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Schrödinger–Poisson systems with a general critical nonlinearity

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We consider a Schrödinger–Poisson system involving a general nonlinearity at critical growth and we prove the existence of positive solutions. The Ambrosetti–Rabinowitz condition is not required. We also study the asymptotics of solutions with respect to a parameter.

Keywords: Schrödinger–Poisson systems; variational methods; critical growth.

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1. Introduction and Main Result

We are concerned with the nonlinear Schrödinger–Poisson system

$$\begin{cases} -\Delta u + u + \lambda\phi u = f(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = \lambda u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\lambda > 0$ and the nonlinearity f reaches the critical growth. In the last decade, the Schrödinger–Poisson system has been object of intensive research because of its strong relevance in applications. From a physical point of view, it describes systems of identically charged particles interacting each other in the case where magnetic effects can be neglected. The nonlinear term f models the interaction between the particles and the coupled term ϕu concerns the interaction with the electric field. For more detailed physical aspects of the Schrödinger–Poisson system, we refer the reader to [3, 7, 8, 30] and to the references therein. In recent years, there has been an increasing attention toward systems like (1.1) and the existence of positive solutions, sign-changing solutions, ground states, radial and non-radial solutions and semi-classical states has been investigated. In [17], D'Aprile and Mugnai obtained the existence of a nontrivial radial solution to (1.1) with $f(u) = |u|^{p-2}u$, for $p \in [4, 6)$. In [19], D'Aprile proved that system (1.1) admits a non-radial solution for $f(u) = |u|^{p-2}u$, with $p \in (4, 6)$. In [6], by using the concentration compactness principle, Azzollini and Pomponio obtained the existence of a ground state solution to (1.1) with $f(u) = |u|^{p-2}u$, for $p \in (3, 6)$. In [27], Ruiz obtained some nonexistence results for

$$\begin{cases} -\Delta u + u + \lambda\phi u = |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, & \text{in } \mathbb{R}^3 \end{cases} \tag{1.2}$$

and established the relation between the existence of the positive solutions to system (1.2) and the parameters $p \in (2, 6)$ and $\lambda > 0$. Moreover, if $\lambda \geq \frac{1}{4}$, the author showed that $p = 3$ is a critical value for the existence of the positive solutions. For $p \in (2, 3)$, Ruiz [28] investigated the existence of radial ground states to system (1.2) and obtained the different behavior of the solutions depending on p as $\lambda \to 0$. We also would like to cite some works [18, 26], where system (1.2) was considered as $\lambda \to 0$. In [18, 26], the authors were concerned with the semi-classical states for system (1.2). Precisely, the authors studied the existence of radial positive solutions concentrating around a sphere. Recently, some works were focused on the existence of sign-changing solutions to (1.1) with $f(u) = |u|^{p-2}u$. By using a gluing method, Kim and Seok [24] proved the existence of sign-changing solutions with a prescribed number of nodal domains for (1.1) with $p \in (4, 6)$. Subsequently, Ianni [20] obtained a similar result for $p \in [4, 6)$. More recently, Wang and Zhou [31] considered the non-autonomous system

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.3}$$

Under suitable conditions on V, they proved the existence of least energy sign-changing solutions to system (1.3) with $p \in (4, 6)$ by minimizing over the sign-changing Nehari manifold. For further works on the non-autonomous Schrödinger–Poisson system, we also would like to mention [2, 13, 23, 25, 34] and the references therein.

The works discussed above mainly focus on the study of system (1.1) with the very special nonlinearity $f(u) = |u|^{p-2}u$. In [4], Azzollini *et al.* were concerned with the existence of a positive radial solution to system (1.1) under the effect of a general nonlinear term, see also [5, 21]. Precisely, let $g(u) = -u + f(u)$, then

Theorem A (see [4]). *Suppose*

$(H1)$ $g(s) \in C(\mathbb{R}, \mathbb{R})$;
$(H2)$ $-\infty < \liminf_{s\to 0} \frac{g(s)}{s} \leq \limsup_{s\to 0} \frac{g(s)}{s} = -m < 0$;
$(H3)$ $\limsup_{s\to\infty} \frac{g(s)}{s^5} \leq 0$;
$(H4)$ *there exists* $\xi_0 > 0$ *such that* $G(\xi_0) := \int_0^{\xi} g(s)\mathrm{d}s > 0$.

Then there exists $\lambda_0 > 0$ *such that* (1.1) *admits a positive radial solution for* $\lambda \in (0, \lambda_0)$.

