



H^s versus C^0 -weighted minimizers

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Abstract. We study a class of semi-linear problems involving the fractional Laplacian under subcritical or critical growth assumptions. We prove that, for the corresponding functional, local minimizers with respect to a C^0 -topology weighted with a suitable power of the distance from the boundary are actually local minimizers in the natural H^s -topology.

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1. Introduction and main result

Let Ω be a bounded domain in \mathbb{R}^N , $N \geq 2$, with $C^{1,1}$ boundary $\partial\Omega$, and $s \in (0, 1)$. We consider the following boundary value problem driven by the fractional Laplacian operator

$$\begin{cases} (-\Delta)^s u = f(x, u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (1.1)$$

The fractional Laplacian operator is defined by

$$(-\Delta)^s u(x) := C(N, s) \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \quad x \in \mathbb{R}^N,$$

where $C(N, s)$ is a suitable positive normalization constant. The nonlinearity $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory mapping which satisfies the growth condition

$$|f(x, t)| \leq a(1 + |t|^{q-1}) \quad \text{a.e. in } \Omega \text{ and for all } t \in \mathbb{R} \quad (a > 0, 1 \leq q \leq 2_s^*) \quad (1.2)$$

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(here $2_s^* := 2N/(N - 2s)$ is the fractional critical exponent). Condition (1.2) is referred to as a *subcritical* or *critical* growth if $q < 2_s^*$ or $q = 2_s^*$, respectively.

For existence and multiplicity results for problem (1.1) via variational methods, see [25–28]. Concerning regularity and non-existence of solutions, we refer the reader to [6, 7, 9, 22–24] and to the references therein. Although the fractional Laplacian operator $(-\Delta)^s$, and more generally pseudodifferential operators, have been a classical topic of functional analysis since long ago, the interest for such operator has constantly increased in the last few years. Nonlocal operators such as $(-\Delta)^s$ naturally arise in continuum mechanics, phase transition phenomena, population dynamics and game theory, as they are the typical outcome of stochastic stabilization of Lévy processes, see e.g. the work of Caffarelli [8] and the references therein.

Problem (1.1) admits a variational formulation. For any measurable function $u : \mathbb{R}^N \rightarrow \mathbb{R}$ we define the Gagliardo seminorm by setting

$$[u]_s^2 := \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} dx dy,$$

and we introduce the fractional Sobolev space

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : [u]_s < \infty\},$$

which is a Hilbert space. We also define a closed subspace

$$X(\Omega) = \{u \in H^s(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}. \tag{1.3}$$

Due to the fractional Sobolev inequality, $X(\Omega)$ is a Hilbert space with inner product

$$\langle u, v \rangle_X = \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy, \tag{1.4}$$

which induces a norm $\|\cdot\|_X = [\cdot]_s$. Set for all $u \in X(\Omega)$

$$\Phi(u) := \frac{\|u\|_X^2}{2} - \int_{\Omega} F(x, u) dx,$$

where

$$F(x, t) = \int_0^t f(x, \tau) d\tau, \quad x \in \Omega, \quad t \in \mathbb{R}.$$

Then, $\Phi \in C^1(X(\Omega))$ and all its critical points are (up to a normalization constant depending on s and N , which we will neglect henceforth) weak solutions of (1.1), namely they satisfy

$$\langle u, v \rangle_X = \int_{\Omega} f(x, u)v dx, \quad \text{for all } v \in X(\Omega). \tag{1.5}$$

In the framework of variational methods, local minimizers of the energy Φ play a fundamental rôle. In a number of situations, one singles out particular solutions arising as constrained minimizers of the energy functional in order-defined subsets of $X(\Omega)$. Since usually the latters have empty interior, it is a nontrivial task to prove that such constrained minimizers are actually unconstrained local minimizers of the energy in the whole $X(\Omega)$.

This issue was analyzed by Brezis and Nirenberg [5] for the semilinear problem

$$\begin{cases} -\Delta u = f(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

They observe that the $C^1(\overline{\Omega})$ topology gives rise to nonempty interiors for most of such order-defined subsets. By the Hopf lemma, constrained minimizers solutions can be seen to lie in the C^1 -interior of the constraint set. The key point which they proved is that local minima with respect to the C^1 -topology remain so in the H^1 -one, despite the latter being much weaker than the former. Thus the constrained minimization procedure gives rise to solutions *which are also unconstrained local minimizers*.

This method was not only fruitfully applied to obtain a huge number of multiplicity results for the semilinear problem (1.6), but also extended to cover a wide range of variational equations.

In the present paper, we aim to develop all the tools needed to reproduce this technique in the fractional setting. In doing so we will gather a number of more or less known results for the fractional Laplacian, including weak and strong maximum principles, a Hopf lemma, and *a priori* estimates for the weak solution of problems of the type (1.1). We will provide a proof for those results for which only a statement was available, or strengthen the conclusions with respect to existing literature. In some cases, we will generalize results known only for special cases such as linear problems, eigenvalue problems, or positive solutions. Detailed discussion will be made for each result. We will then prove that being a local minimizer for Φ with respect to a suitable weighted C^0 -norm, is equivalent to being an $X(\Omega)$ -local minimizer. Particular attention will be paid to the *critical case*, i.e., $q = 2_s^*$ in (1.2), which presents a twofold difficulty: a loss of compactness which prevents minimization of Φ , and the lack of uniform *a priori* estimates for the weak solutions of (1.1). Finally we will give three different applications of this result to nonlocal semilinear problem: a sub-supersolution principle for local minimizers, a multiplicity result for singular nonlinearities, and a multiplicity result for smooth ones.

In order to state the local minimization result, we now describe the natural topology corresponding the C^1 -one above. Define $\delta : \overline{\Omega} \rightarrow \mathbb{R}_+$ by

$$\delta(x) := \text{dist}(x, \mathbb{R}^N \setminus \Omega), \quad x \in \overline{\Omega}, \tag{1.7}$$

and consider problem (1.1) with $f(x, u) = f(x)$ and $f \in L^\infty(\Omega)$. Ros Oton and Serra in [22] proved that a solution u to (1.1) is such that $u/\delta^s \in C^\alpha(\overline{\Omega})$. Thus, a natural topology for the fractional problem (1.1) seems to be the one of

$$C_\delta^0(\overline{\Omega}) = \left\{ u \in C^0(\overline{\Omega}) : \frac{u}{\delta^s} \text{ admits a continuous extension to } \overline{\Omega} \right\}$$

with norm $\|u\|_{0,\delta} = \|u/\delta^s\|_\infty$. Our main result establishes that indeed local minimizers of Φ in $C_\delta^0(\overline{\Omega})$ and in $X(\Omega)$ coincide:

Theorem 1.1. *Let Ω be a bounded $C^{1,1}$ domain, $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ a Carathéodory function satisfying (1.2), and $u_0 \in X(\Omega)$. Then, the following assertions are equivalent:*

- (i) *there exists $\rho > 0$ such that $\Phi(u_0 + v) \geq \Phi(u_0)$ for all $v \in X(\Omega) \cap C_\delta^0(\overline{\Omega})$, $\|v\|_{0,\delta} \leq \rho$,*
- (ii) *there exists $\varepsilon > 0$ such that $\Phi(u_0 + v) \geq \Phi(u_0)$ for all $v \in X(\Omega)$, $\|v\|_X \leq \varepsilon$.*

Notice that, contrary to the result of [5] in the local case $s = 1$, there is no relationship between the topologies of X and $C_\delta^0(\overline{\Omega})$.

