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Uniqueness of ground states for a class of quasi-linear elliptic equations

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Abstract. We prove the uniqueness of radial positive solutions to a class of quasi-linear elliptic problems containing in particular the quasi-linear Schrödinger equation.

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1 Introduction and main results

Let $N \geq 3$ and $1 < p < (N+2)/(N-2)$. As known, the uniqueness [8], up to translations, of the ground state solutions of the Schrödinger equation

$$\mathrm{i}\psi_t + \Delta\psi + |\psi|^{p-1}\psi = 0 \quad \text{in } \mathbb{R}^N$$

plays an important role in the study of its dynamical features such as orbital stability [4] and soliton dynamics [3]. Recently, many contributions were devoted to the physically relevant quasi-linear Schrödinger equations

$$\mathrm{i}\psi_t + \Delta\psi + \psi\Delta|\psi|^2 + |\psi|^{p-1}\psi = 0 \quad \text{in } (0,\infty) \times \mathbb{R}^N, \tag{1.1}$$

for $1 < p < (3N+2)/(N-2)$. We refer the reader to the papers [5–7, 9–14] and to the references therein both for mathematical results and physical background of these equations. Despite they admit various physical applications, only a few rigorous mathematical studies have been carried out in the last decade. Especially in connections with the stability issues investigated in [5, 7], the problem of obtaining a uniqueness result for the ground state solutions of (1.1) is of particular relevance. To the authors's knowledge, two contributions have been addressed to this issue so far, namely [1, 15]. In [15], the uniqueness is obtained in the restricted range $1 < p < (N+2)/(N-2)$ and the result is perturbative in nature, namely it is obtained when the term $\psi\Delta|\psi|^2$ in (1.1) is replaced by $\lambda\psi\Delta|\psi|^2$ for values of $\lambda > 0$

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sufficiently small. In [1], the authors get uniqueness of a class of ground states of (1.1) in the range $1 < p < (3N+2)/(N-2)$. If $a : \mathbb{R} \to \mathbb{R}$ is the function defined by $a(s) = 2s^2 + 1$, then equation (1.1) rewrites as

$$\mathrm{i}\psi_t + \mathrm{div}(a(\psi)D\psi) - \frac{1}{2}a'(\psi)|D\psi|^2 + |\psi|^{p-1}\psi = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^N. \tag{1.2}$$

The goal of this manuscript is to get, for a suitable class of functions $a \in C^2(\mathbb{R})$, the uniqueness of radial standing waves $\psi(x,t) = e^{\mathrm{i}mt}u(x)$, $u > 0$, of equation (1.2), yielding to the quasi-linear elliptic problem in $\mathbb{R}^N$

$$-\mathrm{div}(a(u)Du) + \frac{1}{2}a'(u)|Du|^2 + mu = u^p \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

Physically speaking, the value of the coefficient $m > 0$ of the linear term has to be interpreted as a frequency. It is readily seen by Sobolev embedding that, if the function a behaves like the power $|s|^k$ at infinity, then $1 < p < ((k+1)N+2)/(N-2)$ is the optimal range for existence of nontrivial solutions of (1.3) in

$$X := \big\{u \in H^1(\mathbb{R}^N) : a(u)|Du|^2 \in L^1(\mathbb{R}^N)\big\}. \tag{1.4}$$

For the existence of ground state solutions to (1.3) in the polynomial quadratic case, we refer to [7, 14]. The main results of the paper are the following

Theorem 1.1. *Let $N \geq 3$, $a_1 > 0$ and $\psi \in C^2(\mathbb{R})$. Assume that there exist an integer $2 < k < 2p$ such that*

$$1 < p < \frac{(k+1)N+2}{N-2},$$

and $a(s) = a_1|s|^k + \psi(s)$ with

$$\inf_{s\geq 0}\psi(s) > 0, \quad \inf_{s\geq 0}\big(k\psi(s) - s\psi'(s)\big) \geq 0. \tag{1.5}$$

Furthermore, assume that

$$\lim_{s\to+\infty} s^{\frac{2-k}{2}}\psi(s) = 0, \quad \lim_{s\to+\infty} s^{\frac{4-k}{2}}\psi'(s) = 0, \quad \lim_{s\to+\infty} s^{\frac{6-k}{2}}\psi''(s) = 0. \tag{1.6}$$

Then there exists an $m_0 > 0$ such that the problem

$$\begin{cases} -\mathrm{div}(a(u)Du) + \frac{1}{2}a'(u)|Du|^2 + mu = u^p & \text{in } \mathbb{R}^N, \\ u > 0, & \text{in } \mathbb{R}^N, \\ u \text{ is radially symmetric,} \end{cases} \tag{1.7}$$

admits a unique solution $u \in X \cap C^2(\mathbb{R}^N)$, up to translations, for all $m \geq m_0$.

Theorem 1.2. *Let $N \geq 3$, $a_1 > 0$ and $\psi \in C^2(\mathbb{R})$. Assume that there exist an integer $0 < k \leq 2$ such that*

$$1 < p < \frac{(k+1)N+2}{N-2},$$

and $a(s) = a_1|s|^k + \psi(s)$, where ψ satisfies (1.5). Furthermore,

$$0 < \lim_{s\to+\infty} \psi(s) < +\infty, \qquad \lim_{s\to+\infty} s\psi'(s) = \lim_{s\to+\infty} s^2\psi''(s) = 0. \tag{1.8}$$

Then there exists an $m_0 > 0$ such that problem (1.7) admits a unique solution $u \in X \cap C^2(\mathbb{R}^N)$, up to translations, for all $m \geq m_0$.

Of course, in the particular but physically relevant case when $k = 2$, $a_1 = 2$ and $\psi(s) \equiv 1$, one gets uniqueness of ground states of the quasi-linear Schrödinger equation for $1 < p < (3N+2)/(N-2)$

$$-\Delta u - u\Delta u^2 + mu = u^p \quad \text{in } \mathbb{R}^N, \qquad u > 0 \quad \text{in } \mathbb{R}^N,$$

for m large. Essentially, thinking to more general situations where a nonlinearity $q(u)$ is considered in place of u^p, what we conjecture that should play a role is the fact that $\sqrt{a(s)}/q(s)$ goes to zero for $s \to +\infty$, namely the source q is dominant upon the quasi-linear diffusion $\sqrt{a(s)}$. In the study of high-frequency standing wave solutions $\psi(x,t) = e^{imt}u(x)$ to (1.2) the value of $m > 0$ can be taken large, so that the condition in the statement of Theorems 1.1 and 1.2 is satisfied.