(H_1)–(H_4) are known as Berestycki–Lions conditions, introduced in [9]. There, the authors showed that these conditions are almost necessary and sufficient for the existence of ground states to the nonlinear scalar field equation $-\Delta u = g(u)$, with $u \in H^1(\mathbb{R}^N)$, $N \geq 3$.

We remark that in the literature described above, only the *subcritical* case was considered. A natural question arises on whether results like Theorem A holds if f is at *critical growth*. In fact, in [32], Zhang, obtained the following.

Theorem B (see [32]). *Suppose* $f \in C(\mathbb{R}, \mathbb{R})$ *is odd and*

(g_1) $\lim_{s\to 0} \frac{f(s)}{s} = 0$,
(g_2) $\lim_{s\to +\infty} \frac{f(s)}{s^5} = K > 0$,
(g_3) *There exists* $D > 0$ *and* $q \in (2, 6)$ *such that* $f(s) \geq Ks^5 + Ds^{q-1}$, *for all* $s > 0$,
(g_4) *There exists* $\gamma > 2$ *such that* $0 < \gamma \int_0^s f(\tau)\mathrm{d}\tau \leq sf(s)$, *for all* $s \neq 0$.

Then (i) (1.1) *has a positive radial solution for small* $\lambda > 0$ *if* $q \in (2, 4]$ *with* D *large enough, or* $q \in (4, 6)$; (ii) *if* $\gamma > 3$, (1.1) *admits a ground state solution for any* $\lambda > 0$ *provided* $q \in (2, 4]$ *with* D *large enough, or* $q \in (4, 6)$.

The author was able to obtain this existence result via a truncation argument. Condition (g_4) implies that f is superlinear and is the so-called Ambrosetti–Rabinowitz condition, usually involved in guaranteeing the boundedness of (PS)-sequences.

The aim of this paper is to study the existence of the positive solutions to system (1.1) involving a more general critical nonlinearity compared to that allowed in Theorem B. In particular, the Ambrosetti–Rabinowitz condition is not required.

We shall assume that the following hypotheses on f:

(f_1) $f : \mathbb{R} \to \mathbb{R}$ is continuous, $f = 0$ on $\mathbb{R}^-$ and $\lim_{s\to 0} \frac{f(s)}{s} = 0$.
(f_2) $\limsup_{s\to +\infty} \frac{f(s)}{s^5} \leq 1$.
(f_3) There exist $\mu > 0$ and $q \in (2, 6)$ such that $f(s) \geq \mu s^{q-1}$ for all $s \geq 0$.

Assumption (f_2) implies f has (possibly) a critical growth at infinity and the limit of $f(s)/s^5$ at $+\infty$ may fail to exist. Moreover, there exists $\kappa > 0$ such that

$$f(s) \leq \frac{1}{2}s + \kappa s^5 \quad \text{for all } s \geq 0. \tag{1.4}$$

Before stating the main result, we fix some notations. In the sequel, $\mathcal{S}$ and $\mathcal{C}_q$ denote the best constants of Sobolev embeddings $\mathcal{D}^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$,

$$\mathcal{S}\left(\int_{\mathbb{R}^3} |u|^6 \mathrm{d}x\right)^{\frac{1}{3}} \leq \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x, \quad \text{for all } u \in \mathcal{D}^{1,2}(\mathbb{R}^3),$$

$$\mathcal{C}_q\left(\int_{\mathbb{R}^3} |u|^q \mathrm{d}x\right)^{\frac{2}{q}} \leq \int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^2) \mathrm{d}x, \quad \text{for all } u \in H^1(\mathbb{R}^3).$$

Our main result is the following.

Theorem 1.1. *Suppose that f satisfies (f_1)–(f_3).*

(i) *There exists $\lambda_0 > 0$ such that, for every $\lambda \in (0, \lambda_0)$, system* (1.1) *admits a nontrivial positive solution $(u_\lambda, \phi_\lambda)$, provided that*

$$\mu > \left[\frac{3q-6}{2q\mathcal{S}^{\frac{3}{2}}}\right]^{\frac{q-2}{2}} \mathcal{C}_q^{\frac{q}{2}}.$$

(ii) *Along a subsequence, $(u_\lambda, \phi_\lambda)$ converges to $(u, 0)$ in $H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ as $\lambda \to 0$, where u is a ground state solution to the limit problem*

$$-\Delta u + u = f(u), \quad u \in H^1(\mathbb{R}^3).$$

The rest of the paper is devoted to prove Theorem 1.1. Since we are concerned with system (1.1) with a more general nonlinear term f, the problem becomes more thorny and tough in applying variational methods. In fact, due to the lack of the Ambrosetti–Rabinowitz condition, the boundedness of (PS)-sequence is not easy to be obtained. To overcome this difficulty, we will adopt a local deformation argument from Byeon and Jeanjean [10] to get a bounded (PS)-sequence. Due to the presence of the nonlocal term ϕu, a crucial modification on the min–max value is needed. We will define another min–max value C_λ (see Sec. 3), where all paths are required to be uniformly bounded with respect to λ. Similar arguments can be found in [14].