The paper has the following structure: in Sect. 2 we establish some preliminary results, including the weak and strong maximum principles, and a fractional Hopf lemma; in Sect. 3 we prove *a priori* bounds for non-local problems, both in the subcritical and the critical cases; in Sect. 4 we prove Theorem 1.1; in Sect. 5 we give some applications of our main result; and in Sect. 6 we discuss possible extensions and developments.

Remark 1.2. After completing the present work, we became aware of an interesting paper of Barrios et al. [2], where a special case of Theorem 1.1 is obtained and used to study fractional boundary value problems involving pure power type nonlinearities with critical growth.

2. Preliminary results

In this section we will state and prove some basic results about weak (super)solutions of non-local boundary value problems.

For δ as in (1.7), we define the weighted Hölder-type spaces ($\alpha \in (0, 1)$)

$$\begin{aligned} C_\delta^0(\overline{\Omega}) &:= \left\{ u \in C^0(\overline{\Omega}) : \frac{u}{\delta^s} \text{ admits a continuous extension to } \overline{\Omega} \right\}, \\ C_\delta^{0,\alpha}(\overline{\Omega}) &:= \left\{ u \in C^0(\overline{\Omega}) : \frac{u}{\delta^s} \text{ admits a } \alpha\text{-Hölder continuous extension to } \overline{\Omega} \right\}, \end{aligned} \tag{2.1}$$

endowed with the norms

$$\|u\|_{0,\delta} := \left\| \frac{u}{\delta^s} \right\|_\infty, \quad \|u\|_{\alpha,\delta} := \|u\|_{0,\delta} + \sup_{x,y \in \overline{\Omega}, x \neq y} \frac{|u(x)/\delta(x)^s - u(y)/\delta(y)^s|}{|x - y|^\alpha},$$

respectively. Clearly, any function $u \in C_\delta^0(\overline{\Omega})$ vanishes on $\partial\Omega$, so it can be naturally extended by 0 on $\mathbb{R}^N \setminus \overline{\Omega}$. In this way, we will always consider elements of $C_\delta^0(\overline{\Omega})$ as defined on the whole \mathbb{R}^N . Moreover, by virtue of Ascoli’s theorem, the embedding $C_\delta^{0,\alpha}(\overline{\Omega}) \hookrightarrow C_\delta^0(\overline{\Omega})$ is compact.

The Hilbert space $X(\Omega)$ has been defined in (1.3), with inner product (1.4). The embedding $X(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for all $q \in [1, 2_s^*]$ and compact if $q \in [1, 2_s^*)$ (see [13, Theorem 7.1]). We will set

$$X(\Omega)_+ = \{u \in X(\Omega) : u \geq 0 \text{ a.e. in } \Omega\},$$

the definition of $H^s(\mathbb{R}^N)_+$ being analogous. For all $t \in \mathbb{R}$ we set

$$t_\pm = \max\{\pm t, 0\}.$$

Besides, for all $x \in \mathbb{R}^N$, $r > 0$ we denote by $B_r(x)$ (respectively, $\overline{B}_r(x)$) the open (respectively, closed) ball of radius r centered at x in \mathbb{R}^N . Similarly, $B_\rho^X(u)$, $\overline{B}_\rho^X(u)$ ($B_\rho^\delta(u)$, $\overline{B}_\rho^\delta(u)$) will denote an open and a closed ball, respectively, in $X(\Omega)$ (in $C_\delta^0(\overline{\Omega})$) centered at u with radius ρ . Finally, C will denote a positive constant whose value may change case by case.

We consider the following linear equation with general Dirichlet condition:

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \\ u = g & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.2}$$

where $f \in L^\infty(\Omega)$ and $g \in H^s(\mathbb{R}^N)$. We say that $u \in H^s(\mathbb{R}^N)$ is a *weak supersolution* of (2.2) if $u \geq g$ a.e. in $\mathbb{R}^N \setminus \Omega$ and the following holds for all $v \in X(\Omega)_+$:

$$\int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy \geq \int_{\Omega} f v dx.$$

The definition of a *weak subsolution* is analogous. Clearly, $u \in H^s(\mathbb{R}^N)$ is a weak solution of (2.2) if it is both a weak supersolution and a weak subsolution (this definition of a weak solution agrees with (1.5)). These definitions will be used throughout the paper.

From [12, proof of Theorem 1.1, Remark 4.2] we have the following bound.

Theorem 2.1. *Let $u \in H^s(\mathbb{R}^N)$ be a weak subsolution of (2.2) with $f = 0$. Then, there exists a constant $C = C(N, s)$ such that for any $k \in \mathbb{R}$, $x_0 \in \Omega$, $r > 0$ such that $B_r(x_0) \subseteq \Omega$, we have*

$$\operatorname{ess\,sup}_{B_{r/2}(x_0)} u \leq k + \operatorname{Tail}((u - k)_+; x_0, r/2) + C \left(\int_{B_r(x_0)} (u - k)_+^2 dx \right)^{\frac{1}{2}},$$

where the nonlocal tail of $v \in H^s(\mathbb{R}^N)$ at x_0 is defined by

$$\operatorname{Tail}(v; x_0, r) := r^{2s} \int_{\mathbb{R}^N \setminus B_r(x_0)} \frac{|v(x)|}{|x - x_0|^{N+2s}} dx.$$

The following lemma follows slightly modifying the proof of [22, Lemma 3.2]:

Lemma 2.2. *If $0 < r < R$, $f = 0$, and $g \in H^s(\mathbb{R}^N)$ is such that*

$$g(x) = \begin{cases} 1 & \text{if } x \in \overline{B}_r(0) \\ 0 & \text{if } x \in \mathbb{R}^N \setminus B_R(0), \end{cases}$$

then there exist $c = c(r, R) > 0$ and a weak solution $\varphi \in H^s(\mathbb{R}^N)$ of (2.2) with $f = 0$ in the domain $B_R(0) \setminus \overline{B}_r(0)$, such that a.e. in \mathbb{R}^N

$$\varphi(x) \geq c(R - |x|)_+^s.$$

In the following sections we will use the following fundamental regularity estimate proved in [22, Theorem 1.2].

Theorem 2.3. *Let u be a weak solution of (2.2) with $f \in L^\infty(\Omega)$, $g = 0$. Then there exist $\alpha \in (0, \min\{s, 1 - s\})$ such that $u \in C_\delta^{0,\alpha}(\bar{\Omega})$ (see (2.1)) and $C = C(\Omega, N, s)$ such that*

$$\|u\|_{\alpha,\delta} \leq C \|f\|_\infty.$$

We now prove a weak maximum principle for weak supersolutions of problem (2.2). While the non-negativity result is well known, we could not find a statement of the semicontinuity property in the literature.

