

GROUND STATES FOR SCALAR FIELD EQUATIONS WITH ANISOTROPIC NONLOCAL NONLINEARITIES

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ABSTRACT. We consider a class of scalar field equations with anisotropic nonlocal nonlinearities. We obtain a suitable extension of the well-known compactness lemma of Benci and Cerami to this variable exponent setting, and use it to prove that the Palais-Smale condition holds at all level below a certain threshold. We deduce the existence of a ground state when the variable exponent slowly approaches the limit at infinity from below.

1. Introduction and main results. The study of elliptic partial differential equations involving variable exponent has increased rapidly in recent years, partly in connection with the applications of such equations in physics (electrorheological and thermorheological fluids) and computer science (image processing), partly because of the purely mathematical interest into the functional-analytic setting of such equations (variable exponents Lebesgue and Sobolev spaces). We refer the reader to the monograph of Diening *et al.* [5], the survey paper of Fan and Zhao [6] and the references therein for an account of the main features of this subject.

In the present paper we seek *ground states*, namely least energy solutions, for the following nonlocal anisotropic scalar field equation:

$$-\Delta u + V(x)u = \lambda \frac{|u|^{p(x)-2}u}{\int_{\mathbb{R}^N} |u(x)|^{p(x)} dx}, \quad u \in H^1(\mathbb{R}^N). \quad (1)$$

Here, $N \geq 2$, $V \in L^\infty(\mathbb{R}^N)$ is a weight function satisfying

$$\lim_{|x| \rightarrow \infty} V(x) = V^\infty > 0, \quad (2)$$

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and the variable exponent $p \in C(\mathbb{R}^N)$ satisfies

$$2 < p^- := \inf_{x \in \mathbb{R}^N} p(x) \leq \sup_{x \in \mathbb{R}^N} p(x) =: p^+ < 2^*, \tag{3}$$

$$\lim_{|x| \rightarrow \infty} p(x) = p^\infty \tag{4}$$

($2^* = 2N/(N - 2)$ if $N \geq 3$ and $2^* = \infty$ if $N = 2$). Equation (1) is the Euler-Lagrange equation for the constrained C^1 functional $J|_{\mathcal{M}}$, where we denote for all $u \in H^1(\mathbb{R}^N)$

$$J(u) := \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) dx,$$

$$I(u) := \inf \left\{ \gamma > 0 : \int_{\mathbb{R}^N} \left| \frac{u(x)}{\gamma} \right|^{p(x)} \frac{dx}{p(x)} \leq 1 \right\},$$

and the C^1 Hilbert manifold (see Lemma 2.3 below)

$$\mathcal{M} := \{u \in H^1(\mathbb{R}^N) : I(u) = 1\},$$

namely $u \in H^1(\mathbb{R}^N)$ solves (1) if and only if u is a critical point of $J|_{\mathcal{M}}$ with $J(u) = \lambda$ (see Section 2 below for details). In particular, the ground states of (1) are the minimizers of $J|_{\mathcal{M}}$ and the corresponding energy level is

$$\lambda_1 := \inf_{u \in \mathcal{M}} J(u).$$

In other terms, (1) has a ground state if and only if λ_1 is attained.

The constant exponent case $p(x) \equiv p^\infty \in (2, 2^*)$ of equation (1) has been studied extensively for more than three decades (see Bahri and Lions [1] for a detailed account). Ground states are quite well understood in this case. Set for all $u \in H^1(\mathbb{R}^N)$

$$I^\infty(u) := \left[\int_{\mathbb{R}^N} |u(x)|^{p^\infty} \frac{dx}{p^\infty} \right]^{\frac{1}{p^\infty}}, \quad \mathcal{M}^\infty := \{u \in H^1(\mathbb{R}^N) : I^\infty(u) = 1\}.$$

The infimum

$$\tilde{\lambda}_1 := \inf_{u \in \mathcal{M}^\infty} J(u)$$

is not attained in general. Turning now to the asymptotic case when both p and V are constant, we note that the functional

$$J^\infty(u) := \int_{\mathbb{R}^N} (|\nabla u|^2 + V^\infty u^2) dx, \quad u \in H^1(\mathbb{R}^N)$$

attains its infimum

$$\lambda_1^\infty := \inf_{u \in \mathcal{M}^\infty} J^\infty(u) > 0 \tag{5}$$

at a positive radial function w_1^∞ (see Berestycki and Lions [3] and Byeon *et al.* [4]). Moreover, such minimizer is unique up to translations (see Kwong [8]). By (2) and the translation invariance of J^∞ , one can easily see that $\tilde{\lambda}_1 \leq \lambda_1^\infty$, and $\tilde{\lambda}_1$ is attained if this inequality is strict (see Lions [9, 10]).

In this paper we give sufficient conditions on the weight V and the exponent p for the existence of a ground state of (1). Precisely, we shall prove the following result:

Theorem 1.1. *Assume that $V \in L^\infty(\mathbb{R}^N)$ satisfies (2) and $p \in C(\mathbb{R}^N)$ satisfies (3) and (4). Then $-\infty < \lambda_1 \leq \lambda_1^\infty$. If*

$$\lambda_1 < \left(\frac{p^-}{p^\infty} \right)^{2/p^\infty} \lambda_1^\infty, \quad (6)$$

then λ_1 is attained at a positive minimizer $w_1 \in C^1(\mathbb{R}^N)$.

In particular, there exists a positive ground state if $p^\infty = p^-$ and $\lambda_1 < \lambda_1^\infty$ (hence our result is consistent with the constant exponent case). Hypothesis (6) is *global*, but it can be assured by making convenient *local* assumptions on p , for instance when $p(x)$ slowly approaches p^∞ from below as $|x| \rightarrow \infty$:

Theorem 1.2. *Assume that $V \in L^\infty(\mathbb{R}^N)$ satisfies (2) and $p \in C(\mathbb{R}^N)$ satisfies (3) and (4). Let $\psi \in C^1(\mathbb{R}^+, \mathbb{R}_0^+)$ be a mapping such that $\psi(r) \rightarrow \infty$ as $r \rightarrow \infty$ and the function $e^{-\psi(\cdot)} \in H^1(\mathbb{R}^N)$, and let $R > 0$. Then there exists a $a > 0$ such that, if*

$$p(x) \leq p^\infty - \frac{a}{\psi(|x|)}, \quad |x| \geq R, \quad (7)$$

then λ_1 is attained.

