)\,\mathrm{d}s = \frac{1}{\varepsilon}, \qquad \int_0^\infty s\mu_\varepsilon(s)\,\mathrm{d}s = 1, \tag{2.7}$$

$$\int_0^\infty \beta_\sigma(s)\,\mathrm{d}s = \frac{1}{\sigma}, \qquad \int_0^\infty s\beta_\sigma(s)\,\mathrm{d}s = 1. \tag{2.8}$$

We assume that $a(s) = a_\tau(s)$, with $\tau \in [0,1]$, where $a_\tau : \mathbb{R}^+ \to \mathbb{R}^+$ is a smooth concave function. We put $\nu_\tau(s) = -a_\tau''(s)$, where $\nu_\tau \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ satisfies

$$\nu_\tau(s) \geqslant 0, \qquad \nu_\tau'(s) \leqslant 0, \quad \forall s \in \mathbb{R}^+, \tag{2.9}$$

$$\nu_\tau'(s) + \delta_3\nu_\tau(s) \leqslant 0, \qquad \forall s \in \mathbb{R}^+, \tag{2.10}$$

for some $\delta_3 > 0$. Furthermore we assume that the map $\{\tau \mapsto \nu_\tau\}$ is increasing and there exist two functions $\phi, \psi \in C^0(\mathbb{R}^+)$ with $\phi \geqslant 0$, $\psi \geqslant 0$ and $\phi(0) = \psi(0) = 0$, such that

$$a_\tau(0) = \phi(\tau), \quad \forall \tau \in [0,1], \qquad \|\nu_\tau\|_{L^1(\mathbb{R}^+)} \leqslant \psi(\tau), \quad \forall \tau \in [0,1]. \tag{2.11}$$

2.2. *The scale of phase-spaces*

Let Ω be a smooth bounded subset of $\mathbb{R}^2$. The symbols $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ stand for the norm and the inner product on $L^2(\Omega)$, respectively. We define the positive operator A on $L^2(\Omega)$ by $A = -\Delta$ with domain $\mathcal{D}(A) = H_0^1(\Omega) \cap H^2(\Omega)$, and we introduce the scale of Hilbert spaces $H^m = \mathcal{D}(A^{m/2})$, $m \in \mathbb{R}$, endowed with the inner products $\langle u_1, u_2\rangle_{H^m} = \langle A^{m/2}u_1, A^{m/2}u_2\rangle$. We now consider the weighted Hilbert spaces

$$\mathcal{M}^m_{\tau,\varepsilon} = L^2_{\mu_\varepsilon}(\mathbb{R}^+, H^{m+1}) \cap L^2_{\nu_\tau}(\mathbb{R}^+, H^m), \qquad \mathcal{Q}^m_\sigma = L^2_{\beta_\sigma}(\mathbb{R}^+, H^{m+1}), \quad m \in \mathbb{R},$$

endowed, respectively, with the inner products

$$\begin{aligned}
\langle \eta_1, \eta_2\rangle_{\mathcal{M}^m_{\tau,\varepsilon}} &= \int_0^\infty \mu_\varepsilon(s)\big\langle A^{(1+m)/2}\eta_1(s), A^{(1+m)/2}\eta_2(s)\big\rangle\,\mathrm{d}s \\
&\quad + \int_0^\infty \nu_\tau(s)\big\langle A^{m/2}\eta_1(s), A^{m/2}\eta_2(s)\big\rangle\,\mathrm{d}s, \\
\langle \xi_1, \xi_2\rangle_{\mathcal{Q}^m_\sigma} &= \int_0^\infty \beta_\sigma(s)\big\langle A^{(1+m)/2}\xi_1(s), A^{(1+m)/2}\xi_2(s)\big\rangle\,\mathrm{d}s,
\end{aligned}$$

and we introduce the product spaces

$$\mathcal{H}^m_{\sigma,\tau,\varepsilon} = \begin{cases} H^{m+2} \times H^m \times H^m \times \mathcal{M}^m_{\tau,\varepsilon} \times \mathcal{Q}^{m+1}_\sigma, & \text{if } \sigma > 0 \text{ and } \tau > 0 \text{ or } \varepsilon > 0, \\ H^{m+2} \times H^m \times H^m \times \mathcal{M}^m_{\tau,\varepsilon}, & \text{if } \sigma = 0 \text{ and } \tau > 0 \text{ or } \varepsilon > 0, \\ H^{m+2} \times H^m \times H^m \times \mathcal{Q}^{m+1}_\sigma, & \text{if } \sigma > 0 \text{ and } \tau = \varepsilon = 0, \\ H^{m+2} \times H^m \times H^m, & \text{if } \sigma = \tau = \varepsilon = 0, \end{cases}$$

that will be normed by

$$\|(u,u_t,\vartheta,\eta,\xi)\|^2_{\mathcal{H}^m_{\sigma,\tau,\varepsilon}} = \|u\|^2_{H^{m+2}} + \|u_t\|^2_{H^m} + \|\vartheta\|^2_{H^m} + \|\eta\|^2_{\mathcal{M}^m_{\tau,\varepsilon}} + \|\xi\|^2_{\mathcal{Q}^{m+1}_\sigma}.$$

In particular $\mathcal{H}^0_{\sigma,\tau,\varepsilon}$ is the extended phase-space on which we shall construct the dynamical system associated with (1.1). Throughout the paper, when $\sigma = \tau = \varepsilon = 0$, we shall agree to interpret the five entries vector $z = (u, u_t, \vartheta, \eta, \xi)$ just as the triplet (u, u_t, ϑ).

2.3. The problem setting

In order to formulate the problem in a suitable history space setting, let $T_{\tau,\varepsilon}$ and T_σ be the linear operators on $\mathcal{M}^0_{\tau,\varepsilon}$ and $\mathcal{Q}^1_\sigma$ respectively, defined as

$$T_{\tau,\varepsilon}\eta = -\eta_s, \quad \eta \in \mathcal{D}(T_{\tau,\varepsilon}), \qquad T_\sigma\xi = -\xi_s, \quad \xi \in \mathcal{D}(T_\sigma),$$

where

$$\mathcal{D}(T_{\tau,\varepsilon}) = \{\eta \in \mathcal{M}^0_{\tau,\varepsilon}\colon \eta_s \in \mathcal{M}^0_{\tau,\varepsilon}, \eta(0) = 0\},$$
$$\mathcal{D}(T_\sigma) = \{\xi \in \mathcal{Q}^1_\sigma\colon \xi_s \in \mathcal{Q}^1_\sigma, \xi(0) = 0\},$$

and η_s (resp. ξ_s) stands for the distributional derivative of η (resp. ξ) with respect to the internal variable s. Notice that $T_{\tau,\varepsilon}$ (resp. T_σ) is the infinitesimal generator of the right-translation semigroup on $\mathcal{M}^0_{\tau,\varepsilon}$ (resp. $\mathcal{Q}^1_\sigma$). Moreover, on account of (2.3) and (2.9),

$$\langle T_{\tau,\varepsilon}\eta,\eta\rangle_{\mathcal{M}^0_{\tau,\varepsilon}} = \frac{1}{2}\int_0^\infty \mu'_\varepsilon(s)\|A^{1/2}\eta(s)\|^2\,\mathrm{d}s + \frac{1}{2}\int_0^\infty \nu'_\tau(s)\|\eta(s)\|^2\,\mathrm{d}s \leqslant 0, \tag{2.12}$$

$$\langle T_\sigma\xi,\xi\rangle_{\mathcal{Q}^1_\sigma} = \frac{1}{2}\int_0^\infty \beta'_\sigma(s)\|A\xi(s)\|^2\,\mathrm{d}s \leqslant 0, \tag{2.13}$$

for $\eta \in \mathcal{D}(T_{\tau,\varepsilon})$ and $\xi \in \mathcal{D}(T_\sigma)$. Following the well-established past history approach (see, e.g. [6]), we introduce the so-called past histories of ϑ and u,

$$\eta^t(s) = \int_0^s \vartheta(t-y)\,\mathrm{d}y, \qquad \xi^t(s) = u(t) - u(t-s), \quad (s,t) \in \mathbb{R}^+ \times \mathbb{R}^+.$$

Differentiating these variables leads to further equations ruling the evolution of η and ξ

$$\eta^t_t = -\eta^t_s + \vartheta(t), \qquad \xi^t_t = -\xi^t_s + u_t(t), \quad t \in \mathbb{R}^+.$$

We are now in the right position to introduce the formulation of the problems. On account of the normalization conditions and of the notation previously introduced, for any $\sigma, \tau, \varepsilon \in [0,1]$, given $(u_0, u_1, \vartheta_0, \eta_0, \xi_0)$ in $\mathcal{H}^0_{\sigma,\tau,\varepsilon}$, find $(u, u_t, \vartheta, \eta, \xi) \in C([0,\infty), \mathcal{H}^0_{\sigma,\tau,\varepsilon})$ solution to

$$\begin{cases} u_{tt} + \displaystyle\int_0^\infty \beta_\sigma(s) A^2 \xi(s)\,\mathrm{d}s + A(Au - \vartheta) = 0, \\ \vartheta_t + \phi(\tau)\vartheta + \displaystyle\int_0^\infty \nu_\tau(s)\eta(s)\,\mathrm{d}s + \int_0^\infty \mu_\varepsilon(s) A\eta(s)\,\mathrm{d}s + Au_t = 0, \\ \eta_t = T_{\tau,\varepsilon}\eta + \vartheta, \\ \xi_t = T_\sigma \xi + u_t, \end{cases} \tag{$\mathcal{P}_{\sigma,\tau,\varepsilon}$}$$

for $t \in \mathbb{R}^+$, with initial condition $(u(0), u_t(0), \vartheta(0), \eta^0, \xi^0) = (u_0, u_1, \vartheta_0, \eta_0, \xi_0)$. Similarly, we introduce the limiting problem (formally corresponding to the case $\sigma = \tau = \varepsilon = 0$). Given $(u_0, u_1, \vartheta_0) \in \mathcal{H}^0_{0,0,0}$, find $(u, u_t, \vartheta) \in C([0,\infty), \mathcal{H}^0_{0,0,0})$ solution to

$$\begin{cases} u_{tt} + A^2 u_t + A(Au - \vartheta) = 0, \\ \vartheta_t + A\vartheta + Au_t = 0, \end{cases} \tag{$\mathcal{P}_{0,0,0}$}$$

for $t \in \mathbb{R}^+$, which fulfills the initial conditions $(u(0), u_t(0), \vartheta(0)) = (u_0, u_1, \vartheta_0)$. The above problems are abstract reformulations of the initial and boundary value problems associated with (1.1) and (1.2).

2.4. Well-posedness

System $(\mathcal{P}_{\sigma,\tau,\varepsilon})$ allows us to provide a description of the solutions in terms of a strongly continuous semigroup of operators on $\mathcal{H}^0_{\sigma,\tau,\varepsilon}$. Indeed, setting

$$\zeta(t) = \big(u(t), v(t), \vartheta(t), \eta^t, \xi^t\big)^\top,$$

the problem rewrites as

$$\frac{\mathrm{d}}{\mathrm{d}t}\zeta = \mathcal{L}\zeta, \quad \zeta(0) = \zeta_0,$$

where $\mathcal{L}$ is the linear operator defined by

$$\mathcal{L}\begin{pmatrix} u \\ v \\ \vartheta \\ \eta \\ \xi \end{pmatrix} = \begin{pmatrix} v \\ -\int_0^\infty \beta_\sigma(s) A^2 \xi(s)\,\mathrm{d}s - A(Au - \vartheta) \\ -\phi(\tau)\vartheta - \int_0^\infty \nu_\tau(s)\eta(s)\,\mathrm{d}s - \int_0^\infty \mu_\varepsilon(s) A\eta(s)\,\mathrm{d}s - Av \\ \vartheta + T_{\tau,\varepsilon}\eta \\ v + T_\sigma \xi \end{pmatrix} \tag{2.14}$$

with domain

$$\mathcal{D}(\mathcal{L}) = \left\{ z \in \mathcal{H}^0_{\sigma,\tau,\varepsilon} \;\middle|\; \begin{array}{l} Au - \vartheta \in H^2 \\ v \in H^2, \vartheta \in H^1 \\ \displaystyle\int_0^\infty \mu_\varepsilon(s) A\eta(s)\,\mathrm{d}s \in H^0 \\ \displaystyle\int_0^\infty \nu_\tau(s)\eta(s)\,\mathrm{d}s \in H^0 \\ \displaystyle\int_0^\infty \beta_\sigma(s) A^2\xi(s)\,\mathrm{d}s \in H^0 \\ \eta \in \mathcal{D}(T_{\tau,\varepsilon}), \xi \in \mathcal{D}(T_\sigma) \end{array} \right\}.$$

By virtue of (2.12) and (2.13), it is readily seen that $\mathcal{L}$ is a dissipative operator. We will tacitly extend the definition of $S_{\sigma,\tau,\varepsilon}(t)$ to the case $\sigma = \tau = \varepsilon = 0$ which is well known. Of course, in this case the solution semigroup is a three-component vector only. We now assume that (2.1), (2.2), (2.4), (2.5) and (2.9)–(2.11) hold true. If $\sigma > 0$, $\tau > 0$ and $\varepsilon > 0$, following the proof of [4], Theorem 2.1, we obtain the following theorem.

Theorem 2.1. *System* $(\mathcal{P}_{\sigma,\tau,\varepsilon})$ *defines a* C_0*-semigroup* $S_{\sigma,\tau,\varepsilon}(t)$ *of contractions on* $\mathcal{H}^0_{\sigma,\tau,\varepsilon}$.

