Advances in Mathematical Sciences and Applications
Vol.16, No.1 (2006), pp.15-31

Gakkōtosho
Tokyo, Japan

EXPONENTIAL STABILITY AND SINGULAR LIMIT FOR A LINEAR THERMOELASTIC PLATE WITH MEMORY EFFECTS

Maurizio Grasselli and Marco Squassina
Dipartimento di Matematica "F.Brioschi"
Politecnico di Milano
Via Bonardi 9, I-20133 Milano, Italy
(maurizio.grasselli@mate.polimi.it)
(marco.squassina@mate.polimi.it)

Abstract. We consider a linear model of a thermoelastic plate where the heat flux depends solely on the past history of the temperature gradient through a memory kernel k. The resulting system consists of a fourth-order evolution equation, governing the vertical deflection u, which is coupled with a hyperbolic integrodifferential equation for the temperature field ϑ. The former one contains the term $-\omega\Delta u_{tt}$, $\omega > 0$, that accounts for rotational inertia effects. If this term is missing, it is known that the system, endowed with Navier boundary conditions, is not exponentially stable. Here we prove that its presence restores the exponential stability. Moreover, rescaling k by a time relaxation $\varepsilon > 0$, we obtain a closeness estimate between the solution to the system characterized by ε and ω and the solution to the limiting system formally obtained by setting $\varepsilon = \omega = 0$, namely, the classical linear thermoelastic plate model.

Communicated by Editors; Received June 26, 2008.
The first author was partially supported by the Italian MIUR PRIN Research Project *Aspetti Teorici e Applicativi di Equazioni a Derivate Parziali* and by the Italian MIUR FIRB Research Project *Analisi di Equazioni a Derivate Parziali, Lineari e Non Lineari: Aspetti Metodologici, Modellistica, Applicazioni.* The second author was supported by the MIUR PRIN Research Project *Metodi Variazionali e Topologici nello Studio dei Fenomeni Nonlineari* and by the *Istituto Nazionale di Alta Matematica*
AMS Subject Classification 35B25, 35B40, 45K05, 47D03, 74F05, 74K20

1 Introduction

Let Ω be a bounded planar domain with smooth boundary $\partial\Omega$. Suppose that Ω is occupied, for all time t, by a thin homogeneous isotropic elastic plate. Denoting by u its vertical deflection and by ϑ the temperature variation field, we suppose that the evolution of the pair (u, ϑ) is governed by the following integrodifferential system

$$\begin{cases} u_{tt} - \omega\Delta u_{tt} + \Delta(\Delta u + \vartheta) = 0, \\ \vartheta_t + \displaystyle\int_0^\infty k(s)\big[c\vartheta(t-s) - \Delta\vartheta(t-s)\big]ds - \Delta u_t = 0, \end{cases} \tag{1.1}$$

in $\Omega\times\mathbb{R}^+$, where $\mathbb{R}^+ = (0, \infty)$. Here $\omega \geq 0$, $c \geq 0$, and $k : [0, \infty) \to \mathbb{R}$ is a smooth positive bounded convex function which vanishes at infinity. Moreover, all the other physical constants have been set equal to 1. Observe that, if k coincides with the Dirac mass at 0, then system (1.1) formally becomes the well-known model of linear thermoelastic plate

$$\begin{cases} u_{tt} - \omega\Delta u_{tt} + \Delta(\Delta u + \vartheta) = 0, \\ \vartheta_t + c\vartheta - \Delta\vartheta - \Delta u_t = 0. \end{cases} \tag{1.2}$$

In absence of rotational inertia effects, i.e., $\omega = 0$, system (1.2) is essentially parabolic and its exponential stability was proved with various kinds of boundary conditions (see, for instance, [11, 13, 14, 15, 16, 19, 20] and their references). On the other hand, in the case $\omega > 0$, system (1.2) is weakly hyperbolic and the solutions propagate singularities. Then the proof of exponential stability presents some technical difficulties (cf. [1, 12, 17, 21] and references therein). For a detailed comparison between the two cases, the reader is referred to [18].

Going back to system (1.1), the case $\omega = 0$ was considered and carefully justified from the physical viewpoint in [6]. However, in that paper, not only the heat conduction law is of hereditary type (see [9]), but also the constitutive assumption for the thermal power contains a memory term characterized by a relaxation kernel $a \geq 0$. This implies the presence of a dissipative term $a(0)\vartheta$ in the second equation. The exponential stability proved in [6] essentially depends on such a term to the point that the authors conjectured its failure in the case $a \equiv 0$ (see [6, Remark 5.1]). This fact was indeed proved in [7]. More precisely, system (1.1) with $\omega = 0$ was endowed with Navier boundary conditions

$$\begin{aligned} u(t) = \Delta u(t) = 0 \qquad &\text{on } \partial\Omega,\ t \geq 0, \\ \vartheta(t) = 0 \qquad &\text{on } \partial\Omega,\ t \in \mathbb{R}, \end{aligned}$$

and initial conditions

$$\begin{aligned} (u(0), u_t(0), \vartheta(0)) = (u_0, u_1, \vartheta_0) \qquad &\text{in } \Omega, \\ \vartheta(-s) = \psi(s) \qquad &\text{in } \Omega\times\mathbb{R}^+, \end{aligned}$$

where the initial data $u_0, u_1, \vartheta_0 : \Omega \to \mathbb{R}$ and $\psi : \Omega\times\mathbb{R}^+ \to \mathbb{R}$ are assigned functions. Then, using the so-called past history approach (see [8] and references therein), it was proved that system (1.1) with $\omega = 0$ generates a strongly continuous semigroup $S_{0,\varepsilon}(t)$ acting on an appropriate (extended) phase space such that any trajectory goes to zero as time goes to infinity [7, Thm. 4.1]. However, $S_{0,\varepsilon}(t)$ *fails* to be exponentially stable, no matter how fast the memory kernel squeezes at infinity, provided that its growth around the origin is suitably controlled [7, Thm. 5.4]. It is worth pointing out that if the set of initial data is restricted to null initial past histories, i.e., $\psi \equiv 0$, it can be proved that the energy of the system (1.1) with $\omega \geq 0$ exponentially decays to 0 provided that k satisfies reasonable assumptions. This was done in [4] for a clamped plate, assuming $c = 0$. Therefore, as remarked in [7], the presence of nonvanishing initial past history might play a discriminating role in the stability of a system with memory effects.

The first result of the present paper, consists in showing that if $\omega > 0$, then problem (1.1) generates an exponentially stable semigroup with a decay rate proportional to ω itself (see Theorem 3.2). This will be achieved by using a technique first devised in [10, 22] whose main idea is essentially that of building a suitable perturbation of the first order energy functional. Thus the presence of rotational inertia restores the exponential stability, a pleasant feature from the physical viewpoint.

The second result is concerned with the closeness between the solutions to problem (1.1) and the solutions to the system (1.2) endowed with the same boundary and initial conditions (but the one for the past history of ϑ). We proceed in the spirit of [2] (see also [3, 5]) by replacing the memory kernel k by a suitable rescaling k_ε, namely,

$$k_\varepsilon(s) = \frac{1}{\varepsilon} k\left(\frac{s}{\varepsilon}\right), \qquad \forall\, s \in \mathbb{R}^+, \tag{1.3}$$

where $\varepsilon \in (0, 1]$ is a time relaxation. Note that k_ε approaches the Dirac mass at 0 as ε goes to zero, in the sense of distribution. Thanks to the first result, a convenient reformulation of system (1.1), with k_ε in place of k, generates an exponentially stable semigroup $S_{\omega,\varepsilon}(t)$ on a certain phase space. Then, denoting by $S_{0,0}(t)$ the semigroup generated by system (1.2), we establish an estimate of the difference between two different trajectories, in terms of ε and ω, which holds on any bounded time interval (see Theorem 4.1). In other words, the estimate basically says that the solution to a system like (1.1), which is hyperbolic and propagates data singularities, is close to the (arbitrarily smooth) solution to a parabolic system like (1.2) with $\omega = 0$, provided that ε and ω are small enough. Clearly, the limit process as ε going to zero is singular, for the information on the past history of the temperature field ϑ gets completely lost (see [2] for a complete discussion). As we shall see, this implies that the closeness has always to be understood for time intervals which do not contain 0.