Under the assumptions of Theorems 1.1–1.2, by exploiting the sign and symmetry properties of the set of ground state solutions, following as a variant of [7, Theorem 1.3], we have the following

Theorem 1.3. *There exists an $m_0 > 0$ such that, for all $m \geq m_0$, the ground state solutions $u \in X$ to*

$$-\mathrm{div}(a(u)Du) + \frac{1}{2}a'(u)|Du|^2 + mu = |u|^{p-1}u \quad \text{in } \mathbb{R}^N$$

are unique up to translations, positive, radially symmetric, decreasing and exponentially decaying.

According to the achievements established in the recent papers [5, 7], for the Schrödinger equation (1.2), the orbital stability range for the standing waves solutions associated with the problem ($\gamma > 0$)

$$\min_{\|u\|_{L^2(\mathbb{R}^N)}=\gamma} \mathscr{E}(u), \quad \mathscr{E}(u) = \frac{1}{2}\int_{\mathbb{R}^N} a(u)|Du|^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}, \tag{1.9}$$

is $1 < p < k+1+4/N$, which is contained in the optimal range of the statements of Theorems 1.1–1.2. We point out that the uniqueness, up to translations, of the solutions to problem (1.7) does not easily yield uniqueness of the solutions to the minimization problem (1.9). This further conclusion, which is currently unavailable, would of course be rather useful for both analytical and numerical purposes.

2 Dual semi-linear problem

Problem (1.7) is formally associated with the functional

$$X \ni u \mapsto \frac{1}{2}\int_{\mathbb{R}^N} a(u)|Du|^2 + \frac{m}{2}\int_{\mathbb{R}^N} |u|^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}.$$

Let $a \in C^2(\mathbb{R})$ be a function such that $a(s) \geq \nu$ for any $s \in \mathbb{R}$, for some $\nu > 0$. Let $g : \mathbb{R} \to \mathbb{R}$ denote the (strictly increasing) solution to the Cauchy problem

$$g'(s) = \frac{1}{\sqrt{a(g(s))}}, \quad g(0) = 0. \tag{2.1}$$

The solution is global due to the coercivity of a.

Lemma 2.1. *The function g is uniquely defined, $g \in C^3(\mathbb{R})$ and invertible. Moreover we have*

$$|g(s)| \leq \frac{1}{\sqrt{\nu}}|s|, \quad \textit{for every } s \in \mathbb{R}, \tag{2.2}$$

$$\lim_{s\to 0}\frac{g(s)}{s} = c_0, \quad \textit{and} \quad \lim_{t\to 0}\frac{g^{-1}(t)}{t} = \frac{1}{c_0}, \tag{2.3}$$

where $c_0 := \frac{1}{\sqrt{a(0)}} > 0$.

Proof. Since $a(s) \geq \nu$ for every $s \in \mathbb{R}$, from (2.1) we have that $|g'(s)| \leq \frac{1}{\sqrt{\nu}}$ for all $s \in \mathbb{R}$. Integrating, we get (2.2). Furthermore, from (2.1), we have

$$\lim_{s\to 0}\frac{g(s)}{s} = \lim_{s\to 0}\frac{1}{\sqrt{a(g(s))}} = \frac{1}{\sqrt{a(0)}}.$$

From (2.1) g is invertible and (2.3) follows. □

Using g, we can related, as in [1,6], problem (1.7) to the following semi-linear equation:

$$-\Delta v + m\frac{g(v)}{\sqrt{a(g(v))}} = \frac{|g(v)|^{p-1}g(v)}{\sqrt{a(g(v))}} \quad \text{in } \mathbb{R}^N, \qquad v > 0 \quad \text{in } \mathbb{R}^N, \tag{2.4}$$

and its associated functional $I : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$I(v) = \frac{1}{2}\int_{\mathbb{R}^N} |Dv|^2 - \int_{\mathbb{R}^N} H(v), \tag{2.5}$$

where we have set

$$H(s) = \int_0^s h(t)dt, \quad h(t) = \frac{|g(t)|^{p-1}g(t) - mg(t)}{\sqrt{a(g(t))}}.$$

Consider now the following condition, namely there exists $a_1 > 0$ such that

$$\lim_{s\to+\infty} \frac{a(s)}{s^k} = a_1. \tag{2.6}$$

Lemma 2.2. *Let g be as defined in* (2.1) *and assume condition* (2.6). *Then, we have*

$$\lim_{s\to+\infty} \frac{g(s)}{s^{\frac{2}{k+2}}} = c_1, \quad c_1 := \left(\frac{k+2}{2}\frac{1}{\sqrt{a_1}}\right)^{\frac{2}{k+2}}. \tag{2.7}$$

Furthermore, it holds

$$\lim_{t\to+\infty} \frac{g^{-1}(t)}{t^{\frac{k+2}{2}}} = \left(\frac{1}{c_1}\right)^{\frac{k+2}{2}} \tag{2.8}$$

and

$$\lim_{s\to+\infty} s(g')^2 = \begin{cases} 0 & \textit{if } k > 2, \\ \frac{1}{a_1 c_1^k} & \textit{if } k = 2, \\ +\infty & \textit{if } 0 < k < 2. \end{cases} \tag{2.9}$$

Proof. From the monotonicity of g, and from problem (2.1) we have $g(s) \to +\infty$ as $s \to +\infty$ and $g'(s) \to 0$ as $s \to +\infty$. Then, using (2.6), we have

$$\lim_{s\to+\infty} \frac{g(s)}{s^{\frac{2}{k+2}}} = \lim_{s\to+\infty} \frac{g'(s)}{\frac{2}{k+2}s^{\frac{2}{k+2}-1}} = \frac{k+2}{2}\lim_{s\to+\infty} \frac{s^{\frac{k}{k+2}}}{\sqrt{a(g)}}$$

$$= \frac{k+2}{2}\lim_{s\to+\infty}\sqrt{\frac{s^{\frac{2k}{k+2}}}{\frac{a(g)}{g^k}g^k}} = \frac{k+2}{2}\frac{1}{\sqrt{a_1}}\lim_{s\to+\infty}\left(\frac{s^{\frac{2}{k+2}}}{g(s)}\right)^{\frac{k}{2}}.$$

In particular, the limit on the left hand side cannot be equal to 0 or $+\infty$. In turn, we get

$$\left(\lim_{s\to+\infty} \frac{g(s)}{s^{\frac{2}{k+2}}}\right)^{1+\frac{k}{2}} = \frac{k+2}{2}\frac{1}{\sqrt{a_1}},$$

and the claim follows. Moreover, we have

$$\lim_{t\to+\infty}\frac{g^{-1}(t)}{t^{\frac{k+2}{2}}}=\lim_{s\to+\infty}\frac{s}{g(s)^{\frac{k+2}{2}}}=\lim_{s\to+\infty}\left[\frac{s^{\frac{2}{k+2}}}{g(s)}\right]^{\frac{k+2}{2}}=\left(\frac{1}{c_1}\right)^{\frac{k+2}{2}},$$

completing the proof of (2.8). Finally, observing that

$$\begin{aligned}\lim_{s\to+\infty}s(g')^2&=\lim_{s\to+\infty}\frac{s}{a(g)}=\lim_{s\to+\infty}\frac{s}{g^k\frac{a(g)}{g^k}}\\&=\frac{1}{a_1}\lim_{s\to+\infty}\frac{s}{\left(\frac{g(s)}{c_1 s^{\frac{2}{k+2}}}\right)^k c_1^k s^{\frac{2k}{k+2}}}=\frac{1}{a_1c_1^k}\lim_{s\to+\infty}s^{\frac{2-k}{k+2}}\end{aligned}$$

then (2.9) follows. □

For the sake of completeness, we recall the following two propositions.