The paper is organized as follows.

In Sec. 2, we consider the functional framework and some preliminary results.

In Sec. 3, we construct the min–max level.

In Sec. 4, we use a local deformation argument to give the proof of Theorem 1.1.

Notations

- $\|u\|_p := (\int_{\mathbb{R}^3} |u|^p \mathrm{d}x)^{1/p}$ for $p \in [1, \infty)$.
- $\|u\| := (\|u\|_2^2 + \|\nabla u\|_2^2)^{1/2}$ for $u \in H^1(\mathbb{R}^3)$.
- $H^1_r(\mathbb{R}^3)$ is the subspace of $H^1(\mathbb{R}^3)$ of radially symmetric functions.
- $\mathcal{D}^{1,2}(\mathbb{R}^3) := \{u \in L^{2^*}(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}$.

2. Preliminaries and Functional Setting

We recall that, for $u \in H^1(\mathbb{R}^3)$, the Lax–Milgram theorem implies that there exists a unique $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ such that $-\Delta\phi = \lambda u^2$ with

$$\phi_u(x) := \lambda \int_{\mathbb{R}^3} \frac{u^2(y)}{4\pi|x-y|} \mathrm{d}y. \tag{2.1}$$

Setting

$$T(u) := \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x,$$

we summarize some properties of $\phi_u, T(u)$, which will be used later.

Lemma 2.1 (see [27]). *For any $u \in H^1(\mathbb{R}^3)$, we have*

(1) $\phi_u : H^1(\mathbb{R}^3) \mapsto \mathcal{D}^{1,2}(\mathbb{R}^3)$ *is continuous and maps bounded sets into bounded sets.*
(2) $\phi_u \geq 0$, $T(u) \leq c\lambda\|u\|^4$ *for some* $c > 0$.
(3) *If* $u_n \to u$ *weakly in* $H^1(\mathbb{R}^3)$, *then* $\phi_{u_n} \to \phi_u$ *weakly in* $\mathcal{D}^{1,2}(\mathbb{R}^3)$.
(4) *If* $u_n \to u$ *weakly in* $H^1(\mathbb{R}^3)$, *then* $T(u_n) = T(u) + T(u_n - u) + o(1)$.
(5) *If* u *is a radial function, so is* ϕ_u.

Substituting (2.1) into (1.1), we can rewrite (1.1) in the following equivalent equation

$$-\Delta u + u + \lambda\phi_u u = f(u), \quad u \in H^1(\mathbb{R}^3). \tag{2.2}$$

We define the energy functional $\Gamma_\lambda : H^1(\mathbb{R}^3) \to \mathbb{R}$ by

$$\Gamma_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)\mathrm{d}x + \frac{\lambda}{4} \int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(u)\mathrm{d}x,$$

with $F(t) = \int_0^t f(s)\mathrm{d}s$. It is standard to show that Γ_λ is of class C^1 on $H^1(\mathbb{R}^3)$. Since we are concerned with the positive solutions of (1.1), from now on, we can assume that $f(s) = 0$ for every $s \leq 0$. It is readily proved that any critical point of Γ_λ is nonnegative and, by the maximum principle, it is strictly positive. Moreover, it is easy to verify that $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ is a solution of (1.1) if and only if $u \in H^1(\mathbb{R}^3)$ is a critical point of the functional Γ_λ. If $\lambda = 0$, problem (2.2) becomes

$$-\Delta u + u = f(u), \quad u \in H^1(\mathbb{R}^3), \tag{2.3}$$

which will be referred as the limit problem of (2.2). In general, if a problem is well-behaved and undergoes a small perturbation, then one may expect that the perturbed problem has a solution near the solutions of the original problem. Then if λ is small, it is natural to find a solution of (2.2) in some neighborhood of the solutions to the limit problem (2.3), which will play a crucial rǒle in the study of perturbed problem (2.2). In the following, we study some properties of the limit problem (2.3). First, we show the existence of the ground states of the limit problem (2.3).