3. Exponential stability of $S_{\sigma,\tau,\varepsilon}(t)$

In this section we prove that, for any $\tau \in [0,1]$, the semigroup $S_{\sigma,\tau,\varepsilon}(t)$ is exponentially stable on $\mathcal{H}^0_{\sigma,\tau,\varepsilon}$, admitting a decay rate proportional to $\phi(\tau)$ when $\tau > 0$. In this case, the exponential stability is actually already known from [4] in the elastic case with a nonvanishing kernel a (recall that if a vanishes the exponential stability fails as shown in [5]). However, this result was proven via spectral analysis arguments, without detecting a precise decay rate. Here, we exploit a technique first introduced in [9, 15], namely, and we obtain the decay estimate for a suitably defined perturbation of the energy functional $\mathcal{E}:\mathbb{R}^+ \to \mathbb{R}$, defined by $\mathcal{E}(t) = \|S_{\sigma,\tau,\varepsilon}(t)\|^2_{\mathcal{H}^0_{\sigma,\tau,\varepsilon}}$. We can thus provide a decay rate which shows the role played by the kernels a and h.

The main result of this section is the following theorem.

Theorem 3.1. *Assume that* (2.1)–(2.5) *and* (2.9), (2.10) *hold. Then there exist* $\Theta > 0$, $d_0 > 0$ *and* $\varsigma > 0$, *independent of* σ, τ *and* ε, *such that for any* $\tau \in [0,1]$

$$\mathcal{E}(t) \leqslant \varsigma\mathcal{E}(0)\mathrm{e}^{-(\phi(\tau)+d_0)\Theta t}, \quad \forall t \geqslant 0. \tag{3.1}$$

Remark 3.2. A careful analysis of the proof of Theorem 3.1 shows that the constant d_0 and Θ can be explicitly calculated. In particular, d_0 accounts for the viscoelastic effects.

Proof of Theorem 3.1. Let $\tau \in [0,1]$ and let $0 < \rho_\flat < \rho_\sharp < 1$ to be chosen later. Then, for all $t \geqslant 0$, consider the following perturbation $\mathcal{F}_1$ of the energy functional $\mathcal{E}$

$$\mathcal{F}_1(t) = \mathcal{E}(t) + \rho_\flat\Theta_\flat(t) + \rho_\sharp\Theta_\sharp(t), \quad \Theta_\flat(t) = \langle u_t(t), u(t)\rangle,\ \Theta_\sharp(t) = -\sigma\langle u_t(t), \xi^t\rangle_{\mathcal{Q}^{-1}_\sigma}.$$

We denote by C a generic positive constant independent of $\rho_\flat, \rho_\sharp$ and $\sigma, \tau, \varepsilon$ which may vary from line to line even within the same formula. Observe that, by (2.8), there holds

$$\left|\Theta_\flat(t)\right| + \left|\Theta_\sharp(t)\right| \leqslant C\left(\left\|Au(t)\right\|^2 + \left\|u_t(t)\right\|^2 + \left\|\xi^t\right\|^2_{\mathcal{Q}^1_\sigma}\right) \leqslant C\mathcal{E}(t).$$

Therefore, up to choosing $\rho_\flat$ and $\rho_\sharp$ sufficiently small, we have $\frac{1}{2}\mathcal{F}_1(t) \leqslant \mathcal{E}(t) \leqslant 2\mathcal{F}_1(t)$, so that $\mathcal{E}$ and $\mathcal{F}_1$ turn out to be equivalent for what concerns the energy decay estimate. Let us now multiply the first equation of $(\mathcal{P}_{\sigma,\tau,\varepsilon})$ by u_t in H^0, the second by ϑ in H^0, the third by η in $\mathcal{M}^0_{\tau,\varepsilon}$, the fourth by ξ in $\mathcal{Q}^1_\sigma$ and add the resulting identities. This yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) \leqslant &-\frac{\delta_1}{\varepsilon}\left\|\eta^t\right\|^2_{L^2_{\mu_\varepsilon}(\mathbb{R}^+,H^1)} - \delta_3\left\|\eta^t\right\|^2_{L^2_{\nu_\tau}(\mathbb{R}^+,H^0)} - \frac{\delta_2}{2\sigma}\left\|\xi^t\right\|^2_{\mathcal{Q}^1_\sigma}\\ &+ \frac{1}{2}\int_0^\infty \beta'_\sigma(s)\left\|A\xi^t(s)\right\|^2\,\mathrm{d}s - 2\phi(\tau)\left\|\vartheta(t)\right\|^2,\end{aligned}$$

by virtue of inequalities (2.4), (2.5), (2.10), (2.12), (2.13) and integration by parts. Besides, by direct computation, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Theta_\flat(t) &= \|u_t(t)\|^2 - \|Au(t)\|^2 + \langle\vartheta(t), Au(t)\rangle - \langle\xi^t, u(t)\rangle_{\mathcal{Q}^1_\sigma},\\ \frac{\mathrm{d}}{\mathrm{d}t}\Theta_\sharp(t) &= -\sigma\left\langle u_{tt}(t), \xi^t\right\rangle_{\mathcal{Q}^{-1}_\sigma} - \sigma\left\langle u_t(t), T_\sigma\xi^t\right\rangle_{\mathcal{Q}^{-1}_\sigma} - \left\|u_t(t)\right\|^2,\end{aligned}$$

where, in the last identity, we have used formula (2.8) once again. Then, on account of the obtained formulas for the derivatives of $\mathcal{E}$, $\Theta_\flat$ and $\Theta_\sharp$, we deduce

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_1(t) \leqslant &-\min\left\{\frac{\delta_1}{\varepsilon}, \delta_3\right\}\left\|\eta^t\right\|^2_{\mathcal{M}^0_{\tau,\varepsilon}} - \frac{\delta_2}{2\sigma}\left\|\xi^t\right\|^2_{\mathcal{Q}^1_\sigma} + \frac{1}{2}\int_0^\infty \beta'_\sigma(s)\left\|A\xi^t(s)\right\|^2\,\mathrm{d}s\\ &- 2\phi(\tau)\left\|\vartheta(t)\right\| + \rho_\flat\left\|u_t(t)\right\|^2 - \rho_\flat\left\|Au(t)\right\|^2 + \rho_\flat\langle\vartheta(t), Au(t)\rangle\\ &- \rho_\flat\left\langle\xi^t, u(t)\right\rangle_{\mathcal{Q}^1_\sigma} - \rho_\sharp\sigma\left\langle u_{tt}(t), \xi^t\right\rangle_{\mathcal{Q}^{-1}_\sigma} - \rho_\sharp\sigma\left\langle u_t(t), T_\sigma\xi^t\right\rangle_{\mathcal{Q}^{-1}_\sigma} - \rho_\sharp\left\|u_t(t)\right\|^2.\end{aligned}$$

Therefore, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_1(t) \leqslant &-\rho_\flat\left\|Au(t)\right\|^2 - (\rho_\sharp - \rho_\flat)\left\|u_t(t)\right\|^2 - 2\phi(\tau)\left\|\vartheta(t)\right\|^2\\ &- \min\left\{\frac{\delta_1}{\varepsilon}, \delta_3\right\}\left\|\eta^t\right\|^2_{\mathcal{M}^0_{\tau,\varepsilon}} - \frac{\delta_2}{2\sigma}\left\|\xi^t\right\|^2_{\mathcal{Q}^1_\sigma} + \frac{1}{2}\int_0^\infty \beta'_\sigma(s)\left\|A\xi^t(s)\right\|^2\,\mathrm{d}s + \mathcal{J}(t),\end{aligned}$$

where we have set

$$\mathcal{J}(t) = \rho_\flat\left\langle\vartheta(t), Au(t)\right\rangle - \rho_\flat\left\langle u(t), \xi^t\right\rangle_{\mathcal{Q}^1_\sigma} + \rho_\sharp\sigma\left\langle u_t(t), \xi^t_s\right\rangle_{\mathcal{Q}^{-1}_\sigma} - \rho_\sharp\sigma\left\langle u_{tt}(t), \xi^t\right\rangle_{\mathcal{Q}^{-1}_\sigma}.$$

Notice that, we have

$$
\begin{aligned}
&\langle \vartheta(t), Au(t)\rangle \leqslant \|\vartheta(t)\|^2 + \frac{1}{4}\|Au(t)\|^2,\\
&-\langle u(t), \xi^t\rangle_{\mathcal{Q}^1_\sigma} \leqslant C\rho_\flat \|Au(t)\|^2 + \frac{\delta_2}{8\rho_\flat\sigma}\|\xi^t\|^2_{\mathcal{Q}^1_\sigma},\\
&\sigma\langle u_t(t), \xi^t_s\rangle_{\mathcal{Q}^{-1}_\sigma} \leqslant C\rho_\sharp \|u_t(t)\|^2 - \frac{1}{2\rho_\sharp}\int_0^\infty \beta'_\sigma(s)\|A\xi^t(s)\|^2\,\mathrm{d}s,\\
&-\sigma\langle u_{tt}(t), \xi^t\rangle_{\mathcal{Q}^{-1}_\sigma} = \sigma\langle u(t), \xi^t\rangle_{\mathcal{Q}^1_\sigma} - \sigma\langle \vartheta(t), \xi^t\rangle_{\mathcal{Q}^0_\sigma} + \sigma\left\|\int_0^\infty \beta_\sigma(s)A\xi^t(s)\,\mathrm{d}s\right\|^2\\
&\qquad\qquad \leqslant C\rho_\sharp\|Au(t)\|^2 + C\rho_\sharp\|\vartheta(t)\|^2 + \frac{\delta_2}{8\rho_\sharp\sigma}\|\xi^t\|^2_{\mathcal{Q}^1_\sigma} + C\|\xi^t\|^2_{\mathcal{Q}^1_\sigma}.
\end{aligned}
$$

By the above inequalities, it follows

$$
\begin{aligned}
\mathcal{J}(t) \leqslant{}& \left(\frac{\rho_\flat}{4} + C\rho_\flat^2 + C\rho_\sharp^2\right)\|Au(t)\|^2 + C\rho_\sharp^2\|u_t(t)\|^2 + (\rho_\flat + C\rho_\sharp^2)\|\vartheta(t)\|^2\\
&+ \left(\frac{\delta_2}{4\sigma} + C\rho_\sharp\right)\|\xi^t\|^2_{\mathcal{Q}^1_\sigma} - \frac{1}{2}\int_0^\infty \beta'_\sigma(s)\|A\xi^t(s)\|^2\,\mathrm{d}s.
\end{aligned}
$$

Therefore, we conclude that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_1(t) + \left(\frac{3\rho_\flat}{4} - C\rho_\flat^2 - C\rho_\sharp^2\right)\|Au(t)\|^2\\
&\quad + (\rho_\sharp - \rho_\flat - C\rho_\sharp^2)\|u_t(t)\|^2 + (2\phi(\tau) - \rho_\flat - C\rho_\sharp^2)\|\vartheta(t)\|^2\\
&\quad + \min\left\{\frac{\delta_1}{\varepsilon}, \delta_3\right\}\|\eta^t\|^2_{\mathcal{M}^0_\varepsilon} + \frac{\delta_2 - C\rho_\sharp\sigma}{4\sigma}\|\xi^t\|^2_{\mathcal{Q}^1_\sigma} \leqslant 0.
\end{aligned}
$$

Choosing $\rho_\flat = \bar\rho_\flat\phi(\tau)$ and $\rho_\sharp = \bar\rho_\sharp\phi(\tau)$, where the positive constants $\bar\rho_\flat$ and $\bar\rho_\sharp$ are independent of σ, τ and ε, we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_1(t) + \phi(\tau)\left(\frac{3\bar\rho_\flat}{4} - C\bar\rho_\flat^2 - C\bar\rho_\sharp^2\right)\|Au(t)\|^2\\
&\quad + \phi(\tau)(\bar\rho_\sharp - \bar\rho_\flat - C\bar\rho_\sharp^2)\|u_t(t)\|^2 + \phi(\tau)(2 - \bar\rho_\flat - C\bar\rho_\sharp^2)\|\vartheta(t)\|^2\\
&\quad + \phi(\tau)\min\{\delta_1, \delta_3\}C\|\eta^t\|^2_{\mathcal{M}^0_{\tau,\varepsilon}} + \phi(\tau)\frac{\delta_2 - C\bar\rho_\sharp}{4}C\|\xi^t\|^2_{\mathcal{Q}^1_\sigma} \leqslant 0.
\end{aligned}
$$

Then, fixing $\bar\rho_\flat$ and $\bar\rho_\sharp$ so small that

$$
\Lambda = \min\left\{\frac{3\bar\rho_\flat}{4} - C\bar\rho_\flat^2 - C\bar\rho_\sharp^2, \bar\rho_\sharp - \bar\rho_\flat - C\bar\rho_\sharp^2, 2 - \bar\rho_\flat - C\bar\rho_\sharp^2, \min\{\delta_1, \delta_3\}C, \frac{\delta_2 - C\bar\rho_\sharp}{4}C\right\} > 0,
$$

and $\mathcal{E}$ controls and it is controlled by $\mathcal{F}_1$, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_1(t)+\frac{\Lambda}{2}\phi(\tau)\mathcal{F}_1(t)\leqslant 0,\quad \forall t\geqslant 0. \tag{3.2}$$

By arguing as above, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Theta_\sharp(t)\leqslant -\frac{1}{2}\|u_t\|^2+\frac{1}{32}\|Au\|^2+\frac{1}{16}\|\vartheta\|^2+\frac{C}{\sigma}\int_0^\infty \beta_\sigma(s)\big\|A\xi^t(s)\big\|^2\,\mathrm{d}s. \tag{3.3}$$