We stress that the mentioned estimate also works for $\omega = 0$, focusing solely on the limit for ε going to zero (see Corollary 4.2). In this case the result becomes stronger since the estimate holds true with constants which are independent of the time interval size, so that we can control the differences between two trajectories for any time $t > 0$ (it has to be noticed that, as far as we are aware, the first *uniform in time* singular-like control was obtained in [5] for an electromagnetic system with memory effects). This fact implies that system (1.1) with $\omega = 0$, which lacks of exponential stability for all $\varepsilon > 0$, can be made arbitrarily closed, as $\varepsilon \to 0$, to the limit model (1.2), which instead exhibits exponential decay. Therefore, loosely speaking, the more the memory kernel gets peaked around the origin, the less the trajectories of system (1.1) present deviation from the exponential stability of system (1.2).

The content of the paper is organized as follows. In Section 2 we introduce the basic notation and tools as well as the formulation of the problems in the proper functional setting. Section 3 is devoted to prove the exponential decay of the solutions to system (1.1), for every $\varepsilon \in [0,1]$, provided that $\omega > 0$. Finally, in Section 4, we demonstrate the closeness estimate between the strongly continuous semigroups associated with systems (1.1) and (1.2) when ω and ε tend to zero.

Finally, we point out that our choice of boundary conditions clearly simplifies the functional setup and some technical argument. However, we do think that the present results can be quite easily extended, at least, to the supported (or clamped) plate, taking advantage, for instance, of the results proved in [18] for system (1.2).

2 Functional setup and well-posedness

Given a real normed space $\mathcal{H}$, we denote by $B_{\mathcal{H}}(R)$ the closed ball in $\mathcal{H}$ of radius $R \geq 0$ centered at zero. Let us define the positive operators A and B on $(L^2(\Omega), \langle \cdot, \cdot \rangle, \| \cdot \|)$ by

$$A = -\Delta \qquad \text{and} \qquad B = c\mathbb{I} - \Delta,$$

with domain $\mathcal{D}(A) = \mathcal{D}(B) = H_0^1(\Omega) \cap H^2(\Omega)$. Then we introduce the scale of Hilbert spaces

$$H^r = \mathcal{D}(A^{r/2}), \qquad r \in \mathbb{R},$$

endowed with the inner products

$$\langle u_1, u_2 \rangle_{H^r} = \langle A^{r/2} u_1, A^{r/2} u_2 \rangle.$$

Of course the operator B induces an equivalent inner product in H^r. Naming λ_1 the first eigenvalue of A, for any $s > r$, we also have the inequalities

$$\|A^{r/2} w\| \leq {\lambda_1}^{(r-s)/2} \|A^{s/2} w\|, \qquad \forall\, w \in \mathcal{D}(A^{s/2}). \tag{2.1}$$

We assume that $k : \mathbb{R}^+ \to \mathbb{R}^+$ is a smooth decreasing function satisfying, for the sake of simplicity, the normalization conditions

$$\int_0^\infty k(s) ds = 1, \qquad k(0) = 1. \tag{2.2}$$

Then we set

$$\mu(s) = -k'(s), \qquad \forall\, s \in \mathbb{R}^+,$$

where μ is supposed to satisfy

$$\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \tag{2.3}$$

$$\mu(s) \geq 0, \qquad \forall\, s \in \mathbb{R}^+, \tag{2.4}$$

$$\mu'(s) \leq 0, \qquad \forall\, s \in \mathbb{R}^+. \tag{2.5}$$

Note that the above assumptions entail that μ cannot be identically zero.

Also, for any $\varepsilon \in (0,1]$, we define the function (cf. (1.3))

$$\mu_\varepsilon(s) = \frac{1}{\varepsilon^2}\mu\left(\frac{s}{\varepsilon}\right) = -k'_\varepsilon(s),$$

and we observe that, for $\varepsilon \in (0,1]$,

$$\int_0^\infty \mu_\varepsilon(s)ds = \frac{1}{\varepsilon} \qquad \text{and} \qquad \int_0^\infty s\mu_\varepsilon(s)ds = 1, \tag{2.6}$$

thanks to (2.2).

We now consider the weighted Hilbert spaces

$$\mathcal{M}^r_\varepsilon = L^2_{\mu_\varepsilon}(\mathbb{R}^+, H^{r+1}), \qquad r \in \mathbb{R},$$

equipped with the inner products

$$\langle \eta_1, \eta_2 \rangle_{\mathcal{M}^r_\varepsilon} = \int_0^\infty \mu_\varepsilon(s) \langle B^{(1+r)/2}\eta_1(s), B^{(1+r)/2}\eta_2(s)\rangle ds$$

and we introduce the product spaces

$$\mathcal{H}^r_{\omega,\varepsilon} = \begin{cases} H^{r+2} \times H^{r+1} \times H^r \times \mathcal{M}^r_\varepsilon, & \text{if } \varepsilon > 0,\ \omega > 0, \\ H^{r+2} \times H^r \times H^r \times \mathcal{M}^r_\varepsilon, & \text{if } \varepsilon > 0,\ \omega = 0, \\ H^{r+2} \times H^{r+1} \times H^r, & \text{if } \varepsilon = 0,\ \omega > 0, \\ H^{r+2} \times H^r \times H^r, & \text{if } \varepsilon = 0,\ \omega = 0, \end{cases}$$

endowed with the norms

$$\|(u, u_t, \vartheta, \eta)\|^2_{\mathcal{H}^r_{\omega,\varepsilon}} = \begin{cases} \|u\|^2_{H^{r+2}} + \|u_t\|^2_{H^r} + \omega\|u_t\|^2_{H^{r+1}} + \|\vartheta\|^2_{H^r} + \|\eta\|^2_{\mathcal{M}^r_\varepsilon}, & \text{if } \varepsilon > 0, \\ \|u\|^2_{H^{r+2}} + \|u_t\|^2_{H^r} + \omega\|u_t\|^2_{H^{r+1}} + \|\vartheta\|^2_{H^r}, & \text{if } \varepsilon = 0. \end{cases}$$

In particular, the space $\mathcal{H}^0_{\omega,\varepsilon}$ will be the extended phase-space on which we construct the dynamical system associated with (1.1). Throughout the paper, when $\varepsilon = 0$, we shall agree to interpret the four entries vector $z = (u, u_t, \vartheta, \eta)$ just as the triplet (u, u_t, ϑ). Let T_ε be the linear operator on $\mathcal{M}^0_\varepsilon$ with domain

$$\mathcal{D}(T_\varepsilon) = \left\{\eta \in \mathcal{M}^0_\varepsilon \;\middle|\; \eta_s \in \mathcal{M}^0_\varepsilon,\ \eta(0) = 0\right\},$$

defined by

$$T_\varepsilon \eta = -\eta_s, \qquad \eta \in \mathcal{D}(T_\varepsilon),$$

where η_s stands for the distributional derivative of η with respect to the internal variable s. T_ε is the infinitesimal generator of the right-translation semigroup on $\mathcal{M}^0_\varepsilon$. Notice that, on account of (2.5), there holds

$$\langle T_\varepsilon \eta, \eta \rangle_{\mathcal{M}^0_\varepsilon} = \int_0^\infty \mu'(s)\|B^{1/2}\eta(s)\|^2 ds \le 0, \qquad \forall\, \eta \in \mathcal{D}(T_\varepsilon). \tag{2.7}$$