Theorem 2.4. *If $u \in H^s(\mathbb{R}^N)$ is a weak supersolution of (2.2) with $f = 0$ and $g \in H^s(\mathbb{R}^N)_+$, then $u \geq 0$ a.e. in Ω and u admits a lower semi-continuous representative in Ω .*

Proof. First we prove that $u \in H^s(\mathbb{R}^N)_+$. Since $u \geq g \geq 0$ a.e. in $\mathbb{R}^N \setminus \Omega$, we have $u_- \in X(\Omega)_+$. So, the elementary inequality $(a - b)(a_- - b_-) \leq -(a_- - b_-)^2$, $a, b \in \mathbb{R}$, yields

$$0 \leq \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(u_-(x) - u_-(y))}{|x - y|^{N+2s}} dx dy \leq -[u_-]_s^2,$$

hence $u_- = 0$, namely $u \in H^s(\mathbb{R}^N)_+$. Now we find a lower semi-continuous function u^* such that $u(x) = u^*(x)$ a.e. in Ω . Set for all $x_0 \in \mathbb{R}^N$

$$u^*(x_0) = \operatorname{ess\,liminf}_{x \rightarrow x_0} u(x_0).$$

Since $u \in H^s(\mathbb{R}^N)_+$ we have $u^* \geq 0$ a.e. and u^* is lower semi-continuous in Ω . Now assume that $x_0 \in \Omega$ is a Lebesgue point for u and define $u(x_0)$ accordingly, noting that

$$u(x_0) := \lim_{r \rightarrow 0^+} \int_{B_r(x_0)} u dx \geq \lim_{r \rightarrow 0^+} \operatorname{ess\,inf}_{B_r(x_0)} u = u^*(x_0).$$

To prove the reverse inequality, we apply Theorem 2.1 to the function $-u$ (which is a weak subsolution of (2.2)) with $k = -u(x_0)$ and get

$$\begin{aligned} \operatorname{ess\,sup}_{B_{r/2}(x_0)}(-u) &\leq -u(x_0) + \operatorname{Tail}((u(x_0) - u)_+; x_0, r/2) \\ &+ C \left(\int_{B_r(x_0)} (u(x_0) - u(x))_+^2 dx \right)^{\frac{1}{2}}. \end{aligned}$$

Letting $r \rightarrow 0^+$, since x_0 is a Lebesgue point we have

$$\lim_{r \rightarrow 0^+} \left(\int_{B_r(x_0)} (u(x_0) - u(x))_+^2 dx \right)^{\frac{1}{2}} = 0.$$

Besides, by the Hölder inequality we have

$$\begin{aligned} &\operatorname{Tail}((u(x_0) - u)_+; x_0, r/2) \\ &\leq r^{2s} \left(\int_{\mathbb{R}^N \setminus B_r(x_0)} \frac{(u(x_0) - u(x))_+^2}{|x_0 - x|^{N+2s}} dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N \setminus B_r(x_0)} \frac{1}{|x_0 - x|^{N+2s}} dx \right)^{\frac{1}{2}} \\ &\leq Cr^s \left(\int_{\mathbb{R}^N} \frac{(u(x_0) - u(x))_+^2}{|x_0 - x|^{N+2s}} dx \right)^{\frac{1}{2}} \rightarrow 0 \quad \text{as } r \rightarrow 0^+, \end{aligned}$$

since, being $u \in H^s(\mathbb{R}^N)$, the integral is finite for a.e. $x_0 \in \Omega$. So we have

$$\lim_{r \rightarrow 0^+} \operatorname{ess\,sup}_{B_{r/2}(x_0)}(-u) \leq -u(x_0),$$

i.e. $u^*(x_0) \geq u(x_0)$ for a.e. Lebesgue point $x_0 \in \Omega$ for u , and hence for a.e. $x_0 \in \Omega$. \square

Henceforth any weak supersolution to (2.2), with $f = 0$, will be identified with its lower semi-continuous regularization, and any weak subsolution with its upper semi-continuous regularization, so that their value at any point is well defined.

By means of Theorem 2.4 and Lemma 2.2 we can prove the following strong maximum principle.

Theorem 2.5. *If $u \in H^s(\mathbb{R}^N) \setminus \{0\}$ is a weak supersolution of (2.2) with $f = 0$ and $g \geq 0$ a.e. in \mathbb{R}^N , then $u > 0$ in Ω .*

Proof. We argue by contradiction, assuming that u vanishes at some point of Ω . We recall that, by Theorem 2.4, $u \geq 0$ in Ω and u is lower semi-continuous. So, assuming without loss of generality that Ω is connected, the set

$$\Omega_+ = \{x \in \Omega : u(x) > 0\}$$

is open, nonempty and has a boundary in Ω . Pick $x_1 \in \partial\Omega_+ \cap \Omega$ and set $\delta(x_1) =: 2R > 0$. By lower semi-continuity and $u \geq 0$, we get $u(x_1) = 0$. We can find $x_0 \in \Omega_+ \cap B_R(x_1)$, and some $r \in (0, R)$ such that $u(x) \geq u(x_0)/2$ for all $x \in B_r(x_0)$. Let $\varphi \in H^s(\mathbb{R}^N)$ be as in Lemma 2.2, and set for all $x \in \mathbb{R}^N$

$$w(x) = u(x) - \frac{u(x_0)}{2}\varphi(x - x_0).$$

It is easily seen that $w \in H^s(\mathbb{R}^N)$ is a weak supersolution of (2.2) in the domain $B_R(x_0) \setminus \overline{B_r}(x_0)$, with $g = 0$. Hence, by Theorem 2.4 we have $w \geq 0$ a.e. in $B_R(x_0) \setminus \overline{B_r}(x_0)$. In particular, noting that $x_1 \in B_R(x_0) \setminus \overline{B_r}(x_0)$, we see that

$$u(x_1) \geq \frac{u(x_0)}{2}\varphi(x_1 - x_0) \geq \frac{u(x_0)}{2}c(R - |x_1 - x_0|)_+^s > 0$$

by Lemma 2.2, a contradiction. \square

Remark 2.6. It is worth noting that strong maximum principle type results for the fractional Laplacian were already known. A statement for smooth s -harmonic functions can be found in [10, Proposition 2.7]. The strong maximum principle was proved by Silvestre for distributional supersolutions but under a stronger semicontinuity and compactness condition, see [30, Proposition 2.17]. In [19, Lemma 12] the strong maximum principle was proved for viscosity supersolutions of the fractional p -Laplacian in the case $s < 1 - 1/p$. Recently in [3, Theorem A.1] a weaker statement ($u > 0$ almost everywhere without semicontinuity assumptions) has been proved through a logarithmic lemma for weak supersolutions of the fractional p -Laplacian.

We can now prove a fractional Hopf lemma. This has been first stated by Caffarelli et al. [10, Proposition 2.7] for smooth s -harmonic functions.

Lemma 2.7. *If $u \in H^s(\mathbb{R}^N) \setminus \{0\}$ is a weak supersolution of (2.2) with $f = 0$ and $g \geq 0$ a.e. in \mathbb{R}^N , then there exists $C = C(u) > 0$ such that $u(x) \geq C\delta(x)^s$ for all $x \in \bar{\Omega}$.*

Proof. Let

$$\Omega_h = \{x \in \bar{\Omega} : \delta(x) \leq h\}.$$

We know from Theorems 2.4, 2.5 that u is lower semi-continuous and $u(x) > 0$ in Ω . Hence, by reducing $C > 0$ if necessary, we only need to prove the lower bound on Ω_h , where it holds

$$\inf_{\bar{\Omega} \setminus \Omega_h} u = m_h > 0. \tag{2.3}$$

By classical results (see Aikawa et al. [1]) we know that $C^{1,1}$ -regularity of $\partial\Omega$ provides a uniform interior sphere condition. This in turn implies that there exists a sufficiently small $h > 0$ such that if $l \in (0, 2h]$ and $x \in \Omega_{2h}$

$$\delta(x) = l \iff B_l(x) \subseteq \Omega \tag{2.4}$$

and the metric projection $\Pi : \Omega_{2h} \rightarrow \partial\Omega$ is well defined. We fix such an h and for arbitrary $x_0 \in \Omega_h$ set $x_1 = \Pi(x_0)$, $x_2 = x_1 - 2h\nu(x_1)$, where $\nu : \partial\Omega \rightarrow \mathbb{R}^N$ is the outward unit vector. Then $\delta(x_2) \leq 2h$ by construction and through (2.4) we have $B_{2h}(x_2) \subseteq \Omega$, which forces $\delta(x_2) = 2h$. Let $\varphi \in H^s(\mathbb{R}^N)$ be defined as in Lemma 2.2 with $R = 2h$ and $r = h$ and set $v(x) = m_h\varphi(x - x_2)$ as per (2.3). For all $x \in \bar{B}_h(x_2) \subseteq \Omega \setminus \Omega_h$ we have