Setting now

$$K(t)=-\varepsilon\big\langle\vartheta(t),\eta^t\big\rangle_{\mathcal{M}^{-1}_{0,\varepsilon}},$$

multiplying the second equation of $(\mathcal{P}_{\sigma,\tau,\varepsilon})$ by $\int_0^\infty \mu_\varepsilon(s)\eta^t(s)\,\mathrm{d}s$ and recalling (2.7), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}K(t)=\varepsilon&\left\|\int_0^\infty \mu_\varepsilon(s)A^{1/2}\eta^t(s)\,\mathrm{d}s\right\|^2+\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon\left\langle Au,\int_0^\infty \mu_\varepsilon(s)\eta^t(s)\,\mathrm{d}s\right\rangle\\ &-\varepsilon\left\langle Au,\int_0^\infty \mu_\varepsilon(s)\eta^t_t(s)\,\mathrm{d}s\right\rangle+\varepsilon\left\langle\int_0^\infty \nu_\tau(s)\eta^t(s)\,\mathrm{d}s,\int_0^\infty \mu_\varepsilon(s)\eta^t(s)\,\mathrm{d}s\right\rangle\\ &-\varepsilon\left\langle\vartheta,\int_0^\infty \mu'_\varepsilon(s)\eta^t(s)\,\mathrm{d}s\right\rangle-\|\vartheta\|^2.\end{aligned}$$

Therefore, setting

$$K_2(t)=K(t)-\varepsilon\left\langle Au,\int_0^\infty \mu_\varepsilon(s)\eta^t(s)\,\mathrm{d}s\right\rangle,$$

we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}K_2(t)=\varepsilon&\left\|\int_0^\infty \mu_\varepsilon(s)A^{1/2}\eta^t(s)\,\mathrm{d}s\right\|^2-\varepsilon\left\langle Au,\int_0^\infty \mu'_\varepsilon(s)\eta^t(s)\,\mathrm{d}s\right\rangle-\langle Au,\vartheta\rangle\\ &+\varepsilon\left\langle\int_0^\infty \nu_\tau(s)\eta^t(s)\,\mathrm{d}s,\int_0^\infty \mu_\varepsilon(s)\eta^t(s)\,\mathrm{d}s\right\rangle-\varepsilon\left\langle\vartheta,\int_0^\infty \mu'_\varepsilon(s)\eta^t(s)\,\mathrm{d}s\right\rangle-\|\vartheta\|^2.\end{aligned}$$

Notice that we get

$$\begin{aligned}-\varepsilon\left\langle\vartheta,\int_0^\infty \mu'_\varepsilon(s)\eta^t(s)\,\mathrm{d}s\right\rangle&\leqslant\frac{1}{2}\big\|\vartheta(t)\big\|^2+\frac{1}{2}\left[\int_0^\infty \frac{-\varepsilon\mu'_\varepsilon(s)}{\mu^{1/2}_\varepsilon(s)}\mu^{1/2}_\varepsilon(s)\big\|\eta^t(s)\big\|\,\mathrm{d}s\right]^2\\ &\leqslant\frac{1}{2}\big\|\vartheta(t)\big\|^2+\frac{1}{2}\int_0^\infty \frac{\varepsilon^2(\mu'_\varepsilon(s))^2}{\mu_\varepsilon(s)}\,\mathrm{d}s\int_0^\infty \mu_\varepsilon(s)\big\|\eta^t(s)\big\|^2\,\mathrm{d}s\\ &\leqslant\frac{1}{2}\big\|\vartheta(t)\big\|^2+\frac{C}{\varepsilon}\int_0^\infty \mu_\varepsilon(s)\big\|\eta^t(s)\big\|^2\,\mathrm{d}s.\end{aligned}$$

Moreover,

$$\varepsilon\Big\langle\int_0^\infty \nu_\tau(s)\eta^t(s)\,\mathrm{d}s, \int_0^\infty \mu_\varepsilon(s)\eta^t(s)\,\mathrm{d}s\Big\rangle$$
$$\leqslant \frac{\varepsilon}{2}\Big\|\int_0^\infty \nu_\tau(s)\eta^t(s)\,\mathrm{d}s\Big\|^2 + \frac{\varepsilon}{2}\Big\|\int_0^\infty \mu_\varepsilon(s)\eta^t(s)\,\mathrm{d}s\Big\|^2$$
$$\leqslant \psi(\tau)\int_0^\infty \nu_\tau(s)\big\|\eta^t(s)\big\|^2\,\mathrm{d}s + \int_0^\infty \mu_\varepsilon(s)\big\|A^{1/2}\eta^t(s)\big\|^2\,\mathrm{d}s.$$

Hence we deduce the following inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}K_2(t) \leqslant -\frac{1}{2}\|\vartheta\|^2 + \frac{C}{\varepsilon}\int_0^\infty \mu_\varepsilon(s)\big\|A^{1/2}\eta^t(s)\big\|^2\,\mathrm{d}s$$
$$+ \psi(\tau)\int_0^\infty \nu_\tau(s)\big\|\eta^t(s)\big\|^2\,\mathrm{d}s + \frac{1}{4}\|Au\|^2 - \langle Au, \vartheta\rangle. \tag{3.4}$$

By multiplying the first equation of $(\mathcal{P}_{\sigma,\tau,\varepsilon})$ by u, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_t, u) \leqslant \|u_t\|^2 + \frac{1}{2\sigma}\int_0^\infty \beta_\sigma(s)\big\|A\xi^t(s)\big\|^2\,\mathrm{d}s - \frac{1}{2}\|Au\|^2 + \langle\vartheta, Au\rangle.$$

Whence, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[K_2(t) + (u_t, u)\big] \leqslant -\frac{1}{4}\|Au\|^2 - \frac{1}{2}\|\vartheta\|^2 + \|u_t\|^2 + C\int_0^\infty \nu_\tau(s)\big\|\eta^t(s)\big\|^2\,\mathrm{d}s$$
$$+ \frac{1}{2\sigma}\int_0^\infty \beta_\sigma(s)\big\|A\xi^t(s)\big\|^2\,\mathrm{d}s + \frac{C}{\varepsilon}\int_0^\infty \mu_\varepsilon(s)\big\|A^{1/2}\eta^t(s)\big\|^2\,\mathrm{d}s.$$

Using now inequality (3.3) and setting $K_3(t) = 4\Theta_\sharp(t) + K_2(t) + (u_t, u)$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}K_3(t) \leqslant -\frac{1}{8}\|Au\|^2 - \frac{1}{4}\|\vartheta\|^2 - \|u_t\|^2 + C\int_0^\infty \nu_\tau(s)\big\|\eta^t(s)\big\|^2\,\mathrm{d}s$$
$$+ \frac{C}{\sigma}\int_0^\infty \beta_\sigma(s)\big\|A\xi^t(s)\big\|^2\,\mathrm{d}s + \frac{C}{\varepsilon}\int_0^\infty \mu_\varepsilon(s)\big\|A^{1/2}\eta^t(s)\big\|^2\,\mathrm{d}s.$$

Since, as can be readily checked, it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) \leqslant -\frac{\delta_1}{\varepsilon}\big\|\eta^t\big\|^2_{L^2_{\mu_\varepsilon}(\mathbb{R}^+,H^1)} - \delta_3\big\|\eta^t\big\|^2_{L^2_{\nu_\tau}(\mathbb{R}^+,H^0)} - \frac{\delta_2}{\sigma}\big\|\xi^t\big\|^2_{\mathcal{Q}^1_\sigma},$$

setting $\mathcal{F}_2(t) = N\mathcal{E}(t) + K_3(t)$ with N sufficiently large and independent of ε and σ, it is readily seen that $\mathcal{F}_2$ controls and it is controlled by the energy and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_2(t) + d_0\mathcal{F}_2(t) \leqslant 0, \quad \forall t \geqslant 0, \tag{3.5}$$

for some positive constant d_0 independent of ε and σ. Therefore, by combining inequalities (3.2) and (3.5) and setting $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$ it follows that $\mathcal{F}$ is equivalent to the energy and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t) + C(\phi(\tau) + d_0)\mathcal{F}(t) \leqslant 0, \quad \forall t \geqslant 0.$$

By the Gronwall Lemma we obtain the desired inequality (3.1).

Remark 3.3. We know that

$$\lim_{t\to\infty} \| S_{\sigma,0,\varepsilon}(t)z \|_{\mathcal{H}_{\sigma,0,\varepsilon}} = 0, \quad \forall z \in \mathcal{H}_{\sigma,0,\varepsilon},$$

provided that μ satisfies a mild summability condition (see Theorem A.1 in the Appendix). If, in addition, we assume that $\beta \equiv 0$ and the memory kernel μ does not grow too rapidly around the origin, i.e. $\sqrt{s}\mu(s) \to 0$ for $s \to 0$, then the first order energy fails to vanish through an exponential law of decay (cf. [5], Theorem 5.4, as well as Theorem A.2 in the Appendix for a more general situation). For the case general $\beta \neq 0$, we refer the reader to the Appendix for a discussion on the lack of exponential stability for a class of abstract linear thermoelastic systems with (possibly) fractional operator powers (cf. Theorem A.3).

4. Closeness between $S_{\sigma,\tau,\varepsilon}(t)$ and $S_{0,0,0}(t)$

The aim of this section is to establish, following a pattern recently initiated in [1], a precise quantitative estimate of the closeness between the *non analytic semigroup* $S_{\sigma,\tau,\varepsilon}(t)$ and the *analytic semigroup* $S_{0,0,0}(t)$ (see [10]) in the norm of any extended phase-space $\mathcal{H}^m_{\sigma,\tau,\varepsilon}$, for $m \geqslant 0$, as the parameters σ, τ and ε converge to zero, provided that the initial data are chosen inside a suitable regular bounded subset of $\mathcal{H}^m_{\sigma,\tau,\varepsilon}$ (see also [2,3]). In [7] a similar analysis was carried on in the case $\sigma = \tau = 0$, for a plate model which accounts for the rotational inertia term $-\Delta u_{tt}$ in the equation ruling the vertical deflection. We point out that, along the convergence process, μ_ε and β_σ behave in a singular fashion since $\|\mu_\varepsilon\|_{L^1(\mathbb{R}^+)} \to \infty$ and $\|\beta_\sigma\|_{L^1(\mathbb{R}^+)} \to \infty$ for ε and σ going to zero, whereas the kernel ν_τ satisfies $\|\nu_\tau\|_{L^1(\mathbb{R}^+)} \to 0$ as τ vanishes.

4.1. Discussion of the results

Throughout the section we will assume that, whenever $\sigma > 0$, $\tau > 0$ and $\varepsilon > 0$, conditions (2.1), (2.2), (2.4), (2.5) and (2.9)–(2.11) hold true. In order to perform a comparison between the five component semigroup $S_{\sigma,\tau,\varepsilon}(t)$ and the three component (for $\sigma = \tau = \varepsilon = 0$) limiting semigroup $S_{0,0,0}(t)$, we need to introduce, for any $m \geqslant 0$, the following lifting and projection maps

$$\mathbb{L}_{\sigma,\tau,\varepsilon} : \mathcal{H}^m_{0,0,0} \to \mathcal{H}^m_{\sigma,\tau,\varepsilon},$$
$$\mathbb{P} : \mathcal{H}^m_{\sigma,\tau,\varepsilon} \to \mathcal{H}^m_{0,0,0},$$
$$\mathbb{Q}_{\tau,\varepsilon} : \mathcal{H}^m_{\sigma,\tau,\varepsilon} \to \mathcal{M}^m_{\tau,\varepsilon},$$
$$\mathbb{Q}_{\sigma} : \mathcal{H}^m_{\sigma,\tau,\varepsilon} \to \mathcal{Q}^{m+1}_{\sigma},$$

defined, respectively, by

$$\mathbb{L}_{\sigma,\tau,\varepsilon}(u,u_t,\vartheta)=\begin{cases}(u,u_t,\vartheta,0,0), & \text{if } \sigma>0 \text{ and } \tau>0 \text{ or } \varepsilon>0,\\ (u,u_t,\vartheta,0), & \text{if } \sigma=0 \text{ or } \tau=\varepsilon=0,\\ (u,u_t,\vartheta), & \text{if } \sigma=\tau=\varepsilon=0,\end{cases}$$

and by

$$\mathbb{P}(u,u_t,\vartheta,\eta,\xi)=(u,u_t,\vartheta),$$
$$\mathbb{Q}_{\tau,\varepsilon}(u,u_t,\vartheta,\eta,\xi)=\eta,$$
$$\mathbb{Q}_{\sigma}(u,u_t,\vartheta,\eta,\xi)=\xi.$$

In the case $\tau>0$, if z denotes the initial data, taken inside any bounded subset of $\mathcal{H}^{2m+4}_{\sigma,\tau,\varepsilon}$, we will prove the convergence of $S_{\sigma,\tau,\varepsilon}(t)z$ towards $\mathbb{L}_{\sigma,\tau,\varepsilon}S_{0,0,0}(t)\mathbb{P}z$ in the $\mathcal{H}^{m}_{\sigma,\tau,\varepsilon}$-norm over any *finite-time* interval of the form $[t_0,T]$ with $t_0>0$ (cf. Theorem 4.5). More precisely, as a by-product of Theorem 4.5, we will prove that

$$\lim_{\substack{\sigma\to0^+\\ \tau\to0^+\\ \varepsilon\to0^+}}\sup_{t\in[t_0,T]}\|S_{\sigma,\tau,\varepsilon}(t)z-\mathbb{L}_{\sigma,\tau,\varepsilon}S_{0,0,0}(t)\mathbb{P}z\|_{\mathcal{H}^{m}_{\sigma,\tau,\varepsilon}}=0,$$

for every $R\geqslant0$, $T>t_0>0$ and $z\in B_{\mathcal{H}^{2m+4}_{\sigma,\tau,\varepsilon}}(R)$.