For example, given $R > 0$, we can find $a > 0$ such that, if

$$p(x) \leq p^\infty - ae^{-|x|}, \quad |x| \geq R,$$

then λ_1 is attained. We note that the only assumption on V in Theorem 1.2 is (2).

Finally, we address the problem of *symmetry* of minimizers. Apparently, the best we can achieve under the assumption of radial symmetry of the data V and p is axial symmetry of all ground states:

Corollary 1.3. *Assume that $N \geq 3$, $V \in L^\infty(\mathbb{R}^N)$ satisfies (2) and $p \in C(\mathbb{R}^N)$ satisfies (3) and (4), and both are radially symmetric in \mathbb{R}^N . Moreover, assume that (6) holds. Then, for every minimizer w there exist a line L through 0 and a function $\tilde{w} : L \times \mathbb{R}^+ \rightarrow \mathbb{R}$ such that*

$$w(x) = \tilde{w}(P_L(x), |x - P_L(x)|), \quad x \in \mathbb{R}^N,$$

where $P_L : \mathbb{R}^N \rightarrow L$ denotes the projection onto L .

As in the constant exponent case, the main difficulty is the lack of compactness inherent in this problem, which originates from the invariance of \mathbb{R}^N under the action of the noncompact group of translations, and manifests itself in the noncompactness of the embedding of $H^1(\mathbb{R}^N)$ into the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^N)$. This in turn implies that the manifold \mathcal{M} is not weakly closed in $H^1(\mathbb{R}^N)$ and that $J|_{\mathcal{M}}$ does not satisfy the Palais-Smale compactness condition (shortly (PS)_c) at all energy levels $c \in \mathbb{R}$. We will use the concentration compactness principle of Lions (see [9–11]), expressed as a suitable profile decomposition for (PS) sequences of $J|_{\mathcal{M}}$, to overcome these difficulties. Developing this argument, we will also prove an extension to the variable exponent case of the compactness lemma of Benci and Cerami [2] (see Proposition 3.4 below).

The paper has the following structure: in Section 2 we introduce the mathematical background and establish some technical lemmas; in Section 3 we prove that (PS)_c holds for all c below a threshold level; and in Section 4 we deliver the proofs of our main results.

2. Preliminaries. We consider $H^1(\mathbb{R}^N)$ endowed with the norm defined by

$$\|u\|^2 := \int_{\mathbb{R}^N} \left(|\nabla u(x)|^2 + V^\infty |u(x)|^2 \right) dx, \quad u \in H^1(\mathbb{R}^N),$$

which is equivalent to the standard norm. Clearly we have $J \in C^1(H^1(\mathbb{R}^N))$ with

$$\langle J'(u), v \rangle = 2 \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x) uv) dx, \quad u, v \in H^1(\mathbb{R}^N).$$

We recall some basic features from the theory of variable exponent Lebesgue space, referring the reader to the book of Diening *et al.* [5] for further information. Let $p \in C(\mathbb{R}^N)$ satisfy (3) and (4). The space $L^{p(\cdot)}(\mathbb{R}^N)$ contains all the measurable functions $u : \mathbb{R}^N \rightarrow \mathbb{R}$ such that

$$\rho(u) := \int_{\mathbb{R}^N} |u(x)|^{p(x)} dx < \infty.$$

This is a reflexive Banach space under the following modified Luxemburg norm, introduced by Franzina and Lindqvist [7]:

$$\|u\|_{p(\cdot)} := \inf \left\{ \gamma > 0 : \int_{\mathbb{R}^N} \left| \frac{u(x)}{\gamma} \right|^{p(x)} \frac{dx}{p(x)} \leq 1 \right\}, \quad u \in L^{p(\cdot)}(\mathbb{R}^N).$$

The following relation will be widely used in our study:

$$p^- \min \left\{ \|u\|_{p(\cdot)}^+, \|u\|_{p(\cdot)}^- \right\} \leq \rho(u) \leq p^+ \max \left\{ \|u\|_{p(\cdot)}^+, \|u\|_{p(\cdot)}^- \right\}. \tag{8}$$

It can be proved noting that, for all $u \in L^{p(\cdot)}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \left| \frac{u(x)}{\|u\|_{p(\cdot)}} \right|^{p(x)} \frac{dx}{p(x)} = 1. \tag{9}$$

Analogously, for all $q > 1$ we endow the constant exponent Lebesgue space $L^q(\mathbb{R}^N)$ with the norm

$$\|u\|_q^q := \int_{\mathbb{R}^N} |u(x)|^q \frac{dx}{q}, \quad u \in L^q(\mathbb{R}^N).$$

By (3) and [5, Theorem 3.3.11], the embedding $L^{p^+}(\mathbb{R}^N) \cap L^{p^-}(\mathbb{R}^N) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^N)$ is continuous. So, by the Sobolev embedding theorem, also $H^1(\mathbb{R}^N) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^N)$ is continuous.

We set

$$\sigma(u) := \max \{ \|u\|_{p(\cdot)}^{p^- - 1}, \|u\|_{p(\cdot)}^{p^+ - 1} \}, \quad u \in L^{p(\cdot)}(\mathbb{R}^N),$$

and we prove the following properties that will be used later:

Lemma 2.1. *For all $u, v \in L^{p(\cdot)}(\mathbb{R}^N)$ we have*

- (i) $\int_{\mathbb{R}^N} |u(x)|^{p(x)-1} |v(x)| dx \leq p^+ \sigma(u) \|v\|_{p(\cdot)};$
- (ii) $|\rho(u) - \rho(v)| \leq (p^+)^2 (\sigma(u) + \sigma(v)) \|u - v\|_{p(\cdot)}.$

