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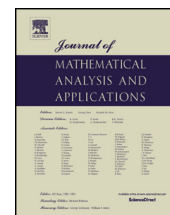# ANALYSIS AND
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1/2-Laplacian problems with exponential nonlinearity

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A R T I C L E   I N F O

A B S T R A C T

By exploiting a suitable Trudinger–Moser inequality for fractional Sobolev spaces, we obtain existence and multiplicity of solutions for a class of one-dimensional nonlocal equations with fractional diffusion and nonlinearity at exponential growth.

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1. Introduction and results

Since the seminal results by Trudinger [13] and Moser [9] on embeddings of exponential type for the Sobolev spaces $H_0^1(\Omega)$ with $\Omega \subset \mathbb{R}^2$, many contributions have appeared related to applications of these results to semi-linear elliptic partial differential equations such as

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with $f(t) \sim e^{4\pi t^q}$ as $t \to +\infty$ for some $0 < q \leqslant 2$. Here the case $q < 2$ is considered as a subcritical growth, while the case $q = 2$ is known as the critical case with respect to the Trudinger–Moser inequality (see [9, Theorem 1])

$$\sup_{\substack{u \in H_0^1(\Omega) \\ \|\nabla u\|_{L^2(\Omega)} \leqslant 1}} \int_\Omega e^{4\pi u^2} \leqslant C|\Omega|. \tag{1.2}$$

For existence and multiplicity of solutions for problems like (1.1) via techniques of critical point theory, we refer the reader to de Figueiredo, Miyagaki & Ruf [4] and to the references therein. Many extensions of

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inequality (1.2) have been achieved, for spaces $W_0^{1,n}(\Omega)$ with $\Omega \subset \mathbb{R}^n$ and also to higher order Sobolev spaces (see Adams [1]). As a consequence, quasi-linear problems involving the n-Laplacian on domains $\Omega \subset \mathbb{R}^n$ or the linear biharmonic operator Δ^2 for functions of $W_0^{2,2}(\Omega)$ on domains $\Omega \subset \mathbb{R}^4$ can be studied. Focusing the attention on nonlinear problems at exponential growth involving linear diffusion, if the dimension four is natural for the biharmonic operator Δ^2 and dimension two is natural for the Laplacian $-\Delta$, the natural setting for the fractional diffusion $(-\Delta)^{1/2}$ is dimension one.

Fractional Sobolev spaces are well known since the beginning of the last century, especially in the framework of harmonic analysis. More recently, after the paper of Caffarelli & Silvestre [3], a large amount of papers were written on problems involving the fractional diffusion $(-\Delta)^s$, $0 < s < 1$. Due to its nonlocal nature, working on bounded domains suggests the functions to be defined on the whole $\mathbb{R}^n$ and that the problems are formulated as follows (see Servadei & Valdinoci [12]):

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.3}$$

For the functional framework of fractional Sobolev spaces and fractional Laplacian, we refer the reader to the survey of Di Nezza, Palatucci & Valdinoci [5]. Equations like (1.3) appear in fractional quantum mechanics in the study of particles on stochastic fields modeled by Lévy processes, which occur widely in physics and biology and have recently attracted much interest. One dimensional cases have been studied by Weinstein [15].

In the present paper we will prove the existence and multiplicity of solutions for a Dirichlet problem driven by the 1/2-Laplacian operator of the following type:

$$\begin{cases} (-\Delta)^{1/2} u = f(u) & \text{in } (0,1), \\ u = 0 & \text{in } \mathbb{R} \setminus (0,1), \end{cases} \tag{P}$$

equivalently written in $(0,1)$ as the nonlocal equation

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^2}\, dy + f\big(u(x)\big) = 0.$$

It is natural to work on the space

$$X = \big\{ u \in H^{1/2}(\mathbb{R}):\ u = 0 \text{ in } \mathbb{R} \setminus (0,1) \big\}, \quad \|u\|_X = [u]_{H^{1/2}(\mathbb{R})} \tag{1.4}$$

where $[\cdot]_{H^{1/2}(\mathbb{R})}$ denotes the Gagliardo semi-norm (see Proposition 2.3). For this space, we state (see Corollary 2.4) and exploit the following Trudinger–Moser type inequality: there exists $0 < \omega \leqslant \pi$ such that for all $0 < \alpha < 2\pi\omega$ we can find $K_\alpha > 0$ such that

$$\int_0^1 e^{\alpha u^2}\, dx \leqslant K_\alpha, \quad \text{for all } u \in X,\ \|u\|_X \leqslant 1. \tag{1.5}$$

We list below our hypotheses on the nonlinearity f in the *subcritical* case:

H Let $f \in C(\mathbb{R})$ be a function such that $f(0) = 0$ and denote

$$F(t) = \int_0^t f(\tau)\, d\tau, \quad \text{for all } t \in \mathbb{R}.$$

Moreover, assume that there exist $t_0, M > 0$ such that:

(i) $0 < F(t) \leqslant M|f(t)|$, for all $|t| \geqslant t_0$;

(ii) $0 < 2F(t) \leqslant f(t)t$, for all $t \neq 0$;

(iii) $\limsup_{t \to 0} \frac{F(t)}{t^2} < \frac{\lambda_1}{4\pi}$ (λ_1 provided by Proposition 2.2 below);

(iv) $\lim_{|t| \to \infty} \frac{|f(t)|}{e^{\alpha t^2}} = 0$, for all $\alpha > 0$.