The first three components of the solution $\mathbb{P}S_{\sigma,\tau,\varepsilon}(t)z$ are shown to converge to $S_{0,0,0}(t)\mathbb{P}z$ in the $\mathcal{H}^{m}_{0,0,0}$-norm on $[0,T]$, whereas the history components η^t and ξ^t vanish on $[t_0,\infty]$ in the $\mathcal{M}^{m}_{\tau,\varepsilon}$-norm and $\mathcal{Q}^{m+1}_{\sigma}$-norm respectively, due to the presence of possibly nonzero initial histories η_0 and ξ_0 (cf. Lemma 4.2).

Besides, in the case $\tau=0$, the singular limit estimate strengthens. Indeed, it turns out to hold on *infinite-time* intervals far away from zero, uniformly with respect to initial data lying inside any ball of $\mathcal{H}^{2m+4}_{\sigma,0,\varepsilon}$, namely we get

$$\lim_{\substack{\sigma\to0^+\\ \varepsilon\to0^+}}\sup_{z\in B_{\mathcal{H}^{2m+4}_{\sigma,0,\varepsilon}}(R)}\sup_{t\geqslant t_0}\|S_{\sigma,0,\varepsilon}(t)z-\mathbb{L}_{\sigma,0,\varepsilon}S_{0,0,0}(t)\mathbb{P}z\|_{\mathcal{H}^{m}_{\sigma,0,\varepsilon}}=0,$$

for every $R\geqslant0$ and $t_0>0$ (cf. Theorem 4.7). Of course, to achieve these results, the role played by the exponential stability of the limiting semigroup $S_{0,0,0}(t)$ will be important.

4.2. *Some preliminary facts*

Before stating the main results of the section, we need a few preliminary results.

Lemma 4.1. *Let $m\geqslant0$, $R\geqslant0$ and $z\in B_{\mathcal{H}^{m}_{\sigma,\tau,\varepsilon}}(R)$. Then there exists $K_R\geqslant0$ such that $\|S_{\sigma,\tau,\varepsilon}(t)z\|_{\mathcal{H}^{m}_{\sigma,\tau,\varepsilon}}\leqslant K_R$ for all $t\geqslant0$.*

Proof. By (2.12) and (2.13), it suffices to multiply the equations of $(\mathcal{P}_{\sigma,\tau,\varepsilon})$ by u_t in H^m, by ϑ in H^m, by η in $\mathcal{M}^{m}_{\tau,\varepsilon}$ and by ξ in $\mathcal{Q}^{m+1}_{\sigma}$, respectively, and add the resulting equations. □

The vanishing of the histories components η^t and ξ^t of $S_{\sigma,\tau,\varepsilon}(t)$ is issued in the following lemma.

Lemma 4.2. *Let $m \geqslant 0$, $R \geqslant 0$ and $z = (u_0, u_1, \vartheta_0, \eta_0, \xi_0) \in B_{\mathcal{H}^m_{\sigma,\tau,\varepsilon}}(R)$. Then there exists $K_R \geqslant 0$ such that the following facts hold:*

(a) *for every $\varepsilon > 0$ and $t \geqslant 0$,*

$$\|\eta^t\|_{L^2_{\mu_\varepsilon}(\mathbb{R}^+,H^{m+1})} \leqslant \|\eta_0\|_{L^2_{\mu_\varepsilon}(\mathbb{R}^+,H^{m+1})}\mathrm{e}^{-\delta_1 t/(4\varepsilon)} + K_R\sqrt{\varepsilon}; \tag{4.1}$$

(b) *for every $\tau > 0$ and $t \geqslant 0$,*

$$\|\eta^t\|_{L^2_{\nu_\tau}(\mathbb{R}^+,H^{m})} \leqslant \|\eta_0\|_{L^2_{\nu_\tau}(\mathbb{R}^+,H^{m})}\mathrm{e}^{-\delta_3 t/4} + K_R\sqrt{\psi(\tau)}; \tag{4.2}$$

(c) *for every $\sigma > 0$ and $t \geqslant 0$,*

$$\|\xi^t\|_{\mathcal{Q}^{m+1}_\sigma} \leqslant \|\xi_0\|_{\mathcal{Q}^{m+1}_\sigma}\mathrm{e}^{-\delta_2 t/(4\sigma)} + K_R\sqrt{\sigma}.$$

Proof. By arguing as in [1], Lemma 5.4, we immediately get assertions (a) and (c). Let C denote a generic positive constant depending on R which may even vary from line to line within the same equation. By multiplying the equation of η by η in $L^2_{\nu_\tau}(\mathbb{R}^+, H^m)$, and taking (2.10), (2.11) and Lemma 4.1 into account, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\|^2_{L^2_{\nu_\tau}(\mathbb{R}^+,H^m)} + \delta_3\|\eta\|^2_{L^2_{\nu_\tau}(\mathbb{R}^+,H^m)} &\leqslant C\int_0^\infty \nu_\tau(s)\|A^{m/2}\eta(s)\|\,\mathrm{d}s\\ &\leqslant C\Big(\int_0^\infty \nu_\tau(s)\,\mathrm{d}s\Big)^{1/2}\Big(\int_0^\infty \nu_\tau(s)\|A^{m/2}\eta(s)\|^2\,\mathrm{d}s\Big)^{1/2}\\ &\leqslant C\sqrt{\psi(\tau)}\|\eta\|_{L^2_{\nu_\tau}(\mathbb{R}^+,H^m)} \leqslant \frac{\delta_3}{2}\|\eta\|^2_{L^2_{\nu_\tau}(\mathbb{R}^+,H^m)} + C\psi(\tau),\end{aligned}$$

so that, by the Gronwall lemma, we immediately obtain (b). □

Definition 4.3. For every $m \geqslant 0$, $\eta_0 \in \mathcal{M}^m_{\tau,\varepsilon}$ and $\xi_0 \in \mathcal{Q}^{m+1}_\sigma$, let us set for every $t \geqslant 0$

$$\varUpsilon^m_{\sigma,\tau,\varepsilon}(t) = \|\eta_0\|_{L^2_{\mu_\varepsilon}(\mathbb{R}^+,H^{m+1})}\mathrm{e}^{-\delta_1 t/(4\varepsilon)} + \|\eta_0\|_{L^2_{\nu_\tau}(\mathbb{R}^+,H^{m})}\mathrm{e}^{-\delta_3 t/4} + \|\xi_0\|_{\mathcal{Q}^{m+1}_\sigma}\mathrm{e}^{-\delta_2 t/(4\sigma)}. \tag{4.3}$$

Furthermore, we introduce the maps $\varPi_\flat : [0,1]^3 \to \mathbb{R}^+$ and $\varPi_\sharp : [0,1] \to \mathbb{R}^+$,

$$\begin{aligned}&\varPi_\flat(\sigma,\tau,\varepsilon) = \sqrt[4]{\varepsilon} + \sqrt[4]{\sigma} + \sqrt[4]{\psi(\tau)},\\ &\varPi_\sharp(\tau) = \sqrt{\psi(\tau)} + \sqrt{\phi(\tau)}.\end{aligned}$$

Observe that $\varPi_\flat$ and $\varPi_\sharp$ are continuous with $\varPi_\flat(0,0,0) = \varPi_\sharp(0) = 0$.

Proposition 4.4. *For every $m \geqslant 0$, $R \geqslant 0$, $\eta_0 \in B_{\mathcal{M}^m_{\tau,\varepsilon}}(R)$, $\xi_0 \in B_{\mathcal{Q}^{m+1}_\sigma}(R)$ and $t_0 > 0$*

$$\lim_{\substack{\sigma\to 0^+\\ \tau\to 0^+\\ \varepsilon\to 0^+}} \sup_{t\geqslant t_0} \varUpsilon^m_{\sigma,\tau,\varepsilon}(t) = 0.$$

Proof. The first and third summands of $\varUpsilon^m_{\sigma,\tau,\varepsilon}$ vanish exponentially. Moreover, observe that, by (2.11) and since $\{\tau \mapsto \nu_\tau\}$ is increasing, $\|\eta_0\|_{L^2_{\nu_\tau}(\mathbb{R}^+,H^m)}$ converges to zero by the Monotone Convergence Theorem. □

4.3. Case $\tau > 0$: the convergence estimate

We are now ready to state the main result of the section, which gives a convergence estimate of $S_{\sigma,\tau,\varepsilon}(t)$ towards $S_{0,0,0}(t)$ in the norm of $\mathcal{H}^m_{\sigma,\tau,\varepsilon}$, for any $m \geqslant 0$, over finite-time intervals.

Theorem 4.5. *For every $m \geqslant 0$, $R \geqslant 0$, $T > 0$ and $z \in B_{\mathcal{H}^{2m+4}_{\sigma,\tau,\varepsilon}}(R)$, there exist two constants $K_R \geqslant 0$ and $Q_{R,T} \geqslant 0$ such that*

$$\|S_{\sigma,\tau,\varepsilon}(t)z - \mathbb{L}_{\sigma,\tau,\varepsilon}S_{0,0,0}(t)\mathbb{P}z\|_{\mathcal{H}^m_{\sigma,\tau,\varepsilon}} \leqslant \varUpsilon^m_{\sigma,\tau,\varepsilon}(t) + K_R\varPi_\flat(\sigma,\tau,\varepsilon) + Q_{R,T}\varPi_\sharp(\tau),$$

for every $t \in [0,T]$.

Remark 4.6. The regularity assumption on the initial data z could be relaxed to get a rougher convergence estimate on finite-time intervals. On the other hand, the price one has to pay is that also the constant K_R which appears in the above theorem would depend on the time interval. With the higher regularity that we require, instead, we are able to exploit the exponential stability of the limiting semigroup $S_{0,0,0}(t)$ and to have, at least in the case $\tau = 0$, the convergence estimate holding uniformly in time. So, for $m = 0$, we get a convergence estimate for the thermoviscoelastic model starting with initial data having *four* levels of regularity above the regularity of the base phase-space. In the thermoelastic plate model considered in [7] (essentially, w.r.t. our notation, the case when $\sigma = \tau = 0$) one needs to go just *two* levels of regularity above. Finally, in the case of single parabolic and hyperbolic equations with memory (cf. [1,2]) it suffices to require *one* level of regularity above the basic regularity to get a time dependent control.

Proof of Theorem 4.5. Let $m \geqslant 0$, $R \geqslant 0$ and $z = (u_0, u_1, \vartheta_0, \eta_0, \xi_0) \in B_{\mathcal{H}^{2m+4}_{\sigma,\tau,\varepsilon}}(R)$. Since

$$S_{\sigma,\tau,\varepsilon}(t)z = \big(\mathbb{P}S_{\sigma,\tau,\varepsilon}(t)z, \mathbb{Q}_{\tau,\varepsilon}S_{\sigma,\tau,\varepsilon}(t)z, \mathbb{Q}_\sigma S_{\sigma,\tau,\varepsilon}(t)z\big), \quad t \geqslant 0,$$

we get the assertion if we prove that, for $T > 0$, there exist $K_R \geqslant 0$ and $Q_{R,T} \geqslant 0$ with

$$\|\mathbb{P}S_{\sigma,\tau,\varepsilon}(t)z - S_{0,0,0}(t)\mathbb{P}z\|_{\mathcal{H}^m_{0,0,0}} \leqslant K_R\varPi_\flat(\sigma,\tau,\varepsilon) + Q_{R,T}\varPi_\sharp(\tau), \tag{4.4}$$

$$\|\mathbb{Q}_{\tau,\varepsilon}S_{\sigma,\tau,\varepsilon}(t)z\|_{\mathcal{M}^m_{\tau,\varepsilon}} + \|\mathbb{Q}_\sigma S_{\sigma,\tau,\varepsilon}(t)z\|_{\mathcal{Q}^{m+1}_\sigma} \leqslant \varUpsilon^m_{\sigma,\tau,\varepsilon}(t) + K_R\big(\sqrt{\varepsilon} + \sqrt{\sigma} + \sqrt{\psi(\tau)}\big), \tag{4.5}$$

for every $t \in [0,T]$. By combining inequalities (a)–(c) of Lemma 4.2, we immediately obtain (4.5). Then, we turn to the proof of inequality (4.4). Let us set

$$\begin{aligned}
&\bar{u}(t) = \hat{u}(t) - u(t),\\
&\bar{u}_t(t) = \hat{u}_t(t) - u_t(t),\\
&\bar{\vartheta}(t) = \hat{\vartheta}(t) - \vartheta(t),\\
&\bar{\eta}^t = \hat{\eta}^t - \eta^t,\\
&\bar{\xi}^t = \hat{\xi}^t - \xi^t,
\end{aligned}$$

where $(\hat{u}, \hat{u}_t, \hat{\vartheta}, \hat{\eta}, \hat{\xi})$ denotes the solution to $(\mathcal{P}_{\sigma,\tau,\varepsilon})$ with initial data z, while (u, u_t, ϑ) stands for the solution to $\mathcal{P}_{0,0,0}$ with initial data $\mathbb{P}z$. Besides, η^t (resp. ξ^t) denotes the solution at time t of the Cauchy problem in $\mathcal{M}^0_{\tau,\varepsilon}$ (resp. $\mathcal{Q}^1_\sigma$)

$$\begin{cases} \eta_t = T_{\tau,\varepsilon}\eta + \vartheta, & t > 0,\\ \eta^0 = \eta_0, \end{cases} \qquad \begin{cases} \xi_t = T_\sigma \xi + u_t, & t > 0,\\ \xi^0 = \xi_0. \end{cases}$$