Proof. We prove (i). Taking $a = (|u(x)|/\|u\|_{p(\cdot)})^{p(x)-1}$, $b = |v(x)|/\|v\|_{p(\cdot)}$ and $q = p(x)$ in the well-known Young's inequality

$$ab \leq \left(1 - \frac{1}{q}\right) a^{q/(q-1)} + \frac{1}{q} b^q \quad a, b \geq 0, q > 1, \tag{10}$$

and integrating over \mathbb{R}^N gives (note that $\|u\|_{p(\cdot)}^{p(x)-1} \leq \sigma(u)$ in \mathbb{R}^N)

$$\begin{aligned} & \frac{1}{\sigma(u) \|v\|_{p(\cdot)}} \int_{\mathbb{R}^N} |u(x)|^{p(x)-1} |v(x)| dx \\ & \leq \int_{\mathbb{R}^N} \left| \frac{u(x)}{\|u\|_{p(\cdot)}} \right|^{p(x)-1} \frac{|v(x)|}{\|v\|_{p(\cdot)}} dx \\ & \leq \int_{\mathbb{R}^N} \left(1 - \frac{1}{p(x)}\right) \left| \frac{u(x)}{\|u\|_{p(\cdot)}} \right|^{p(x)} dx + \int_{\mathbb{R}^N} \left| \frac{v(x)}{\|v\|_{p(\cdot)}} \right|^{p(x)} \frac{dx}{p(x)} \\ & \leq p^+, \end{aligned}$$

the last inequality following from (9). Now we prove (ii). Taking $a = |u(x)|$, $b = |v(x)|$, $q = p(x)$ in the elementary inequality

$$|a^q - b^q| \leq q(a^{q-1} + b^{q-1})|a - b|, \quad a, b \geq 0, q \geq 2,$$

and integrating over \mathbb{R}^N gives

$$\begin{aligned} |\rho(u) - \rho(v)| & \leq \int_{\mathbb{R}^N} \left| |u(x)|^{p(\cdot)} - |v(x)|^{p(\cdot)} \right| dx \\ & \leq p^+ \left(\int_{\mathbb{R}^N} |u(x)|^{p(x)-1} ||u(x)| - |v(x)|| dx + \int_{\mathbb{R}^N} |v(x)|^{p(x)-1} ||u(x)| - |v(x)|| dx \right) \\ & \leq (p^+)^2 (\sigma(u) + \sigma(v)) \|u - v\|_{p(\cdot)}, \end{aligned}$$

the last inequality following from (i). □

From Lemma 2.1 (ii) it follows that, if $(u_k), (v_k)$ are bounded sequences in $L^{p(\cdot)}(\mathbb{R}^N)$, then there exists $C > 0$ such that

$$|\rho(u_k) - \rho(v_k)| \leq C \|u_k - v_k\|_{p(\cdot)}, \quad k \in \mathbb{N}.$$

The case $V(x) \equiv V^\infty, p(x) \equiv p^\infty$ represents a limit case for (1). We set

$$\rho^\infty(u) = \int_{\mathbb{R}^N} |u(x)|^{p^\infty} dx, \quad \sigma^\infty(u) = \|u\|_{p^\infty}^{p^\infty-1}, \quad u \in L^{p^\infty}(\mathbb{R}^N),$$

so Lemma 2.1 gives for all $u, v \in L^{p^\infty}(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} |u(x)|^{p^\infty-1} |v(x)| dx \leq p^\infty \sigma^\infty(u) \|v\|_{p^\infty}, \tag{11}$$

$$|\rho^\infty(u) - \rho^\infty(v)| \leq (p^\infty)^2 (\sigma^\infty(u) + \sigma^\infty(v)) \|u - v\|_{p^\infty}. \tag{12}$$

Moreover, we have the following asymptotic laws in the presence of translations:

Lemma 2.2. *If $u \in H^1(\mathbb{R}^N)$, (y_k) is a sequence in \mathbb{R}^N with $|y_k| \rightarrow \infty$ and we set $u_k = u(\cdot - y_k)$, then*

- (i) $\rho(u_k) \rightarrow \rho^\infty(u)$,
- (ii) $I(u_k) \rightarrow I^\infty(u)$,
- (iii) $J(u_k) \rightarrow J^\infty(u)$.

Proof. We prove (i). For all $k \in \mathbb{N}$, the change of variable $z = x - y_k$ gives

$$\rho(u_k) = \int_{\mathbb{R}^N} |u_k(x)|^{p(x)} dx = \int_{\mathbb{R}^N} |u(z)|^{p(z+y_k)} dz.$$

Since $p(\cdot + y_k) \rightarrow p^\infty$ by (4) and $|u(z)|^{p(z+y_k)} \leq |u(z)|^{p^-} + |u(z)|^{p^+}$, the last integral converges to $\rho^\infty(u)$ by the dominated convergence theorem.

We prove (ii). As in the proof of (i), for all $\gamma > 0$

$$\int_{\mathbb{R}^N} \left| \frac{u_k(x)}{\gamma} \right|^{p(x)} \frac{dx}{p(x)} \rightarrow \int_{\mathbb{R}^N} \left| \frac{u(z)}{\gamma} \right|^{p^\infty} \frac{dz}{p^\infty} = \left(\frac{\|u\|_{p^\infty}}{\gamma} \right)^{p^\infty}. \tag{13}$$

If $\|u_k\|_{p(\cdot)} \not\rightarrow \|u\|_{p^\infty}$, then there exists $\varepsilon_0 > 0$ such that, on a renumbered subsequence, either $\|u_k\|_{p(\cdot)} \leq \|u\|_{p^\infty} - \varepsilon_0$ or $\|u_k\|_{p(\cdot)} \geq \|u\|_{p^\infty} + \varepsilon_0$. In the former case, $\|u\|_{p^\infty} \geq \varepsilon_0$ and, taking ε_0 smaller if necessary, we may assume that this inequality is strict. Then

$$\int_{\mathbb{R}^N} \left(\frac{|u_k(x)|}{\|u\|_{p^\infty} - \varepsilon_0} \right)^{p(x)} \frac{dx}{p(x)} \leq \int_{\mathbb{R}^N} \left(\frac{|u_k(x)|}{\|u_k\|_{p(\cdot)}} \right)^{p(x)} \frac{dx}{p(x)} = 1.$$

Passing to the limit as $k \rightarrow \infty$, (13) implies

$$\left(\frac{\|u\|_{p^\infty}}{\|u\|_{p^\infty} - \varepsilon_0} \right)^{p^\infty} \leq 1,$$

a contradiction. The latter case leads to a similar contradiction.