Proposition 2.2. *Suppose that f satisfies* (f_1)–(f_3), *then the limit problem* (2.3) *has a ground state* $u \in H_r^1(\mathbb{R}^3)$, *provided that*

$$\mu > \left[\frac{3q-6}{2q\mathcal{S}^{\frac{3}{2}}}\right]^{\frac{q-2}{2}} \mathcal{C}_q^{\frac{q}{2}}. \tag{2.4}$$

Remark 2.3. In [1] the authors established the existence of the ground state solutions for the nonlinear scalar field equation involving critical growth in $\mathbb{R}^N$, for $N \geq 2$. In particular, assuming that f satisfies (f_1)–(f_3) and an additional condition

(f_4) $\quad sf(s) - 2F(s) \geq 0$ for all $s \geq 0$, where $F(s) = \int_0^s f(\tau)\mathrm{d}\tau$,

the authors proved that (2.3) has a ground state. We remark that (f_4) can be removed.

To prove Proposition 2.2, we will use the following notations.

$$\mathcal{M} := \left\{u \in H_r^1(\mathbb{R}^3)\backslash\{0\} : \int_{\mathbb{R}^3} G(u)\mathrm{d}x = 1\right\},$$

$$\mathcal{P} := \left\{u \in H_r^1(\mathbb{R}^3)\backslash\{0\} : 6\int_{\mathbb{R}^3} G(u)\mathrm{d}x = \int_{\mathbb{R}^3} |\nabla u|^2\mathrm{d}x\right\},$$

where $G(t) = F(t) - \frac{1}{2}t^2$. $\mathcal{P}$ is the so-called Pohozǎev manifold. It follows, from (f_3), that there exists $\xi > 0$ such that $G(\xi) > 0$. Then it is easy to check that $\mathcal{M} \neq \emptyset$ and $\mathcal{P} \neq \emptyset$. Define

$$M := \frac{1}{2}\inf_{u\in\mathcal{M}} \int_{\mathbb{R}^3} |\nabla u|^2\mathrm{d}x, \quad p := \inf_{u\in\mathcal{P}} I(u),$$

and the Mountain Pass value

$$b := \inf_{\gamma\in\Upsilon} \max_{0\leq t\leq 1} I(\gamma(t)),$$

where $\Upsilon = \{\gamma \in C([0,1], H_r^1(\mathbb{R}^3)) : \gamma(0) = 0, I(\gamma(1)) < 0\}$ and

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^2)\mathrm{d}x - \int_{\mathbb{R}^3} F(u)\mathrm{d}x.$$

Lemma 2.4. *Let f satisfy* (f_1)–(f_3) *and* (2.4). *Then* $0 < M < \frac{\sqrt[3]{6}}{2}\mathcal{S}$ *and* $p < \frac{1}{3}\mathcal{S}^{\frac{3}{2}}$.

Proof. Obviously $M \in [0, \infty)$. We claim that $M > 0$. Assume by contradiction that it is $M = 0$. Then there exists $\{u_n\}_n \subset \mathcal{M}$ such that $\|\nabla u_n\|_2 \to 0$ as $n \to \infty$. By Sobolev's embedding theorem, $\|u_n\|_6 \to 0$ as $n \to \infty$. Thus, it follows from (1.4) that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^3} G(u_n)\mathrm{d}x \leq \limsup_{n\to\infty} \frac{\kappa}{6}\int_{\mathbb{R}^3} |u_n|^6\mathrm{d}x = 0,$$

a contradiction, proving the claim. We now claim that $p \leq b$. It suffices to prove that

$$\gamma([0,1]) \cap \mathcal{P} \neq \emptyset \quad \text{for all } \gamma \in \Upsilon,$$

whose proof is similar to that in [22, Lemma 4.1]. Let

$$P(u) = \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x - 6 \int_{\mathbb{R}^3} G(u) \mathrm{d}x.$$

Then by (1.4) it is easy to know that there exists $\rho_0 > 0$ such that

$$P(u) > 0 \quad \text{if } 0 < \|u\| \leq \rho_0. \tag{2.5}$$

For any $\gamma \in \Upsilon$, $P(\gamma(0)) = 0$ and $P(\gamma(1)) \leq 6I(\gamma(1)) < 0$. Thus, there exists $t_0 \in (0,1)$ such that $P(\gamma(t_0)) = 0$ with $\|\gamma(t_0)\| > \rho_0$, which implies $\gamma([0,1]) \cap \mathcal{P} \neq \emptyset$. We now use an idea from Coleman–Glazer–Martin [15] to prove that $p = \frac{2\sqrt{3}}{9} M^{\frac{3}{2}}$. Define $\Phi : \mathcal{M} \to \mathcal{P}$: $(\Phi(u))(x) = u(\frac{x}{t_u})$, where $t_u = \sqrt{6}/6 \|\nabla u\|_2$. Then Φ is a bijection. For $u \in H^1(\mathbb{R}^3)$, let us set

$$T_0(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x, \quad V(u) = \int_{\mathbb{R}^3} G(u) \mathrm{d}x.$$