Proposition 2.3. *Suppose $a(s)|s|^{-k}\to a_1$ as $s\to\infty$ and $1<p<\frac{(k+1)N+2}{N-2}$. Then $I\in C^1(H^1(\mathbb{R}^N))$.*

Proof. Since $a(s)|s|^{-k}\to a_1$ as $s\to\infty$, arguing as for the proof of (2.7) yields

$$\lim_{s\to\infty}\frac{|g(s)|}{|s|^{\frac{2}{k+2}}}=c_1.$$

Taking into account (2.3), and recalling that $\frac{2(p+1)}{k+2}<\frac{2N}{N-2}$, we have

$$\lim_{s\to 0}\frac{|h(s)|}{|s|}=\lim_{s\to 0}\frac{||g(s)|^{p-1}g(s)-mg(s)|}{|s|\sqrt{a(g(s))}}=\frac{m}{\sqrt{a(0)}}\lim_{s\to 0}\frac{|g(s)|}{|s|}=\frac{m}{a(0)},$$

and

$$\begin{aligned}\lim_{s\to\infty}\frac{|h(s)|}{|s|^{2^*-1}}&=\lim_{s\to\infty}\frac{||g(s)|^{p-1}g(s)-mg(s)|}{|s|^{2^*-1}\sqrt{a(g(s))}}\\&=\lim_{s\to\infty}\frac{|g(s)|^p}{|s|^{2^*-1}\sqrt{a(g(s))}}=\frac{c_1^{p-k/2}}{\sqrt{a_1}}\lim_{s\to\infty}\frac{|s|^{\frac{2p}{k+2}}}{|s|^{2^*-1}|s|^{\frac{k}{k+2}}}=0.\end{aligned}$$

This yields the assertion by [2, Theorem A.VI]. □

Proposition 2.4. *Suppose $a(s)|s|^{-k}\to a_1$ as $s\to\infty$ and $1<p<\frac{(k+1)N+2}{N-2}$. Let $v\in H^1(\mathbb{R}^N)$ be a nontrivial critical point of I, $v>0$ and let $u=g(v)$. Then u is a positive classical solution of* (1.7).

Proof. If $v \in H^1(\mathbb{R}^N)$, $v > 0$, is a critical point for I, then v is a positive weak solution of problem (2.4). From regularity theory (see e.g. [2, Lemma 1]) it follows that $v \in C^2(\mathbb{R}^N)$. Hence, from regularity and monotonicity of g we obtain that $u \in C^2(\mathbb{R}^N)$ and $u = g(v) > 0$ in $\mathbb{R}^N$. Moreover

$$Dv = (g^{-1})'(u)Du = \sqrt{a(u)}Du$$

and

$$\Delta v = (g^{-1})''(u)\,|Du|^2 + (g^{-1})'(u)\Delta u = \sqrt{a(u)}\Delta u + \frac{a'(u)}{2\sqrt{a(u)}}\,|Du|^2 .$$

Using (2.4), we get

$$\frac{1}{\sqrt{a(u)}}\left[-a(u)\Delta u - \frac{1}{2}a'(u)\,|Du|^2 + mu - u^p\right] = 0,$$

so that u is a classical solution of (1.7). □

Next we prove that uniqueness in $H^1(\mathbb{R}^N)$ for (2.4) yields uniqueness in X for the original problem, where X is as defined in (1.4).

Lemma 2.5. *Assume that condition* (2.6) *holds and that* $1 < p < \frac{(k+1)N+2}{N-2}$. *Furthermore, assume that* (2.4) *has at most one positive radial solution*

$$v \in H^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$$

up to translations. Then (1.7) *admits at most one positive radial solution*

$$u = g(v) \in X \cap C^2(\mathbb{R}^N)$$

up to translations.

Proof. Let $u \in X \cap C^2(\mathbb{R}^N)$ with $u > 0$ and set $v = g^{-1}(u)$. Then $v \in C^2(\mathbb{R}^N)$, $v > 0$, and

$$|Dv|^2 = |(g^{-1})'|^2|Du|^2 = a(u)|Du|^2 \in L^1(\mathbb{R}^N).$$

By Lemmas 2.1–2.2, we have

$$|g^{-1}(s)| \le C|s| + C|s|^{\frac{k+2}{2}}$$

(hence $|g^{-1}(s)| \le C|s| + C|s|^{(k+2)\frac{N}{N-2}}$), yielding

$$\int_{\mathbb{R}^N} v^2 \le C\int_{\mathbb{R}^N} u^2 + C\int_{\mathbb{R}^N} u^{(k+2)\frac{N}{N-2}}.$$

Finally, for $\rho > 0$ large, we obtain

$$\int_{\mathbb{R}^N} u^{(k+2)\frac{N}{N-2}} \leq C_k \left(\int_{\mathbb{R}^N} u^k \, |Du|^2 \right)^{\frac{N}{N-2}}$$
$$\leq C_k \left(\rho^k \int_{\{u \leq \rho\}} |Du|^2 + C \int_{\{u \geq \rho\}} a(u) \, |Du|^2 \right)^{\frac{N}{N-2}}.$$

Whence, we conclude that $v \in H^1(\mathbb{R}^N)$. Assume now that u_1 and u_2 are two positive radial solutions to problem (1.7) in the space $X \cap C^2(\mathbb{R}^N)$. Then, setting $v_1 := g^{-1}(u_1)$ and $v_2 := g^{-1}(u_2)$, from the first part of the proof we learn that $v_1, v_2 \in H^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ and $v_1, v_2 > 0$. Mimicking the proof of Proposition 2.4 (argue in the reversed order), it is easy to see that v_1 and v_2 solve problem (2.4). Of course, being $u_1 = u_1(|x|)$ and $u_2 = u_2(|x|)$, we have $v_1 = v_1(|x|)$ and $v_2 = v_2(|x|)$. From the uniqueness (up to translations) for the positive radial $H^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ solutions of (2.4), it follows $v_1(\cdot) = v_2(\cdot + x_0)$, for some $x_0 \in \mathbb{R}^N$. But then, in turn, $u_1(\cdot) = g(v_1(\cdot)) = g(v_2(\cdot + x_0)) = u_2(\cdot + x_0)$, concluding the proof. □