Following the past history approach (see, e.g., [8] and references therein), we introduce the so-called integrated past history of ϑ,

$$\eta^t(s) = \int_0^s \vartheta(t-y)dy, \qquad (s,t) \in \mathbb{R}^+ \times \mathbb{R}^+. \tag{2.8}$$

Differentiating (2.8) leads to a further equation ruling the evolution of η

$$\eta_t^t(s) = -\eta_s^t(s) + \vartheta(t), \qquad t \in \mathbb{R}^+.$$

Also, in view of (2.8), the initial-boundary conditions for η read as

$$\eta^0(s) = \int_0^s \vartheta(-y)dy, \qquad \eta^t(0) = 0, \qquad \forall\, t \geq 0.$$

We are now in the position to introduce the rigorous formulation of our problem.
For $\varepsilon \in [0,1]$ and $\omega \geq 0$, given $(u_0, u_1, \vartheta_0, \eta_0) \in \mathcal{H}^0_{\omega,\varepsilon}$, find $(u, u_t, \vartheta, \eta) \in C([0,\infty), \mathcal{H}^0_{\omega,\varepsilon})$ solution to

$$\begin{cases} u_{tt} + \omega A u_{tt} + A(Au - \vartheta) = 0, \\ \vartheta_t + \displaystyle\int_0^\infty \mu_\varepsilon(s) B\eta(s) ds + Au_t = 0, \\ \eta_t = T_\varepsilon \eta + \vartheta, \end{cases} \tag{$\mathcal{P}_{\omega,\varepsilon}$}$$

for $t \in \mathbb{R}^+$, which fulfills the initial conditions $(u(0), u_t(0), \vartheta(0), \eta^0) = (u_0, u_1, \vartheta_0, \eta_0)$.

Similarly, we introduce the formal limiting problem.
Given $(u_0, u_1, \vartheta_0) \in \mathcal{H}^0_{0,0}$, find $(u, u_t, \vartheta) \in C([0,\infty), \mathcal{H}^0_{0,0})$ solution to

$$\begin{cases} u_{tt} + A(Au - \vartheta) = 0, \\ \vartheta_t + B\vartheta + Au_t = 0, \end{cases} \tag{$\mathcal{P}_{0,0}$}$$

for $t \in \mathbb{R}^+$, which fulfills the initial conditions $(u(0), u_t(0), \vartheta(0)) = (u_0, u_1, \vartheta_0)$.
The above problems are abstract reformulation of the initial and boundary value problems associated with (1.1) and (1.2) (cf. Introduction). In particular, system $\mathcal{P}_{\omega,\varepsilon}$ allows us to provide a description of the solutions in terms of a strongly continuous (i.e., C^0) semigroup of operators. Indeed, setting $\zeta(t) = (u(t), v(t), \vartheta(t), \eta^t)^\top$, we can rewrite the first problem as

$$\begin{cases} \frac{d}{dt}\zeta = \mathcal{L}\zeta, \\ \zeta(0) = \zeta_0, \end{cases}$$

where $\mathcal{L}$ is the linear operator defined by

$$\mathcal{L}\begin{pmatrix} u \\ v \\ \vartheta \\ \eta \end{pmatrix} = \begin{pmatrix} v \\ -(\mathbb{I} + \omega A)^{-1} A(Au - \vartheta) \\ -Av - \int_0^\infty \mu_\varepsilon(s) B\eta(s) ds \\ \vartheta + T_\varepsilon \eta \end{pmatrix}$$

with domain

$$\mathcal{D}(\mathcal{L}) = \left\{ z \in \mathcal{H}^0_{\omega,\varepsilon} \;\middle|\; \begin{array}{l} Au - \vartheta \in H^2 \\ v \in H^2 \\ \vartheta \in H^1 \\ \int_0^\infty \mu_\varepsilon(s) B\eta(s) ds \in H^0 \\ \eta \in \mathcal{D}(T_\varepsilon) \end{array} \right\}.$$

By virtue of (2.7), $\mathcal{L}$ is a dissipative operator due to memory effects, namely

$$\langle \mathcal{L}w, w \rangle_{\mathcal{H}^0_{\omega,\varepsilon}} \leq 0,$$

for all $w \in \mathcal{D}(\mathcal{L})$. Then, arguing, e.g., as in [6], we have the following well-posedness result.

Theorem 2.1. *Let* (2.3)-(2.5) *hold if* $\varepsilon > 0$. *Then, for every* $\omega \geq 0$ *and* $\varepsilon \in [0, 1]$, *system* $\mathcal{P}_{\omega,\varepsilon}$ *defines a* C_0*-semigroup* $S_{\omega,\varepsilon}(t)$ *of contractions on the phase-space* $\mathcal{H}^0_{\omega,\varepsilon}$.

Remark 2.2. In the above theorem, we tacitly extend the definition of $S_{\omega,\varepsilon}(t)$ to the case $\varepsilon = 0$ which is well known (cf. Introduction). Of course, in this case we remind that the solution semigroup has three components only.

3 Exponential stability for $\omega > 0$

We already mentioned that, in absence of the rotational inertia term ($\omega = 0$), if the memory kernel is not allowed to grow too rapidly around the origin, although each trajectory of $\mathcal{P}_{0,\varepsilon}$ squeezes to zero as time goes to infinity, the associated semigroup $S_{0,\varepsilon}(t)$ on $\mathcal{H}^0_{0,\varepsilon}$ lacks of exponentially stability. More precisely, there holds [7, Thm. 5.4]

Theorem 3.1. *Let* $\varepsilon \in (0, 1]$ *and assume that* μ *satisfies* (2.3)-(2.5) *and*

$$\lim_{s \to 0} \sqrt{s}\, \mu_\varepsilon(s) = 0.$$

Then the semigroup $S_{0,\varepsilon}(t)$ *on* $\mathcal{H}^0_{0,\varepsilon}$ *is not exponentially stable.*

The aim of the present section is to prove that, on the contrary, for all $\omega > 0$, the semigroup $S_{\omega,\varepsilon}(t)$ on $\mathcal{H}^0_{\omega,\varepsilon}$ is exponentially stable, with a decay rate which is proportional to ω. As we anticipated in the Introduction, we use a technique first introduced in [10, 22], namely, we obtain the decay estimate for a suitably defined perturbation of the energy functional $\mathcal{E} : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\mathcal{E}(t) = \|S_{\omega,\varepsilon}(t)\|^2_{\mathcal{H}^0_{\omega,\varepsilon}}.$$

Without loss of generality, we shall restrict the attention to the case $\omega \in (0, 1]$.
The main result of this section is

Theorem 3.2. *Let (2.3) and (2.4) hold if $\varepsilon > 0$. In addition, assume that*

$$\mu'(s) + \delta\mu(s) \leq 0, \tag{3.1}$$

for some $\delta > 0$. Then, for every $\omega \in (0, 1]$ and $\varepsilon \in [0, 1]$, there exist two positive constants Θ and ς, both independent of ε and ω, such that

$$\mathcal{E}(t) \leq \varsigma\mathcal{E}(0)e^{-\omega\Theta t}, \qquad \forall\, t \geq 0.$$

Remark 3.3. The energy exponential decay rate which appears in Theorem 3.2 breaks down in the case $\omega = 0$, according to the lack of exponential stability (cf. Thm. 3.1). Also, observe that (3.1) entails the exponential decay of μ.