These problems reconstruct the missing components of the limiting semigroup $S_{0,0,0}(t)$ which are needed in order to perform the comparison argument (cf. [1,2]). Then, it can be readily checked that $(\bar{u}, \bar{u}_t, \bar{\vartheta}, \bar{\eta}, \bar{\xi})$ solves the system

$$\begin{cases}
\bar{u}_{tt} + A^2\bar{u} + \displaystyle\int_0^\infty \beta_\sigma(s)A^2\hat{\xi}(s)\,\mathrm{d}s - A^2u_t - A\bar{\vartheta} = 0,\\
\bar{\vartheta}_t + \phi(\tau)\bar{\vartheta} + \phi(\tau)\vartheta + \displaystyle\int_0^\infty \nu_\tau(s)\hat{\eta}(s)\,\mathrm{d}s + \int_0^\infty \mu_\varepsilon(s)A\hat{\eta}(s)\,\mathrm{d}s - A\vartheta + A\bar{u}_t = 0,\\
\bar{\eta}_t = T_{\tau,\varepsilon}\bar{\eta} + \bar{\vartheta},\\
\bar{\xi}_t = T_\sigma\bar{\xi} + \bar{u}_t,\\
\big(\bar{u}(0), \bar{u}_t(0), \bar{\vartheta}(0), \bar{\eta}^0, \bar{\xi}^0\big) = (0,0,0,0,0).
\end{cases}$$

By multiplying the first equation by $\bar{u}_t$ in H^m, the second by $\bar{\vartheta}$ in H^m, the third by $\bar{\eta}$ in $\mathcal{M}^m_{\tau,\varepsilon}$ and the fourth by $\bar{\xi}$ in $\mathcal{Q}^{m+1}_\sigma$ we obtain, respectively,

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\big\|A^{(m+2)/2}\bar{u}\big\|^2 + \big\|A^{m/2}\bar{u}_t\big\|^2\big) + \langle\hat{\xi}, \bar{u}_t\rangle_{\mathcal{Q}^{m+1}_\sigma}\\
&\quad - \big\langle A^{(m+2)/2}u_t, A^{(m+2)/2}\bar{u}_t\big\rangle - \big\langle A^{(m+1)/2}\bar{\vartheta}, A^{(m+1)/2}\bar{u}_t\big\rangle = 0,\\
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|A^{m/2}\bar{\vartheta}\big\|^2 + \phi(\tau)\big\|A^{m/2}\bar{\vartheta}\big\|^2 + \phi(\tau)\big\langle A^{m/2}\vartheta, A^{m/2}\bar{\vartheta}\big\rangle\\
&\quad + \langle\hat{\eta}, \bar{\vartheta}\rangle_{\mathcal{M}^m_{\tau,\varepsilon}} - \big\langle A^{(m+1)/2}\vartheta, A^{(m+1)/2}\bar{\vartheta}\big\rangle + \big\langle A^{(m+1)/2}\bar{u}_t, A^{(m+1)/2}\bar{\vartheta}\big\rangle = 0,\\
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{\eta}\|^2_{\mathcal{M}^m_{\tau,\varepsilon}} - \langle T_{\tau,\varepsilon}\bar{\eta}, \bar{\eta}\rangle_{\mathcal{M}^m_{\tau,\varepsilon}} - \langle\bar{\eta}, \bar{\vartheta}\rangle_{\mathcal{M}^m_{\tau,\varepsilon}} = 0,\\
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{\xi}\|^2_{\mathcal{Q}^{m+1}_\sigma} - \langle T_\sigma\bar{\xi}, \bar{\xi}\rangle_{\mathcal{Q}^{m+1}_\sigma} - \langle\bar{\xi}, \bar{u}_t\rangle_{\mathcal{Q}^{m+1}_\sigma} = 0.
\end{aligned}$$

Taking (2.12) and (2.13) into account, and adding the above identities, we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\|A^{(m+2)/2}\bar{u}\|^2 + \|A^{m/2}\bar{u}_t\|^2 + \|A^{m/2}\bar{\vartheta}\|^2 + \|\bar{\eta}\|^2_{\mathcal{M}^m_{\tau,\varepsilon}} + \|\bar{\xi}\|^2_{\mathcal{Q}^{m+1}_\sigma}\big) \leqslant 2I_\varepsilon + 2J_\sigma + 2K_\tau,$$

where we have set

$$I_\varepsilon(t) = -\int_0^\infty \mu_\varepsilon(s)\big\langle A^{(m+1)/2}\eta^t(s), A^{(m+1)/2}\bar{\vartheta}(t)\big\rangle\,\mathrm{d}s + \big\langle A^{(m+1)/2}\vartheta(t), A^{(m+1)/2}\bar{\vartheta}(t)\big\rangle,$$

$$J_\sigma(t) = -\int_0^\infty \beta_\sigma(s)\big\langle A^{(m+2)/2}\xi^t(s), A^{(m+2)/2}\bar{u}_t(t)\big\rangle\,\mathrm{d}s + \big\langle A^{(m+2)/2}u_t(t), A^{(m+2)/2}\bar{u}_t(t)\big\rangle,$$

$$K_\tau(t) = -\int_0^\infty \nu_\tau(s)\big\langle A^{m/2}\eta^t(s), A^{m/2}\bar{\vartheta}(t)\big\rangle\,\mathrm{d}s - \phi(\tau)\big\langle A^{m/2}\vartheta(t), A^{m/2}\bar{\vartheta}(t)\big\rangle.$$

We stress, for later use, that the above term $K_\tau(t)$ appears under the assumption that $\tau > 0$, whereas we would simply have $K_0(t) = 0$ for all $t \geqslant 0$ in the case $\tau = 0$, since $\nu_0 = 0$ and $\phi(0) = 0$. We shall write, for all $t \geqslant 0$, $I_\varepsilon(t) = \sum_{j=1}^5 I_j(t)$ and $J_\sigma(t) = \sum_{j=1}^5 J_j(t)$, being the I_js and the J_js defined, respectively, by

$$I_1(t) = \int_{\sqrt{\varepsilon}}^\infty s\mu_\varepsilon(s)\big\langle A^{(m+1)/2}\vartheta(t), A^{(m+1)/2}\bar{\vartheta}(t)\big\rangle\,\mathrm{d}s,$$

$$I_2(t) = -\int_{\sqrt{\varepsilon}}^\infty \mu_\varepsilon(s)\big\langle A^{(m+1)/2}\eta^t(s), A^{(m+1)/2}\bar{\vartheta}(t)\big\rangle\,\mathrm{d}s,$$

$$I_3(t) = -\int_{\min\{\sqrt{\varepsilon},t\}}^{\sqrt{\varepsilon}} \mu_\varepsilon(s)\big\langle A^{(m+1)/2}\eta_0(s-t), A^{(m+1)/2}\bar{\vartheta}(t)\big\rangle\,\mathrm{d}s,$$

$$I_4(t) = \int_{\min\{\sqrt{\varepsilon},t\}}^{\sqrt{\varepsilon}} (s-t)\mu_\varepsilon(s)\big\langle A^{(m+1)/2}\vartheta(t), A^{(m+1)/2}\bar{\vartheta}(t)\big\rangle\,\mathrm{d}s,$$

$$I_5(t) = \int_0^{\sqrt{\varepsilon}} \mu_\varepsilon(s)\bigg[\int_0^{\min\{s,t\}} \big\langle A^{(m+1)/2}\big(\vartheta(t)-\vartheta(t-y)\big), A^{(m+1)/2}\bar{\vartheta}(t)\big\rangle\,\mathrm{d}y\bigg]\,\mathrm{d}s,$$

and

$$J_1(t) = \int_{\sqrt{\sigma}}^\infty s\beta_\sigma(s)\big\langle A^{(m+2)/2}u_t(t), A^{(m+2)/2}\bar{u}_t(t)\big\rangle\,\mathrm{d}s,$$

$$J_2(t) = -\int_{\sqrt{\sigma}}^\infty \beta_\sigma(s)\big\langle A^{(m+2)/2}\xi^t(s), A^{(m+2)/2}\bar{u}_t(t)\big\rangle\,\mathrm{d}s,$$

$$J_3(t) = -\int_{\min\{\sqrt{\sigma},t\}}^{\sqrt{\sigma}} \beta_\sigma(s)\big\langle A^{(m+2)/2}\xi_0(s-t), A^{(m+2)/2}\bar{u}_t(t)\big\rangle\,\mathrm{d}s,$$

$$J_4(t) = \int_{\min\{\sqrt{\sigma},t\}}^{\sqrt{\sigma}} (s-t)\beta_\sigma(s)\big\langle A^{(m+2)/2}u_t(t), A^{(m+2)/2}\bar{u}_t(t)\big\rangle\,\mathrm{d}s,$$

$$J_5(t) = \int_0^{\sqrt{\sigma}} \beta_\sigma(s)\bigg[\int_0^{\min\{s,t\}} \big\langle A^{(m+2)/2}\big(u_t(t)-u_t(t-y)\big), A^{(m+2)/2}\bar{u}_t(t)\big\rangle\,\mathrm{d}y\bigg]\,\mathrm{d}s.$$

In the following, we shall denote by $C \geqslant 0$ a generic constant which may even vary from line to line and may depend on R, but it is independent of σ, τ and ε. By virtue of Lemma 4.1, we have $\|S_{\sigma,\tau,\varepsilon}(t)z\|_{\mathcal{H}^{2m+4}_{\sigma,\tau,\varepsilon}} \leqslant C$ for all $t \geqslant 0$. In particular,

$$\|A^{m+3}\hat{u}(t)\| + \|A^{m+2}\hat{u}_t(t)\| + \|A^{m+2}\hat{\vartheta}(t)\| + \|\hat{\eta}^t\|_{\mathcal{M}^{2m+4}_{\tau,\varepsilon}} + \|\hat{\xi}^t\|_{\mathcal{Q}^{2m+5}_{\sigma}} \leqslant C, \tag{4.6}$$

$$\|A^{m+3}u(t)\| + \|A^{m+2}u_t(t)\| + \|A^{m+2}\vartheta(t)\| + \|\eta^t\|_{\mathcal{M}^{2m+4}_{\tau,\varepsilon}} + \|\xi^t\|_{\mathcal{Q}^{2m+5}_{\sigma}} \leqslant C, \tag{4.7}$$

for all $t \geqslant 0$. Furthermore, since $S_{0,0,0}(t)$ is exponentially stable, there exists $\varpi > 0$ with

$$\|Au(t)\| + \|u_t(t)\| + \|\vartheta(t)\| \leqslant C\mathrm{e}^{-\varpi t}, \quad \forall t \geqslant 0. \tag{4.8}$$

Concerning the treatment of the terms I_js and J_js, we will proceed on the line of [1,2] but strengthening the estimates, whenever it is possible, through the first order energy exponential decay furnished by (4.8). Observe first that, due to (2.4)–(2.6),

$$\int_{\sqrt{\varepsilon}}^{\infty} s\mu_\varepsilon(s)\,\mathrm{d}s \leqslant C\varepsilon, \quad \forall \varepsilon > 0, \qquad \int_{\sqrt{\sigma}}^{\infty} s\beta_\sigma(s)\,\mathrm{d}s \leqslant C\sigma, \quad \forall \sigma > 0. \tag{4.9}$$

Hence, by (4.6)–(4.9), we immediately get

$$I_1(t) = \int_{\sqrt{\varepsilon}}^{\infty} s\mu_\varepsilon(s)\langle \vartheta(t), A^{m+1}\bar{\vartheta}(t)\rangle\,\mathrm{d}s \leqslant C\varepsilon\|A^{m+1}\bar{\vartheta}(t)\|\|\vartheta(t)\| \leqslant C\varepsilon\mathrm{e}^{-\varpi t}, \quad \forall t \geqslant 0,$$

$$J_1(t) = \int_{\sqrt{\sigma}}^{\infty} s\beta_\sigma(s)\langle u_t(t), A^{m+2}\bar{u}_t(t)\rangle\,\mathrm{d}s \leqslant C\sigma\|A^{m+2}\bar{u}_t(t)\|\|u_t(t)\| \leqslant C\sigma\mathrm{e}^{-\varpi t}, \quad \forall t \geqslant 0.$$

Let us now prove that there holds

$$\|\eta^t\|^2_{L^2_{\mu_\varepsilon}(\mathbb{R}^+,H^{m+1})} \leqslant C\mathrm{e}^{-\delta_1 t/\varepsilon} + C\sqrt{\varepsilon}\mathrm{e}^{-\varpi t}, \quad \forall t \geqslant 0, \tag{4.10}$$

$$\|\xi^t\|^2_{\mathcal{Q}^{m+1}_{\sigma}} \leqslant C\mathrm{e}^{-\delta_2 t/\sigma} + C\sqrt{\sigma}\mathrm{e}^{-\varpi t}, \quad \forall t \geqslant 0, \tag{4.11}$$

Indeed, arguing as in [1], Lemma 5.4, we readily obtain

$$\|\eta^t\|_{L^2_{\mu_\varepsilon}(\mathbb{R}^+,H^{2m+2})} \leqslant C\mathrm{e}^{-\delta_1 t/(4\varepsilon)} + C\sqrt{\varepsilon}, \quad \forall t \geqslant 0,$$

$$\|\xi^t\|_{\mathcal{Q}^{2m+3}_{\sigma}} \leqslant C\mathrm{e}^{-\delta_2 t/(4\sigma)} + C\sqrt{\sigma}, \quad \forall t \geqslant 0.$$

Whence, by multiplying the equation for η times η in $L^2_{\mu_\varepsilon}(\mathbb{R}^+, H^{m+1})$, in light of (4.8),