3 General computations

Motivated by the conclusion of Lemma 2.5, in order to get the desired uniqueness result, in this section we study the uniqueness of positive radial solution of (2.4). To this end, we let

$$h(v) := \frac{|g(v)|^{p-1} g(v) - m g(v)}{\sqrt{a(g(v))}},$$

so that we can rewrite (2.4) as $-\Delta v = h(v)$ in $\mathbb{R}^N$. In this way, we have that $h(s) \leq 0$ on $(0, s_0]$, where

$$s_0 := g^{-1}\left(m^{\frac{1}{p-1}}\right), \tag{3.1}$$

and $h(s) > 0$ for all $s > s_0$. Since we have $a \in C^2(\mathbb{R})$, it follows that $g \in C^3(\mathbb{R})$ and $h \in C^2(0, +\infty)$. Then, letting $\mathscr{K}_h(s) = \frac{sh'(s)}{h(s)}$, the uniqueness result of Serrin and Tang [16, Theorem 1], tells us that problem (2.4) admits a unique positive radial solution if

$$\mathscr{K}_h'(s) \leq 0, \quad \text{for every } s \geq s_0. \tag{3.2}$$

Observe that it holds

$$\mathscr{K}_h'(s) = \frac{(h'(s) + sh''(s))h(s) - s(h'(s))^2}{h(s)^2}.$$

Then we only need to evaluate the quantity

$$K(s) := \big(h'(s) + sh''(s)\big)h(s) - s\big(h'(s)\big)^2.$$

We start by making some calculations. Using (2.1) and the fact that $g(s) > 0$ for $s > 0$, we have that

$$\begin{aligned}
h(s) &= gg'(g^{p-1} - m),\\
h'(s) &= (p-1)g^{p-1}(g')^2 + (g^{p-1} - m)((g')^2 + gg''),\\
h''(s) &= p(p-1)g^{p-2}(g')^3 + 3(p-1)g^{p-1}g'g''\\
&\qquad + (g^{p-1} - m)(3g'g'' + gg''').
\end{aligned}$$

Then, we have

$$\begin{aligned}
K(s) &= \big(h'(s) + sh''(s)\big)h(s) - s\big(h'(s)\big)^2\\
&= sgg'(g^{p-1} - m)\Big[p(p-1)g^{p-2}(g')^3 + 3(p-1)g^{p-1}g'g''\\
&\qquad\qquad + (g^{p-1} - m)(3g'g'' + gg''')\Big]\\
&\quad + gg'(g^{p-1} - m)\Big[(p-1)g^{p-1}(g')^2 + (g^{p-1} - m)((g')^2 + gg'')\Big]\\
&\quad - s\Big[(p-1)g^{p-1}(g')^2 + (g^{p-1} - m)((g')^2 + gg'')\Big]^2\\
&= (g^{p-1} - m)^2\Big[sgg'(3g'g'' + gg''') + gg'((g')^2 + gg'')\\
&\qquad\qquad - s((g')^4 + g^2(g'')^2 + 2g(g')^2g'')\Big]\\
&\quad + (g^{p-1} - m)\Big[sgg'(p(p-1)g^{p-2}(g')^3 + 3(p-1)g^{p-1}g'g'')\\
&\qquad\qquad + (p-1)gg'g^{p-1}(g')^2\\
&\quad - 2(p-1)sg^{p-1}(g')^2((g')^2 + gg'')\Big] - (p-1)^2sg^{2p-2}(g')^4\\
&= (g^{p-1} - m)^2\Big[sg^2g'g''' + sg(g')^2g'' - s(g')^4 - sg^2(g'')^2\\
&\qquad\qquad + g(g')^3 + g^2g'g''\Big]\\
&\quad + (g^{p-1} - m)(p-1)g^{p-1}(g')^2\Big[(p-2)s(g')^2 + sgg'' + gg'\Big]\\
&\quad - (p-1)^2sg^{2p-2}(g')^4.
\end{aligned}$$

Using $g'' = -\frac{1}{2}a'(g)(g')^4$ and $g''' = -\frac{1}{2}a''(g)(g')^5 + (a'(g))^2(g')^7$, we have

$$
\begin{aligned}
K(s) = (g^{p-1}-m)^2\Big[& sg^2g'\Big(-\frac{1}{2}a''(g)(g')^5 + (a'(g))^2(g')^7\Big) \\
& + sg(g')^2\Big(-\frac{1}{2}a'(g)(g')^4\Big) \\
& - s(g')^4 - sg^2\Big(-\frac{1}{2}a'(g)(g')^4\Big)^2 \\
& + g(g')^3 + g^2g'\Big(-\frac{1}{2}a'(g)(g')^4\Big)\Big] \\
& + (g^{p-1}-m)(p-1)g^{p-1}(g')^2\Big[(p-2)s(g')^2 \\
& \quad + sg\Big(-\frac{1}{2}a'(g)(g')^4\Big) + gg'\Big] \\
& - (p-1)^2sg^{2p-2}(g')^4 \\
= (g^{p-1}-m)^2(g')^3\Big[& -\frac{1}{2}a''(g)sg^2(g')^3 + (a'(g))^2sg^2(g')^5 \\
& - \frac{1}{2}a'(g)sg(g')^3 - sg' - \frac{1}{4}(a'(g))^2sg^2(g')^5 \\
& + g - \frac{1}{2}a'(g)g^2(g')^2\Big] \\
& + (g^{p-1}-m)(p-1)g^{p-1}(g')^3\Big[(p-2)sg' - \frac{1}{2}a'(g)sg(g')^3 + g\Big] \\
& - (p-1)^2sg^{2p-2}(g')^4.
\end{aligned}
$$