Proof of Theorem 3.2. We limit ourself to furnish the proof for $\varepsilon > 0$, the corresponding proof for $\varepsilon = 0$ being easily recoverable and well-known (cf. Introduction). Let $0 < \nu < \rho < \sigma < 1$ to be chosen later on, and define the energy perturbation functional

$$\mathcal{F}(t) = \mathcal{E}(t) + \rho\Upsilon(t) + \nu\Lambda(t) + \sigma\Pi(t),$$

where we have set

$$\begin{aligned}
\Upsilon(t) &= \omega\langle u_t(t), \vartheta(t)\rangle + \langle u_t(t), A^{-1}\vartheta(t)\rangle, \\
\Lambda(t) &= \langle u_t(t), u(t)\rangle + \omega\langle A^{1/2}u_t(t), A^{1/2}u(t)\rangle, \\
\Pi(t) &= -\varepsilon\langle\vartheta(t), \eta^t\rangle_{\mathcal{M}_\varepsilon^{-1}}.
\end{aligned}$$

From now on, we denote by C a generic positive constant independent of ρ, ν, σ and ω, ε which may even vary from line to line within the same equation. Observe that, by means of (2.6), we have

$$|\Pi(t)| \leq C\Big(\|\vartheta(t)\|^2 + \|\eta^t\|^2_{\mathcal{M}_\varepsilon^0}\Big).$$

Then, by Schwarz and Young inequalities and (2.1), up to choosing the constants ρ, ν and σ sufficiently small, we have

$$\frac{1}{2}\mathcal{F}(t) \leq \mathcal{E}(t) \leq 2\mathcal{F}(t), \tag{3.2}$$

so that $\mathcal{E}$ and $\mathcal{F}$ turn out to be equivalent for what concerns the energy decay estimate.

Let us now multiply the first equation of $\mathcal{P}_{\omega,\varepsilon}$ by u_t in H^0, the second by ϑ in H^0, the third by η in $\mathcal{M}_\varepsilon^0$ and add the resulting identities. This yields

$$\frac{d}{dt}\mathcal{E}(t) = 2\langle T_\varepsilon\eta, \eta\rangle_{\mathcal{M}_\varepsilon^0} \leq -\frac{\delta}{2\varepsilon}\|\eta^t\|^2_{\mathcal{M}_\varepsilon^0} + \frac{1}{2}\int_0^\infty \mu_\varepsilon'(s)\|B^{1/2}\eta^t(s)\|^2 ds,$$

by virtue of (2.7), (3.1), and integration by parts. Besides, by direct computation,

$$\begin{aligned}
\frac{d}{dt}\Upsilon(t) &= \omega\langle u_{tt}(t), \vartheta(t)\rangle + \omega\langle u_t(t), \vartheta_t(t)\rangle \\
&\quad + \langle u_{tt}(t), A^{-1}\vartheta(t)\rangle + \langle u_t(t), A^{-1}\vartheta_t(t)\rangle, \\
\frac{d}{dt}\Lambda(t) &= \|u_t(t)\|^2 + \omega\|A^{1/2}u_t(t)\|^2 + \langle u(t), u_{tt}(t) + \omega A u_{tt}(t)\rangle \\
&= \|u_t(t)\|^2 + \omega\|A^{1/2}u_t(t)\|^2 - \|Au(t)\|^2 + \langle\vartheta(t), Au(t)\rangle, \\
\frac{d}{dt}\Pi(t) &= -\varepsilon\langle\vartheta_t(t), \eta^t\rangle_{\mathcal{M}_\varepsilon^{-1}} - \varepsilon\langle\vartheta(t), T_\varepsilon\eta^t\rangle_{\mathcal{M}_\varepsilon^{-1}} - \|\vartheta(t)\|^2,
\end{aligned}$$

where, in the last identity, we have exploited formula (2.6) once again. Also, by testing with $A^{-1}\vartheta$ the equation for u of system $\mathcal{P}_{\omega,\varepsilon}$, we get

$$\langle u_{tt}(t), A^{-1}\vartheta(t)\rangle + \langle Au(t), \vartheta(t)\rangle + \omega\langle u_{tt}(t), \vartheta(t)\rangle = \|\vartheta(t)\|^2.$$

Furthermore, by multiplying in H^0 the equation for ϑ of system $\mathcal{P}_{\omega,\varepsilon}$ by ωu_t, $\omega\vartheta$ and $A^{-1}u_t$ respectively, we conclude that

$$\begin{aligned}
\omega\langle u_t(t), \vartheta_t(t)\rangle + \omega\langle u_t(t), \eta^t\rangle_{\mathcal{M}_\varepsilon^0} + \omega\|A^{1/2}u_t(t)\|^2 &= 0, \\
\omega\langle\vartheta_t(t), \vartheta(t)\rangle + \omega\langle\vartheta(t), \eta^t\rangle_{\mathcal{M}_\varepsilon^0} + \omega\langle A^{1/2}u_t(t), A^{1/2}\vartheta(t)\rangle &= 0, \\
\langle A^{-1}\vartheta_t(t), u_t(t)\rangle + \langle A^{-1}u_t(t), \eta^t\rangle_{\mathcal{M}_\varepsilon^0} + \|u_t(t)\|^2 &= 0.
\end{aligned}$$

Hence, by combining the previous identities, we reach

$$\begin{aligned}
\frac{d}{dt}\Upsilon(t) &= -\langle\vartheta(t), Au(t)\rangle + \|\vartheta(t)\|^2 - \omega\langle u_t(t), \eta^t\rangle_{\mathcal{M}_\varepsilon^0} \\
&\quad - \omega\|A^{1/2}u_t(t)\|^2 - \langle A^{-1}u_t(t), \eta^t\rangle_{\mathcal{M}_\varepsilon^0} - \|u_t(t)\|^2.
\end{aligned}$$

On account of the obtained formulas for the derivatives of $\mathcal{E}$, Υ, Λ, and Π, we deduce

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}(t) &\le -\frac{\delta}{2\varepsilon}\|\eta^t\|^2_{\mathcal{M}_\varepsilon^0} + \frac{1}{2}\int_0^\infty \mu_\varepsilon'(s)\|B^{1/2}\eta^t(s)\|^2 ds + \nu\|u_t(t)\|^2 + \omega\nu\|A^{1/2}u_t(t)\|^2 \\
&\quad - \nu\|Au(t)\|^2 - (\rho-\nu)\langle\vartheta(t), Au(t)\rangle + \rho\|\vartheta(t)\|^2 - \rho\omega\langle u_t(t), \eta^t\rangle_{\mathcal{M}_\varepsilon^0} \\
&\quad - \rho\omega\|A^{1/2}u_t(t)\|^2 - \rho\langle A^{-1}u_t(t), \eta^t\rangle_{\mathcal{M}_\varepsilon^0} - \rho\|u_t(t)\|^2 \\
&\quad - \sigma\varepsilon\langle\vartheta_t(t), \eta^t\rangle_{\mathcal{M}_\varepsilon^{-1}} - \sigma\varepsilon\langle\vartheta(t), T_\varepsilon\eta^t\rangle_{\mathcal{M}_\varepsilon^{-1}} - \sigma\|\vartheta(t)\|^2.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}(t) &\le -\nu\|Au(t)\|^2 - (\rho-\nu)\|u_t(t)\|^2 - (\rho-\nu)\omega\|A^{1/2}u_t(t)\|^2 \\
&\quad - (\sigma-\rho)\|\vartheta(t)\|^2 - \frac{\delta}{2\varepsilon}\|\eta^t\|^2_{\mathcal{M}_\varepsilon^0} + \frac{1}{2}\int_0^\infty \mu_\varepsilon'(s)\|B^{1/2}\eta^t(s)\|^2 ds + \mathcal{J}(t),
\end{aligned}$$

where we have set

$$\mathcal{J}(t) = -(\rho-\nu)\langle\vartheta(t), Au(t)\rangle - \rho\omega\langle u_t(t), \eta^t\rangle_{\mathcal{M}^0_\varepsilon} - \rho\langle A^{-1}u_t(t), \eta^t\rangle_{\mathcal{M}^0_\varepsilon}$$
$$- \sigma\varepsilon\langle\vartheta_t(t), \eta^t\rangle_{\mathcal{M}^{-1}_\varepsilon} + \sigma\varepsilon\langle\vartheta(t), \eta^t_s\rangle_{\mathcal{M}^{-1}_\varepsilon}.$$