By a (weak) solution of problem (P) we mean a function $u \in X$ satisfying (3.1) (see Section 3). The following are our main results:

Theorem 1.1. *If **H** hold, then (P) has a nontrivial solution $u \in H^{1/2}(\mathbb{R})$. If in addition f is odd, then (P) has infinitely many solutions in $H^{1/2}(\mathbb{R})$.*

On symmetric domains, we also have the following result:

Theorem 1.2. *If **H** hold, then the problem*

$$\begin{cases} (-\Delta)^{1/2}u = f(u) & in\ (-1, 1), \\ u = 0 & in\ \mathbb{R} \setminus (-1, 1), \end{cases}$$

has an even nontrivial solution $u \in H^{1/2}(\mathbb{R})$ decreasing on $\mathbb{R}^+$.

Now we turn to the *critical* case, under the following assumptions:

H′ Assume **H**(i)–(iii) and:

(iv) there exists $0 < \alpha_0 < 2\pi\omega$ such that

$$\lim_{|t| \to \infty} \frac{|f(t)|}{e^{\alpha t^2}} = \begin{cases} \infty & if\ 0 < \alpha < \alpha_0, \\ 0 & if\ \alpha > \alpha_0; \end{cases}$$

(v) there exists $\psi \in X$ such that $\|\psi\|_X = 1$ and

$$\sup_{t \in \mathbb{R}^+} \left(\frac{t^2}{4\pi} - \int_0^1 F(t\psi)\, dx \right) < \frac{\omega}{2\alpha_0}.$$

For this case we have the following result:

Theorem 1.3. *If **H′** hold, then (P) has a nontrivial solution $u \in H^{1/2}(\mathbb{R})$.*

These results establish a one dimensional fractional counterpart (with the additional information of symmetry and monotonicity of the solution in Theorem 1.2 for symmetric domains) of the results of [4] for the local case in dimension two. As far as the critical case is concerned, typically when $f(t) \sim e^{\alpha_0 t^2}$ as $t \to \infty$, it is still unclear how to detect suitable (concentrating) optimizing sequences in X for the fractional Trudinger–Moser inequality (1.5). However, we can prove that in this case the functional associated to the problem satisfies the Palais–Smale condition at each level $c < \omega/(2\alpha_0)$ and that the problem has a nontrivial solution under the additional hypothesis **H′**(v). We point out that, with a similar machinery, existence and multiplicity of solutions for fractional non-autonomous problems like

$$\begin{cases} (-\Delta)^{1/2}u = f(x, u) & in\ (a, b), \\ u = 0 & in\ \mathbb{R} \setminus (a, b), \end{cases}$$

can be obtained under suitable assumptions on $f : (a, b) \times \mathbb{R} \to \mathbb{R}$.

2. Preliminaries

First we recall some basic facts about the 1/2-Laplacian operator and the related function space $H^{1/2}(\mathbb{R})$, following mainly [5]. For all $s \in (0,1)$, all measurable u and all $x \in \mathbb{R}$ we set

$$(-\Delta)^s u(x) = -\frac{C_s}{2} \int_{\mathbb{R}} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{1+2s}} \, dy,$$

with the constant

$$C_s = \left[\int_{\mathbb{R}} \frac{1 - \cos(\xi)}{|\xi|^{1+2s}} \, d\xi \right]^{-1}$$

(see [5, Lemma 3.3]). We focus on the case $s = 1/2$. Note that $C_{1/2} = \pi^{-1}$. We define

$$H^{1/2}(\mathbb{R}) = \left\{ u \in L^2(\mathbb{R}) : \int_{\mathbb{R}^2} \frac{(u(x) - u(y))^2}{|x - y|^2} \, dx \, dy < \infty \right\}$$

and for all $u \in H^{1/2}(\mathbb{R})$ we introduce the Gagliardo seminorm

$$[u]_{H^{1/2}(\mathbb{R})} = \left[\int_{\mathbb{R}^2} \frac{(u(x) - u(y))^2}{|x - y|^2} \, dx \, dy \right]^{\frac{1}{2}}$$

and the norm

$$\|u\|_{H^{1/2}(\mathbb{R})} = \left(\|u\|_{L^2(\mathbb{R})}^2 + [u]_{H^{1/2}(\mathbb{R})}^2 \right)^{\frac{1}{2}}.$$

We know that $(H^{1/2}(\mathbb{R}), \|\cdot\|_{H^{1/2}(\mathbb{R})})$ is a Hilbert space. Moreover, by [5, Proposition 3.6]

$$\left\|(-\Delta)^{1/4} u\right\|_{L^2(\mathbb{R})} = (2\pi)^{-\frac{1}{2}} [u]_{H^{1/2}(\mathbb{R})}, \quad \text{for all } u \in H^{1/2}(\mathbb{R}). \tag{2.1}$$

Our main tool is a fractional Trudinger–Moser inequality (see Ozawa [10, Theorem 1] and Kozono, Sato & Wadade [7, Theorem 1.1]):

Theorem 2.1. *There exists $0 < \omega \leqslant \pi$ with the following property: for all $0 < \alpha < \omega$ there exists $H_\alpha > 0$ such that*

$$\int_{\mathbb{R}} \left(e^{\alpha u^2} - 1 \right) dx \leqslant H_\alpha \|u\|_{L^2(\mathbb{R})}^2,$$

for every $u \in H^{1/2}(\mathbb{R})$ with $\|(-\Delta)^{1/4} u\|_{L^2(\mathbb{R})} \leqslant 1$.