Finally we prove (iii). We have for all $k \in \mathbb{N}$

$$J(u_k) = \int_{\mathbb{R}^N} (|\nabla u_k|^2 + V(x) u_k^2) dx = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(z + y_k) u^2) dz,$$

where $z = x - y_k$. Since $V(\cdot + y_k) \rightarrow V^\infty$ by (2) and $V \in L^\infty(\mathbb{R}^N)$, the last integral converges to $J^\infty(u)$ by the dominated convergence theorem. \square

As a consequence of the previous results, the functional $I : H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ is well defined and continuous. We now address the question of differentiability of I . We set

$$\langle A(u), v \rangle = \frac{\int_{\mathbb{R}^N} \left| \frac{u(x)}{I(u)} \right|^{p(x)-2} \frac{u(x)}{I(u)} v(x) dx}{\int_{\mathbb{R}^N} \left| \frac{u(x)}{I(u)} \right|^{p(x)} dx}, \quad u \in H^1(\mathbb{R}^N) \setminus \{0\}, v \in H^1(\mathbb{R}^N).$$

Lemma 2.3. $I \in C^1(H^1(\mathbb{R}^N) \setminus \{0\})$ with $I'(u) = A(u)$ for all $u \in H^1(\mathbb{R}^N) \setminus \{0\}$.

Proof. We fix $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ and $v \in H^1(\mathbb{R}^N)$. Reasoning as in [7, Lemma A.1], we see that

$$\lim_{\varepsilon \rightarrow 0^+} \frac{I(u + \varepsilon v) - I(u)}{\varepsilon} = \langle A(u), v \rangle.$$

Moreover, setting $w = u/I(u) \in H^1(\mathbb{R}^N)$, by Lemma 2.1 (i) we have

$$|\langle A(u), v \rangle| \leq \frac{\rho^+(w)}{\rho(w)} \|v\|_{p(\cdot)},$$

which, together with the continuous embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^N)$, yields $A(u) \in H^{-1}(\mathbb{R}^N)$. So, I is Gâteaux differentiable in u with $I'(u) = A(u)$.

Now we prove that $I' : H^1(\mathbb{R}^N) \rightarrow H^{-1}(\mathbb{R}^N)$ is continuous. Let (u_k) be a sequence in $H^1(\mathbb{R}^N) \setminus \{0\}$ such that $u_k \rightarrow u \neq 0$ in $H^1(\mathbb{R}^N)$, in particular we have $u_k \rightarrow u$ and in $L^{p(\cdot)}(\mathbb{R}^N)$. Set $w_k = u_k/I(u_k)$, $w = u/I(u)$, hence $w_k \rightarrow w$ both in $H^1(\mathbb{R}^N)$ in $L^{p(\cdot)}(\mathbb{R}^N)$. Besides, Lemma 2.1 (ii) implies $\rho(w_k) \rightarrow \rho(w) > 0$. Set also

$$z_k = \left| |w_k|^{p(x)-2} w_k - |w|^{p(x)-2} w \right|^{1/(p(x)-1)}.$$

It is easily seen that $z_k \in L^{p(\cdot)}(\mathbb{R}^N)$ and $\sigma(z_n) \rightarrow 0$. By Lemma 2.1 (i), for all $k \in \mathbb{N}$ and all $v \in H^1(\mathbb{R}^N)$ we have

$$\begin{aligned} |\langle I'(u_k) - I'(u), v \rangle| &= \left| \frac{\int_{\mathbb{R}^N} |w_k|^{p(x)-2} w_k v \, dx}{\rho(w_k)} - \frac{\int_{\mathbb{R}^N} |w|^{p(x)-2} w v \, dx}{\rho(w)} \right| \\ &\leq \frac{\int_{\mathbb{R}^N} |z_k|^{p(x)-1} |v| \, dx}{\rho(w_k)} + \int_{\mathbb{R}^N} |w|^{p(x)-1} |v| \, dx \left| \frac{1}{\rho(w_k)} - \frac{1}{\rho(w)} \right| \\ &\leq \left(\frac{p^+ \sigma(z_n)}{\rho(w_k)} + \frac{p^+ \sigma(w) |\rho(w_k) - \rho(w)|}{\rho(w_k) \rho(w)} \right) \|v\|_{p(\cdot)}. \end{aligned}$$

By the convergences above and the continuous embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^N)$, we can find a sequence (ξ_k) in \mathbb{R} , independent of v , such that $\xi_k \rightarrow 0^+$ and

$$|\langle I'(u_k) - I'(u), v \rangle| \leq \xi_k \|v\|,$$

thus $I'(u_k) \rightarrow I'(u)$ in $H^{-1}(\mathbb{R}^N)$. □

In particular, as 1 is a regular value of I , \mathcal{M} turns out to be a C^1 Hilbert manifold. By the Lagrange multiplier rule, $u \in \mathcal{M}$ is a critical point of $J|_{\mathcal{M}}$ if and only if there exists $\mu \in \mathbb{R}$ such that

$$J'(u) = \mu I'(u) \quad \text{in } H^{-1}(\mathbb{R}^N),$$

that is (recalling that $I(u) = 1$), if and only if u is a (weak) solution of (1) with $\lambda = \mu/2$. Moreover, testing (1) with u yields $J(u) = \lambda$.

3. A compactness result. In this section we prove that $J|_{\mathcal{M}}$ satisfies $(PS)_c$ whenever $c \in \mathbb{R}$ lies below a certain threshold level. The main technical tool that we will use for handling the convergence matters is the following profile decomposition due to Solimini [13] for bounded sequences in $H^1(\mathbb{R}^N)$ (the present statement can be found in the book of Tintarev and Fieseler [15, Corollary 3.3]).

Proposition 3.1. *Let (u_k) be a bounded sequence in $H^1(\mathbb{R}^N)$, and assume that there is a constant $\delta > 0$ such that, if $u_k(\cdot + y_k) \rightharpoonup w \neq 0$ on a renumbered subsequence for some sequence (y_k) in \mathbb{R}^N with $|y_k| \rightarrow \infty$, then $\|w\| \geq \delta$. Then there exist $m \in \mathbb{N}$, $w^{(1)}, \dots, w^{(m)} \in H^1(\mathbb{R}^N)$, and sequences $(y_k^{(1)}), \dots, (y_k^{(m)})$ in \mathbb{R}^N , $y_k^{(1)} = 0$ for all $k \in \mathbb{N}$, $w^{(n)} \neq 0$ for all $2 \leq n \leq m$, such that, on a renumbered subsequence,*

- (i) $u_k(\cdot + y_k^{(n)}) \rightharpoonup w^{(n)}$;
- (ii) $|y_k^{(n)} - y_k^{(l)}| \rightarrow \infty$ for all $n \neq l$;
- (iii) $\sum_{n=1}^m \|w^{(n)}\|^2 \leq \liminf_k \|u_k\|^2$;
- (iv) $u_k - \sum_{n=1}^m w^{(n)}(\cdot - y_k^{(n)}) \rightarrow 0$ in $L^q(\mathbb{R}^N)$ for all $q \in (2, 2^*)$.