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\eta^t\|^2_{L^2_{\mu_\varepsilon}(\mathbb{R}^+,H^{m+1})} + \frac{\delta_1}{\varepsilon}\|\eta^t\|^2_{L^2_{\mu_\varepsilon}(\mathbb{R}^+,H^{m+1})} \\
&\quad\leqslant 2\|\vartheta(t)\|\int_0^\infty \mu_\varepsilon(s)\|A^{m+1}\eta^t(s)\|\,\mathrm{d}s \\
&\quad\leqslant \frac{2}{\sqrt{\varepsilon}}\|\vartheta(t)\|\|\eta^t\|^2_{L^2_{\mu_\varepsilon}(\mathbb{R}^+,H^{2m+2})} \leqslant \frac{C}{\sqrt{\varepsilon}}\mathrm{e}^{-(\varpi+\delta_1/(4\varepsilon))t} + C\mathrm{e}^{-\varpi t},
\end{aligned}$$

which yields (4.10) via the Gronwall Lemma. In a similar fashion, again by (4.8), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\xi^t\|^2_{\mathcal{Q}^{m+1}_\sigma} + \frac{\delta_2}{\sigma}\|\xi^t\|^2_{\mathcal{Q}^{m+1}_\sigma} \leqslant 2\|u_t(t)\| \int_0^\infty \beta_\sigma(s)\|A^{m+2}\xi^t(s)\|\,\mathrm{d}s$$
$$\leqslant \frac{2}{\sqrt{\sigma}}\|u_t(t)\|\|\xi^t\|_{\mathcal{Q}^{2m+3}_\sigma} \leqslant \frac{C}{\sqrt{\sigma}}\mathrm{e}^{-(\varpi+\delta_2/(4\sigma))t} + C\mathrm{e}^{-\varpi t},$$

which yields inequality (4.11). By means of (4.6), (4.7), (4.9) and (4.10), (4.11) we have

$$I_2(t) \leqslant C\int_{\sqrt{\varepsilon}}^\infty \mu_\varepsilon(s)\|A^{(m+1)/2}\eta^t(s)\|\,\mathrm{d}s$$
$$\leqslant C\sqrt{\varepsilon}\|\eta^t\|_{L^2_{\mu_\varepsilon}(\mathbb{R}^+,H^{m+1})} \leqslant C\sqrt{\varepsilon}\mathrm{e}^{-\delta_1 t/(2\varepsilon)} + C\sqrt{\varepsilon}\mathrm{e}^{-\varpi t/2}, \quad \forall t \geqslant 0,$$
$$J_2(t) \leqslant C\int_{\sqrt{\sigma}}^\infty \beta_\sigma(s)\|A^{(m+2)/2}\xi^t(s)\|\,\mathrm{d}s$$
$$\leqslant C\sqrt{\sigma}\|\xi^t\|_{\mathcal{Q}^{m+1}_\sigma} \leqslant C\sqrt{\sigma}\mathrm{e}^{-\delta_2 t/(2\sigma)} + C\sqrt{\sigma}\mathrm{e}^{-\varpi t/2}, \quad \forall t \geqslant 0.$$

Taking (2.4), (2.5), (2.7), (2.8) and (4.6), (4.7) into account, we get, for $t < \sqrt{\varepsilon}$,

$$I_3(t) \leqslant C\int_t^{\sqrt{\varepsilon}} \mu_\varepsilon(s)\|A^{(m+1)/2}\eta_0(s-t)\|\,\mathrm{d}s$$
$$\leqslant C\mathrm{e}^{-\delta_1 t/\varepsilon}\left(\int_0^\infty \mu_\varepsilon(s)\,\mathrm{d}s\right)^{1/2}\|\eta_0\|_{L^2_{\mu_\varepsilon}(\mathbb{R}^+,H^{m+1})} \leqslant \frac{C}{\sqrt{\varepsilon}}\mathrm{e}^{-\delta_1 t/\varepsilon}, \quad \forall t \geqslant 0,$$
$$J_3(t) \leqslant C\int_t^{\sqrt{\sigma}} \beta_\sigma(s)\|A^{(m+2)/2}\xi_0(s-t)\|\,\mathrm{d}s$$
$$\leqslant C\mathrm{e}^{-\delta_2 t/\sigma}\left(\int_0^\infty \beta_\sigma(s)\,\mathrm{d}s\right)^{1/2}\|\xi_0\|_{\mathcal{Q}^{m+1}_\sigma} \leqslant \frac{C}{\sqrt{\sigma}}\mathrm{e}^{-\delta_2 t/\sigma}, \quad \forall t \geqslant 0.$$

Arguing in a similar fashion, there holds

$$I_4(t) \leqslant C\mathrm{e}^{-\delta_1 t/\varepsilon}\int_0^\infty s\mu_\varepsilon(s)\,\mathrm{d}s = C\mathrm{e}^{-\delta_1 t/\varepsilon}, \quad \forall t \geqslant 0,$$
$$J_4(t) \leqslant C\mathrm{e}^{-\delta_2 t/\sigma}\int_0^\infty s\beta_\sigma(s)\,\mathrm{d}s = C\mathrm{e}^{-\delta_2 t/\sigma}, \quad \forall t \geqslant 0.$$

Observe now that

$$\|\partial_t S_{0,0,0}(t)\mathbb{P}z\|_{\mathcal{H}^0_{0,0,0}} \leqslant C, \quad \forall t \geqslant 0. \tag{4.12}$$

Indeed, from the equations of $\mathcal{P}_{0,0,0}$, by virtue of (4.7), we deduce

$$\|\vartheta_t(t)\| \leqslant \|A\vartheta(t)\| + \|Au_t(t)\| \leqslant C, \quad \forall t \geqslant 0,$$
$$\|u_{tt}(t)\| \leqslant \|A^2u(t)\| + \|A^2u_t(t)\| + \|A\vartheta(t)\| \leqslant C, \quad \forall t \geqslant 0,$$

so that (4.12) readily follows. In particular (4.12) yields

$$\begin{aligned}\|u_t(t)-u_t(t-y)\|+\|\vartheta(t)-\vartheta(t-y)\| &\leqslant \|S_{0,0,0}(t-y)(S_{0,0,0}(y)\mathbb{P}z-\mathbb{P}z)\|_{\mathcal{H}^0_{0,0,0}}\\ &\leqslant C\mathrm{e}^{-\varpi t}\int_0^y \|\partial_t S_{0,0,0}(\varsigma)\mathbb{P}z\|_{\mathcal{H}^0_{0,0,0}}\,\mathrm{d}\varsigma\\ &\leqslant C\mathrm{e}^{-\varpi t}y,\end{aligned}$$

for every $t \geqslant 0$ and $y \in [0,t]$. Hence, by (2.7), (2.8) and (4.6), (4.7), we obtain

$$\begin{aligned}I_5(t) &\leqslant \|A^{m+1}\bar{\vartheta}(t)\|\int_0^{\sqrt{\varepsilon}}\mu_\varepsilon(s)\int_0^{\min\{s,t\}}\|\vartheta(t)-\vartheta(t-y)\|\,\mathrm{d}y\,\mathrm{d}s\\ &\leqslant C\sqrt{\varepsilon}\mathrm{e}^{-\varpi t},\quad \forall t\geqslant 0,\\ J_5(t) &\leqslant \|A^{m+2}\bar{u}_t(t)\|\int_0^{\sqrt{\sigma}}\beta_\sigma(s)\int_0^{\min\{s,t\}}\|u_t(t)-u_t(t-y)\|\,\mathrm{d}y\,\mathrm{d}s\\ &\leqslant C\sqrt{\sigma}\mathrm{e}^{-\varpi t},\quad \forall t\geqslant 0.\end{aligned}$$

We now turn to the estimate of K_τ. Taking condition (2.11) as well as (b) of Lemma 4.2 into account, we obtain

$$\begin{aligned}K_\tau(t) &= -\int_0^\infty \nu_\tau(s)\langle A^{m/2}\eta^t(s), A^{m/2}\bar{\vartheta}(t)\rangle\,\mathrm{d}s - \phi(\tau)\langle A^{m/2}\vartheta(t), A^{m/2}\bar{\vartheta}(t)\rangle,\\ &\leqslant C\int_0^\infty \nu_\tau(s)\|A^{m/2}\eta^t(s)\|\,\mathrm{d}s + C\phi(\tau)\\ &\leqslant C\Big(\int_0^\infty \nu_\tau(s)\,\mathrm{d}s\Big)^{1/2}\Big(\int_0^\infty \nu_\tau(s)\|A^{m/2}\eta(s)\|^2\,\mathrm{d}s\Big)^{1/2} + C\phi(\tau)\\ &\leqslant C\sqrt{\psi(\tau)}\|\eta_0\|_{L^2_{\nu_\tau}(\mathbb{R}^+,H^m)}\mathrm{e}^{-\delta_3 t/4} + C\psi(\tau) + C\phi(\tau)\\ &\leqslant C\sqrt{\psi(\tau)}\mathrm{e}^{-\delta_3 t/4} + C\psi(\tau) + C\phi(\tau),\quad \forall t\geqslant 0.\end{aligned}$$

Therefore, collecting the previous inequalities, we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbb{P}S_{\sigma,\tau,\varepsilon}(t)z - S_{0,0,0}(t)\mathbb{P}z\|^2_{\mathcal{H}^m_{0,0,0}} \leqslant \varphi_1(t)+\varphi_2(t),\quad \forall t\geqslant 0,$$

where we have set

$$\begin{aligned}\varphi_1(t) &= C\Big[(\sqrt{\varepsilon}+\sqrt{\sigma})\mathrm{e}^{-\varpi t/2} + \frac{1}{\sqrt{\varepsilon}}\mathrm{e}^{-\delta_1 t/(2\varepsilon)} + \frac{1}{\sqrt{\sigma}}\mathrm{e}^{-\delta_2 t/(2\sigma)}\Big],\\ \varphi_2(t) &= C\sqrt{\psi(\tau)}\mathrm{e}^{-\delta_3 t/4} + C\psi(\tau) + C\phi(\tau).\end{aligned}$$

Notice that, there holds

$$\int_0^t \varphi_1(\varsigma)\,\mathrm{d}\varsigma \leqslant C\big(\sqrt{\varepsilon}+\sqrt{\sigma}\big), \quad \forall t \geqslant 0,$$

$$\int_0^t \varphi_2(\varsigma)\,\mathrm{d}\varsigma \leqslant C\sqrt{\psi(\tau)} + C\big(\psi(\tau)+\phi(\tau)\big)t, \quad \forall t \geqslant 0.$$

Consequently, by integrating the above differential inequality in time, we can find two constants $K_R \geqslant 0$ and $Q_{R,T} \geqslant 0$ such that

$$\big\|\big(\bar{u}(t), \bar{u}_t(t), \bar{\vartheta}(t)\big)\big\|_{\mathcal{H}^m_{0,0,0}} \leqslant K_R \Pi_\flat(\sigma,\tau,\varepsilon) + Q_{R,T}\Pi_\sharp(\tau), \quad \forall t \geqslant 0,$$

which proves (4.4). The proof is now complete. □

4.4. Case $\tau = 0$: uniform in time estimate

As a straightforward but important corollary of the main Theorem 4.5, in the case $\tau = 0$, we obtain the following

Theorem 4.7. *For every $m \geqslant 0$, $R \geqslant 0$ and $z \in B_{\mathcal{H}^{2m+4}_{\sigma,0,\varepsilon}}(R)$ there exists $K_R \geqslant 0$ with*

$$\begin{aligned}&\big\|S_{\sigma,0,\varepsilon}(t)z - \mathbb{L}_{\sigma,0,\varepsilon}S_{0,0,0}(t)\mathbb{P}z\big\|_{\mathcal{H}^m_{\sigma,0,\varepsilon}}\\ &\quad \leqslant \|\eta_0\|_{\mathcal{M}^m_{0,\varepsilon}}\mathrm{e}^{-\delta_1 t/(4\varepsilon)} + \|\xi_0\|_{\mathcal{Q}^{m+1}_\sigma}\mathrm{e}^{-\delta_2 t/(4\sigma)} + K_R\big(\sqrt[4]{\varepsilon}+\sqrt[4]{\sigma}\big),\end{aligned}$$

for every $t \in [0,\infty)$.

Proof. It suffices to retrace the steps in the proof of Theorem 4.5, taking into account that $\Pi_\flat(\sigma,0,\varepsilon) = \sqrt[4]{\varepsilon}+\sqrt[4]{\sigma}$ and $K_0(t) = 0$ for every $t \geqslant 0$, which in turn yields $Q_{R,T} = 0$. □

Appendix: Failure of exponential decay

Here we want to analyze some variants of our model with no energy relaxation (i.e., $a \equiv 0$), in order to show that the relaxation of the heat flux and/or of the strain may not ensure the exponential stability of the corresponding semigroup.

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain and let $\alpha, \sigma \geqslant 0$. We consider the thermoelastic system with memory in an abstract setting

$$\begin{cases} u_{tt} + \displaystyle\int_0^\infty h'(s)A^2 u(t-s)\,\mathrm{d}s + A^2 u - A^\sigma \vartheta = 0,\\ \vartheta_t + \displaystyle\int_0^\infty k(s)A^\alpha \vartheta(s-t)\,\mathrm{d}s + A^\sigma u_t = 0.\end{cases}$$

Notice that the corresponding memory free model, studied, e.g., in [12],

$$\begin{cases} u_{tt} + A^2 u - A^\sigma \vartheta = 0,\\ \vartheta_t + A^\alpha \vartheta + A^\sigma u_t = 0,\end{cases}$$

includes, for $A = -\Delta$, as particular cases:

- *thermoelastic plates*, for $\alpha = \sigma = 1$;
- *viscoelasticity*, for $\alpha = 0$ and $\sigma = 1$.