Then for $u \in \mathcal{M}$, $I(\Phi(u)) = t_u T_0(u) - t_u^3 V(u) = \sqrt{6}/18 \|\nabla u\|_2^3$. Thus,

$$\inf_{u \in \mathcal{P}} I(u) = \inf_{u \in \mathcal{M}} I(\Phi(u)) = \frac{\sqrt{6}}{18} \inf_{u \in \mathcal{M}} \|\nabla u\|_2^3,$$

which implies $p = \frac{2\sqrt{3}}{9} M^{\frac{3}{2}}$. Finally, similar to that in [1], taking $\psi \in H_r^1(\mathbb{R}^3)$ with $\psi \geq 0$, $\|\psi\|_q^2 = \mathcal{C}_q^{-1}$ and $\|\psi\| = 1$, then

$$b \leq \max_{t \geq 0} I(t\psi) \leq \max_{t \geq 0} \left(\frac{t^2}{2} - \mu \frac{t^q}{q} \|\psi\|_q^q \right) = \frac{q-2}{2q} \mu^{-\frac{2}{q-2}} \mathcal{C}_q^{\frac{q}{q-2}}.$$

Thus, by virtue of (2.4), we have $p < \frac{1}{3} \mathcal{S}^{\frac{3}{2}}$ and, in turn, $M < \frac{\sqrt[3]{6}}{2} \mathcal{S}$. $\square$

In the following, we will show that p can be achieved. This implies that the limit problem (2.3) admits a ground state solution. Similar to that in [9], it is enough to prove that M can be achieved. Now, we give the following Brezis–Lieb Lemma.

Lemma 2.5. *Let $h \in C(\mathbb{R}^3 \times \mathbb{R})$ and suppose that*

$$\lim_{t \to 0} \frac{h(x,t)}{t} = 0 \quad \text{and} \quad \limsup_{|t| \to \infty} \frac{|h(x,t)|}{|t|^5} < \infty, \tag{2.6}$$

uniformly in $x \in \mathbb{R}^3$. If $u_n \to u_0$ weakly in $H^1(\mathbb{R}^3)$ and $u_n \to u_0$ a.e. in $\mathbb{R}^3$, then

$$\int_{\mathbb{R}^3} (H(x,u_n) - H(x,u_n - u_0) - H(x,u_0)) \mathrm{d}x = o(1),$$

where $H(x,t) = \int_0^t h(x,s) \mathrm{d}s$.

Proof. The proof is standard. For the subcritical case, we refer to Coti Zelati and Rabinowitz [16]. For any fixed $\delta > 0$, set $\Omega_n(\delta) := \{x \in \mathbb{R}^3 : |u_n(x) - u_0(x)| \leq \delta\}$.

Then

$$\begin{aligned}
&\int_{\mathbb{R}^3}(H(x,u_n)-H(x,u_n-u_0)-H(x,u_0))\mathrm{d}x\\
&\quad=\int_{\mathbb{R}^3\setminus\Omega_n(\delta)}(H(x,u_n)-H(x,u_n-u_0)-H(x,u_0))\mathrm{d}x\\
&\qquad+\int_{\Omega_n(\delta)}(H(x,u_n)-H(x,u_0))\mathrm{d}x-\int_{\Omega_n(\delta)}H(x,u_n-u_0)\mathrm{d}x\\
&\quad:=J_1+J_2+J_3.
\end{aligned}$$