Since $y_k^{(1)} = 0$ for all $k \in \mathbb{N}$, by (ii) we have $|y_k^{(n)}| \rightarrow \infty$ for all $2 \leq n \leq m$. Moreover, by (iv), (3) and [5, Theorem 3.3.11], we also have

$$u_k - \sum_{n=1}^m w^{(n)}(\cdot - y_k^{(n)}) \rightarrow 0 \quad \text{in } L^{p(\cdot)}(\mathbb{R}^N). \tag{14}$$

Next we show that the sublevel sets of $J|_{\mathcal{M}}$ are bounded. Set

$$S^a = \{u \in \mathcal{M} : J(u) \leq a\}, \quad a \in \mathbb{R}.$$

Lemma 3.2. *For all $a \in \mathbb{R}$, S^a is bounded.*

Proof. By (2), we can find $R > 0$ such that $|V(x) - V^\infty| < V^\infty/2$ for all $|x| > R$. So, $V^\infty/2 - V \leq 0$ outside the ball $B_R(0)$. For all $u \in S^a$, we have

$$\begin{aligned} \frac{\|u\|^2}{2} &\leq \int_{\mathbb{R}^N} \left(|\nabla u|^2 + \frac{V^\infty}{2} u^2 \right) dx \leq a + \int_{\mathbb{R}^N} \left(\frac{V^\infty}{2} - V(x) \right) u^2 dx \\ &\leq a + \int_{B_R(0)} \left(\frac{V^\infty}{2} - V(x) \right) u^2 dx \leq a + \left(\frac{V^\infty}{2} + \|V\|_\infty \right) \int_{B_R(0)} u^2 dx. \end{aligned}$$

Since $u \in \mathcal{M}$, and $L^{p(\cdot)}(B_R(0))$ is continuously embedded in $L^2(B_R(0))$ (see [5, Corollary 3.3.4]), the last integral is bounded. So, S^a is bounded. \square

Now, let (u_k) be a (PS) $_c$ -sequence for $J|_{\mathcal{M}}$ for some $c \in \mathbb{R}$, namely $J(u_k) \rightarrow c$ and there exists a sequence (μ_k) in \mathbb{R} such that $J'(u_k) - \mu_k I'(u_k) \rightarrow 0$ in $H^{-1}(\mathbb{R}^N)$. So we have

$$-\Delta u_k + V(x)u_k = \frac{\mu_k}{2\rho(u_k)} |u_k|^{p(x)-2} u_k + o(1). \tag{15}$$

Testing (15) with u_k , we easily get $\mu_k/2 \rightarrow c$. Besides, since $u_k \in \mathcal{M}$, by (3) we have $p^- \leq \rho(u_k) \leq p^+$, whence, on a renumbered subsequence, $\rho(u_k) \rightarrow \rho_0$ for some $\rho_0 \in [p^-, p^+]$.

We prove some technical properties of (u_k) :

Lemma 3.3. *Let (u_k) be as above and $w \in H^1(\mathbb{R}^N)$:*

(i) *if $u_k \rightharpoonup w$ on a renumbered subsequence, then*

$$-\Delta w + V(x)w = \frac{c}{\rho_0} |w|^{p(x)-2} w;$$

(ii) *if $u_k(\cdot + y_k) \rightharpoonup w$ on a renumbered subsequence for some sequence (y_k) in \mathbb{R}^N with $|y_k| \rightarrow \infty$, then*

$$\Delta w + V^\infty w = \frac{c}{\rho_0} |w|^{p^\infty-2} w.$$

Proof. We prove (i). By the density of $C_0^\infty(\mathbb{R}^N)$ in $H^1(\mathbb{R}^N)$, it suffices to show that

$$\int_{\mathbb{R}^N} (\nabla w \cdot \nabla v + V(x)wv) dx = \frac{c}{\rho_0} \int_{\mathbb{R}^N} |w|^{p(x)-2} wv dx, \quad v \in C_0^\infty(\mathbb{R}^N). \tag{16}$$

We have $\text{supp } v \subset \Omega$ for some bounded domain $\Omega \subset \mathbb{R}^N$. Testing (15) with v gives

$$\int_{\Omega} (\nabla u_k \cdot \nabla v + V(x)u_k v) dx = \frac{\mu_k}{2\rho(u_k)} \int_{\Omega} |u_k|^{p(x)-2} u_k v dx + o(1).$$

Since $V \in L^\infty(\mathbb{R}^N)$ and $u_k \rightharpoonup w$, we have

$$\int_{\Omega} (\nabla u_k \cdot \nabla v + V(x)u_k v) dx \rightarrow \int_{\mathbb{R}^N} (\nabla w \cdot \nabla v + V(x)wv) dx.$$

Besides, $\mu_k/(2\rho(u_k)) \rightarrow c/\rho_0$. Finally, by compactness of the embedding $H^1(\Omega) \hookrightarrow L^{p^+}(\Omega)$, on a renumbered subsequence we have $u_k \rightarrow w$ in $L^{p^+}(\Omega)$ and $u_k(x) \rightarrow w(x)$ a.e. in Ω . By (3), (10) we have a.e. in Ω

$$|u_k|^{p(x)-1}|v| \leq (1 + |u_k|^{p^+-1})|v| \leq |v| + \left(1 - \frac{1}{p^+}\right)|u_k|^{p^+} + \frac{1}{p^+}|v|^{p^+},$$

hence by the generalized dominated convergence theorem

$$\int_{\Omega} |u_k|^{p(x)-2}u_k w \, dx \rightarrow \int_{\mathbb{R}^N} |w|^{p(x)-2}w v \, dx,$$

proving (i). We prove (ii). As above, we only need to show that for all $v \in C_0^\infty(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} (\nabla w \cdot \nabla v + V^\infty w v) \, dx = \frac{c}{\rho_0} \int_{\mathbb{R}^N} |w|^{p^\infty-2}w v \, dx. \tag{17}$$