$$u(x) \geq m_h = v(x),$$

so $u - v$ is a weak supersolution of (2.2) in $B_{2h}(x_2) \setminus \bar{B}_h(x_2)$ with $f = g = 0$. By Theorem 2.4 we have $u \geq v$ in $B_{2h}(x_2) \setminus \bar{B}_h(x_2)$. In particular, we have

$$u(x_0) \geq v(x_0) \geq C(2h - |x_0 - x_2|)^s = C\delta(x_0)^s,$$

with $C > 0$ depending on h, m_h and Ω , which concludes the proof. □

3. A priori bounds

In this section we prove some a priori bounds for the weak solutions of problem (1.1), both in the subcritical and critical cases. We will use an adaptation of the classical Moser iteration technique. A similar method was used by Brasco et al. [4, Theorem 3.3] for the first eigenfunctions of the fractional Laplacian (in fact, for a more general, nonlinear operator, see Sect. 6 below), while most L^∞ -bounds for nonlocal equations are based on a different method, see [15, 17, 29]. A fractional version of De Giorgi’s iteration method was developed by Mingione [21]. We introduce some notation: for all $t \in \mathbb{R}$ and $k > 0$, we set

$$t_k = \operatorname{sgn}(t) \min\{|t|, k\}. \tag{3.1}$$

The Moser method in the fractional setting is based on the following elementary inequality:

Lemma 3.1. *For all $a, b \in \mathbb{R}$, $r \geq 2$, and $k > 0$ we have*

$$(a - b)(a|a|_k^{r-2} - b|b|_k^{r-2}) \geq \frac{4(r - 1)}{r^2}(a|a|_k^{\frac{r}{2}-1} - b|b|_k^{\frac{r}{2}-1})^2.$$

Proof. By the symmetry of the inequality, we may assume $a \geq b$. We set for all $t \in \mathbb{R}$

$$h(t) = \begin{cases} \operatorname{sgn}(t)|t|^{\frac{r}{2}-1} & \text{if } |t| < k \\ \frac{2}{r}\operatorname{sgn}(t)k^{\frac{r}{2}-1} & \text{if } |t| \geq k. \end{cases}$$

It is readily seen that

$$\int_b^a h(t) dt = \frac{2}{r}(a|a|_k^{\frac{r}{2}-1} - b|b|_k^{\frac{r}{2}-1})$$

and, since $4(r - 1) \leq r^2$, a similar computation gives

$$\int_b^a h(t)^2 dt \leq \frac{1}{r - 1}(a|a|_k^{r-2} - b|b|_k^{r-2}).$$

Now, the Schwartz inequality yields

$$\left(\int_b^a h(t) dt \right)^2 \leq (a - b) \int_b^a h(t)^2 dt,$$

which is the conclusion. □

We prove an L^∞ -bound on the weak solutions of (1.1) (in the subcritical case such bound is uniform):

Theorem 3.2. *If f satisfies (1.2), then for any weak solution $u \in X(\Omega)$ of (1.1) we have $u \in L^\infty(\Omega)$. Moreover, if $q < 2_s^*$ in (1.2), then there exists a function $M \in C(\mathbb{R}^+)$, only depending on the constants in (1.2), N , s and Ω , such that*

$$\|u\|_\infty \leq M(\|u\|_{2_s^*}).$$

Proof. Let $u \in X(\Omega)$ be a weak solution of (1.1) and set $\gamma = (2_s^*/2)^{1/2}$. For all $r \geq 2$, $k > 0$, the mapping $t \mapsto t|t|_k^{r-2}$ is Lipschitz in \mathbb{R} , hence $u|u|_k^{r-2} \in X(\Omega)$. We apply the fractional Sobolev inequality, Lemma 3.1, test (1.5) with $u|u|_k^{r-2}$, and we use (1.2) to obtain

$$\begin{aligned} \|u|u|_k^{\frac{r}{2}-1}\|_{2_s^*}^2 &\leq C\|u|u|_k^{\frac{r}{2}-1}\|_X^2 \leq \frac{Cr^2}{r-1}\langle u, u|u|_k^{r-2} \rangle_X \\ &\leq Cr \int_\Omega |f(x, u)||u||u|_k^{r-2} dx \\ &\leq Cr \int_\Omega (|u||u|_k^{r-2} + |u|^q|u|_k^{r-2}) dx, \end{aligned} \tag{3.2}$$

for some $C > 0$ independent of $r \geq 2$ and $k > 0$. Applying the Fatou Lemma as $k \rightarrow \infty$ yields

$$\|u\|_{\gamma^{2r}} \leq Cr^{\frac{1}{r}} \left(\int_\Omega (|u|^{r-1} + |u|^{r+q-2}) dx \right)^{\frac{1}{r}} \tag{3.3}$$

(where the right hand side may be ∞). Our aim is to develop from (3.3) a suitable bootstrap argument to prove that $u \in L^p(\Omega)$ for all $p \geq 1$. We define recursively a sequence $\{r_n\}$ by choosing $\mu > 0$ and setting

$$r_0 = \mu, \quad r_{n+1} = \gamma^2 r_n + 2 - q.$$

The only fixed point of $t \rightarrow \gamma^2 t + 2 - q$ is

$$\mu_0 = \frac{q - 2}{\gamma^2 - 1},$$

so we have $r_n \rightarrow +\infty$ iff $\mu > \mu_0$. We now split the proof into the subcritical and critical cases.

- *Subcritical case:* $q < 2_s^*$. We fix

$$\mu = 2_s^* + 2 - q > \max\{2, \mu_0\}, \tag{3.4}$$

and bootstrap on the basis of (3.3). Since $r_0 + q - 2 = 2_s^*$, we have $u \in L^{r_0+q-2}(\Omega)$ (in particular $u \in L^{r_0-1}(\Omega)$). Hence, choosing $r = r_0$ in (3.3), we obtain a finite right hand side, so $u \in L^{\gamma^2 r_0}(\Omega) = L^{r_1+q-2}(\Omega)$, and so on. Iterating this argument and noting that $r \mapsto r^{1/r}$ is bounded in $[2, \infty)$, for all $n \in \mathbb{N}$ we have $u \in L^{\gamma^2 r_n}(\Omega)$ and

$$\|u\|_{\gamma^2 r_n} \leq H(n, \|u\|_{2_s^*})$$

(henceforth, H will denote a continuous function of one or several real variables, whose definition may change case by case). By (3.4) we know that $\gamma^2 r_n \rightarrow \infty$ as $n \rightarrow \infty$, so for all $p \geq 1$ we can find $n \in \mathbb{N}$ such that $\gamma^2 r_n \geq p$. Applying Hölder inequality, for all $p \geq 1$ we have $u \in L^p(\Omega)$ and

$$\|u\|_p \leq H(p, \|u\|_{2_s^*}). \tag{3.5}$$

The L^p -bound above is not yet enough to prove our assertion, as the right hand side may not be bounded as $p \rightarrow \infty$. Thus, we need to improve (3.5) to a uniform L^p -bound. Set $\gamma' = \gamma/(\gamma - 1)$ and notice that from (3.5) and Hölder inequality it follows

$$\|1 + |u|^{q-1}\|_{\gamma'} \leq H(\|u\|_{2_s^*}).$$

Therefore, for any $r \geq 2$ we have

$$\begin{aligned} \int_{\Omega} (|u|^{r-1} + |u|^{r+q-2}) \, dx &\leq \|1 + |u|^{q-1}\|_{\gamma'} \| |u|^{r-1} \|_{\gamma} \leq H(\|u\|_{2_s^*}) \|u\|_{\gamma(r-1)}^{r-1} \\ &\leq H(\|u\|_{2_s^*}) |\Omega|^{\frac{1}{\gamma r}} \|u\|_{\gamma r}^{r-1}. \end{aligned}$$