Notice that, by the Young inequality, there holds

$$\big|\langle\vartheta(t), Au(t)\rangle\big| \leq \|\vartheta(t)\|^2 + \frac{1}{4}\|Au(t)\|^2,$$
$$\big|\langle u_t(t), \eta^t\rangle_{\mathcal{M}^0_\varepsilon}\big| \leq C\rho\omega\|A^{1/2}u_t(t)\|^2 + \frac{\delta}{12\rho\omega\varepsilon}\|\eta^t\|^2_{\mathcal{M}^0_\varepsilon},$$
$$\big|\langle A^{-1}u_t(t), \eta^t\rangle_{\mathcal{M}^0_\varepsilon}\big| \leq C\rho\|u_t(t)\|^2 + \frac{\delta}{12\rho\varepsilon}\|\eta^t\|^2_{\mathcal{M}^0_\varepsilon}.$$

Moreover, integrating by parts, we get

$$\varepsilon\langle\vartheta(t), \eta^t_s\rangle_{\mathcal{M}^{-1}_\varepsilon} \leq C\sigma\|\vartheta(t)\|^2 - \frac{1}{2\sigma}\int_0^\infty \mu'_\varepsilon(s)\|B^{1/2}\eta^t(s)\|^2 ds.$$

Also, we have

$$\big|\varepsilon\langle\vartheta_t(t), \eta^t\rangle_{\mathcal{M}^{-1}_\varepsilon}\big| \leq \langle Au_t(t), \eta^t\rangle_{\mathcal{M}^{-1}_\varepsilon} + \Big\|\int_0^\infty \mu_\varepsilon(s)B^{1/2}\eta^t(s)ds\Big\|^2$$
$$\leq C\sigma\|A^{1/2}u_t(t)\|^2 + \frac{\delta}{12\sigma\varepsilon}\|\eta^t\|^2_{\mathcal{M}^0_\varepsilon} + C\|\eta^t\|^2_{\mathcal{M}^0_\varepsilon}.$$

By the above inequalities, it follows

$$\mathcal{J}(t) \leq \frac{\rho-\nu}{4}\|Au(t)\|^2 + C\rho^2\|u_t(t)\|^2 + C\Big(\rho^2 + \frac{\sigma^2}{\omega}\Big)\omega\|A^{1/2}u_t(t)\|^2$$
$$+ (C\sigma^2 + \rho - \nu)\|\vartheta(t)\|^2 + \Big(\frac{\delta}{4\varepsilon} + C\sigma\Big)\|\eta^t\|^2_{\mathcal{M}^0_\varepsilon} - \frac{1}{2}\int_0^\infty \mu'_\varepsilon(s)\|B^{1/2}\eta^t(s)\|^2 ds.$$

Therefore, we conclude that

$$\frac{d}{dt}\mathcal{F}(t) + \frac{5\nu-\rho}{4}\|Au(t)\|^2 + (\rho-\nu-C\rho^2)\|u_t(t)\|^2$$
$$+ \Big(\rho-\nu-C\rho^2 - \frac{C\sigma^2}{\omega}\Big)\omega\|A^{1/2}u_t(t)\|^2$$
$$+ (\sigma - 2\rho + \nu - C\sigma^2)\|\vartheta(t)\|^2 + \frac{\delta - C\sigma\varepsilon}{4\varepsilon}\|\eta^t\|^2_{\mathcal{M}^0_\varepsilon} \leq 0.$$

Choosing now $\rho = \rho_0\omega$, $\nu = \nu_0\omega$, and $\sigma = \sigma_0\omega$, yields

$$\frac{d}{dt}\mathcal{F}(t) + \omega\frac{5\nu_0-\rho_0}{4}\|Au(t)\|^2 + \omega(\rho_0-\nu_0-C\rho_0^2)\|u_t(t)\|^2$$
$$+ \omega(\rho_0-\nu_0-C\rho_0^2 - C\sigma_0^2)\omega\|A^{1/2}u_t(t)\|^2$$
$$+ \omega(\sigma_0 - 2\rho_0 + \nu_0 - C\sigma_0^2)\|\vartheta(t)\|^2 + \frac{\delta - C\sigma_0\omega\varepsilon}{4\varepsilon}\|\eta^t\|^2_{\mathcal{M}^0_\varepsilon} \leq 0.$$

Then, fixing ρ_0, ν_0 and σ_0 so small that (3.2) holds and

$$\gamma = \min\left\{\frac{5\nu_0 - \rho_0}{4}, \rho_0 - \nu_0 - C\rho_0^2 - C\sigma_0^2, \sigma_0 - 2\rho_0 + \nu_0 - C\sigma_0^2, \frac{\delta - C\sigma_0}{4}\right\} > 0,$$

in light of (3.2), it follows that

$$\frac{d}{dt}\mathcal{F}(t) + \frac{\omega\gamma}{2}\mathcal{F}(t) \leq 0, \qquad \forall\, t \geq 0.$$

By the Gronwall lemma, using again (3.2), we get the assertion with $\Theta = \frac{\gamma}{2}$ and $\varsigma = 4$. $\square$

4 The Singular Limit Estimate

The main goal of this section is to establish, in the spirit of [2, 3] (see also [5]), a quantitative estimate of the closeness between the semigroups $S_{\omega,\varepsilon}(t)$ and $S_{0,0}(t)$, when both the parameters ω and ε tend to zero (or when $\omega = 0$ and ε goes to zero), provided that the initial data are taken inside a suitable bounded subset of the extended phase-space $\mathcal{H}^0_{\omega,\varepsilon}$. From now on, in the case $\varepsilon > 0$, we will always assume (2.3)-(2.4) and (3.1).

In order to properly compare the four component solution ($\varepsilon > 0$) with the three-component solution ($\varepsilon = 0$), we need to introduce the lifting and projection maps

$$\mathbb{L}_{\omega,\varepsilon} : \mathcal{H}^0_{\omega,0} \to \mathcal{H}^0_{\omega,\varepsilon}, \qquad \mathbb{P}_\omega : \mathcal{H}^0_{\omega,\varepsilon} \to \mathcal{H}^0_{\omega,0}, \qquad \mathbb{Q}_{\omega,\varepsilon} : \mathcal{H}^0_{\omega,\varepsilon} \to \mathcal{M}^0_\varepsilon,$$

defined, respectively, by

$$\mathbb{L}_{\omega,\varepsilon}(u, u_t, \vartheta) = \begin{cases} (u, u_t, \vartheta, 0), & \text{if } \varepsilon > 0, \\ (u, u_t, \vartheta), & \text{if } \varepsilon = 0, \end{cases}$$

and

$$\mathbb{P}_\omega(u, u_t, \vartheta, \eta) = (u, u_t, \vartheta) \qquad \text{and} \qquad \mathbb{Q}_{\omega,\varepsilon}(u, u_t, \vartheta, \eta) = \eta.$$

If z denotes the initial data, we prove the convergence of $S_{\omega,\varepsilon}(t)z$ towards $\mathbb{L}_{\omega,\varepsilon}S_{0,0}(t)\mathbb{P}_\omega z$ in the $\mathcal{H}^0_{\omega,\varepsilon}$-norm. To be more precise, the first three components of the solution $\mathbb{P}_\omega S_{\omega,\varepsilon}(t)z$ are shown to converge to $S_{0,0}(t)\mathbb{P}_\omega z$ in the $\mathcal{H}^0_{\omega,0}$-norm, whereas the history component η^t vanishes in the $\mathcal{M}^0_\varepsilon$-norm on all time intervals $[\tau, T]$, with $\tau > 0$, due to the presence of a possibly nonvanishing initial history. In addition, when $\omega = 0$, the coefficients appearing in the estimate no longer depend on the time interval, so that in turn we obtain a closeness control over the whole $\mathbb{R}^+$.