We do not possess an explicit formula for the optimal constant ω, and neither we know whether the inequality above holds for $\alpha = \omega$.

Now we turn to the space X, defined in (1.4). Clearly the only constant function in X is 0, so the seminorm $[\cdot]_{H^{1/2}(\mathbb{R})}$ turns out to be a norm on X, which we denote by $\|\cdot\|_X$. We have the following Poincaré-type inequality:

Proposition 2.2. *There exists $\lambda_1 > 0$ such that for all $u \in X$*

$$\|u\|_{L^2(0,1)} \leqslant \lambda_1^{-\frac{1}{2}} \|u\|_X.$$

Moreover, equality holds for some $u \in X$ with $\|u\|_{L^2(0,1)} = 1$.

Proof. We set

$$S = \left\{ u \in X \colon \|u\|_{L^2(0,1)} = 1 \right\}$$

and equivalently prove that

$$\inf_{u \in S} \|u\|_X^2 = \lambda_1 > 0. \tag{2.2}$$

Clearly $\lambda_1 \geqslant 0$. We first prove that λ_1 is attained in S. Let $(u_n) \subset S$ be a minimizing sequence for (2.2). In particular, $\sup_{n \in \mathbb{N}} [u_n]^2_{H^{1/2}(\mathbb{R})} < \infty$ and (u_n) is bounded in $L^2(0,1)$. In light of [5, Theorem 7.1], there exists $u \in L^2(0,1)$ such that, up to a subsequence, $u_n \to u$ in $L^2(0,1)$. We extend u by setting $u(x) = 0$ for all $x \in \mathbb{R} \setminus (0,1)$, so $u \in L^2(\mathbb{R})$ and $u_n \to u$ a.e. in $\mathbb{R}$. Fatou's lemma yields

$$\int_{\mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x-y|^2} \, dx \leqslant \liminf_n \int_{\mathbb{R}^2} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^2} \, dx = \lambda_1,$$

hence $u \in X$. Moreover, $\|u\|_{L^2(0,1)} = 1$, hence $u \in S$, in particular $u \neq 0$ and $\|u\|_X^2 = \lambda_1 > 0$. $\quad \square$

Due to Proposition 2.2, we can prove further properties of X:

Proposition 2.3. $(X, \|\cdot\|_X)$ *is a Hilbert space.*

Proof. Clearly the norm $\|\cdot\|_X$ is induced by an inner product, defined for all $u, v \in X$ by

$$\langle u, v \rangle_X = \int_{\mathbb{R}^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^2} \, dx \, dy.$$

Moreover, by Proposition 2.2 we have for all $u \in X$

$$\|u\|_X \leqslant \|u\|_{H^{1/2}(\mathbb{R})} \leqslant \left(\lambda_1^{-1} + 1 \right)^{\frac{1}{2}} \|u\|_X. \tag{2.3}$$

So, completeness of X follows at once from that of $H^{1/2}(\mathbb{R})$. $\quad \square$

We specialize Theorem 2.1 to the space X:

Corollary 2.4. *For all $0 < \alpha < 2\pi\omega$ there exists $K_\alpha > 0$ such that*

$$\int_0^1 e^{\alpha u^2} \, dx \leqslant K_\alpha$$

for all $u \in X$, $\|u\|_X \leqslant 1$.

Proof. Fix $u \in X$ with $\|u\|_X \leqslant 1$. Set $v = (2\pi)^{1/2}u$, then $v \in H^{1/2}(\mathbb{R})$ and by (2.1) we have $\|(-\Delta)^{1/4}v\|_{L^2(\mathbb{R})} \leqslant 1$. Set $\tilde{\alpha} = (2\pi)^{-1}\alpha$, so $0 < \tilde{\alpha} < \omega$ and by Theorem 2.1 and Proposition 2.2 we have

$$\int_0^1 e^{\alpha u^2}\, dx = \int_{\mathbb{R}} \left[e^{\tilde{\alpha}v^2} - 1 \right] dx + 1$$

$$\leqslant H_{\tilde{\alpha}}\|v\|_{L^2(0,1)}^2 + 1 \leqslant \frac{2\pi H_{\tilde{\alpha}}}{\lambda_1} + 1 := K_\alpha,$$

which concludes the proof. $\square$

We point out an important consequence of the results above:

Proposition 2.5. $e^{u^2} \in L^1(0,1)$ *for every* $u \in X$.