Since

$$
\begin{aligned}
(g^{p-1}-m)g^{p-1} &= (g^{p-1}-m)^2 + m(g^{p-1}-m), \\
g^{2p-2} &= (g^{p-1}-m)^2 + 2m(g^{p-1}-m) + m^2,
\end{aligned}
$$

we have

$$
\begin{aligned}
K(s) = (g^{p-1}-m)^2(g')^3\Big[& -\frac{1}{2}a''(g)sg^2(g')^3 + \frac{3}{4}(a'(g))^2sg^2(g')^5 \\
& - \frac{1}{2}a'(g)sg(g')^3 - sg' + g - \frac{1}{2}a'(g)g^2(g')^2\Big] \\
& + (g^{p-1}-m)^2(p-1)(g')^3\Big[(p-2)sg' - \frac{1}{2}a'(g)sg(g')^3 + g\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\quad + m(g^{p-1}-m)(p-1)(g')^3\Big[(p-2)sg' - \frac{1}{2}a'(g)sg(g')^3 + g\Big] \\
&\quad - ((g^{p-1}-m)^2 + 2m(g^{p-1}-m) + m^2)(p-1)^2 s(g')^4 \\
&= (g^{p-1}-m)^2(g')^3\Big[-\frac{1}{2}a''(g)sg^2(g')^3 + \frac{3}{4}(a'(g))^2 sg^2(g')^5 \\
&\qquad\qquad - \frac{1}{2}a'(g)sg(g')^3 - sg' + g - \frac{1}{2}a'(g)g^2(g')^2 \\
&\qquad\qquad + (p-1)(p-2)sg' - \frac{1}{2}(p-1)a'(g)sg(g')^3 \\
&\qquad\qquad + (p-1)g - s(p-1)^2 g'\Big] \\
&\quad + m(g^{p-1}-m)(p-1)(g')^3\Big[(p-2)sg' - \frac{1}{2}a'(g)sg(g')^3 \\
&\qquad\qquad + g - 2(p-1)sg'\Big] \\
&\quad - m^2(p-1)^2 s(g')^4 \\
&= (g^{p-1}-m)^2(g')^3\Big[-\frac{1}{2}a''(g)sg^2(g')^3 + \frac{3}{4}(a'(g))^2 sg^2(g')^5 \\
&\qquad\qquad - \frac{1}{2}a'(g)g(g')^2(psg'+g) - psg' + pg\Big] \\
&\quad + m(g^{p-1}-m)(p-1)(g')^3\Big[-\frac{1}{2}a'(g)sg(g')^3 + g - psg'\Big] \\
&\quad - m^2(p-1)^2 s(g')^4 \\
&= (g')^3\Big[(g^{p-1}-m)^2\mathscr{H}_1(s) + m(p-1)(g^{p-1}-m)\mathscr{H}_2(s) \\
&\qquad\qquad - m^2 s(p-1)^2 g'\Big],
\end{aligned}
$$

where we have set

$$
\begin{aligned}
\mathscr{H}_1(s) &:= -\frac{1}{2}a''(g)sg^2(g')^3 + \frac{3}{4}(a'(g))^2 sg^2(g')^5 \\
&\qquad - \frac{1}{2}a'(g)g(g')^2(psg'+g) - psg' + pg,
\end{aligned} \tag{3.3}
$$

$$
\mathscr{H}_2(s) := -\frac{1}{2}a'(g)sg(g')^3 + g - psg'. \tag{3.4}
$$

Notice that, since $g' > 0$, we obtain

$$\begin{aligned} K(s) &= m^2(g')^3\Big[\Big(\frac{g^{p-1}}{m}-1\Big)^2\mathscr{H}_1(s) \\ &\qquad + (p-1)\Big(\frac{g^{p-1}}{m}-1\Big)\mathscr{H}_2(s) - s(p-1)^2 g'\Big] \qquad (3.5) \\ &< m^2(g')^3\Big(\frac{g^{p-1}}{m}-1\Big)\Big[\Big(\frac{g^{p-1}}{m}-1\Big)\mathscr{H}_1(s) + (p-1)\mathscr{H}_2(s)\Big]. \end{aligned}$$

In particular, in order for the function K to be asymptotically negative, it is sufficient to find $s_1 > 0$ and $s_2 > 0$ depending on the data of the problem, except on the value of $m > 0$, such that

$$\mathscr{H}_1(s) \leq 0, \quad \text{for every } s \geq s_1, \qquad \mathscr{H}_2(s) \leq 0, \quad \text{for every } s \geq s_2.$$

Then one chooses the values of m large enough that s_0 (cf. (3.1)) becomes greater than $\max\{s_1, s_2\}$.

4 Proof of Theorems 1.1 and 1.2

Under the assumptions of Theorems 1.1 and 1.2, we now set

$$a(s) = a_0(s) := a_1|s|^k + \psi(s),$$

where $k, a_1 > 0$ and the function $\psi(s) \in C^2(\mathbb{R})$ is bounded below away from zero. Observe that from (1.6) in Theorem 1.1 and from (1.8) in Theorem 1.2 it follows that $\psi(s) = o(s^k)$ as $s \to +\infty$ so that (2.6) is satisfied. In the case $0 < k \leq 2$, (1.8) implies that $\lim_{s\to+\infty} k\psi(s) - s\psi'(s) = k \lim_{s\to+\infty} \psi(s) > 0$.

4.1 Asymptotic estimates

We have $a_0(g_0) = a_1 g_0^k + \psi(g_0)$ and

$$a_0'(g_0) = ka_1 g_0^{k-1} + \psi'(g_0), \quad a_0''(g_0) = k(k-1)a_1 g_0^{k-2} + \psi''(g_0),$$

where g_0 is the function satisfying the Cauchy problem

$$g_0'(s) = \frac{1}{\sqrt{a_1 g_0^k(s) + \psi(g_0)}}, \quad g_0(0) = 0.$$

In particular, $a_1 g_0^k = \frac{1}{(g_0')^2} - \psi(g_0)$. We have the following simple result.

Lemma 4.1. *Assume that the given function ψ satisfies $\psi(s) = o(s^k)$ as $s \to +\infty$, $k\psi(s) - s\psi'(s) \geq 0$ for all $s \geq 0$ and $k\psi(s) - s\psi'(s) \to \alpha$ as $s \to +\infty$ for some $\alpha > 0$ in the case $0 < k \leq 2$. Consider the function $G_0 : \mathbb{R}^+ \to \mathbb{R}$ defined by setting*

$$G_0(s) = s - \frac{2}{k+2}\frac{g_0}{g_0'} = s - \frac{2}{k+2} g_0 \sqrt{a_0(g_0)}.$$

Then, for any $k > 0$, G_0 is nondecreasing and satisfies $G_0(s) \geq 0$ for every $s \geq 0$ and $G_0(s) > 0$ eventually for $s > 0$ large. Moreover, we have the limit

$$\lim_{s \to +\infty} \frac{s(g_0')^2}{G_0(s)} = \begin{cases} 0 & \text{if } k \geq 2, \\ \frac{2-k}{\alpha} & \text{if } 0 < k < 2. \end{cases} \tag{4.1}$$