Noting that $r \mapsto |\Omega|^{1/(\gamma r)}$ is bounded in $[2, \infty)$, we see that

$$\int_{\Omega} (|u|^{r-1} + |u|^{r+q-2}) \, dx \leq H(\|u\|_{2_s^*}) \|u\|_{\gamma r}^{r-1}.$$

The inequality above can be used in (3.3) to obtain the following estimate:

$$\|u\|_{\gamma^2 r}^r \leq H(\|u\|_{2_s^*}) \|u\|_{\gamma r}^{r-1}.$$

Setting $v = u/H(\|u\|_{2_s^*})$ and $r = \gamma^{n-1}$ ($\gamma^{n-1} \geq 2$ for $n \in \mathbb{N}$ big enough), we have the following nonlinear recursive relation:

$$\|v\|_{\gamma^{n+1}} \leq \|v\|_{\gamma^n}^{1-\gamma^{1-n}}$$

which, iterated, provides

$$\|v\|_{\gamma^n} \leq \|v\|_{\gamma}^{\prod_{i=0}^{n-2} (1-\gamma^{-i})} \quad n \in \mathbb{N}.$$

It is easily seen that the sequence $(\prod_{i=0}^{n-2}(1 - \gamma^{-i}))$ is bounded in \mathbb{R} , so for all $n \in \mathbb{N}$ we have

$$\|v\|_{\gamma^n} \leq H(\|u\|_{2_s^*}).$$

Going back to u , and recalling that $\gamma^n \rightarrow \infty$ as $n \rightarrow \infty$, we find $M \in C(\mathbb{R}_+)$ such that for all $p \geq 1$

$$\|u\|_p \leq M(\|u\|_{2_s^*}),$$

i.e., from classical results in functional analysis, $u \in L^\infty(\Omega)$ and

$$\|u\|_\infty \leq M(\|u\|_{2_s^*}). \tag{3.6}$$

- *Critical case:* $q = 2_s^*$. We start from (3.2), with $r = q + 1 > 2$, and fix $\sigma > 0$ such that $Cr\sigma < 1/2$. Then there exists $K_0 > 0$ (depending on u) such that

$$\left(\int_{\{|u|>K_0\}} |u|^q dx \right)^{1-\frac{2}{q}} \leq \sigma. \tag{3.7}$$

By Hölder inequality and (3.7) we have

$$\begin{aligned} \int_\Omega |u|^q |u|_k^{r-2} dx &\leq K_0^{q+r-2} |\{|u| \leq K_0\}| + \int_{\{|u|>K_0\}} |u|^q |u|_k^{r-2} dx \\ &\leq K_0^{q+r-2} |\Omega| + \left(\int_\Omega (u^2 |u|_k^{r-2})^{\frac{q}{2}} dx \right)^{\frac{2}{q}} \left(\int_{\{|u|>K_0\}} |u|^q dx \right)^{1-\frac{2}{q}} \\ &\leq K_0^{q+r-2} |\Omega| + \sigma \|u\|_k^{\frac{r}{2}-1} \|u\|_q^2. \end{aligned}$$

Recalling that $Cr\sigma < 1/2$, and that (3.2) holds, we obtain

$$\frac{1}{2} \|u\|_k^{\frac{q-1}{2}} \|u\|_q^2 \leq Cr(\|u\|_q^q + K_0^{2q-1} |\Omega|).$$

Letting $k \rightarrow \infty$, we have

$$\|u\|_{\frac{q(q+1)}{2}} \leq \tilde{H}(K_0, \|u\|_q)$$

(where, as above, \tilde{H} is a continuous function). Now the bootstrap argument can be applied through (3.3), starting with

$$r_0 = \mu = \frac{q(q+1)}{2} + 2 - q > \mu_0 = 2,$$

since $u \in L^{r_0+q-2}(\Omega)$. The rest of the proof follows *verbatim*, providing in the end $u \in L^\infty(\Omega)$ and

$$\|u\|_\infty \leq \tilde{M}(K_0, \|u\|_{2_s^*}) \tag{3.8}$$

for a convenient function $\tilde{M} \in C(\mathbb{R}^2)$. □

Remark 3.3. In the critical case $q = 2_s^*$, the uniform L^∞ -estimate (3.6) cannot hold true. We introduce the *fractional Talenti functions* by setting for all $\varepsilon > 0$ and $z \in \mathbb{R}^N$

$$\mathcal{T}_{\varepsilon,z}(x) = \left(\frac{\varepsilon}{\varepsilon^2 + |x - z|^2} \right)^{\frac{N-2s}{2}}.$$

It is readily seen that there exists $\Gamma(N, s) > 0$ such that, for all $\varepsilon > 0$ and $z \in \mathbb{R}^N$, $\Gamma(N, s)\mathcal{T}_{\varepsilon, z}$ is a positive solution of the fractional equation

$$(-\Delta)^s u = u^{\frac{N+2s}{N-2s}} \quad \text{in } \mathbb{R}^N, \tag{3.9}$$

Actually, in the local case $s = 1$, Chen et al. [11] have proved that $\mathcal{T}_{\varepsilon, z}$ are the *only* positive solutions of (3.9). We have $\|\mathcal{T}_{\varepsilon, z}\|_\infty \rightarrow \infty$ as $\varepsilon \rightarrow 0$ and, by rescaling, it follows that $\|\mathcal{T}_{\varepsilon, z}\|_{2_s^*}$ is independent of ε . If $z \in \Omega$, ε is very small (so that almost all the mass of $\mathcal{T}_{\varepsilon, z}$ is contained in Ω) and we truncate $\mathcal{T}_{\varepsilon, z}$ so that it is set equal to zero outside Ω , we would find that (3.6) is violated as $\varepsilon \rightarrow 0$. Thus, it seems that the non-uniform estimate (3.8), involving a real number $K_0 > 0$ such that (3.7) holds for a convenient $\sigma > 0$, cannot be improved in general.

4. Proof of Theorem 1.1

Proof that (i) implies (ii). We shall divide the proof into several steps:

Case $u_0 = 0$. We note that $\Phi(u_0) = 0$, so our hypothesis rephrases as

$$\inf_{u \in X(\Omega) \cap \overline{B}_\rho^s(0)} \Phi(u) = 0. \tag{4.1}$$

Again, we consider separately the subcritical and critical cases.