Proof. We follow Trudinger [13]. Choose $0 < \alpha < \omega$ and set for all $t \in \mathbb{R}$

$$\phi(t) = \frac{e^{\alpha t^2} - 1}{H_\alpha} \quad (H_\alpha \text{ defined as in Theorem 2.1}).$$

We introduce the Orlicz norm induced by ϕ putting for all measurable $u : (0,1) \to \mathbb{R}$

$$\|u\|_\phi = \inf\left\{ \gamma > 0 : \int_0^1 \phi\left(\frac{u}{\gamma}\right) dx \leqslant 1 \right\},$$

and the corresponding Orlicz space $L_{\phi^*}(0,1)$, see Krasnosel'skiĭ & Rutickiĭ [8, p. 67] for the definition. We prove (by identifying a function $v \in X$ with its restriction to $(0,1)$) that

$$X \hookrightarrow L_{\phi^*}(0,1) \quad \text{continuously.} \tag{2.4}$$

For all $v \in X \setminus \{0\}$, we set $w = \|v\|_{H^{1/2}(\mathbb{R})}^{-1}v$, so by (2.1)

$$\left\|(-\Delta)^{1/4}w\right\|_{L^2(\mathbb{R})} = \frac{[v]_{H^{1/2}(\mathbb{R})}}{(2\pi)^{1/2}\|v\|_{H^{1/2}(\mathbb{R})}} \leqslant (2\pi)^{-1/2} < 1.$$

So, in light of Theorem 2.1, we have

$$\int_0^1 \phi\left(\frac{v}{\|v\|_{H^{1/2}(\mathbb{R})}}\right) dx = \int_{\mathbb{R}} \frac{e^{\alpha w^2} - 1}{H_\alpha}\, dx \leqslant \|w\|_{L^2(\mathbb{R})}^2 \leqslant 1,$$

hence by (2.3)

$$\|v\|_\phi \leqslant \|v\|_{H^{1/2}(\mathbb{R})} \leqslant \left(\lambda_1^{-1} + 1\right)^{\frac{1}{2}}\|v\|_X.$$

Thus, (2.4) is proved.

Now fix $u \in X$ and set $\tilde{u} = \alpha^{-1/2}u$. By the results of Fiscella, Servadei & Valdinoci [6], we know that $C_c^\infty(0,1)$ is a dense linear subspace of X. So, there exists a sequence (ψ_n) in $C_c^\infty(0,1)$ such that $\psi_n \to \tilde{u}$ in X. By (2.4), we have $\psi_n \to \tilde{u}$ in $L_{\phi^*}(0,1)$ as well. In particular $\tilde{u} \in E_\phi$, namely the closure of the set of

bounded functions of X in $L_{\phi^*}(0,1)$. From a general result on Orlicz spaces (see [8, formula (10.1), p. 81]) it follows that

$$\int_0^1 \phi(\tilde{u})\, dx < \infty,$$

which immediately yields the conclusion. $\square$

We conclude this section with a technical result which we shall use later:

Lemma 2.6. *If (v_n) is a sequence in X with $\|v_n\|_X = 1$ for all $n \in \mathbb{N}$ and $v_n \rightharpoonup v$ in X, $0 < \|v\|_X < 1$, then for all $0 < \alpha < 2\pi\omega$ and all $1 < p < (1 - \|v\|_X^2)^{-1}$ the sequence $(e^{\alpha v_n^2})$ is bounded in $L^p(0,1)$.*

Proof. By applying the generalized Hölder inequality with exponents $\gamma_1, \gamma_2, \gamma_3 > 1$ such that $\gamma_1 \alpha < 2\pi\omega$ and $\gamma_1^{-1} + \gamma_2^{-1} + \gamma_3^{-1} = 1$, we have

$$\int_0^1 e^{p\alpha v_n^2}\, dx = \int_0^1 e^{p\alpha[(v_n - v)^2 + 2(v_n - v)v + v^2]}\, dx$$

$$\leqslant \left[\int_0^1 e^{\gamma_1 p\alpha(v_n - v)^2}\, dx\right]^{\frac{1}{\gamma_1}} \left[\int_0^1 e^{2\gamma_2 p\alpha(v_n - v)v}\, dx\right]^{\frac{1}{\gamma_2}} \left[\int_0^1 e^{\gamma_3 p\alpha v^2}\, dx\right]^{\frac{1}{\gamma_3}}.$$

We estimate the three integrals separately. First we note that

$$\|v_n - v\|_X^2 = 1 - 2\langle v_n, v\rangle_X + \|v\|_X^2 \to 1 - \|v\|_X^2 < \frac{1}{p},$$

so for $n \in \mathbb{N}$ big enough we have $\|v_n - v\|_X^2 < 1/p$. Hence, by Corollary 2.4

$$\int_0^1 e^{\gamma_1 p\alpha(v_n - v)^2}\, dx \leqslant \int_0^1 e^{\gamma_1 \alpha\left(\frac{v_n - v}{\|v_n - v\|_X}\right)^2}\, dx \leqslant K_{\gamma_1 \alpha}.$$

Besides, by Corollary 2.4 and Proposition 2.5 we have for some $c_1 > 0$

$$\int_0^1 e^{2\gamma_2 p\alpha(v_n - v)v}\, dx \leqslant \int_0^1 e^{2\left(\frac{\alpha}{2}\right)^{1/2} \frac{v_n - v}{\|v_n - v\|_X}(c_1 v)}\, dx \leqslant \int_0^1 e^{\frac{\alpha}{2}\left(\frac{v_n - v}{\|v_n - v\|_X}\right)^2 + (c_1 v)^2}\, dx$$

$$\leqslant \left[\int_0^1 e^{\alpha\left(\frac{v_n - v}{\|v_n - v\|_X}\right)^2}\, dx\right]^{1/2} \left[\int_0^1 e^{2c_1^2 v^2}\, dx\right]^{1/2} \leqslant K_\alpha \left[\int_0^1 e^{2c_1^2 v^2}\, dx\right]^{1/2}.$$