We have $\text{supp } v \subset \Omega$ for some bounded domain $\Omega \subset \mathbb{R}^N$. Testing (15) with $v(\cdot - y_k)$ and making the change of variable $z = x - y_k$ gives for all $k \in \mathbb{N}$

$$\int_{\mathbb{R}^N} (\nabla \tilde{u}_k \cdot \nabla v + V(z + y_k) \tilde{u}_k v) \, dz = \frac{\mu_k}{2\rho(u_k)} \int_{\mathbb{R}^N} |\tilde{u}_k|^{p(z+y_k)-2} \tilde{u}_k v \, dz + o(1),$$

where $\tilde{u}_k = u_k(\cdot + y_k)$. Since $\tilde{u}_k \rightarrow w$ and $V(\cdot + y_k) \rightarrow V^\infty$ uniformly on Ω , we have

$$\int_{\mathbb{R}^N} (\nabla \tilde{u}_k \cdot \nabla v + V(z + y_k) \tilde{u}_k v) \, dz \rightarrow \int_{\mathbb{R}^N} (\nabla w \cdot \nabla v + V^\infty w v) \, dx.$$

Besides, $\mu_k/(2\rho(u_k)) \rightarrow c/\rho_0$. Finally, exploiting again (3), (10) as in the proof of (i) we have, on a renumbered subsequence,

$$\int_{\mathbb{R}^N} |\tilde{u}_k|^{p(z+y_k)-2} \tilde{u}_k v \, dz \rightarrow \int_{\mathbb{R}^N} |w|^{p^\infty-2}w v \, dx,$$

which proves (ii). □

The main result of this section is the following extension to the variable exponent case of the compactness lemma of Benci and Cerami [2, Lemma 3.1].

Proposition 3.4. *Let (u_k) be a $(PS)_c$ -sequence for $J|_{\mathcal{M}}$, $c \in \mathbb{R}$. Then there exist $m \in \mathbb{N}$, $w^{(1)}, \dots, w^{(m)} \in H^1(\mathbb{R}^N)$, and sequences $(y_k^{(1)}), \dots, (y_k^{(m)})$ in \mathbb{R}^N , $y_k^{(1)} = 0$ for all $k \in \mathbb{N}$, $w^{(n)} \neq 0$ for all $2 \leq n \leq m$, such that, on a renumbered subsequence, $\rho(u_k) \rightarrow \rho_0$ for some $\rho_0 \in [p^-, p^+]$, and*

- (i) $u_k(\cdot + y_k^{(n)}) \rightharpoonup w^{(n)}$;
- (ii) $|y_k^{(n)} - y_k^{(l)}| \rightarrow \infty$ for all $n \neq l$;
- (iii) $\sum_{n=1}^m \|w^{(n)}\|^2 \leq \liminf_k \|u_k\|^2$;
- (iv) $-\Delta w^{(1)} + V(x)w^{(1)} = c/\rho_0 |w^{(1)}|^{p(x)-2}w^{(1)}$;
- (v) $-\Delta w^{(n)} + V^\infty w^{(n)} = c/\rho_0 |w^{(1)}|^{p^\infty-2}w^{(n)}$, $2 \leq n \leq m$;
- (vi) $J(w^{(1)}) = c/\rho_0 \rho(w^{(1)})$;
- (vii) $J^\infty(w^{(n)}) = c/\rho_0 \rho^\infty(w^{(n)})$, $2 \leq n \leq m$;
- (viii) $\rho(w^{(1)}) + \sum_{n=2}^m \rho^\infty(w^{(n)}) = \rho_0$;
- (ix) $J(w^{(1)}) + \sum_{n=2}^m J^\infty(w^{(n)}) = c$;

$$(x) \quad u_k - \sum_{n=1}^m w^{(n)}(\cdot - y_k^{(n)}) \rightarrow 0 \text{ in } H^1(\mathbb{R}^N).$$

Proof. By Lemma 3.2, the sequence (u_k) is bounded. Passing to a subsequence, we have $\rho(u_k) \rightarrow \rho_0$. We shall apply Proposition 3.1. To this end, set

$$\delta = \left(\frac{p^-(\lambda_1^\infty)^{\frac{p^\infty}{2}}}{p^\infty c} \right)^{\frac{1}{p^\infty - 2}} > 0$$

(in particular, $c > 0$). If $u_k(\cdot + y_k) \rightharpoonup w$ in $H^1(\mathbb{R}^N)$, on a renumbered subsequence, for some sequence (y_k) in \mathbb{R}^N , $|y_k| \rightarrow \infty$ and some $w \neq 0$, then by Lemma 3.3 (ii) and the definition of λ_1^∞ we have, testing with w ,

$$\|w\|^2 = \frac{p^\infty c}{\rho_0} \|w\|_{p^\infty}^{p^\infty} \leq \frac{p^\infty c}{p^-} (\lambda_1^\infty)^{-\frac{p^\infty}{2}} \|w\|^{p^\infty},$$

hence $\|w\| \geq \delta$. Then, by Proposition 3.1 there exist $w^{(n)}$, $(y_k^{(n)})$ ($1 \leq n \leq m$) and a renumbered subsequence (u_k) satisfying (i)-(iii) and

$$u_k - \sum_{n=1}^m w^{(n)}(\cdot - y_k^{(n)}) \rightarrow 0 \text{ in } L^q(\mathbb{R}^N), \quad 2 < q < 2^*.$$

By (14), the convergence above also holds in $L^{p(\cdot)}(\mathbb{R}^N)$. From (i) (ii) and Lemma 3.3 we deduce (iv) and (v). Further, testing (iv) with $w^{(1)}$ and (v) with $w^{(n)}$ yields (vi) and (vii), respectively. We prove now (viii). Set for all $k \in \mathbb{N}$

$$w_k = \sum_{n=1}^m w^{(n)}(\cdot - y_k^{(n)}).$$

Since $\|u_k - w_k\|_{p(\cdot)} \rightarrow 0$ by (14) and $\|u_k\|_{p(\cdot)} = 1$, we have $\|w_k\|_{p(\cdot)} \rightarrow 1$. Since $\rho(u_k) \rightarrow \rho_0$, then $\rho(w_k) \rightarrow \rho_0$ by Lemma 2.1 (ii). Let $\varepsilon > 0$. Since $C_0^\infty(\mathbb{R}^N)$ is dense in $L^{p(\cdot)}(\mathbb{R}^N)$, there exists $\tilde{w}^{(1)} \in C_0^\infty(\mathbb{R}^N)$ such that $\|w^{(1)} - \tilde{w}^{(1)}\|_{p(\cdot)} < \varepsilon$, and since $C_0^\infty(\mathbb{R}^N)$ is dense in $L^{p^\infty}(\mathbb{R}^N)$, for all $2 \leq n \leq m$ there is $\tilde{w}^{(n)} \in C_0^\infty(\mathbb{R}^N)$ with $\|w^{(n)} - \tilde{w}^{(n)}\|_{p^\infty} < \varepsilon$. Let