- *Subcritical case:* $q < 2_s^*$. We argue by contradiction, assuming that there exists a sequence (ε_n) in $(0, \infty)$ such that $\varepsilon_n \rightarrow 0$ and for all $n \in \mathbb{N}$

$$\inf_{u \in \overline{B}_{\varepsilon_n}^X(0)} \Phi(u) = m_n < 0.$$

By (1.2) and the compact embedding $X(\Omega) \hookrightarrow L^q(\Omega)$, the functional Φ is sequentially weakly lower semicontinuous in $X(\Omega)$, hence m_n is attained at some $u_n \in \overline{B}_{\varepsilon_n}^X(0)$ for all $n \in \mathbb{N}$. We claim that, for all $n \in \mathbb{N}$, there exists $\mu_n \leq 0$ such that for all $v \in X(\Omega)$

$$\langle u_n, v \rangle_X - \int_\Omega f(x, u_n)v \, dx = \mu_n \langle u_n, v \rangle_X. \tag{4.2}$$

Indeed, if $u_n \in B_{\varepsilon_n}^X(0)$, then u_n is a local minimizer of Φ in $X(\Omega)$, hence a critical point, so (4.2) holds with $\mu_n = 0$. If $u_n \in \partial B_{\varepsilon_n}^X(0)$, then u_n minimizes Φ restricted to the C^1 -Banach manifold

$$\left\{ u \in X(\Omega) : \frac{\|u\|_X^2}{2} = \frac{\varepsilon_n^2}{2} \right\},$$

so we can find a Lagrange multiplier $\mu_n \in \mathbb{R}$ such that (4.2) holds. More precisely, testing (4.2) with $-u_n$ and recalling that $\Phi(u) \geq \Phi(u_n)$ for all $u \in B_{\varepsilon_n}^X(0)$, we easily get

$$0 \leq \Phi' (u_n)(-u_n) = -\mu_n \|u_n\|_X^2,$$

hence $\mu_n \leq 0$.

Setting $C_n = (1 - \mu_n)^{-1} \in (0, 1]$, we see that for all $n \in \mathbb{N}$ the function $u_n \in X(\Omega)$ is a weak solution of the auxiliary boundary value problem

$$\begin{cases} (-\Delta)^s u = C_n f(x, u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where the nonlinearity satisfies (1.2) uniformly with respect to $n \in \mathbb{N}$. By Theorem 3.2 (and recalling that (u_n) is bounded in $L^{2_s^*}(\Omega)$), there exists $M > 0$ such that for all $n \in \mathbb{N}$ we have $u_n \in L^\infty(\Omega)$ with $\|u_n\|_\infty \leq M$. This, in turn, implies that for all $n \in \mathbb{N}$

$$\|C_n f(\cdot, u_n(\cdot))\|_\infty \leq a(1 + M^{q-1}).$$

Now we apply Theorem 2.3, which assures the existence of $\alpha > 0$ and $C > 0$ such that, for all $n \in \mathbb{N}$, we have $u_n \in C_\delta^{0,\alpha}(\bar{\Omega})$ with $\|u_n\|_{\alpha,\delta} \leq Ca(1 + M^{q-1})$. By the compact embedding $C_\delta^{0,\alpha}(\bar{\Omega}) \hookrightarrow C_\delta^0(\bar{\Omega})$, up to a subsequence, we see that (u_n) is strongly convergent in $C_\delta^0(\bar{\Omega})$, hence (by a simple computation) (u_n) is uniformly convergent in $\bar{\Omega}$. Since $u_n \rightarrow 0$ in $X(\Omega)$, passing to a subsequence, we may assume $u(x) \rightarrow 0$ a.e. in Ω , so we deduce $u_n \rightarrow 0$ in $C_\delta^0(\bar{\Omega})$. In particular, for $n \in \mathbb{N}$ big enough we have $\|u_n\|_{0,\delta} \leq \rho$ together with

$$\Phi(u_n) = m_n < 0,$$

a contradiction to (4.1).

- *Critical case:* $q = 2_s^*$. We need to overcome a twofold difficulty, as the critical growth both prevents compactness (and hence the existence of minimizers of Φ on closed balls of $X(\Omega)$), and does not allow to get immediately a uniform estimate on the L^∞ -norms of solutions of the auxiliary problem. Again we argue by contradiction, assuming that there exist sequences (ε_n) in $(0, \infty)$ and (w_n) in $X(\Omega)$ such that for all $n \in \mathbb{N}$ we have $w_n \in \bar{B}_{\varepsilon_n}^X(0)$ and $\Phi(w_n) < 0$. For all $k > 0$ we define $f_k, F_k : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ by setting for all $(x, t) \in \bar{\Omega} \times \mathbb{R}$

$$f_k(x, t) = f(x, t_k), \quad F_k(x, t) = \int_0^t f_k(x, \tau) d\tau$$

(t_k defined as in (3.1)). Accordingly, we define the functionals $\Phi_k \in C^1(X(\Omega))$ by setting for all $u \in X(\Omega)$

$$\Phi_k(u) = \frac{\|u\|_X^2}{2} - \int_\Omega F_k(x, u) dx.$$

By the dominated convergence Theorem, for all $u \in X(\Omega)$ we have $\Phi_k(u) \rightarrow \Phi(u)$ as $k \rightarrow \infty$. So, for all $n \in \mathbb{N}$ we can find $k_n \geq 1$ such that $\Phi_{k_n}(w_n) < 0$. Since f_k has subcritical growth, for all $n \in \mathbb{N}$ there exists $u_n \in \bar{B}_{\varepsilon_n}^X(0)$ such that

$$\Phi_{k_n}(u_n) = \inf_{u \in \bar{B}_{\varepsilon_n}^X(0)} \Phi_{k_n}(u) \leq \Phi_{k_n}(w_n) < 0.$$

As in the previous case we find a sequence (C_n) in $(0, 1]$ such that u_n is a weak solution of

$$\begin{cases} (-\Delta)^s u = C_n f_{k_n}(x, u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

and the nonlinearities $C_n f_{k_n}$ satisfy (1.2) uniformly with respect to $n \in \mathbb{N}$. We recall that $u_n \rightarrow 0$ in $X(\Omega)$, hence in $L^{2^*_s}(\Omega)$. So, (3.7) holds with $K_0 = 0$ and $n \in \mathbb{N}$ big enough. Therefore, Theorem 3.2 assures that $u_n \in L^\infty(\Omega)$ and that $\|u_n\|_\infty \leq M$ for some $M > 0$ independent of $n \in \mathbb{N}$. Now we can argue as in the subcritical case, proving that (up to a subsequence) $u_n \rightarrow 0$ in $C^0_\delta(\bar{\Omega})$ and uniformly in $\bar{\Omega}$. In particular, for $n \in \mathbb{N}$ big enough we have $\|u_n\|_{0,\delta} \leq \rho$ and $\|u_n\|_\infty \leq 1$, hence

$$\Phi(u_n) = \Phi_{k_n}(u_n) < 0,$$

a contradiction to (4.1).

Case $u_0 \neq 0$. For all $v \in C_c^\infty(\Omega)$, we have in particular $v \in X(\Omega) \cap C^0_\delta(\bar{\Omega})$, so the minimality ensures

$$\Phi'(u_0)(v) = 0, \quad v \in C_c^\infty(\Omega). \tag{4.3}$$

Since $C_c^\infty(\Omega)$ is a dense subspace of $X(\Omega)$ (see Fiscella et al. [14]) and $\Phi'(u_0) \in X(\Omega)^*$, equality (4.3) holds in fact for all $v \in X(\Omega)$, i.e., u_0 is a weak solution of (1.1). By Theorem 3.2, we have $u_0 \in L^\infty(\Omega)$, hence $f(\cdot, u_0(\cdot)) \in L^\infty(\Omega)$. Now Theorem 2.3 implies $u_0 \in C^0_\delta(\bar{\Omega})$. We set for all $(x, t) \in \Omega \times \mathbb{R}$