Finally, clearly

$$\int_0^1 e^{\gamma_3 p\alpha v^2}\, dx < \infty.$$

Thus, $(e^{\alpha v_n^2})$ is bounded in $L^p(0,1)$. $\square$

3. Proofs of Theorems 1.1 and 1.2

In this section we act under **H**. We give our problem a variational formulation by setting for all $u \in X$

$$\varphi(u) = \frac{\|u\|_X^2}{4\pi} - \int_0^1 F(u)\, dx.$$

Proposition 2.5, **H**(i) and **H**(iv) imply that $\varphi \in C^1(X)$. By (2.1), its derivative is given for all $u, v \in X$ by

$$\langle \varphi'(u), v \rangle = \frac{1}{2\pi} \langle u, v \rangle_X - \int_0^1 f(u) v\, dx$$

$$= \int_{\mathbb{R}} (-\Delta)^{1/4} u (-\Delta)^{1/4} v\, dx - \int_0^1 f(u) v\, dx.$$

In particular, if $u \in X$ and $\varphi'(u) = 0$, then for all $v \in X$

$$\int_{\mathbb{R}} (-\Delta)^{1/4} u (-\Delta)^{1/4} v\, dx = \int_0^1 f(u) v\, dx, \tag{3.1}$$

namely u is a (weak) solution of (P).

First we point out some consequences of **H**. By **H**(iv), for all $\alpha > 0$ there exists $c_2 > 0$ such that

$$\left| f(t) \right| \leqslant c_2 e^{\alpha t^2}, \quad \text{for all } t \in \mathbb{R}. \tag{3.2}$$

By virtue of **H**(i), there exists $c_3 > 0$ such that

$$F(t) \geqslant c_3 e^{\frac{|t|}{M}}, \quad \text{for all } |t| \geqslant t_0. \tag{3.3}$$

Finally, by **H**(i) and **H**(ii), for all $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$F(t) \leqslant \varepsilon f(t) t, \quad \text{for all } |t| \geqslant t_\varepsilon. \tag{3.4}$$

The following lemma shows a compactness property of φ:

Lemma 3.1. *φ satisfies the Palais–Smale condition at every level $c \in \mathbb{R}$.*

Proof. Let (u_n) be a sequence in X such that $\varphi(u_n) \to c$ $(c \in \mathbb{R})$ and $\varphi'(u_n) \to 0$ in X^*. We need to show that (u_n) has a convergent subsequence in X. By (3.4), for all $0 < \varepsilon < 1/2$ we can find $c_4 > 0$ such that for all $t \in \mathbb{R}$

$$F(t) \leqslant \varepsilon f(t) t + c_4.$$

For $n \in \mathbb{N}$ big enough we have $\varphi(u_n) \leqslant c + 1$ and $\|\varphi'(u_n)\|_{X^*} \leqslant 1$, so

$$c + 1 \geqslant \frac{\|u_n\|_X^2}{4\pi} - \int_0^1 \left[\varepsilon f(u_n) u_n + c_4 \right] dx = \left(\frac{1}{2} - \varepsilon \right) \frac{\|u_n\|_X^2}{2\pi} + \varepsilon \langle \varphi'(u_n), u_n \rangle - c_4$$

$$\geqslant \left(\frac{1}{2} - \varepsilon \right) \frac{\|u_n\|_X^2}{2\pi} - \varepsilon \|u_n\|_X - c_4.$$

Thus, (u_n) is bounded in X. By Proposition 2.2, (u_n) is bounded in $H^{1/2}(\mathbb{R})$ as well. By [5, Theorem 7.1 and Theorem 6.10], passing to a subsequence we may assume that $u_n \rightharpoonup u$ in both X and $H^{1/2}(\mathbb{R})$, and that $u_n \to u$ in $L^q(0,1)$ for all $q \geqslant 1$ and $u_n(x) \to u(x)$ a.e. in $(0,1)$. In particular, there exists $c_5 > 0$ such that $\|u_n\|_X^2 \leqslant c_5$, for all $n \in \mathbb{N}$. Observe that $(f(u_n))$ is bounded in $L^2(0,1)$. Indeed, by choosing $0 < \alpha < \pi\omega/c_5$, by Corollary 2.4 and (3.2) we get

$$\int_0^1 f^2(u_n)\,dx \leqslant c_2^2 \int_0^1 e^{2\alpha u_n^2}\,dx \leqslant c_2^2 \int_0^1 e^{2\alpha c_5 \left(\frac{u_n}{\|u_n\|_X}\right)^2}\,dx \leqslant c_2^2 K_{2\alpha c_5}. \tag{3.5}$$