The main result of the section is

Theorem 4.1. *For every $R \geq 0$, $T > 0$, and $z \in B_{\mathcal{H}^2_{\omega,\varepsilon}}(R)$, there exist $K_R \geq 0$, independent of T, and $Q_{R,T} \geq 0$ such that*

$$\|S_{\omega,\varepsilon}(t)z - \mathbb{L}_{\omega,\varepsilon}S_{0,0}(t)\mathbb{P}_\omega z\|_{\mathcal{H}^0_{\omega,\varepsilon}} \leq \|\eta_0\|_{\mathcal{M}^0_\varepsilon} e^{-\frac{\delta t}{4\varepsilon}} + Q_{R,T}\sqrt{\omega} + K_R\sqrt[4]{\varepsilon},$$

for every $t \in [0, T]$.

A meaningful and straightforward byproduct of Theorem 4.1 is

Corollary 4.2. *For every $R \geq 0$ and $z \in B_{\mathcal{H}^2_{0,\varepsilon}}(R)$, there exists $K_R \geq 0$ such that*

$$\|S_{0,\varepsilon}(t)z - \mathbb{L}_{0,\varepsilon} S_{0,0}(t)\mathbb{P}_0 z\|_{\mathcal{H}^0_{0,\varepsilon}} \leq \|\eta_0\|_{\mathcal{M}^0_\varepsilon} e^{-\frac{\delta t}{4\varepsilon}} + K_R \sqrt[4]{\varepsilon},$$

for every $t \geq 0$.

Before coming to the proof of Theorem 4.1, we must demonstrate a regularity result, that is,

Lemma 4.3. *Let $m \geq 0$, $R \geq 0$, and $z \in B_{\mathcal{H}^m_{\omega,\varepsilon}}(R)$. Then there exists $M_R \geq 0$ such that*

$$\|S_{\omega,\varepsilon}(t)z\|_{\mathcal{H}^m_{\omega,\varepsilon}} \leq M_R, \qquad \forall\, t \geq 0.$$

Proof. We only give a formal inductive argument. For $m = 0$ the assertion readily follows by multiplying the first equation of $\mathcal{P}_{\omega,\varepsilon}$ by u_t in H^0, the second by ϑ in H^0, the third by η in $\mathcal{M}^0_\varepsilon$ and adding the resulting equations. Let $m \geq 1$ and assume that the property holds for $m-1$. Then, performing similar multiplications by $B^m u_t$ in H^0, by $B^m\vartheta$ in H^0, and by η in $\mathcal{M}^m_\varepsilon$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|B^{m/2}Au\|^2 + (1-c\omega)\|B^{m/2}u_t\|^2 + \omega\|B^{(m+1)/2}u_t\|^2) = \langle A\vartheta, B^m u_t\rangle,\\
&\frac{1}{2}\frac{d}{dt}\|B^{m/2}\vartheta\|^2 = -\langle \eta, \vartheta\rangle_{\mathcal{M}^m_\varepsilon} - \langle Au_t, B^m\vartheta\rangle,\\
&\frac{1}{2}\frac{d}{dt}\|\eta\|^2_{\mathcal{M}^m_\varepsilon} = \langle T_\varepsilon \eta, \eta\rangle_{\mathcal{M}^m_\varepsilon} + \langle \eta, \vartheta\rangle_{\mathcal{M}^m_\varepsilon}.
\end{aligned}$$

Therefore, since operators A and B commute, and $\langle T_\varepsilon\eta, \eta\rangle_{\mathcal{M}^m_\varepsilon} \leq 0$ by virtue of (2.5) (see also (2.7)), adding the above identities we get

$$\frac{d}{dt}(\|B^{m/2}Au\|^2 + (1-c\omega)\|B^{m/2}u_t\|^2 + \omega\|B^{(m+1)/2}u_t\|^2 + \|B^{m/2}\vartheta\|^2 + \|\eta\|^2_{\mathcal{M}^m_\varepsilon}) \leq 0,$$

which yields the assertion thanks to the induction assumption and on account of the equivalence between the norms $\|A^{m/2}\cdot\|$ and $\|B^{m/2}\cdot\|$. $\square$

4.1 Proof of Theorem 4.1

Let $R \geq 0$, $T > 0$, and $z = (u_0, u_1, \vartheta_0, \eta_0) \in B_{\mathcal{H}^2_{\omega,\varepsilon}}(R)$. It is easy to realize that the assertion follows once we prove that there exist $K_R \geq 0$, independent of T, and $Q_{R,T} \geq 0$ such that

$$\|\mathbb{P}_\omega S_{\omega,\varepsilon}(t)z - S_{0,0}(t)\mathbb{P}_\omega z\|_{\mathcal{H}^0_{\omega,0}} \leq Q_{R,T}\sqrt{\omega} + K_R\sqrt[4]{\varepsilon}, \tag{4.1}$$

$$\|\mathbb{Q}_{\omega,\varepsilon} S_{\omega,\varepsilon}(t)z\|_{\mathcal{M}^0_\varepsilon} \leq \|\eta_0\|_{\mathcal{M}^0_\varepsilon} e^{-\frac{\delta t}{4\varepsilon}} + K_R\sqrt{\varepsilon}, \tag{4.2}$$

for every $t \in [0,T]$. Throughout the proof, we shall denote by $C \geq 0$ a generic constant which may even vary from line to line and may depend on R, but it is independent of T, ω, and ε. Let us set

$$\begin{aligned} \bar{u}(t) &= \hat{u}(t) - u(t), \qquad & \bar{u}_t(t) &= \hat{u}_t(t) - u_t(t), \\ \bar{\vartheta}(t) &= \hat{\vartheta}(t) - \vartheta(t), \qquad & \bar{\eta}^t &= \hat{\eta}^t - \eta^t, \end{aligned}$$

where $(\hat{u}, \hat{u}_t, \hat{\vartheta}, \hat{\eta})$ denotes the solution to $\mathcal{P}_{\omega,\varepsilon}$ with initial data z, while (u, u_t, ϑ) stands for the solution to $\mathcal{P}_{0,0}$ with initial data $\mathbb{P}_\omega z$. Besides, η^t is the solution at time t of the following Cauchy problem in $\mathcal{M}^0_\varepsilon$

$$\begin{cases} \eta_t = T_\varepsilon \eta + \vartheta, & t > 0, \\ \eta^0 = \eta_0, \end{cases}$$

which reconstructs the missing component of $S_{0,0}(t)$ (see [2]). Then, we can easily check that $(\bar{u}, \bar{u}_t, \bar{\vartheta}, \bar{\eta})$ solves

$$\begin{cases} \bar{u}_{tt} + \omega A \bar{u}_{tt} + \omega A u_{tt} + A(A\bar{u} - \bar{\vartheta}) = 0, \\ \bar{\vartheta}_t + \displaystyle\int_0^\infty \mu_\varepsilon(s) B \hat{\eta}(s) ds - B\vartheta + A\bar{u}_t = 0, \\ \bar{\eta}_t = T_\varepsilon \bar{\eta} + \bar{\vartheta}, \\ (\bar{u}(0), \bar{u}_t(0), \bar{\vartheta}(0), \bar{\eta}^0) = (0,0,0,0). \end{cases}$$