Proof. We have $G_0(0) = 0$ and, in addition, for every $s \geq 0$ it holds

$$\begin{aligned} G_0'(s) &= 1 - \frac{2}{k+2} - \frac{1}{k+2} g_0 \frac{a_1 k g_0^{k-1} + \psi'(g_0)}{a_1 g_0^k + \psi(g_0)} \\ &= \frac{1}{k+2} \frac{1}{a_1 g_0^k(s) + \psi(g_0)} \\ &\qquad \times \left[k a_1 g_0^k(s) + k\psi(g_0) - k a_1 g_0^k(s) - \psi'(g_0) g_0 \right] \\ &= \frac{1}{k+2} \frac{k\psi(g_0) - \psi'(g_0) g_0}{a_1 g_0^k(s) + \psi(g_0)}, \end{aligned}$$

which readily yields the first assertion. Then from (2.9) and recalling that G_0 is an increasing function, we obtain conclusion (4.1) for $k > 2$. Moreover, in the cases $0 < k \leq 2$, we can write

$$\begin{aligned} G_0'(s) &= \frac{1}{k+2} \frac{k\psi(g_0) - \psi'(g_0) g_0}{a_1 g_0^k(s) \left(1 + \frac{\psi(g_0)}{a_1 g_0^k} \right)} \\ &= \frac{1}{k+2} \frac{k\psi(g_0) - \psi'(g_0) g_0}{a_1 \left(\frac{g_0(s)}{c_1 s^{\frac{2}{k+2}}} \right)^k c_1^k s^{\frac{2k}{k+2}} \left(1 + \frac{\psi(g_0)}{a_1 g_0^k} \right)}. \end{aligned}$$

From equation (2.7) we get $G_0'(s) \geq c s^{-\frac{2k}{k+2}}$ for $s > 0$ large and some $c > 0$. Then, if $0 < k < 2$, $G_0(s) - C_0 > c' s^{(2-k)/(k+2)}$ for $s > 0$ large and $c', C_0 \in \mathbb{R}$ with $c' > 0$. In the case $k = 2$, we have $G_0(s) - C_0 > c' \log s$ for any $s > 0$ large. Taking the limits as $s \to +\infty$, we finally get, for any $0 < k \leq 2$,

$$\lim_{s \to +\infty} G_0(s) = +\infty. \tag{4.2}$$

Then, if $k = 2$, equations (2.9) and (4.2) yield $\frac{s(g_0')^2}{G_0(s)} \to 0$ as $s \to +\infty$. If, instead, $0 < k < 2$, from (4.2) we have

$$
\begin{aligned}
&\lim_{s\to+\infty} \frac{s(g_0')^2}{G_0(s)} \\
&\quad = \lim_{s\to+\infty} \frac{(g_0')^2 + 2sg_0'g_0''}{G_0'(s)} \\
&\quad = (k+2) \lim_{s\to+\infty} \Bigg(\frac{(g_0')^2 + 2sg_0'\left(-\frac{1}{2}(ka_1 g_0^{k-1} + \psi'(g_0))(g_0')^4\right)}{k\psi(g_0) - \psi'(g_0)g_0} \\
&\qquad\qquad \times (a_1 g_0^k + \psi(g_0)) \Bigg) \\
&\quad = (k+2) \lim_{s\to+\infty} \frac{1 - s(ka_1 g_0^{k-1} + \psi'(g_0))(g_0')^3}{k\psi(g_0) - \psi'(g_0)g_0} \\
&\quad = (k+2) \lim_{s\to+\infty} \frac{1 - k\dfrac{s}{g_0}\left(\frac{1}{(g_0')^2} - \psi(g_0)\right)(g_0')^3 - s\psi'(g_0)(g_0')^3}{k\psi(g_0) - \psi'(g_0)g_0} \\
&\quad = (k+2) \lim_{s\to+\infty} \frac{1 - k\dfrac{sg_0'}{g_0} + \dfrac{s(g_0')^3}{g_0}\left(k\psi(g_0) - g_0\psi'(g_0)\right)}{k\psi(g_0) - \psi'(g_0)g_0}.
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
\lim_{s\to+\infty} \frac{sg_0'}{g_0} &= \lim_{s\to+\infty} \frac{s}{\sqrt{a_1}\, g_0^{1+\frac{k}{2}}} \\
&= \lim_{s\to+\infty} \frac{1}{\sqrt{a_1}} \frac{s}{\left(\dfrac{g_0}{c_1 s^{\frac{2}{k+2}}}\right)^{\frac{k+2}{2}} c_1^{\frac{k+2}{2}} s} \\
&= \frac{1}{\sqrt{a_1}} \frac{1}{c_1^{\frac{k+2}{2}}} = \frac{2}{k+2}.
\end{aligned}
$$

Taking into account that $k\psi(s) - s\psi'(s) \to \alpha > 0$ as $s \to +\infty$, for $0 < k < 2$ we conclude

$$
\lim_{s\to+\infty} \frac{s(g_0')^2}{G_0(s)} = \frac{k+2}{\alpha}\left(1 - k\frac{2}{k+2}\right) = \frac{2-k}{\alpha}.
$$

This ends the proof. □

Let us now set, for each $s > 0$ large,

$$\begin{aligned}\mathscr{H}_1(s) &:= -\frac{1}{2}a_0''(g_0)sg_0^2(g_0')^3 + \frac{3}{4}(a_0'(g_0))^2 sg_0^2(g_0')^5\\ &\qquad - \frac{1}{2}a_0'(g_0)g_0(g_0')^2(psg_0' + g_0) - psg_0' + pg_0,\\ \mathscr{H}_2(s) &:= -\frac{1}{2}a_0'(g_0)sg_0(g_0')^3 + g_0 - psg_0'.\end{aligned}$$