$$\tilde{F}(x, t) = F(x, u_0(x) + t) - F(x, u_0(x)) - f(x, u_0(x))t,$$

and for all $v \in X(\Omega)$

$$\tilde{\Phi}(v) = \frac{\|v\|_X^2}{2} - \int_\Omega \tilde{F}(x, v) \, dx.$$

Clearly we have $\tilde{\Phi} \in C^1(X(\Omega))$ and the mapping $\tilde{f} : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ defined by $\tilde{f}(x, t) = \partial_t \tilde{F}(x, t)$ satisfies a growth condition of the type (1.2). Besides, by (4.3), we have for all $v \in X(\Omega)$

$$\begin{aligned} \tilde{\Phi}(v) &= \frac{1}{2} (\|u_0 + v\|_X^2 - \|u_0\|_X^2) - \int_\Omega (F(x, u_0 + v) - F(x, u_0)) \, dx \\ &= \Phi(u_0 + v) - \Phi(u_0), \end{aligned}$$

in particular $\tilde{\Phi}(0) = 0$. Our hypothesis thus rephrases as

$$\inf_{v \in X(\Omega) \cap \bar{B}_\rho^\delta(0)} \tilde{\Phi}(v) = 0$$

and by the previous cases, we can find $\varepsilon > 0$ such that for all $v \in X(\Omega)$, $\|v\|_X \leq \varepsilon$, we have $\tilde{\Phi}(v) \geq 0$, namely $\Phi(u_0 + v) \geq \Phi(u_0)$.

Proof that (ii) implies (i). Suppose by contradiction that there exists a sequence (u_n) which converges to u_0 in $C^0_\delta(\bar{\Omega})$ and $\Phi(u_n) < \Phi(u_0)$. Observe that

$$\int_{\Omega} F(x, u_n) dx \rightarrow \int_{\Omega} F(x, u_0) dx,$$

and this, together with $\Phi(u_n) < \Phi(u_0)$, implies that

$$\limsup_n \|u_n\|_X^2 \leq \|u_0\|_X^2. \tag{4.4}$$

In particular (u_n) is bounded in $X(\Omega)$ and, up to a subsequence, it converges weakly and pointwisely to u_0 . By semicontinuity, (4.4) forces $\|u_n\|_X \rightarrow \|u_0\|_X$, thus $u_n \rightarrow u_0$ in X as $n \rightarrow \infty$, which concludes the proof. \square

5. Applications

In this section we present some existence and multiplicity results for the solutions of problem (1.1), under (1.2) plus some further conditions. In the proofs of such results, Theorem 1.1 will play an essential rôle.

Our first result ensures that, if problem (1.1) admits a weak subsolution and a weak supersolution, then it admits a solution which is also a local minimizer of the energy functional. We define weak super- and subsolutions of (1.1) as in Sect. 2.

Theorem 5.1. *Let $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function satisfying (1.2) and $f(x, \cdot)$ be nondecreasing in \mathbb{R} for a.a. $x \in \Omega$. Suppose that $\bar{u}, \underline{u} \in H^s(\mathbb{R}^N)$ are a weak supersolution and a weak subsolution, respectively, of (1.1) which are not solutions. Then, there exists a solution $u_0 \in X(\Omega)$ of (1.1) such that $\underline{u} \leq u_0 \leq \bar{u}$ a.e. in Ω and u_0 is a local minimizer of Φ on $X(\Omega)$.*

Proof. We first observe that $\underline{u} \leq \bar{u}$ a.e. in \mathbb{R}^N . Indeed, by monotonicity of $f(x, \cdot)$, $\bar{u} - \underline{u}$ is easily seen to be a weak supersolution of (2.2) with $f = g = 0$ and Theorem 2.4 forces $\bar{u} - \underline{u} \geq 0$. We set for all $(x, t) \in \Omega \times \mathbb{R}$

$$\tilde{f}(x, t) := \begin{cases} f(x, \underline{u}(x)) & \text{if } t \leq \underline{u}(x) \\ f(x, t) & \text{if } \underline{u}(x) < t < \bar{u}(x) \\ f(x, \bar{u}(x)) & \text{if } t \geq \bar{u}(x) \end{cases} \quad \tilde{F}(x, t) := \int_0^t \tilde{f}(x, \tau) d\tau$$

and for all $u \in X(\Omega)$

$$\tilde{\Phi}(u) := \frac{\|u\|_X^2}{2} - \int_{\Omega} \tilde{F}(x, u) dx.$$

The functional $\tilde{\Phi} \in C^1(X(\Omega))$ is sequentially weakly lower semicontinuous and coercive, since monotonicity of $f(x, \cdot)$, (1.2) and Hölder inequality imply for all $u \in X(\Omega)$

$$\int_{\Omega} \tilde{F}(x, u) dx \leq \int_{\Omega} (|f(x, \underline{u})| + |f(x, \bar{u})|) |u| dx \leq C(1 + \|\underline{u}\|_q^{q-1} + \|\bar{u}\|_q^{q-1}) \|u\|_X.$$

Let $u_0 \in X(\Omega)$ be a global minimizer of $\tilde{\Phi}$, which therefore solves

$$\begin{cases} (-\Delta)^s u_0 = \tilde{f}(x, u_0) & \text{in } \Omega \\ u_0 = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

By testing the above equation with $(u_0 - \bar{u})_+$ and $(u_0 - \underline{u})_-$, respectively, we easily see that $\underline{u} \leq u_0 \leq \bar{u}$ in Ω . Again by monotonicity and the definition of \tilde{f} , we have, in the weak sense,

$$(-\Delta)^s(\bar{u} - u_0) \geq f(x, \bar{u}) - \tilde{f}(x, u_0) \geq 0$$

in Ω , while $\bar{u} - u_0 \geq 0$ in $\mathbb{R}^N \setminus \Omega$, so $\bar{u} - u_0$ is a weak supersolution of (2.2), nonnegative by Theorem 2.4. It holds $\bar{u} - u_0 \neq 0$, otherwise we would have $\bar{u} \in X(\Omega)$ and, in the weak sense,

$$(-\Delta)^s \bar{u} = \tilde{f}(x, \bar{u}) = f(x, \bar{u})$$

in Ω , against our hypotheses on \bar{u} . By Lemma 2.7, we have $(\bar{u} - u_0)/\delta^s \geq C$ in $\bar{\Omega}$ for some $C > 0$. Similarly we prove that $(u_0 - \underline{u})/\delta^s \geq C$ in $\bar{\Omega}$. Thus, u_0 is a solution of (1.1).

Now we prove that u_0 is a local minimizer of Φ . By Theorems 3.2 and 2.3 we have $u_0 \in C_\delta^0(\bar{\Omega})$. For any $u \in \bar{B}_{C/2}^\delta(u_0)$ we have in $\bar{\Omega}$

$$\frac{\bar{u} - u}{\delta^s} = \frac{\bar{u} - u_0}{\delta^s} + \frac{u_0 - u}{\delta^s} \geq C - \frac{C}{2},$$

in particular $\bar{u} - u > 0$ in Ω . Similarly, $u - \underline{u} > 0$ in Ω , so $\tilde{\Phi}$ agrees with Φ in $\bar{B}_{C/2}^\delta(u_0) \cap X(\Omega)$ and u_0 turns out to be a local minimizer of Φ in $C_\delta^0(\bar{\Omega}) \cap X(\Omega)$. Now, Theorem 1.1 implies that u_0 is a local minimizer of Φ in $X(\Omega)$ as well. \square

We present now a multiplicity theorem for problem (1.1), whose proof combines Theorem 1.1, spectral properties of $(-\Delta)^s$ and Morse-theoretical methods (the fully nonlinear case is examined in [17, Theorem 5.3]). In what follows, $0 < \lambda_{1,s} < \lambda_{2,s} \leq \dots$ will denote the eigenvalues of $(-\Delta)^s$ in $X(\Omega)$ (see [26]).