By reflexivity of $L^2(0,1)$, passing to a subsequence, we have $f(u_n) \rightharpoonup f(u)$ in $L^2(0,1)$. As a consequence, for all $v \in X$ we have

$$\langle \varphi'(u), v \rangle = \frac{1}{2\pi}\langle u, v \rangle_X - \int_0^1 f(u)v\,dx = \lim_n \langle \varphi'(u_n), v \rangle = 0,$$

namely u is a solution of (P). Observe that

$$\lim_n \int_0^1 f(u_n)u_n\,dx = \int_0^1 f(u)u\,dx,$$

since by (3.5) and $f(u_n) \rightharpoonup f(u)$ in $L^2(0,1)$ it holds

$$\left| \int_0^1 f(u_n)u_n\,dx - \int_0^1 f(u)u\,dx \right| \leqslant \|f(u_n)\|_{L^2(0,1)}\|u_n - u\|_{L^2(0,1)} + \left| \int_0^1 \big(f(u_n) - f(u)\big)u\,dx \right|. \tag{3.6}$$

In turn we have

$$\lim_n \frac{\|u_n\|_X^2}{2\pi} = \lim_n \left[\int_0^1 f(u_n)u_n\,dx + \langle \varphi'(u_n), u_n \rangle \right] = \int_0^1 f(u)u\,dx = \frac{\|u\|_X^2}{2\pi},$$

which immediately yields the assertion. $\square$

The following lemmas deal with the mountain pass geometry for φ:

Lemma 3.2. *There exist $\rho, a > 0$ such that $\varphi(u) \geqslant a$ for all $u \in X$ with $\|u\|_X = \rho$.*

Proof. By **H**(iii) there exist $0 < \mu < \lambda_1$ and $\delta > 0$ such that for all $|t| < \delta$ we have $F(t) \leqslant \mu t^2/(4\pi)$. Fix $q > 2$, $0 < \alpha < 2\pi\omega$ and $r > 1$ such that $r\alpha < 2\pi\omega$ as well. By (3.2) there exists $c_6 > 0$ such that for all $|t| \geqslant \delta$ we have $F(t) \leqslant c_6 e^{\alpha t^2}|t|^q$. Summarizing, for all $t \in \mathbb{R}$, we obtain

$$F(t) \leqslant \frac{\mu t^2}{4\pi} + c_6 e^{\alpha t^2}|t|^q.$$

Reasoning as in [5, Theorem 6.10], we see that the embedding $X \hookrightarrow L^{r'q}(0,1)$ $(r' = r/(r-1))$ is continuous. So, by the estimate above, Proposition 2.2 and Corollary 2.4, we have for all $u \in X$, $\|u\|_X \leqslant 1$ (and for a convenient $c_7 > 0$)

$$\varphi(u) \geqslant \frac{\|u\|_X^2}{4\pi} - \int_0^1 \left[\frac{\mu u^2}{4\pi} + c_6 e^{\alpha u^2} |u|^q \right] dx$$

$$\geqslant \left(1 - \frac{\mu}{\lambda_1}\right) \frac{\|u\|_X^2}{4\pi} - c_6 \left(\int_0^1 e^{r\alpha u^2} dx \right)^{1/r} \left(\int_0^1 |u|^{r'q} dx \right)^{1/r'}$$

$$\geqslant \left(1 - \frac{\mu}{\lambda_1}\right) \frac{\|u\|_X^2}{4\pi} - c_7 \|u\|_X^q.$$

Set for all $t \geqslant 0$

$$g(t) = \left(1 - \frac{\mu}{\lambda_1}\right) \frac{t^2}{4\pi} - c_7 t^q.$$

By a straightforward computation we find $0 < \rho < 1$ such that $g(\rho) = a > 0$. So, for all $u \in X$ with $\|u\|_X = \rho$ we have $\varphi(u) \geqslant a$. $\square$

Lemma 3.3. *If $Y \subset X$ is a linear subspace generated by bounded functions and $\dim(Y) < \infty$, then* $\sup_{u \in Y} \varphi(u) < \infty$ *and*

$$\lim_{\substack{\|u\|_X \to \infty \\ u \in Y}} \varphi(u) = -\infty.$$

Proof. Fix $p > 2$. By (3.3), we have $|t|^{-p} F(t) \to \infty$ for $|t| \to \infty$, so we can find $c_8 > 0$ such that for all $t \in \mathbb{R}$ we have $F(t) \geqslant |t|^p - c_8$. Whence, for some $c_9 > 0$, we obtain for all $u \in Y$

$$\varphi(u) \leqslant \frac{\|u\|_X^2}{4\pi} - \|u\|_{L^p(0,1)}^p + c_8 \leqslant \frac{\|u\|_X^2}{4\pi} - c_9 \|u\|_X^p + c_8,$$

which readily yields the assertion. $\square$

Proof of Theorem 1.1 concluded. The existence of one solution follows by applying the Mountain Pass Theorem (see Rabinowitz [11, Theorem 2.2]) to φ and combining Lemmas 3.1, 3.2 and 3.3.

Concerning the multiplicity, we apply [11, Theorem 9.12]. $\square$

Proof of Theorem 1.2 concluded. Given a nonnegative function $u \in X$ and any $H = (a, \infty)$ with $a < 0$, we have the following inequality for the polarization u^H (see Baernstein [2, Theorem 2, p. 58])

$$\int_{\mathbb{R}^2} \frac{(u^H(x) - u^H(y))^2}{|x - y|^2} \, dx \, dy \leqslant \int_{\mathbb{R}^2} \frac{(u(x) - u(y))^2}{|x - y|^2} \, dx \, dy,$$

which implies that $\varphi(u^H) \leqslant \varphi(u)$, for all nonnegative u of X.