By multiplying the first equation of system $\mathcal{P}_{\omega,\varepsilon}$ by $\bar{u}_t$ in H^0, the second by $\bar{\vartheta}$ in H^0, and the third by $\bar{\eta}$ in $\mathcal{M}^0_\varepsilon$, we obtain, respectively,

$$\frac{1}{2}\frac{d}{dt}\big(\|A\bar{u}\|^2 + \|\bar{u}_t\|^2 + \omega\|A^{1/2}\bar{u}_t\|^2\big) + \omega\langle A^{1/2}u_{tt}, A^{1/2}\bar{u}_t\rangle - \langle A^{1/2}\bar{\vartheta}, A^{1/2}\bar{u}_t\rangle = 0,$$

$$\frac{1}{2}\frac{d}{dt}\|\bar{\vartheta}\|^2 + \langle \hat{\eta}, \bar{\vartheta}\rangle_{\mathcal{M}^0_\varepsilon} - \langle B^{1/2}\vartheta, B^{1/2}\bar{\vartheta}\rangle + \langle A^{1/2}\bar{u}_t, A^{1/2}\bar{\vartheta}\rangle = 0,$$

$$\frac{1}{2}\frac{d}{dt}\|\bar{\eta}\|^2_{\mathcal{M}^0_\varepsilon} - \langle T_\varepsilon\bar{\eta}, \bar{\eta}\rangle_{\mathcal{M}^0_\varepsilon} - \langle \bar{\eta}, \bar{\vartheta}\rangle_{\mathcal{M}^0_\varepsilon} = 0.$$

Taking (2.7) and (3.1) into account, and adding the above identities, we end up with

$$\frac{d}{dt}\big(\|A\bar{u}\|^2 + \|\bar{u}_t\|^2 + \omega\|A^{1/2}\bar{u}_t\|^2 + \|\bar{\vartheta}\|^2 + \|\bar{\eta}\|^2_{\mathcal{M}^0_\varepsilon}\big) \leq 2I + 2J,$$

where we have set

$$J(t) = -\omega\langle A^{1/2}u_{tt}(t), A^{1/2}\bar{u}_t(t)\rangle = -\omega\langle u_{tt}(t), A\bar{u}_t(t)\rangle,$$

$$I(t) = -\int_0^\infty \mu_\varepsilon(s)\langle B^{1/2}\eta(s), B^{1/2}\bar{\vartheta}(t)\rangle ds + \langle B^{1/2}\vartheta(t), B^{1/2}\bar{\vartheta}(t)\rangle = \sum_{j=1}^5 I_j(t),$$

being the I_js defined by

$$I_1(t) = \int_{\sqrt{\varepsilon}}^{\infty} s\mu_\varepsilon(s) \big\langle \vartheta(t), B\bar{\vartheta}(t)\big\rangle \, ds,$$

$$I_2(t) = -\int_{\sqrt{\varepsilon}}^{\infty} \mu_\varepsilon(s) \big\langle B^{1/2}\eta^t(s), B^{1/2}\bar{\vartheta}(t)\big\rangle \, ds,$$

$$I_3(t) = -\int_{\min\{\sqrt{\varepsilon},t\}}^{\sqrt{\varepsilon}} \mu_\varepsilon(s) \big\langle B^{1/2}\eta_0(s-t), B^{1/2}\bar{\vartheta}(t)\big\rangle \, ds,$$

$$I_4(t) = \int_{\min\{\sqrt{\varepsilon},t\}}^{\sqrt{\varepsilon}} (s-t)\mu_\varepsilon(s) \big\langle \vartheta(t), B\bar{\vartheta}(t)\big\rangle \, ds,$$

$$I_5(t) = \int_0^{\sqrt{\varepsilon}} \mu_\varepsilon(s) \bigg[\int_0^{\min\{s,t\}} \big\langle \vartheta(t) - \vartheta(t-y), B\bar{\vartheta}(t)\big\rangle \, dy \bigg] ds.$$

We turn to the estimate of the above terms. Notice that by Lemma 4.3 we have

$$\|A^2\hat{u}(t)\| + \|A\hat{u}_t(t)\| + \|B\hat{\vartheta}(t)\| \le C, \tag{4.3}$$

$$\|A^2 u(t)\| + \|Au_t(t)\| + \|B\vartheta(t)\| \le C. \tag{4.4}$$

Moreover, from the exponential stability of $S_{0,0}(t)$, there exists $\kappa > 0$ such that

$$\|Au(t)\| + \|u_t(t)\| + \|\vartheta(t)\| \le Ce^{-\kappa t}, \qquad \forall\, t \ge 0. \tag{4.5}$$

From the equation for ϑ of system $\mathcal{P}_{0,0}$, we obtain by comparison

$$\|\vartheta_t(t)\| \le c\|\vartheta(t)\| + \|A\vartheta(t)\| + \|Au_t(t)\|. \tag{4.6}$$

Besides, adding the equations for u and ϑ of system $\mathcal{P}_{0,0}$ yields

$$u_{tt} = -Au_t - A^2 u - \vartheta_t - c\vartheta,$$

so that we have

$$\|u_{tt}(t)\| \le \|Au_t(t)\| + \|A^2 u(t)\| + \|\vartheta_t(t)\| + c\|\vartheta(t)\|. \tag{4.7}$$

Then, by combining (4.4), (4.6), and (4.7), there holds

$$\|u_{tt}(t)\| \le C.$$

As a consequence, recalling again (4.3) and (4.4), we get

$$J(t) = -\omega\langle u_{tt}(t), A\bar{u}_t(t)\rangle \le \omega\|u_{tt}(t)\|\|A\bar{u}_t(t)\| \le C\omega.$$

For the treatment of the terms I_js, we proceed as in [2] but strengthening the estimates in light of the decay furnished by (4.5). Thus observe that, using the exponential decay implied by (3.1), we have

$$\int_{\sqrt{\varepsilon}}^{\infty} s\mu_\varepsilon(s) ds \le C\varepsilon, \qquad \forall\, \varepsilon > 0, \tag{4.8}$$

$$\int_{\sqrt{\varepsilon}}^{\infty} \mu_\varepsilon(s) ds \le C\sqrt{\varepsilon}, \qquad \forall\, \varepsilon > 0. \tag{4.9}$$

Hence, by (4.3)-(4.5) and (4.8), we immediately infer

$$I_1(t) \le C\varepsilon \|B\bar{\vartheta}(t)\| \|\vartheta(t)\| \le C\varepsilon e^{-\kappa t}$$

Let us now prove that there holds

$$\|\eta^t\|^2_{\mathcal{M}^0_\varepsilon} \le Ce^{-\frac{\delta t}{\varepsilon}} + C\sqrt{\varepsilon}\, e^{-\kappa t}, \qquad \forall\, t \ge 0, \tag{4.10}$$

Indeed, arguing as in the proof of [2, Lemma 5.4], it is readily seen that

$$\|\eta^t\|_{\mathcal{M}^1_\varepsilon} \le Ce^{-\frac{\delta t}{4\varepsilon}} + C\sqrt{\varepsilon}, \qquad \forall\, t \ge 0.$$

On the other hand, by multiplying the equation for η times η in $\mathcal{M}^0_\varepsilon$, in light of (4.5), we infer

$$\begin{aligned}
\frac{d}{dt}\|\eta^t\|^2_{\mathcal{M}^0_\varepsilon} + \frac{\delta}{\varepsilon}\|\eta^t\|^2_{\mathcal{M}^0_\varepsilon} &\le \|\vartheta(t)\| \int_0^\infty \mu_\varepsilon(s)\|B\eta^t(s)\|ds \\
&\le \frac{1}{\sqrt{\varepsilon}}\|\vartheta(t)\|\|\eta^t\|_{\mathcal{M}^1_\varepsilon} \\
&\le \frac{C}{\sqrt{\varepsilon}} e^{-(\kappa+\frac{\delta}{4\varepsilon})t} + Ce^{-\kappa t},
\end{aligned}$$

which yields (4.10) by the Gronwall Lemma. Now, thanks to (4.3), (4.4), (4.9), and (4.10), we obtain

$$\begin{aligned}
I_2(t) &\le C\int_{\sqrt{\varepsilon}}^\infty \mu_\varepsilon(s)\|B^{1/2}\eta^t(s)\|ds \\
&\le C\sqrt{\varepsilon}\|\eta^t\|_{\mathcal{M}^0_\varepsilon} \le C\sqrt{\varepsilon}\, e^{-\frac{\delta t}{2\varepsilon}} + C\sqrt{\varepsilon}\, e^{-\frac{\kappa}{2}t}.
\end{aligned}$$