Theorem 5.2. *Let $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function satisfying*

- (i) $|f(x, t)| \leq a(1 + |t|^{q-1})$ a.e. in Ω and for all $t \in \mathbb{R}$ ($a > 0, 1 < q < 2_s^*$);
- (ii) $f(x, t)t \geq 0$ a.e. in Ω and for all $t \in \mathbb{R}$;
- (iii) $\lim_{t \rightarrow 0} \frac{f(x, t) - b|t|^{r-2}t}{t} = 0$ uniformly a.e. in Ω ($b > 0, 1 < r < 2$);
- (iv) $\limsup_{|t| \rightarrow \infty} \frac{2F(x, t)}{t^2} < \lambda_{1,s}$ uniformly a.e. in Ω .

Then problem (1.1) admits at least three non-zero solutions.

Proof. We define $\Phi \in C^1(X(\Omega))$ as in the Introduction. From (ii),(iii) we immediately see that 0 is a critical point of Φ , which is not a local minimizer by [17, Lemma 5.5]. We introduce two truncated energy functionals, setting for all $(x, t) \in \Omega \times \mathbb{R}$

$$f_\pm(x, t) = f(x, \pm t_\pm), \quad F_\pm(x, t) = \int_0^t f_\pm(x, \tau) d\tau$$

and for all $u \in X(\Omega)$

$$\Phi_{\pm}(u) = \frac{\|u\|_X^2}{2} - \int_{\Omega} F_{\pm}(x, u) \, dx.$$

Clearly f_+ satisfies (1.2). It can be easily seen (see [17, Lemma 5.5]) that there exists $u^+ \in X(\Omega) \setminus \{0\}$ such that

$$\Phi_+(u^+) = \inf_{u \in X(\Omega)} \Phi_+(u).$$

Then, taking into account Theorem 2.4 and (ii), u^+ is a nonnegative weak solution to (1.1). By Theorem 3.2, we have $u^+ \in L^\infty(\Omega)$, so by Theorem 2.3 we deduce $u^+ \in C_\delta^0(\bar{\Omega})$. Moreover, again by (ii), u^+ is a weak supersolution of problem (2.2) with $f = g = 0$, hence by Lemma 2.7 $u^+/\delta^s > 0$ in $\bar{\Omega}$. Now [17, Lemma 5.1] implies that $u^+ \in \text{int}(C_+)$, where

$$C_+ = \{u \in C_\delta^0(\bar{\Omega}) : u(x) \geq 0 \text{ in } \bar{\Omega}\}$$

and the interior is defined with respect to the $C_\delta^0(\bar{\Omega})$ -topology. Let $\rho > 0$ be such that $B_\rho^\delta(u^+) \subset C_+$. Since Φ and Φ_+ agree on $C_+ \cap X(\Omega)$,

$$\Phi(u^+ + v) \geq \Phi(u^+), \quad v \in B_\rho^\delta(0) \cap X(\Omega)$$

and by Theorem 1.1, u^+ is a strictly positive local minimizer for Φ in $X(\Omega)$. Similarly, looking at Φ_- , we can detect another strictly negative local minimizer $u^- \in -\text{int}(C_+)$ of Φ . Now, a Morse-theoretic argument shows that there exists a further critical point $\tilde{u} \in X(\Omega)$ of Φ with $u \notin \{0, u^\pm\}$ (see the proof of [17, Theorem 5.3]). □

We conclude this section with a fractional version of a classical multiplicity result for semilinear problems based on Morse theory:

Theorem 5.3. *Let $f \in C^1(\mathbb{R})$ satisfy*

- (i) $|f'(t)| \leq a(1 + |t|^{q-2})$ for all $t \in \mathbb{R}$ ($a > 0, 1 < q \leq 2_s^*$);
- (ii) $f(t)t \geq 0$ and for all $t \in \mathbb{R}$;
- (iii) $f'(0) > \lambda_{2,s}$ and $f'(0)$ is not an eigenvalue of $(-\Delta)^s$ in $X(\Omega)$;
- (iv) $\limsup_{|t| \rightarrow \infty} \frac{f(t)}{t} < \lambda_{1,s}$.

Then problem (1.1) admits at least four non-zero solutions.

Proof. Due to (i), we have $\Phi \in C^2(X(\Omega))$, and by (iv) Φ is coercive. By (iii), we know that 0 is a nondegenerate critical point of Φ with Morse index $m \geq 2$ (see Li et al. [18, Proposition 1.1]). Therefore, reasoning as in the proof of Theorem 5.2, we find two local minimizers $u^\pm \in \pm C_+$ for Φ , with $u^+ > 0$ and $u^- < 0$ in Ω . Now, the Hess–Kato Theorem and a Morse-theoretic argument provide two further critical points $u_0, u_1 \in X(\Omega) \setminus \{0, u^\pm\}$ (as in Liu and Liu [20, Theorem 1.3]). □

6. Final comments and open questions

Let $p \in (1, \infty)$ and $s \in (0, 1)$. Recently, in [17], quasi-linear problems involving the *fractional p -Laplacian operator* were investigated by applying techniques of Morse theory to the functional

$$\Phi(u) = \frac{1}{p} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dx dy - \int_{\Omega} F(x, u) dx,$$

over the space of functions $u \in W^{s,p}(\mathbb{R}^N)$ with $u = 0$ outside Ω . Critical points of Φ give rise to nonlinear equations whose leading term is the *fractional p -Laplacian*, namely (up to a multiplicative constant)

$$(-\Delta)_p^s u(x) = \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} dy.$$

Recent contributions on the subject of the fractional p -Laplacian operator are also contained in [8, 12, 15, 19].

A natural question is whether a counterpart of Theorem 1.1 holds in this nonlinear setting. This would provide a nonlocal version of the results of García Azorero et al. [16], which extend the Brezis–Nirenberg theorem on local minimizers to nonlinear operators of the p -Laplacian type. Notice that the Moser iteration used in the proof of Theorem 3.2 seems flexible enough to carry over in the nonlinear case (with [4, Lemma C.2] replacing Lemma 3.1). Hence, the main difficulty seems to be the proof of a boundary regularity estimate for the boundedly inhomogeneous fractional p -Laplacian equation as the one of Theorem 2.3.

Another point of interest lies in the fractional Hopf Lemma. As seen in Sect. 5, the main point in focusing to $C_\delta^0(\bar{\Omega})$ local minimizers is the fact that many order-related subsets of $X(\Omega)$ turn out to have nonempty interior with respect to the $C_\delta^0(\bar{\Omega})$ -topology. As mentioned in the Introduction, this is in strong contrast with the features of the topology of $X(\Omega)$, and the main tool to exploit this difference is Lemma 2.7. It would be therefore interesting to explore the validity of such a statement for more general nonlocal operators, and for the fractional p -Laplacian in particular.

Finally, it is worth noting that in [5], the sub-supersolution principle analogous to Theorem 5.1 is proved under a more general hypothesis on the nonlinearity $f(x, t)$, namely

There exists $k \geq 0$ such that for a.e.
 $x \in \Omega$ the map $t \mapsto f(x, t) + kt$ is non-decreasing.

While we considered in Theorem 5.1 only non-decreasing nonlinearities, it seems that with little effort one can obtain the tools needed to treat the latter, more general, case. Indeed, it suffices to prove, for the operator $(-\Delta)^s u + ku$, $k \geq 0$, all the corresponding results of Sect. 2.

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