The existence of an even solution on $(-1, 1)$, decreasing on $(0, 1)$, equal to zero on $\mathbb{R} \setminus (-1, 1)$ follows by the (symmetric) Mountain Pass Theorem of Van Schaftingen [14, Theorem 3.2] applied to the functional φ on X with the V therein chosen as $V = L^2(-1, 1)$, and on account of Lemmas 3.1, 3.2 and 3.3. $\square$

Example 3.4. Fix $1 < q < 2$ and $0 < \mu < \lambda_1/(2\pi)$. Define $f : \mathbb{R} \to \mathbb{R}$ by setting, for all $t \geqslant 0$,

$$f(t) = \begin{cases} \mu t & \text{if } 0 \leqslant t \leqslant 1, \\ \mu t^{q-1} e^{t^q - 1} & \text{if } t > 1, \end{cases}$$

and $f(t) = -f(-t)$ for all $t < 0$. It is easily seen that f is continuous, odd and satisfies **H**. By Theorem 1.1, then, the corresponding problem (P) admits infinitely many solutions.

4. Proof of Theorem 1.3

In this section, we consider the critical case, that is, we act under $\mathbf{H}'$.

An important remark here is that (3.2) holds only for $\alpha > \alpha_0$. We prove that the Palais–Smale condition is satisfied only for levels in a certain range:

Lemma 4.1. *If f satisfies $\mathbf{H}'$, then φ satisfies the Palais–Smale condition at any level $c < \omega/(2\alpha_0)$.*

Proof. Let (u_n) be a sequence in X such that $\varphi(u_n) \to c$ and $\varphi'(u_n) \to 0$ in X^*. Arguing as in the proof of Lemma 3.1, it is readily seen that there exists a positive constant c_{10} such that, for all $n \in \mathbb{N}$,

$$\max\left\{ \|u_n\|_X^2, \int_0^1 f(u_n)u_n\, dx, \int_0^1 F(u_n)\, dx \right\} \leqslant c_{10}.$$

Moreover, up to a subsequence, $u_n \rightharpoonup u$ in X and $u_n \to u$ in $L^q(0,1)$ for all $q \geqslant 1$. Reasoning as in [4, Lemma 2.1] we have $f(u_n) \to f(u)$ in $L^1(0,1)$. Whence, in light of $\mathbf{H}(i)$, it follows that $\int_0^1 F(u_n)\, dx \to \int_0^1 F(u)\, dx$. So we have

$$\frac{\|u_n\|_X^2}{4\pi} \to c + \int_0^1 F(u)\, dx. \tag{4.1}$$

Then, since $\varphi'(u_n) \to 0$, we get

$$\int_0^1 f(u_n)u_n\, dx \to 2\left(c + \int_0^1 F(u)\, dx \right).$$

So, by means of $\mathbf{H}(ii)$, we have

$$c = \frac{1}{2}\lim_n \int_0^1 \left[f(u_n)u_n - 2F(u_n) \right] dx \geqslant 0.$$

Besides, for all $v \in C_c^\infty(0,1)$ we have

$$\langle \varphi'(u), v \rangle = \frac{1}{2\pi}\langle u, v \rangle_X - \int_0^1 f(u)v\, dx = \lim_n \langle \varphi'(u_n), v \rangle = 0.$$

Recalling again the density result of [6], we have $\langle \varphi'(u), v \rangle = 0$ for all $v \in X$, namely u is a solution of (P). By $\mathbf{H}(ii)$ and taking $v = u$ we have

$$\varphi(u) = \frac{1}{2}\left(\frac{\|u\|_X^2}{2\pi} - 2\int_0^1 F(u)\, dx \right) \geqslant \frac{1}{2}\left(\frac{\|u\|_X^2}{2\pi} - \int_0^1 f(u)u\, dx \right) = 0.$$

Summarizing, we have $c \geqslant 0$ and $\varphi(u) \geqslant 0$. Now we distinguish three cases.

(a) If $c = 0$, then by virtue of (4.1) and $\varphi(u) \geqslant 0$, we get

$$\frac{\|u\|_X^2}{4\pi} \geqslant \int_0^1 F(u)\, dx = \lim_n \frac{\|u_n\|_X^2}{4\pi}.$$

Recalling that $u_n \rightharpoonup u$ in X, we conclude that $u_n \to u$ in X.

(b) If $c > 0$, $u = 0$, then the sequence $(f(u_n))$ is bounded in $L^q(0,1)$, for some $q > 1$. Indeed, since $c < \omega/(2\alpha_0)$ we can find $q > 1$, $\varepsilon > 0$ and $\alpha_0 < \alpha < 2\pi\omega$ such that $2\pi(2c + \varepsilon)q\alpha := \beta < 2\pi\omega$. Since $\|u_n\|_X^2 \to 4\pi c$, for $n \in \mathbb{N}$ big enough we have $\|u_n\|_X^2 < 2\pi(2c + \varepsilon)$. So, applying (3.2) and Corollary 2.4 we have

$$\int_0^1 |f(u_n)|^q\, dx \leqslant c_2^q \int_0^1 e^{q\alpha u_n^2}\, dx \leqslant c_2^q \int_0^1 e^{\beta(\frac{u_n}{\|u_n\|_X})^2}\, dx \leqslant c_2^q K_\beta.$$