A.1. Preliminaries and main results

Let $k:\mathbb{R}^+ \to \mathbb{R}^+$ and $h:\mathbb{R}^+ \to \mathbb{R}^+$ be smooth, decreasing, summable functions and set $\mu(s) = -k'(s)$ and $\beta(s) = -h'(s)$, where

$$\mu, \beta \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \tag{A.1}$$

$$\mu(s) \geqslant 0, \qquad \beta(s) \geqslant 0, \quad \forall s \in \mathbb{R}^+, \tag{A.2}$$

$$\mu'(s) \leqslant 0, \qquad \beta'(s) \leqslant 0, \quad \forall s \in \mathbb{R}^+, \tag{A.3}$$

$$0 < \int_0^\infty s^2 \mu(s)\,\mathrm{d}s < \infty, \qquad \int_0^\infty \beta(s)\,\mathrm{d}s > 0. \tag{A.4}$$

Moreover, we assume that A is a (strictly) positive selfadjoint linear operator on $L^2(\Omega)$ with domain $\mathcal{D}(A)$ which admits a diverging sequence of positive eigenvalues $\{\gamma_n\}_{n\geqslant 1}$.

We introduce the scale of Hilbert spaces $H^\alpha = \mathcal{D}(A^{\alpha/2})$, $\alpha \in \mathbb{R}$, endowed with the inner products $\langle u_1, u_2\rangle_{H^\alpha} = \langle A^{\alpha/2}u_1, A^{\alpha/2}u_2\rangle$ and we consider the weighted spaces

$$\mathcal{M}_\alpha = L^2_\mu(\mathbb{R}^+, H^\alpha), \qquad \mathcal{Q} = L^2_\beta(\mathbb{R}^+, H^2), \quad \alpha \in \mathbb{R},$$

endowed, respectively, with the inner products

$$\langle \eta_1, \eta_2\rangle_{\mathcal{M}_\alpha} = \int_0^\infty \mu(s)\langle \eta_1(s), \eta_2(s)\rangle_{H^\alpha}\,\mathrm{d}s, \qquad \langle \xi_1, \xi_2\rangle_{\mathcal{Q}} = \int_0^\infty \beta(s)\langle \xi_1(s), \xi_2(s)\rangle_{H^2}\,\mathrm{d}s.$$

Finally, we introduce the product space

$$\mathcal{H} = H^2 \times H^0 \times H^0 \times \mathcal{M}_\alpha \times \mathcal{Q},$$

endowed with the norm

$$\|(u, u_t, \vartheta, \eta, \xi)\|^2_{\mathcal{H}} = \|u\|^2_{H^2} + \|u_t\|^2_{H^0} + \|\vartheta\|^2_{H^0} + \|\eta\|^2_{\mathcal{M}_\alpha} + \|\xi\|^2_{\mathcal{Q}}.$$

In order to formulate the problem in the history space setting we denote by T and T' the linear operators on $\mathcal{M}_\alpha$ and $\mathcal{Q}$ respectively, defined as

$$T\eta = -\eta_s, \quad \eta \in \mathcal{D}(T), \qquad T'\xi = -\xi_s, \quad \xi \in \mathcal{D}(T'),$$

where $\mathcal{D}(T) = \{\eta \in \mathcal{M}_\alpha\colon \eta_s \in \mathcal{M}_\alpha, \eta(0) = 0\}$ and $\mathcal{D}(T') = \{\xi \in \mathcal{Q}\colon \xi_s \in \mathcal{Q}, \xi(0) = 0\}$, and η_s (resp. ξ_s) stands for the distributional derivative of η (resp. ξ) with respect to the internal variable s. On account of (A.3), we immediately get

$$\langle T\eta, \eta\rangle_{\mathcal{M}_\alpha} \leqslant 0, \qquad \langle T'\xi, \xi\rangle_{\mathcal{Q}} \leqslant 0, \tag{A.5}$$

for $\eta \in \mathcal{D}(T)$ and $\xi \in \mathcal{D}(T')$. Let us introduce the formulation of the problems. On account of the notation introduced above, given $(u_0, u_1, \vartheta_0, \eta_0, \xi_0)$ in $\mathcal{H}$, find $(u, u_t, \vartheta, \eta, \xi) \in C([0,\infty), \mathcal{H})$ solution to

$$\begin{cases} u_{tt} + \int_0^\infty \beta(s) A^2 \xi(s)\,\mathrm{d}s + A^2 u - A^\sigma \vartheta = 0, \\ \vartheta_t + \int_0^\infty \mu(s) A^\alpha \eta(s)\,\mathrm{d}s + A^\sigma u_t = 0, \\ \eta_t = T\eta + \vartheta, \\ \xi_t = T'\xi + u_t, \end{cases} \tag{$\mathcal{P}_{\alpha,\sigma}$}$$

for $t \in \mathbb{R}^+$, with initial conditions

$$\big(u(0), u_t(0), \vartheta(0), \eta^0, \xi^0\big) = (u_0, u_1, \vartheta_0, \eta_0, \xi_0),$$

and abstract boundary conditions

$$u(t) \in \mathcal{D}\big(A^2\big), \qquad \vartheta(t) \in \mathcal{D}\big(A^\alpha\big), \quad t \geqslant 0.$$

System $(\mathcal{P}_{\alpha,\sigma})$ allows us to provide a description of the solutions in terms of a strongly continuous semigroup of operators on $\mathcal{H}$. Indeed, setting $\zeta(t) = (u(t), v(t), \vartheta(t), \eta^t, \xi^t)^\top$, the problem rewrites as

$$\frac{\mathrm{d}}{\mathrm{d}t}\zeta = \mathcal{L}\zeta, \quad \zeta(0) = \zeta_0,$$

where $\mathcal{L}$ is the linear operator defined by

$$\mathcal{L}\begin{pmatrix} u \\ v \\ \vartheta \\ \eta \\ \xi \end{pmatrix} = \begin{pmatrix} v \\ -\int_0^\infty \beta(s) A^2 \xi(s)\,\mathrm{d}s - A^2 u + A^\sigma \vartheta \\ -\int_0^\infty \mu(s) A^\alpha \eta(s)\,\mathrm{d}s - A^\sigma v \\ \vartheta + T\eta \\ v + T'\xi \end{pmatrix} \tag{A.6}$$

with domain

$$\mathcal{D}(\mathcal{L}) = \left\{ z \in \mathcal{H} \;\middle|\; \begin{array}{l} Au \in H^2, v \in H^{2\sigma}, \vartheta \in H^{2\sigma} \\ \int_0^\infty \mu(s) A^\alpha \eta(s)\,\mathrm{d}s \in H^0 \\ \int_0^\infty \beta(s) A^2(s)\,\mathrm{d}s \in H^0 \\ \eta \in \mathcal{D}(T), \xi \in \mathcal{D}(T') \end{array} \right\}.$$

Since, by (A.5), $\mathcal{L}$ is a dissipative operator, arguing, e.g., as in [4], and assuming that (A.1)–(A.3) hold, we learn that $(\mathcal{P}_{\alpha,\sigma})$ induces a C_0-semigroup $S_{\alpha,\sigma}(t)$ of contractions on $\mathcal{H}$.

Assuming that (A.1)–(A.4) hold, we have the following

Theorem A.1 (Pointwise decay). *For every $\alpha, \sigma \geqslant 0$,*

$$\lim_{t\to\infty} \big\| S_{\alpha,\sigma}(t) z_0 \big\|_{\mathcal{H}} = 0, \quad \forall z_0 \in \mathcal{H}.$$

The main result of the Appendix are the following theorems.

Theorem A.2 (Non-exponential decay I). *Assume that* $0 \leqslant \alpha < 2$, $\sigma \geqslant 0$ *and*

$$\mu(s) = \kappa_1 s^{-\omega_1} \mathrm{e}^{-\delta_1 s}, \quad 0 \leqslant \omega_1 < \frac{2-\alpha}{2}, \ \kappa_1, \delta_1 > 0, \qquad \beta(s) = 0.$$

Then $S_{\alpha,\sigma}(t)$ *is not exponentially stable on* $\mathcal{H}$.

Theorem A.3 (Non-exponential decay II). *Assume that for some* $\kappa_1, \kappa_2, \delta_1, \delta_2 > 0$,

$$\mu(s) = \kappa_1 s^{-\omega_1} \mathrm{e}^{-\delta_1 s}, \qquad \beta(s) = \kappa_2 s^{-\omega_2} \mathrm{e}^{-\delta_2 s}.$$

Furthermore, suppose that

$$0 \leqslant \alpha < 2, \qquad 0 \leqslant \sigma < 1, \qquad \alpha \leqslant 2\sigma,$$

and that

$$\frac{2\sigma - \alpha}{2} \leqslant \omega_1 < \frac{2-\alpha}{2}, \qquad 0 \leqslant \omega_2 \leqslant \omega_1 - \frac{2\sigma - \alpha}{2}.$$

Then $S_{\alpha,\sigma}(t)$ *is not exponentially stable on* $\mathcal{H}$.

Remark A.4. The condition $\alpha < 2\sigma$ on the A powers is not new in thermoelasticity. It appears for instance (among other restrictions) in the study of smoothing/non-smoothing properties for a class of abstract (memory free) thermoelastic systems (cf. [12]).

Remark A.5. The memory kernels of Theorems A.2 and A.3 also satisfy the extra summability condition (A.4). Hence, any trajectory of the system goes to zero, but with an arbitrarily slow decay rate, according to the chosen initial data.

The rest of the Appendix is devoted to the proof of the above results.

A.2. Proof of Theorem A.1

In order to prove the result, we shall exploit the following sufficient condition for the (pointwise) decay to zero of any trajectory of a linear gradient system (cf., e.g., [5], Theorem A.2 and Corollary A.3).

Lemma A.6. *Let* $S(t)$ *be a linear gradient system on a Banach space* $\mathcal{H}$, *let* $z_0 \in \mathcal{H}$ *and assume that*

$$\bigcup_{t \geqslant 0} S(t) z_0 \text{ is relatively compact in } \mathcal{H}.$$

Then

$$\lim_{t \to \infty} S(t) z_0 = 0.$$

The same holds if the hypotheses are satisfied for all $z_0 \in \mathcal{X}$, *with* $\mathcal{X}$ *dense subset of* $\mathcal{H}$.

By exploiting (A.1)–(A.3) and (A.5) and observing that, by (A.4), μ and β cannot be identically equal to zero, it is easily seen that $S_{\alpha,\sigma}(t)$ is a gradient system on $\mathcal{H}$ (argue, e.g., as in [5], Proposition 3.2). We shall also set

$$\mathcal{M}^1_\alpha = L^2_\mu(\mathbb{R}^+, H^{\alpha+1}), \qquad \mathcal{Q}^1 = L^2_\beta(\mathbb{R}^+, H^3), \qquad \mathcal{H}^1 = H^3 \times H^1 \times H^1 \times \mathcal{M}^1_\alpha \times \mathcal{Q}^1.$$

If $\mathcal{L}$ denotes the linear operator defined in (A.6), since the space $\mathcal{D}(\mathcal{L}) \cap \mathcal{H}^1$ is dense in $\mathcal{H}$, according to Lemma A.6 it is sufficient to check the assumptions for a fixed $z_0 \in \mathcal{D}(\mathcal{L}) \cap \mathcal{H}^1$. Let $C = C(z_0)$ denote a generic positive constant. It is readily seen that $\|S_{\alpha,\sigma}(t)z_0\|_{\mathcal{H}^1} \leqslant C$ for all $t \geqslant 0$. Indeed, by (A.5) it suffices to multiply the equations of $(\mathcal{P}_{\alpha,\sigma})$ by u_t in H^1, by ϑ in H^1, by η in $\mathcal{M}^1_\alpha$ and by ξ in $\mathcal{Q}^1$ respectively and add the resulting equations. Let us consider the sets

$$\mathcal{C}_1 = \overline{\bigcup_{t\geqslant 1} \eta^t}^{\mathcal{M}_\alpha} \quad \text{and} \quad \mathcal{C}_2 = \overline{\bigcup_{t\geqslant 1} \xi^t}^{\mathcal{Q}}.$$

We claim that $\mathcal{C}_1 \times \mathcal{C}_2 \subset \mathcal{M}^1_\alpha \times \mathcal{Q}^1$ is compactly embedded into $\mathcal{M}_\alpha \times \mathcal{Q}$. To this aim, we recall the following compactness result (see, e.g., [5], Lemma 2.1) for the spaces $\mathcal{M}^1_\alpha \times \mathcal{Q}^1$. Assume that $\mathcal{C}_1 \subset \mathcal{M}^1_\alpha$ and $\mathcal{C}_2 \subset \mathcal{Q}^1$ satisfy:

(i) $\sup_{\eta\in\mathcal{C}_1} \|\eta\|_{\mathcal{M}^1_\alpha} < \infty$ and $\sup_{\eta\in\mathcal{C}_1} \|\eta_s\|_{\mathcal{M}_\alpha} < \infty$,

(ii) $\sup_{\xi\in\mathcal{C}_2} \|\xi\|_{\mathcal{Q}^1} < \infty$ and $\sup_{\xi\in\mathcal{C}_2} \|\xi_s\|_{\mathcal{Q}} < \infty$,