$$\tilde{w}_k = \sum_{n=1}^m \tilde{w}^{(n)}(\cdot - y_k^{(n)}).$$

By (ii) and Lemma 2.2 (ii) we have

$$\begin{aligned} \|w_k - \tilde{w}_k\|_{p(\cdot)} &\leq \sum_{n=1}^m \left\| w^{(n)}(\cdot - y_k^{(n)}) - \tilde{w}^{(n)}(\cdot - y_k^{(n)}) \right\|_{p(\cdot)} \\ &\rightarrow \left\| w^{(1)} - \tilde{w}^{(1)} \right\|_{p(\cdot)} + \sum_{n=2}^m \left\| w^{(n)} - \tilde{w}^{(n)} \right\|_{p^\infty} \leq m\varepsilon. \end{aligned}$$

So,

$$\limsup_k \|w_k - \tilde{w}_k\|_{p(\cdot)} \leq m\varepsilon.$$

Since $\|w_k\|_{p(\cdot)} \rightarrow 1$ and $\rho(w_k) \rightarrow \rho_0$, then by Lemma 2.1 (ii) we can find a constant $C > 0$ such that

$$\limsup_k |\rho(\tilde{w}_k) - \rho_0| \leq C\varepsilon.$$

On the other hand, for all sufficiently large k , the (compact) supports of $\tilde{w}^{(n)}(\cdot - y_k^{(n)})$ are pairwise disjoint by (ii) and hence

$$\rho(\tilde{w}_k) = \sum_{n=1}^m \rho(\tilde{w}^{(n)}(\cdot - y_k^{(n)})) \rightarrow \rho(\tilde{w}^{(1)}) + \sum_{n=2}^m \rho^\infty(\tilde{w}^{(n)})$$

by (ii) and Lemma 2.2 (i). So

$$\left| \rho(\tilde{w}^{(1)}) + \sum_{n=2}^m \rho^\infty(\tilde{w}^{(n)}) - \rho_0 \right| \leq C\varepsilon.$$

By Lemma 2.1 (ii) and (12) we have

$$\left| \rho(w^{(1)}) - \rho(\tilde{w}^{(1)}) \right| \leq C\varepsilon, \quad \left| \rho^\infty(w^{(n)}) - \rho^\infty(\tilde{w}^{(n)}) \right| \leq C\varepsilon, \quad 2 \leq n \leq m.$$

Since $\varepsilon > 0$ is arbitrary, (viii) follows. Adding (vi) and (vii) and substituting (viii), we get (ix). We conclude by proving (x). Set $v_k = u_k - w_k$ and $\tilde{u}_k = u_k - w^{(1)}$ for all $k \in \mathbb{N}$. Note that both (\tilde{u}_k) and (v_k) are bounded in $H^1(\mathbb{R}^N)$. By (15), (iv) and (v) we have

$$\begin{aligned} & -\Delta v_k + V^\infty v_k \\ &= (V^\infty - V(x))\tilde{u}_k + \frac{\mu_k}{2\rho(u_k)} |u_k|^{p(x)-2} u_k \\ & - \frac{c}{\rho_0} \left(|w^{(1)}|^{p(x)-2} w^{(1)} + \sum_{n=2}^m |w^{(n)}(x - y_k^{(n)})|^{p^\infty-2} w^{(n)}(c - y_k^{(n)}) \right) + \eta_k, \end{aligned}$$

for a sequence (η_k) in $H^{-1}(\mathbb{R}^N)$ with $\eta_k \rightarrow 0$ in $H^{-1}(\mathbb{R}^N)$. Testing with v_k and using Lemma 2.1 (i) and (11) gives

$$\begin{aligned} \|v_k\|^2 &\leq \int_{\mathbb{R}^N} |(V(x) - V^\infty)\tilde{u}_k v_k| dx \\ &+ C((\sigma(u_k) + 1) \|v_k\|_{p(\cdot)} + \|v_k\|_{p^\infty}) + o(\|v_k\|). \end{aligned}$$

Since $\|u_k\|$ is bounded, so are $\|\tilde{u}_k\|$, $\sigma(u_k)$, and $\|v_k\|$. If $\|v_k\| \not\rightarrow 0$, then there exists $\varepsilon_0 > 0$ such that, on a renumbered subsequence, $\|v_k\| \geq \varepsilon_0$. By (2), there exists $R > 0$ such that

$$\int_{B_R(0)^c} |(V(x) - V^\infty)\tilde{u}_k v_k| dx \leq 2 \sup_{x \in B_R(0)^c} |V(x) - V^\infty| \|\tilde{u}_k\|_2 \|v_k\|_2 \leq \frac{\varepsilon_0^2}{2}.$$

Then, from the equation above, Hölder inequality, Proposition 3.1 (iv) and (14),

$$\frac{\varepsilon_0^2}{2} \leq C(\|v_k\|_{L^2(B_R(0))} + \|v_k\|_{p^\infty} + \|v_k\|_{p(\cdot)}) + o(1) \rightarrow 0,$$

a contradiction. Thus, (x) is proved. □

Now we prove that $J|_{\mathcal{M}}$ satisfies $(PS)_c$ whenever c lies below a threshold level:

Theorem 3.5. *Assume that $V \in L^\infty(\mathbb{R}^N)$ satisfies (2) and $p \in C(\mathbb{R}^N)$ satisfies (3) and (4). Then $J|_{\mathcal{M}}$ satisfies $(PS)_c$ for all*

$$c < \left(\frac{p^-}{p^\infty} \right)^{2/p^\infty} \lambda_1^\infty. \tag{18}$$

Proof. Let c satisfy condition (18) and let $u_k \in \mathcal{M}$ be a $(PS)_c$ sequence for $J|_{\mathcal{M}}$. Then u_k admits a renumbered subsequence that satisfies the conclusions of Proposition 3.4. Let us set

$$t_1 = \rho(w^{(1)})/\rho_0, \quad t_n = \rho^\infty(w^{(n)})/\rho_0, \quad \text{for } n = 2, \dots, m.$$