Recalling that $u_n \to 0$ in $L^{q'}(0,1)$ $(q' = q/(q-1))$ and that

$$0 \leqslant \int_0^1 f(u_n)u_n\, dx \leqslant \|f(u_n)\|_{L^q(0,1)}\|u_n\|_{L^{q'}(0,1)},$$

from $\varphi'(u_n) \to 0$ we have immediately

$$\lim_n \frac{\|u_n\|_X^2}{2\pi} = \lim_n \int_0^1 f(u_n)u_n\, dx = 0,$$

whence $u_n \to 0$ in X. Thus $\varphi(u_n) \to 0 < c$, a contradiction.

(c) If $c > 0$, $u \neq 0$, then we prove that $\varphi(u) = c$. This equality yields the strong convergence by means of (4.1). We know that $\varphi(u) \leqslant c$, so by contradiction assume $\varphi(u) < c$. Then

$$\|u_n\|_X^2 \to 4\pi\left(c + \int_0^1 F(u)\, dx\right) > \|u\|_X^2.$$

Set $v_n = \|u_n\|_X^{-1}u_n$ and $v = (4\pi c + 4\pi \int_0^1 F(u)\, dx)^{-1/2}u$. So we have $\|v_n\|_X = 1$, $0 < \|v\|_X < 1$ and $v_n \rightharpoonup v$ in X. Since $c < \omega/(2\alpha_0)$, we can find $q > 1$, $\alpha_0 < \alpha < 2\pi\omega$ such that $qc < \omega/(2\alpha)$, hence (recall $\varphi(u) \geqslant 0$)

$$2q\alpha < \frac{\omega}{c - \varphi(u)}.$$

We have

$$\lim_n q\alpha\|u_n\|_X^2 = 4\pi q\alpha\left(c + \int_0^1 F(u)\, dx\right) < 2\pi\omega \frac{c + \int_0^1 F(u)\, dx}{c - \varphi(u)}.$$

We can choose $p > 1$, $0 < \gamma < 2\pi\omega$ such that

$$p < \frac{c + \int_0^1 F(u)\, dx}{c - \varphi(u)} = \frac{1}{1 - \|v\|_X^2}$$

and for $n \in \mathbb{N}$ big enough

$$q\alpha\|u_n\|_X^2 < p\gamma.$$

Since $\gamma < 2\pi\omega$, by Lemma 2.6 the sequence $(e^{\gamma v_n^2})$ is bounded in $L^p(0,1)$, so

$$\int_0^1 |f(u_n)|^q \, dx \leqslant c_2^q \int_0^1 e^{q\alpha u_n^2} \, dx \leqslant c_2^q \int_0^1 e^{p(\gamma v_n^2)} \, dx,$$

which proves that $(f(u_n))$ is bounded in $L^q(0,1)$. Passing if necessary to a subsequence, we have $f(u_n) \rightharpoonup f(u)$ in $L^q(0,1)$ while $u_n \to u$ in $L^{q'}(0,1)$. So,

$$\left| \int_0^1 f(u_n)u_n \, dx - \int_0^1 f(u)u \, dx \right| \leqslant \|f(u_n)\|_{L^q(0,1)}\|u_n - u\|_{L^{q'}(0,1)} + \left| \int_0^1 (f(u_n) - f(u))u \, dx \right|,$$

hence

$$\lim_n \int_0^1 f(u_n)u_n \, dx = \int_0^1 f(u)u \, dx.$$

As above, this yields $u_n \to u$ in X. This in turn implies $\varphi(u) = c$, a contradiction.

This concludes the proof. $\quad\square$

Proof of Theorem 1.3 concluded. The conclusions of Lemmas 3.2 and 3.3 still hold, with small changes in the proofs. Moreover, if we fix $t > 0$ such that $\varphi(t\psi) < \varphi(0)$ and denote by Γ the set of continuous paths in X joining 0 and $t\psi$ and set

$$c = \inf_{\gamma \in \Gamma} \max_{\tau \in [0,1]} \varphi(\gamma(\tau)),$$

by $\mathbf{H}'(\mathrm{v})$ we see that $c < \omega/(2\alpha_0)$. Thus, by Lemma 4.1, φ satisfies the Palais–Smale condition at level c. By the Mountain Pass Theorem, then, (P) has a nontrivial solution. $\quad\square$

Example 4.2. Fix $0 < \mu < \lambda_1/(2\pi)$, $0 < \alpha_0 < 2\pi\omega$. Define $f : \mathbb{R} \to \mathbb{R}$ by setting, for all $t \geqslant 0$,

$$f(t) = \begin{cases} \mu t & \text{if } 0 \leqslant t \leqslant 1, \\ \mu t e^{\alpha_0(t^2-1)} & \text{if } t > 1, \end{cases}$$

and $f(t) = -f(-t)$ for all $t < 0$. It is easily seen that f is continuous and satisfies $\mathbf{H}'(\mathrm{i})$–(iv). If there exists $\psi \in X$ satisfying $\mathbf{H}'(\mathrm{v})$, then by Theorem 1.3 the corresponding problem (P) admits a nontrivial solution.

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