By conditions (2.6), for any $\rho>0$, there exists $C_\rho>0$ such that $|h(x,t)|\leq\rho|t|+C_\rho|t|^5$ for all $(x,t)\in\mathbb{R}\times\mathbb{R}^3$. Then

$$\begin{aligned}
|J_3|&\leq\int_{\Omega_n(\delta)}\left(\frac{\rho}{2}|u_n-u_0|^2+\frac{C_\rho}{6}|u_n-u_0|^6\right)\mathrm{d}x\\
&\leq\left(\frac{\rho}{2}+\frac{C_\rho}{6}\delta^4\right)\int_{\mathbb{R}^3}|u_n-u_0|^2\mathrm{d}x,
\end{aligned}$$

and

$$\begin{aligned}
|J_2|&\leq\int_{\Omega_n(\delta)}[\rho(|u_n|+|u_0|)+C_\rho(|u_n|+|u_0|)^5]|u_n-u_0|\mathrm{d}x\\
&\leq\rho\left(\int_{\mathbb{R}^3}(|u_n|+|u_0|)^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}|u_n-u_0|^2\mathrm{d}x\right)^{\frac{1}{2}}\\
&\quad+C_\rho\left(\int_{\mathbb{R}^3}(|u_n|+|u_0|)^6\mathrm{d}x\right)^{\frac{5}{6}}\left(\int_{\Omega_n(\delta)}|u_n-u_0|^6\mathrm{d}x\right)^{\frac{1}{6}}\\
&\leq\rho\left(\int_{\mathbb{R}^3}(|u_n|+|u_0|)^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}|u_n-u_0|^2\mathrm{d}x\right)^{\frac{1}{2}}\\
&\quad+C_\rho\delta^{\frac{2}{3}}\left(\int_{\mathbb{R}^3}(|u_n|+|u_0|)^6\mathrm{d}x\right)^{\frac{5}{6}}\left(\int_{\mathbb{R}^3}|u_n-u_0|^2\mathrm{d}x\right)^{\frac{1}{6}}.
\end{aligned}$$

Then, since $\{u_n\}_n$ is bounded in $H^1(\mathbb{R}^3)$, for every $\varepsilon>0$, there exist $\rho,\delta>0$ such that $|J_2|+|J_3|\leq\varepsilon/2$, for all $n\geq1$. On the other hand,

$$\begin{aligned}
J_1&=\int_{B_R(0)\setminus\Omega_n(\delta)}(H(x,u_n)-H(x,u_n-u_0)-H(x,u_0))\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3\setminus(\Omega_n(\delta)\cup B_R(0))}(H(x,u_n)-H(x,u_n-u_0)-H(x,u_0))\mathrm{d}x\\
&:=K_1+K_2,
\end{aligned}$$

where $B_R(0)=\{x\in\mathbb{R}^N:|x|<R\}, R>0$. Noting that

$$|K_2|\leq\int_{\mathbb{R}^3\setminus B_R(0)}[\rho(|u_n|+|u_0|)+C_\rho(|u_n|+|u_0|)^5]|u_0|\mathrm{d}x+\int_{\mathbb{R}^3\setminus B_R(0)}H(x,u_0)\mathrm{d}x,$$

there exists $R > 0$ with $|K_2| \leq \varepsilon/4$, for all $n \geq 1$. Recall that $u_n \to u_0$ a.e. in $\mathbb{R}^3$, then it follows from the Severini–Egoroff theorem that u_n converges to u_0 in measure in $B_R(0)$, which implies

$$\lim_{n\to 0} |B_R(0)\backslash\Omega_n(\delta)| = 0.$$

In turn $|K_1| \leq \varepsilon/4$ for n large. Then $|J_1| \leq \varepsilon/2$ for n large and the proof is complete. □

Proof of Proposition 2.2. The proof is similar to that of [33]. We may assume that there exists $\{u_n\} \subset H_r^1(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3} G(u_n)\mathrm{d}x = 1$ and $\int_{\mathbb{R}^3} |\nabla u_n|^2\mathrm{d}x \to 2M$, as $n \to \infty$. By (f_1) and (f_2), $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^3)$. Thus there is $u_0 \in H_r^1(\mathbb{R}^3)$ such that, up to a subsequence, $u_n \to u_0$ weakly in $H_r^1(\mathbb{R}^3)$. Then

$$\int_{\mathbb{R}^3} |\nabla u_n|^2\mathrm{d}x = \int_{\mathbb{R}^3} |\nabla u_0|^2\mathrm{d}x + \int_{\mathbb{R}^3} |\nabla u_n - \nabla u_0|^2\mathrm{d}x + o(1).$$

Moreover, by Lemma 2.5, we have

$$\int_{\mathbb{R}^3} G(u_n)\mathrm{d}x = \int_{\mathbb{R}^3} G(u_0)\mathrm{d}x + \int_{\mathbb{R}^3} G(u_n - u_0)\mathrm{d}x + o(1).$$