(iii) $\lim_{x\to\infty} \Big[\sup_{\eta\in\mathcal{C}_1} \mathbb{T}_\eta(x)\Big] = 0$ and $\lim_{x\to\infty} \Big[\sup_{\xi\in\mathcal{C}_2} \mathbb{T}_\xi(x)\Big] = 0$,

where the tails functions $\mathbb{T}_\eta$ and $\mathbb{T}_\xi$ are defined by

$$\mathbb{T}_\eta(x) = \int_{(0,1/x)\cup(x,\infty)} \mu(s)\big\|A^{\alpha/2}\eta(s)\big\|^2\,\mathrm{d}s, \quad x \geqslant 1,$$
$$\mathbb{T}_\xi(x) = \int_{(0,1/x)\cup(x,\infty)} \beta(s)\big\|A\xi(s)\big\|^2\,\mathrm{d}s, \quad x \geqslant 1.$$

Then $\mathcal{C}_1 \times \mathcal{C}_2$ is relatively compact in $\mathcal{M}_\alpha \times \mathcal{Q}$. Indeed, by simply mimicking the proofs of [5], Lemmas 4.3 and 4.4, exploiting the representation formulas for η^t and ξ^t

$$\eta^t(s) = \begin{cases} \displaystyle\int_0^s \vartheta(t-y)\,\mathrm{d}y, & 0 < s \leqslant t, \\ \displaystyle\eta_0(s-t) + \int_0^t \vartheta(t-y)\,\mathrm{d}y, & s > t, \end{cases}$$
$$\xi^t(s) = \begin{cases} u(t) - u(t-s), & 0 < s \leqslant t, \\ \xi_0(s-t) + u(t) - u(0), & s > t, \end{cases}$$

it is readily seen that (i)–(iii) are fulfilled (we point out that the addition summability assumption (A.4) on μ pops up in the proof of (iii) for η^t). Now, consider the set

$$\mathcal{K} = B_{H^3\times H^1\times H^1}(C) \times \mathcal{C}_1 \times \mathcal{C}_2.$$

Then, $\mathcal{K}$ is compact in $\mathcal{H}$ being $B_{H^3\times H^1\times H^1}(C)$ compact in $H^2 \times H^0 \times H^0$ and $\mathcal{C}_1 \times \mathcal{C}_2$ compact in $\mathcal{M}_\alpha \times \mathcal{Q}$. Moreover, by construction, there holds $S_{\alpha,\sigma}(t)z_0 \in \mathcal{K}$ for every $t \geqslant 0$. Therefore, by Lemma A.6, we have $S_{\alpha,\sigma}(t)z_0 \to 0$ in $\mathcal{H}$ as $t \to \infty$. □

A.3. Proof of Theorems A.2 and A.3

To prove the results, we shall exploit the following classical result due to Prüss [14].

Lemma A.7. *Let $S(t) = \mathrm{e}^{t\mathcal{L}}$ be a C_0-semigroup of contractions on a Hilbert space $\mathcal{H}$. Then $S(t)$ is exponentially stable if and only if $\mathrm{i}\mathbb{R}$ belongs to the resolvent set of $\mathcal{L}$, and there exists $\varepsilon > 0$ such that*

$$\inf_{\lambda\in\mathbb{R}}\big\|(\mathrm{i}\lambda\mathbb{I} - \mathcal{L})z\big\|_{\mathcal{H}} \geqslant \varepsilon\|z\|_{\mathcal{H}}, \quad \forall z \in \mathcal{D}(\mathcal{L}).$$

We start with the proof of Theorem A.3, for the proof of Theorem A.2 is just a simple by-product. Let $\mathcal{L}$ be the linear operator defined in (A.6). For $\lambda \in \mathbb{R}$ and for $\tilde{z} = (0, 0, 0, \tilde{\eta}, \tilde{\xi})^\top \in \mathcal{H}$, we consider the complex equation $(\mathrm{i}\lambda\mathbb{I} - \mathcal{L})z = \tilde{z}$, which explicitly writes as

$$\begin{cases} \mathrm{i}\lambda u - v = 0, \\ \mathrm{i}\lambda v + \displaystyle\int_0^\infty \beta(s)A^2\xi(s)\,\mathrm{d}s + A^2u - A^\sigma\vartheta = 0, \\ \mathrm{i}\lambda\vartheta + \displaystyle\int_0^\infty \mu(s)A^\alpha\eta(s)\,\mathrm{d}s + A^\sigma v = 0, \\ \mathrm{i}\lambda\xi - v + \xi_s = \tilde{\xi}, \\ \mathrm{i}\lambda\eta - \vartheta + \eta_s = \tilde{\eta}. \end{cases}$$

We shall denote by $\{\gamma_n\}$ the sequence of (positive) eigenvalues of A and by $\{w_n\}$ the corresponding sequence of normalized eigenvectors. We choose

$$\tilde{\eta}(s) = \tilde{\eta}_n(s) = \gamma_n^{-\alpha/2}w_n, \qquad \tilde{\xi}(s) = \tilde{\xi}_n(s) = \varLambda_n w_n.$$

where $\varLambda_n = \varLambda_n(\gamma_n)$ will be suitably chosen later on. If $\tilde{z}_n = (0, 0, 0, \tilde{\eta}_n, \tilde{\xi}_n)^\top$, then it holds

$$\|\tilde{z}_n\|_{\mathcal{H}}^2 = k_0 + h_0(\gamma_n\varLambda_n)^2, \quad \text{for all } n \in \mathbb{N}, \tag{A.7}$$

where we have set

$$k_0 = \int_0^\infty \mu(s)\,\mathrm{d}s, \qquad h_0 = \int_0^\infty \beta(s)\,\mathrm{d}s.$$

We shall prove the assertion by applying Lemma A.7, arguing by contradiction. To this aim, we find a sequence $\{\lambda_n\}$ in $\mathbb{R}$ and a corresponding solution z_n such that $\|z_n\|_{\mathcal{H}} \to \infty$, as $n \to \infty$. We search for a solution $z = (u, v, \vartheta, \eta, \xi)^\top$ of the form

$$u = u_n = pw_n, \qquad v = v_n = qw_n, \qquad \vartheta = \vartheta_n = rw_n,$$
$$\eta = \eta_n = \varphi w_n, \qquad \xi = \xi_n = \psi w_n,$$

where $p, q, r \in \mathbb{C}$, $\varphi \in H^1_\mu(\mathbb{R}^+)$ and $\psi \in H^1_\beta(\mathbb{R}^+)$, with $\varphi(0) = \psi(0) = 0$. Whence, the above system leads to the following equations

$$\begin{cases} \mathrm{i}\lambda p - q = 0, \\ \gamma_n^2 p - \lambda^2 p - \gamma_n^\sigma r + \gamma_n^2 \displaystyle\int_0^\infty \beta(s)\psi(s)\,\mathrm{d}s = 0, \\ \mathrm{i}\lambda r + \mathrm{i}\lambda\gamma_n^\sigma p + \gamma_n^\alpha \displaystyle\int_0^\infty \mu(s)\varphi(s)\,\mathrm{d}s = 0, \\ \mathrm{i}\lambda\varphi(s) - r + \varphi_s(s) = \dfrac{1}{\gamma_n^{\alpha/2}}, \\ \mathrm{i}\lambda\psi(s) - q + \psi_s(s) = \Lambda_n. \end{cases}$$

Imposing $\varphi(0) = \psi(0) = 0$, we can integrate the last two equations, getting

$$\varphi(s) = \frac{1}{\mathrm{i}\lambda}\bigl(r + \gamma_n^{-\alpha/2}\bigr)\bigl(1 - \mathrm{e}^{-\mathrm{i}\lambda s}\bigr),$$
$$\psi(s) = \frac{1}{\mathrm{i}\lambda}(q + \Lambda_n)\bigl(1 - \mathrm{e}^{-\mathrm{i}\lambda s}\bigr).$$

Then, we are led to the following system

$$\begin{cases} \mathrm{i}\lambda r + \mathrm{i}\lambda\gamma_n^\sigma p + \frac{\gamma_n^\alpha}{\mathrm{i}\lambda}\bigl(r + \gamma_n^{-\alpha/2}\bigr)\bigl(k_0 - c(\lambda)\bigr) = 0, \\ \gamma_n^2 p - \lambda^2 p - \gamma_n^\sigma r + \frac{\gamma_n^2}{\mathrm{i}\lambda}(\mathrm{i}\lambda p + \Lambda_n)\bigl(h_0 - b(\lambda)\bigr) = 0, \end{cases} \tag{A.8}$$

being $c(\lambda)$ and $b(\lambda)$ the Laplace transform of the kernels μ and β respectively,

$$c(\lambda) = \int_0^\infty \mu(s)\mathrm{e}^{-\mathrm{i}\lambda s}\,\mathrm{d}s, \quad \lambda \in \mathbb{R}^+,$$
$$b(\lambda) = \int_0^\infty \beta(s)\mathrm{e}^{-\mathrm{i}\lambda s}\,\mathrm{d}s, \quad \lambda \in \mathbb{R}^+.$$

We now impose the conditions

$$\mathrm{i}\lambda r + \mathrm{i}\lambda\gamma_n^\sigma p + \frac{r k_0 \gamma_n^\alpha}{\mathrm{i}\lambda} = 0, \qquad p = \frac{\gamma_n^\sigma r}{(1 + h_0)\gamma_n^2 - \lambda^2}. \tag{A.9}$$

These yield the fourth order algebraic equation

$$\lambda^4 - \bigl[(1 + h_0)\gamma_n^2 + \gamma_n^{2\sigma} + k_0\gamma_n^\alpha\bigr]\lambda^2 + k_0(1 + h_0)\gamma_n^{\alpha+2} = 0. \tag{A.10}$$

Taking into account that, since $\alpha < 2$ and $\sigma < 1$, we have

$$(1 + h_0)\gamma_n^2 + \gamma_n^{2\sigma} + k_0\gamma_n^\alpha = \mathcal{O}\bigl(\gamma_n^2\bigr), \quad \text{as } n \to \infty,$$

it is easy to realize that (A.10) admits a real positive solution $\lambda = \lambda_n = \lambda_n(\gamma_n) = \mathcal{O}(\gamma_n)$, as $n \to \infty$. Consequently, setting $c_n = c(\lambda_n)$ and $b_n = b(\lambda_n)$, from (A.8), (A.9), we get

$$r = \frac{k_0 - c_n}{\gamma_n^{\alpha/2} c_n} = r_n(\gamma_n),$$
$$p = \frac{\gamma_n^{\sigma} r_n}{(1+h_0)\gamma_n^2 - \lambda_n^2} = p_n(\gamma_n),$$
$$\Lambda_n = \frac{\mathrm{i}\lambda_n p_n b_n}{h_0 - b_n} = \Lambda_n(\gamma_n).$$

Notice that the above quantities depend solely on the eigenvalues γ_n of A. Moreover,

$$\begin{aligned} b_n &= \int_0^\infty \kappa_2 s^{-\omega_2} \mathrm{e}^{-(\mathrm{i}\lambda_n + \delta_2)s}\,\mathrm{d}s = \kappa_2 \lambda_n^{\omega_2 - 1}\left(\mathrm{i} + \frac{\delta_2}{\lambda_n}\right)^{\omega_2 - 1}\Gamma(1-\omega_2) \\ &= \mathcal{O}\big(\lambda_n^{\omega_2-1}\big) = \mathcal{O}\big(\gamma_n^{\omega_2-1}\big), \\ c_n &= \int_0^\infty \kappa_1 s^{-\omega_1} \mathrm{e}^{-(\mathrm{i}\lambda_n + \delta_1)s}\,\mathrm{d}s = \kappa_1 \lambda_n^{\omega_1 - 1}\left(\mathrm{i} + \frac{\delta_1}{\lambda_n}\right)^{\omega_1 - 1}\Gamma(1-\omega_1) \\ &= \mathcal{O}\big(\lambda_n^{\omega_1-1}\big) = \mathcal{O}\big(\gamma_n^{\omega_1-1}\big), \end{aligned}$$

as $n \to \infty$, where Γ is the Gamma function, so that

$$\frac{b_n}{c_n} = \mathcal{O}\big(\gamma_n^{\omega_2 - \omega_1}\big) \quad \text{as } n \to \infty.$$

As a consequence, we obtain

$$\begin{aligned} \gamma_n \Lambda_n(\gamma_n) &= \frac{\mathrm{i}\lambda_n b_n}{h_0 - b_n} \frac{\gamma_n^{\sigma+1}}{(1+h_0)\gamma_n^2 - \lambda_n^2} \frac{k_0 - c_n}{\gamma_n^{\alpha/2} c_n} \\ &= \mathcal{O}\left(\frac{\gamma_n^{\sigma+2-\alpha/2}}{(1+h_0)\gamma_n^2 - \lambda_n^2}\frac{b_n}{c_n}\right) = \mathcal{O}\big(\gamma_n^{\omega_2 - \omega_1 + \sigma - \alpha/2}\big), \end{aligned} \tag{A.11}$$

as $n \to \infty$. By (A.7), (A.11) and the assumptions on $\sigma, \alpha, \omega_1, \omega_2$, we learn that

$$\sup_{n \geqslant 1} \|\tilde{z}_n\|_{\mathcal{H}} < \infty.$$

On the other hand, by the assumptions on σ, α, ω_1, we have $|r_n| \to \infty$ as $n \to \infty$, yielding

$$\|z_n\|_{\mathcal{H}} \geqslant \|\vartheta_n\| = |r_n| \to \infty, \quad \text{as } n \to \infty,$$

which readily yields a contradiction and concludes the proof of Theorem A.3.

The proof of Theorem A.2 simply follows by mimicking the above steps, observing that by assumption we have $h_0 = 0$. In particular $\|\tilde{z}_n\|_{\mathcal{H}} = \sqrt{k_0}$ by (A.7), whereas $\omega_1 < \frac{2-\alpha}{2}$ implies that $\|z_n\|_{\mathcal{H}} \to \infty$ as $n \to \infty$, yielding again the assertion. □

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