Using (3.1) once again and taking advantage of (4.3), (4.4), we get, for $t < \sqrt{\varepsilon}$,

$$\begin{aligned}
I_3(t) &\le C\int_t^{\sqrt{\varepsilon}} \mu_\varepsilon(s)\|B^{1/2}\eta_0(s-t)\|ds \\
&\le Ce^{-\frac{\delta t}{\varepsilon}}\left(\int_0^\infty \mu_\varepsilon(s)ds\right)^{1/2}\|\eta_0\|_{\mathcal{M}^0_\varepsilon} \le \frac{C}{\sqrt{\varepsilon}}\, e^{-\frac{\delta t}{\varepsilon}}.
\end{aligned}$$

Arguing in a similar fashion and using (2.6),

$$I_4(t) \le Ce^{-\frac{\delta t}{\varepsilon}} \int_0^\infty s\mu_\varepsilon(s)ds = Ce^{-\frac{\delta t}{\varepsilon}}.$$

Observe now that, on account of (4.4)-(4.7), we have

$$\begin{aligned}
\|\vartheta(t) - \vartheta(t-y)\| &\le \|S_{0,0}(t-y)(S_{0,0}(y)\mathbb{P}_\omega z - \mathbb{P}_\omega z)\|_{\mathcal{H}^0_{0,0}} \\
&\le Ce^{-\kappa t}\int_0^y \|S_{0,0\,t}(\xi)\|_{\mathcal{H}^0_{0,0}} d\xi \\
&\le Ce^{-\kappa t} y,
\end{aligned}$$

for any $y \in [0,t]$, with $t \in [0,T]$. Hence, by (4.3) and (4.4), we infer, for every $t \in [0,T]$,

$$I_5(t) \le \|B\bar{\vartheta}(t)\| \int_0^{\sqrt{\varepsilon}} \mu_\varepsilon(s) \int_0^{\min\{s,t\}} \|\vartheta(t) - \vartheta(t-y)\| dy ds \le C\sqrt{\varepsilon}\, e^{-\kappa t}.$$

Therefore, by collecting the previous inequalities, we end up with

$$\frac{d}{dt}\big(\|A\bar{u}\|^2 + \|\bar{u}_t\|^2 + \omega\|A^{1/2}\bar{u}_t\|^2 + \|\bar{\vartheta}\|^2 + \|\bar{\eta}\|^2_{\mathcal{M}^0_\varepsilon}\big) \le C\left(\omega + \sqrt{\varepsilon}e^{-\frac{\kappa}{2}t} + \frac{1}{\sqrt{\varepsilon}}\, e^{-\frac{\delta t}{2\varepsilon}}\right).$$

Notice that, for every $t \in [0,T]$,

$$\int_0^t \Big[\sqrt{\varepsilon}\, e^{-\frac{\kappa}{2}\xi} + \frac{1}{\sqrt{\varepsilon}}\, e^{-\frac{\delta\xi}{2\varepsilon}}\Big] d\xi \le C\sqrt{\varepsilon}\,,$$

Consequently, by integrating the above differential inequality on $[0,T]$ we find $Q_{R,T} \ge 0$ and $K_R \ge 0$ independent of T such that

$$\|A\bar{u}\|^2 + \|\bar{u}_t\|^2 + \omega\|A^{1/2}\bar{u}_t\|^2 + \|\bar{\vartheta}\|^2 \le K_R\sqrt{\varepsilon} + Q_{R,T}\omega, \qquad \forall\, t \in [0,T], \tag{4.11}$$

which proves (4.1). Besides, by mimicking the proof of [2, Lemma 5.4], we easily recover inequality (4.2). The proof is now complete.

References

[1] G. Avalos, I. Lasiecka, *Exponential stability of a thermoelastic system with free boundary conditions without mechanical dissipation*, SIAM J. Math. Anal. **29** (1998), 155–182.

[2] M. Conti, V. Pata, M. Squassina, *Singular limit of differential systems with memory*, Indiana Univ. Math. J. (2005), to appear.

[3] M. Conti, V. Pata, M. Squassina, *Singular limit of dissipative hyperbolic equations with memory*, Discrete Contin. Dyn. Syst. (2005), to appear.

[4] M. Fabrizio, B. Lazzari, J.E. Muñoz Rivera, *Asymptotic behaviour of a thermoelastic plate of weakly hyperbolic type*, Differential Integral Equations **13** (2000), 1347–1370.

[5] C. Giorgi, M.G. Naso, V. Pata, *Energy decay of electromagnetic systems with memory*, Math. Models Methods Appl. Sci., to appear.

[6] C. Giorgi, V. Pata, *Stability of linear thermoelastic systems with memory*, Math. Models Methods Appl. Sci. **11** (2001), 627–644.

[7] M. Grasselli, J.E. Muñoz Rivera, V. Pata, *On the energy decay of the linear thermoelastic plate with memory*, J. Math. Anal. Appl., to appear.

[8] M. Grasselli, V. Pata, *Uniform attractors of nonautonomous systems with memory*, in "Evolution Equations, Semigroups and Functional Analysis" (A. Lorenzi and B. Ruf, Eds.), pp.155–178, Progr. Nonlinear Differential Equations Appl. no.50, Birkhäuser, Boston, 2002.

[9] M.E. Gurtin, A.C. Pipkin, *A general theory of heat conduction with finite wave speeds*, Arch. Rational Mech. Anal. **31** (1968), 113–126.

[10] A. Haraux, E. Zuazua, *Decay estimate for some damped hyperbolic equations*, Arch. Rational Mech. Anal. **100** (1988), 191–208.

[11] D.B. Henry, A. Perissinito, O. Lopes, *On the essential spectrum of a semigroup of thermoelasticity*, Nonlinear Anal. **21** (1993), 65–75.

[12] J. Lagnese, *Boundary stabilization of thin plates*, SIAM, Philadelphia, 1989.

[13] K. Liu, Z. Liu, *Exponential stability and analyticity of abstract linear thermoelastic systems*, Z. Angew. Math. Phys. **48** (1997), 885–904.

[14] Z.Y. Liu, M. Renardy, *A note on the equation of a thermoelastic plate*, Appl. Math. Lett. **8** (1995), 1–6.

[15] Z. Liu, S. Zheng, *Semigroups associated with dissipative systems*, Chapman & Hall/CRC, Boca Raton, 1999.

[16] J.U. Kim, *On the energy decay of a linear thermoelastic bar and plate*, SIAM J. Math. Anal. **23** (1992), 889–899.

[17] J.E. Muñoz Rivera, Y. Shibata, *A linear thermoelastic plate equation with Dirichlet boundary conditions*, Math. Meth. Appl. Sci. **20** (1997), 915–932.

[18] J.E. Muñoz Rivera, L.H. Fatori, *Regularizing properties and propagations of singularities for thermoelastic plates*, Math. Meth. Appl. Sci. **21** (1998), 797–821.

[19] J.E.. Muñoz Rivera, R. Racke, *Smoothing properties, decay and global existence of solutions to nonlinear coupled systems of thermoelastic type*, SIAM J. Math. Anal. **26** (1995), 1547–1563.

[20] J.E. Muñoz Rivera, R. Racke, *Large solution and smoothing properties for nonlinear thermoelastic systems*, J. Differential Equations **127** (1996), 454–483.

[21] Y. Shibata, *On the exponential decay of the energy of a linear thermoelastic plate*, Comp. Appl. Math. **13** (1994), 81–102.

[22] E. Zuazua, *Stability and decay for a class of nonlinear hyperbolic problems*, Asymptotic Anal. **1** (1998), 1–28.