It is easy to know that $M = \inf\{T_0(u) : V(u) = 1, u \in H_r^1(\mathbb{R}^3)\}$. Moreover,

$$T_0(u_n) = T_0(v_n) + T_0(u_0) + o(1), \quad V(u_n) = V(v_n) + V(u_0) + o(1),$$

where $v_n = u_n - u_0$. Set $S_n = T_0(v_n)$, $S_0 = T_0(u_0)$, $V(v_n) = \lambda_n$, $V(u_0) = \lambda_0$, we have $\lambda_n = 1 - \lambda_0 + o(1)$ and $S_n = M - S_0 + o(1)$. To prove that u_0 is a minimizer of M, it suffices to prove $\lambda_0 = 1$, which implies $u_n \to u_0$ strongly in $H_r^1(\mathbb{R}^3)$. It is easy to see that

$$T_0(u) \geq M(V(u))^{1/3}, \tag{2.7}$$

for all $u \in H^1(\mathbb{R}^3)$ and $V(u) \geq 0$. As we can see in [33], $\lambda_0 \in [0, 1]$. If $\lambda_0 \in [0, 1)$, then $\lambda_n > 0$ for n large enough. By (2.7), we have that $S_0 \geq M(\lambda_0)^{1/3}$ and $S_n \geq M(\lambda_n)^{1/3}$. This implies

$$\begin{aligned} M &= \lim_{n\to\infty}(S_0 + S_n) \geq \lim_{n\to\infty} M((\lambda_0)^{1/3} + (\lambda_n)^{1/3}) \\ &= M((\lambda_0)^{1/3} + (1-\lambda_0)^{1/3}) \geq M(\lambda_0 + 1 - \lambda_0) = M, \end{aligned}$$

which implies that $\lambda_0 = 0$. So we get that $u_0 = 0$ and $\lim_{n\to\infty} S_n = M$. By (f_1) and (f_2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $F(s) \leq \frac{1}{4}s^2 + C_\varepsilon s^4 + (1+\varepsilon)s^6/6$ for $s \in \mathbb{R}$. Then

$$1 = \lim_{n\to\infty} \lambda_n \leq \frac{1+\varepsilon}{6} \limsup_{n\to\infty} \|v_n\|_6^6,$$

since $\|v_n\|_4 \to 0$, namely $\limsup_{n\to\infty} \|v_n\|_6^2 \geq \sqrt[3]{6}$. Thus

$$M = \frac{1}{2}\limsup_{n\to\infty} \|\nabla v_n\|_2^2 \geq \frac{\mathcal{S}}{2}\limsup_{n\to\infty} \|v_n\|_6^2 \geq \frac{\sqrt[3]{6}}{2}\mathcal{S},$$

which contradicts Lemma 2.4. Therefore, we conclude that $\lambda_0 = 1$. Therefore, $u_0 \in \mathcal{M}$ and $\int_{\mathbb{R}^3} |\nabla u_0|^2 \mathrm{d}x = 2M$. Setting $t_0 = \|\nabla u_0\|_2/\sqrt{6}$, it follows from Coleman–Glazer–Martin [15] that $\omega = u_0(\frac{\cdot}{t_0}) \in \mathcal{P}$ is a ground state solution to problem (2.3). □

Define as $\mathcal{S}_r$ the set of the *radial ground states* U of (2.3). Then $\omega \in \mathcal{S}_r$. Moreover, thanks to Lemma 2.5, similarly as that in [12, 22, 33], we have the following.

Proposition 2.6. (i) $b = I(\omega)$, *namely the Mountain Pass value agrees with the least energy level.*
(ii) $\mathcal{S}_r$ *is compact in* $H^1_r(\mathbb{R}^3)$.

Proof. (i) Obviously, by (f_1)–(f_3) we know that b is well defined. As we can see in the proof of Lemma 2.4 and Proposition 2.2, we get that $p \leq b$ and $p = I(\omega)$. To prove b is the least energy, it suffices to prove $b \leq I(\omega)$. Noting that ω is a ground state solution to (2.3), similar to that in [22], there exists a path $\gamma \in \Gamma$ satisfying $\gamma(0) = 0$, $I(\gamma(1)) < 0$, $\omega \in \gamma([0,1])$ and $\max_{t\in[0,1]} I(\gamma(t)) = I(\omega)$. Thus, $b \leq I(\omega)$.