4.2 Sign of the term $\mathscr{H}_1$

Concerning the term $\mathscr{H}_1$, we have

$$\begin{aligned}\mathscr{H}_1(s) &= -\frac{1}{2}\Big[a_1k(k-1)g_0^{k-2} + \psi''(g_0)\Big]sg_0^2(g_0')^3\\ &\quad + \frac{3}{4}\Big[a_1^2k^2g_0^{2k-2} + (\psi'(g_0))^2 + 2a_1kg_0^{k-1}\psi'(g_0)\Big]sg_0^2(g_0')^5\\ &\quad - \frac{1}{2}\Big[a_1kg_0^{k-1} + \psi'(g_0)\Big]g_0(g_0')^2(psg_0' + g_0) - psg_0' + pg_0\\ &= -\frac{1}{2}k(k-1)(a_1g_0^k)s(g_0')^3 - \frac{1}{2}sg_0^2(g_0')^3\psi''(g_0)\\ &\quad + \frac{3}{4}k^2(a_1^2g_0^{2k})s(g_0')^5 + \frac{3}{4}sg_0^2(g_0')^5(\psi'(g_0))^2\\ &\quad + \frac{3}{2}k(a_1g_0^k)sg_0(g_0')^5\psi'(g_0) - \frac{1}{2}k(a_1g_0^k)(g_0')^2(psg_0' + g_0)\\ &\quad - \frac{1}{2}g_0(g_0')^2(psg_0' + g_0)\psi'(g_0) - psg_0' + pg_0\\ &= -\frac{1}{2}k(k-1)\left(\frac{1}{(g_0')^2} - \psi(g_0)\right)s(g_0')^3 - \frac{1}{2}sg_0^2(g_0')^3\psi''(g_0)\\ &\quad + \frac{3}{4}k^2\left(\frac{1}{(g_0')^4} + \psi^2(g_0) - 2\frac{\psi(g_0)}{(g_0')^2}\right)s(g_0')^5\\ &\quad + \frac{3}{4}sg_0^2(g_0')^5(\psi'(g_0))^2\\ &\quad + \frac{3}{2}k\left(\frac{1}{(g_0')^2} - \psi(g_0)\right)sg_0(g_0')^5\psi'(g_0)\\ &\quad - \frac{1}{2}k\left(\frac{1}{(g_0')^2} - \psi(g_0)\right)(g_0')^2(psg_0' + g_0)\\ &\quad - \frac{1}{2}g_0(g_0')^2(psg_0' + g_0)\psi'(g_0) - psg_0' + pg_0\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2}k(k-1)sg_0' + \frac{3}{4}k^2 sg_0' - \frac{1}{2}k(psg_0' + g_0) - psg_0' + pg_0 \\
&\quad + \frac{1}{2}k(k-1)s(g_0')^3\psi(g_0) - \frac{1}{2}sg_0^2(g_0')^3\psi''(g_0) \\
&\quad + \frac{3}{4}k^2 s(g_0')^5\psi^2(g_0) - \frac{3}{2}k^2 s(g_0')^3\psi(g_0) + \frac{3}{4}sg_0^2(g_0')^5(\psi'(g_0))^2 \\
&\quad + \frac{3}{2}ksg_0(g_0')^3\psi'(g_0) - \frac{3}{2}ksg_0(g_0')^5\psi(g_0)\psi'(g_0) \\
&\quad + \frac{1}{2}k(g_0')^2(psg_0' + g_0)\psi(g_0) - \frac{1}{2}g_0(g_0')^2(psg_0' + g_0)\psi'(g_0) \\
&= \frac{1}{4}\left(k^2 + 2k(1-p) - 4p\right)sg_0' + \frac{1}{2}(2p-k)g_0 \\
&\quad + \frac{1}{2}k\psi(g_0)s(g_0')^3(k-1-3k+p) + \frac{1}{2}kg_0(g_0')^2\psi(g_0) \\
&\quad + \frac{3}{4}k^2 s(g_0')^5\psi^2(g_0) + \mathscr{R}_1(\psi', \psi''),
\end{aligned}$$

where we have set

$$\begin{aligned}
\mathscr{R}_1(\psi', \psi'') := &-\frac{1}{2}sg_0^2(g_0')^3\psi''(g_0) + \frac{3}{4}sg_0^2(g_0')^5(\psi'(g_0))^2 \\
&+ \frac{3}{2}ksg_0(g_0')^3\psi'(g_0) - \frac{3}{2}ksg_0(g_0')^5\psi(g_0)\psi'(g_0) \\
&- \frac{1}{2}g_0(g_0')^2(psg_0' + g_0)\psi'(g_0).
\end{aligned}$$

Then

$$\begin{aligned}
\mathscr{H}_1(s) &= \frac{1}{4}(k+2)(k-2p)\left(sg_0' - \frac{2}{k+2}g_0\right) \\
&\quad + \frac{1}{2}k(p-2k-1)s(g_0')^3\psi(g_0) + \frac{1}{2}kg_0(g_0')^2\psi(g_0) \\
&\quad + \frac{3}{4}k^2 s(g_0')^5\psi^2(g_0) + \mathscr{R}_1(\psi', \psi'') \\
&= \frac{1}{4}(k+2)(k-2p)\left(sg_0' - \frac{2}{k+2}g_0\right) \\
&\quad + \frac{1}{2}k(g_0')^2\psi(g_0)\left((p-2k-1)sg_0' + g_0\right) \\
&\quad + \frac{3}{4}k^2 s(g_0')^5\psi^2(g_0) + \mathscr{R}_1(\psi', \psi'')
\end{aligned}$$

and, since $G_0(s) \geq 0$ for all $s \geq 0$ by Lemma 4.1, for every $s > 0$ we get

$$\begin{aligned}
\mathscr{H}_1(s) \leq\ & \frac{1}{4}(k+2)(k-2p)g_0' G_0(s) \\
& + \frac{1}{2}k(g_0')^2\psi(g_0)\left[(p-2k-1)sg_0' + \frac{k+2}{2}sg_0'\right] \\
& + \frac{3}{4}k^2 s(g_0')^5\psi^2(g_0) + \mathscr{R}_1(\psi', \psi'') \\
=\ & \frac{1}{4}(k+2)(k-2p)g_0' G_0(s) + \frac{1}{4}k(g_0')^2\psi(g_0)(2p-3k)sg_0' \\
& + \frac{3}{4}k^2 s(g_0')^5\psi^2(g_0) + \mathscr{R}_1(\psi', \psi'') \\
=\ & \frac{1}{4}g_0' G_0(s)\left[(k+2)(k-2p) + (2p-3k)k\frac{s(g_0')^2\psi(g_0)}{G_0(s)}\right. \\
& \left. + 3k^2\frac{s(g_0')^4\psi^2(g_0)}{G_0(s)} + 4\frac{\mathscr{R}_1(\psi', \psi'')}{g_0' G_0(s)}\right].
\end{aligned}$$

We shall now distinguish two cases.

Case I ($k > 2$). In this case, exploiting assumptions (1.6), taking into account that (cf. Lemma 2.2),

$$s(g_0')^2 = O\left(g_0^{\frac{2-k}{2}}\right), \quad g_0' = O\left(g_0^{-\frac{k}{2}}\right), \tag{4.3}$$

as $s \to +\infty$ and recalling that G_0 is nondecreasing by virtue of Lemma 4.1, we have

$$\lim_{s\to+\infty} \frac{s(g_0'(s))^2}{G_0(s)}\psi(g_0) = 0, \qquad \lim_{s\to+\infty} \frac{s(g_0'(s))^4}{G_0(s)}\psi^2(g_0) = 0,$$
$$\lim_{s\to+\infty} \frac{\mathscr{R}_1(\psi', \psi'')}{g_0' G(s)} = 0.$$