Then

$$\sum_{n=1}^m t_n = 1 \tag{19}$$

by (viii) of Proposition 3.4, so each $t_n \in [0, 1]$, and $t_n \neq 0$ for $n \geq 2$. For $n = 2, \dots, m$,

$$c t_n = J^\infty(w^{(n)}) \geq \lambda_1^\infty \|w^{(n)}\|_{p^\infty}^2 = \lambda_1^\infty \left(\frac{\rho_0 t_n}{p^\infty}\right)^{2/p^\infty} \geq \lambda_1^\infty \left(\frac{p^- t_n}{p^\infty}\right)^{2/p^\infty}$$

by (vii) of Proposition 3.4 and (5), so

$$t_n \geq \left[\frac{\lambda_1^\infty}{c} \left(\frac{p^-}{p^\infty}\right)^{2/p^\infty} \right]^{p^\infty/(p^\infty-2)} > 1$$

by (18). Then (19) implies $m = 1$ and hence $u_k \rightarrow w^{(1)}$ in $H^1(\mathbb{R}^N)$ by (x) of Proposition 3.4. □

4. Proofs of the main theorems.

4.1. Proof of Theorem 1.1. Since the sublevel sets of $J|_{\mathcal{M}}$ are bounded by Lemma 3.2 and $J|_{\mathcal{M}}$ is clearly bounded on bounded sets, $\lambda_1 > -\infty$. To see that $\lambda_1 \leq \lambda_1^\infty$, let w_1^∞ be the minimizer of J^∞ on \mathcal{M}^∞ mentioned in the introduction, $y_k \in \mathbb{R}^N$, $|y_k| \rightarrow \infty$, and $w_k = w_1^\infty(\cdot - y_k)$. Since $w_k/\|w_k\|_{p(\cdot)} \in \mathcal{M}$,

$$\lambda_1 \leq J\left(\frac{w_k}{\|w_k\|_{p(\cdot)}}\right) = \frac{J(w_k)}{\|w_k\|_{p(\cdot)}^2} \rightarrow \frac{J^\infty(w_1^\infty)}{\|w_1^\infty\|_{p^\infty}^2} = \lambda_1^\infty$$

by (ii) and (iii) of Lemma 2.2, so $\lambda_1 \leq \lambda_1^\infty$. Assume now that (6) holds. Since $J|_{\mathcal{M}}$ satisfies the Palais-Smale condition at the level λ_1 by Theorem 3.5, it has a minimizer w_1 by a standard argument. Then $|w_1|$ is a minimizer too and hence we may assume $w_1 \geq 0$, $w_1 \neq 0$. We note that, for all $v \in H^1(\mathbb{R}^N)$, $v \geq 0$, we have

$$\int_{\mathbb{R}^N} (\nabla w_1 \cdot \nabla v + \|V\|_\infty w_1 v) dx \geq 0,$$

so by the strong maximum principle (see [15, Proposition C.2]) we have $w_1 > 0$. Observe that a solution $u \in H^1(\mathbb{R}^N)$ of (1) satisfies $-\Delta u = g(x, u)$ for a Carathéodory function $g : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\left| \frac{g(x, u)}{u} \right| \leq C + C|u|^{p(x)-2} \leq C + C|u|^{\frac{4}{N-2}}, \quad \text{for some } C = C(V, N, p^+) > 0.$$

Then by standard regularity theory, $u \in C^1(\mathbb{R}^N)$, see e.g. [14, Appendix B]. □

4.2. **Proof of Theorem 1.2.** Since $e^{-\psi(|x|)} / \|e^{-\psi(|x|)}\|_{p(\cdot)} \in \mathcal{M}$,

$$\lambda_1 \leq J\left(\frac{e^{-\psi(|x|)}}{\|e^{-\psi(|x|)}\|_{p(\cdot)}}\right) = \frac{J(e^{-\psi(|x|)})}{\|e^{-\psi(|x|)}\|_{p(\cdot)}^2}. \quad (20)$$

By virtue of condition (7),

$$\rho(e^{-\psi(|x|)}) = \int_{\mathbb{R}^N} e^{-\psi(|x|)p(x)} dx \geq e^a \int_{B_R(0)^c} e^{-p^\infty \psi(|x|)} dx. \quad (21)$$

It follows from (20), (8) and (21) that (6) holds if $a > 0$ is sufficiently large. \square

4.3. **Proof of Corollary 1.3.** If $N \geq 3$, V and p are radially symmetric in \mathbb{R}^N , we can get some symmetry properties of minimizers by applying the results of Mariş [12]. We can equivalently define

$$\mathcal{M} = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u(x)|^{p(x)} \frac{dx}{p(x)} = 1 \right\}.$$

For any hyperplane Π through 0, splitting \mathbb{R}^N in two half-spaces Π^+ and Π^- , and all $u \in H^1(\mathbb{R}^N)$ we define functions $u_{\Pi^+}, u_{\Pi^-} : \mathbb{R}^N \rightarrow \mathbb{R}$ by setting

$$u_{\Pi^+}(x) = \begin{cases} u(x), & \text{if } x \in \Pi^+ \cup \Pi \\ u(2P_\Pi(x) - x), & \text{if } x \in \Pi^- \end{cases},$$

$$u_{\Pi^-}(x) = \begin{cases} u(x), & \text{if } x \in \Pi^- \cup \Pi \\ u(2P_\Pi(x) - x), & \text{if } x \in \Pi^+, \end{cases}$$

where $P_{(\cdot)}$ is the orthogonal projection from \mathbb{R}^N to an affine submanifold (\cdot) . Clearly $u_{\Pi^\pm} \in H^1(\mathbb{R}^N)$, hypothesis **A1** of [12] is satisfied. Since u is of class $C^1(\mathbb{R}^N)$, so hypothesis **A2** holds as well. By [12, Theorem 1] we learn that, for every minimizer w of \tilde{J} , there exists a line L through 0 such that $w(x) = \tilde{w}(P_L(x), |x - P_L(x)|)$ for all $x \in \mathbb{R}^N$, for a convenient function $\tilde{w} : L \times \mathbb{R}^+ \rightarrow \mathbb{R}$. \square

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