Whence, since $p > k/2$, given $\varepsilon \in (0,1)$ there exists an $s_1 > 0$ depending upon a and k (but independent upon the value of m) such that

$$\begin{aligned}
k(2p-3k)\frac{s(g_0'(s))^2}{G_0(s)}\psi(g_0) + 3k^2\frac{s(g_0'(s))^4}{G_0(s)}\psi^2(g_0) + 4\frac{\mathscr{R}_1(\psi', \psi'')}{g_0' G(s)} \\
\leq -\varepsilon(k+2)(k-2p),
\end{aligned}$$

for every $s \geq s_1$. Recalling Lemma 4.1 ($G_0(s) > 0$ eventually for s large), since $p > \frac{k}{2}$, it follows that

$$\mathscr{H}_1(s) \leq \frac{1-\varepsilon}{4}(k+2)(k-2p)g_0'(s)G_0(s) < 0, \quad \text{for every } s \geq s_1.$$

Case II ($0 < k \leq 2$). In this case, notice that $p > 1 \geq k - 1$ holds and by virtue of (4.1) of Lemma 4.1

$$\begin{aligned}
&\lim_{s\to+\infty} k(2p-3k)\frac{s(g_0'(s))^2}{G_0(s)}\psi(g_0) \\
&\quad = \lim_{s\to+\infty} k(2p-3k)\psi(g_0)\cdot \lim_{s\to+\infty}\frac{s(g_0'(s))^2}{G_0(s)} \\
&\quad = (2p-3k)(2-k),
\end{aligned}$$

since, in the notation of Lemma 4.1, and recalling assumptions (1.8), it holds that $\alpha = k\psi_\infty$, where here we have set $\psi_\infty = \lim_{s\to+\infty}\psi(s) \in \mathbb{R}^+$. Of course, in turn,

$$\lim_{s\to+\infty}\frac{s(g_0'(s))^4}{G_0(s)}\psi^2(g_0) = 0.$$

Finally, by arguing as for the previous case, taking into account assumptions (1.8) we have again

$$\lim_{s\to+\infty}\frac{\mathscr{R}_1(\psi',\psi'')}{g_0'G(s)} = 0.$$

In conclusion, there exists an $\tilde{s}_1 > 0$ depending upon a and k (but independent upon m), with

$$\begin{aligned}
&k(2p-3k)\frac{s(g_0'(s))^2}{G_0(s)}\psi(g_0) + 3k^2\frac{s(g_0'(s))^4}{G_0(s)}\psi^2(g_0) + 4\frac{\mathscr{R}_1(\psi',\psi'')}{g_0'G(s)} \\
&\quad \leq (2p-3k)(2-k) - 4\varepsilon k(k-p-1),
\end{aligned}$$

for every $s \geq \tilde{s}_1$. In turn, we get

$$\mathscr{H}_1(s) \leq (1-\varepsilon)k(k-p-1)g_0'(s)G_0(s) < 0, \quad \text{for every } s \geq \tilde{s}_1.$$

In conclusion, in any case, $\mathscr{H}_1$ becomes negative for values of $s > 0$ sufficiently large.

4.3 Sign of the term $\mathscr{H}_2$

Concerning the term (3.4), setting

$$\mathscr{R}_2(\psi') := -\frac{1}{2}\psi'(g_0)sg_0(g_0')^3,$$

we have

$$
\begin{aligned}
\mathscr{H}_2(s) &= -\frac{1}{2}a_0'(g_0)sg_0(g_0')^3 + g_0 - psg_0' \\
&= -\frac{1}{2}ka_1g_0^k s(g_0')^3 + g_0 - psg_0' + \mathscr{R}_2(\psi') \\
&= -\frac{1}{2}k\left(\frac{1}{(g_0')^2} - \psi(g_0)\right)s(g_0')^3 + g_0 - psg_0' + \mathscr{R}_2(\psi') \\
&= g_0 - \frac{1}{2}(k+2p)sg_0' + \frac{1}{2}ks(g_0')^3\psi(g_0) + \mathscr{R}_2(\psi').
\end{aligned}
$$

Then, in light of Lemma 4.1, we get for $s > 0$

$$
\begin{aligned}
\mathscr{H}_2(s) &\le \left(\frac{k+2}{2} - \frac{1}{2}(k+2p)\right)sg_0' + \frac{1}{2}ks(g_0')^3\psi(g_0) + \mathscr{R}_2(\psi') \\
&= (1-p)sg_0' + \frac{1}{2}ks(g_0')^3\psi(g_0) + \mathscr{R}_2(\psi') \\
&= sg_0'\left[(1-p) + \frac{k(g_0')^2\psi(g_0)}{2} + \frac{\mathscr{R}_2(\psi')}{sg_0'}\right].
\end{aligned}
$$

Taking into account assumption (1.6) (for $k > 2$) and (1.8) (for $0 < k \le 2$), and observing again that (4.3) holds, it is readily verified that it always holds

$$
\lim_{s\to+\infty}(g_0')^2\psi(g_0) = 0, \qquad \lim_{s\to+\infty}\frac{\mathscr{R}_2(\psi')}{sg_0'} = 0.
$$

Hence, since $p > 1$, there exists an $s_2 > 0$, depending only upon a and k, such that $\mathscr{H}_2(s) < 0$ for $s \ge s_2$.

Remark 4.2. By virtue of further manipulations of the terms $\mathscr{H}_1$ and $\mathscr{H}_2$, in the case $0 < k < 2$, it is possible to relax assumptions (1.5)–(1.8) by requiring that the following conditions hold:

$$
k\psi(s) - \psi'(s)s \ge 0, \quad \text{for } s \ge 0,
$$

$$
\liminf_{s\to+\infty} k\psi(s) - \psi'(s)s \ge \alpha > 0,
$$

$$
\lim_{s\to+\infty}\frac{k\psi(s) - \psi'(s)s}{s^k} = 0,
$$

$$
\lim_{s\to+\infty}\frac{(k-1)s\psi'(s) - s^2\psi''(s)}{k\psi(s) - \psi'(s)s} < \frac{2k}{2-k}(p+1-k).
$$

4.4 Proof of Theorems 1.1 and 1.2 concluded

By the previous steps, there exists

$$\bar{s} = \max\{s_1, s_2\} > 0,$$

depending upon a but independent upon $m > 0$, such that

$$\mathscr{H}_1(s) < 0 \quad \text{and} \quad \mathscr{H}_2(s) < 0 \quad \text{for any } s \geq \bar{s}.$$

Then, recalling that $s_0 := g^{-1}(m^{1/(p-1)})$ (see formula (3.1)), it is sufficient to take m_0 sufficiently large such that $s_0 \geq \bar{s}$ for every $m \geq m_0$, so that (see formula (3.5)) $K(s) \leq 0$ holds for every $s \geq s_0$, yielding (condition (3.2) if fulfilled) the desired uniqueness for problem (2.4), and in turn for the original problem (1.7) through Lemma 2.5.

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