(ii) We adopt some ideas in [12] to show the compactness of $\mathcal{S}_r$. Similar to that in [12], $\mathcal{S}_r$ is bounded in $H^1_r(\mathbb{R}^3)$. For any $\{u_n\} \subset \mathcal{S}_r$, without loss of generality, we can assume that $u_n \to u_0$ weakly in $H^1_r(\mathbb{R}^3)$ and $u_n \to u_0$ a.e. in $\mathbb{R}^3$. It follows from [11] that $v_n(\cdot) := u_n(\sqrt{M/3}\cdot)$ is a minimizer of $T_0(v)$ on $\{v \in H^1_r(\mathbb{R}^3) : V(v) = 1\}$. This means that $\{v_n\}$ is a positive and radially symmetric minimizing sequence of M. As we can see in the proof of Proposition 2.2, $v_n \to v_0 := u_0(\sqrt{M/3}\cdot)$ strongly in $H^1_r(\mathbb{R}^3)$. Thus, $u_n \to u_0$ strongly in $H^1_r(\mathbb{R}^3)$ and $u_0 \in \mathcal{S}_r$, i.e. $\mathcal{S}_r$ is compact. □

3. The Minimax Level

Let $U \in \mathcal{S}_r$ be arbitrary but fixed. By the Pohozăev identity, for $U_t(x) = U(\frac{x}{t})$ we have

$$I(U_t) = \left(\frac{t}{2} - \frac{t^3}{6}\right) \int_{\mathbb{R}^3} |\nabla U|^2 \mathrm{d}x.$$

Thus, there exists $t_0 > 1$ such that $I(U_t) < -2$ for $t \geq t_0$. Set

$$D_\lambda \equiv \max_{t\in[0,t_0]} \Gamma_\lambda(U_t).$$

Then, by virtue of Lemma 2.1, we get that $D_\lambda \to b$, as $\lambda \to 0$.

Moreover, it is easy to verify the following lemma, which is crucial to define the uniformly bounded set of the mountain passes as previously mentioned.

Lemma 3.1. *There exist* $\lambda_1 > 0$ *and* $\mathcal{C}_0 > 0$, *such that for any* $0 < \lambda < \lambda_1$ *there hold*

$$\Gamma_\lambda(U_{t_0}) < -2, \quad \|U_t\| \leq \mathcal{C}_0, \quad \forall\, t \in (0, t_0], \quad \|u\| \leq \mathcal{C}_0, \quad \forall\, u \in \mathcal{S}_r.$$

Proof. Due to the Pohozǎev identity, as we can see in [12], there exists $C > 0$ such that $\|u\| \le C$ for any $u \in \mathcal{S}_r$. For $U \in \mathcal{S}_r$ fixed above and $t \in (0, t_0]$,

$$\|U_t\|^2 = t\|\nabla U\|_2^2 + t^3\|U\|_2^2 \le (t + t^3)\|U\|^2 \le C^2(t_0 + t_0^3).$$

The second and last part of the assertion hold if $\mathcal{C}_0 = 2t_0^2 C$. For the first part, by Lemma 2.1

$$\Gamma_\lambda(U_{t_0}) \le I(U_{t_0}) + 4c\lambda^2\|U_{t_0}\|^4 \le I(U_{t_0}) + 4c\lambda^2\mathcal{C}_0^4.$$

It follows from $I(U_{t_0}) < -2$ that there exist $\lambda_1 > 0$ with $\Gamma_\lambda(U_{t_0}) < -2$ for any $0 < \lambda < \lambda_1$. The proof is completed. □

Now, for any $\lambda \in (0, \lambda_1)$, we define a min–max value C_λ:

$$C_\lambda = \inf_{\gamma \in \Upsilon_\lambda} \max_{s \in [0, t_0]} \Gamma_\lambda(\gamma(s)),$$

where

$$\Upsilon_\lambda = \{\gamma \in C([0, t_0], H_r^1(\mathbb{R}^3)) : \gamma(0) = 0, \gamma(t_0) = U_{t_0}, \|\gamma(t)\| \le \mathcal{C}_0 + 1, t \in [0, t_0]\}.$$

Obviously, $U_t \in \Upsilon_\lambda$. Moreover, $C_\lambda \le D_\lambda$ for $\lambda \in (0, \lambda_1)$.

Proposition 3.2. $\lim_{\lambda \to 0} C_\lambda = b$.

Proof. It suffices to prove that

$$\liminf_{\lambda \to 0} C_\lambda \ge b.$$

Noting that $\phi_u \ge 0$, we see that for any $\gamma \in \Upsilon_\lambda$, $\tilde{\gamma}(\cdot) = \gamma(t_0 \cdot) \in \Upsilon$. It follows that $C_\lambda \ge b$, concluding the proof. □

4. Proof of Theorem 1.1

Now for $\alpha, d > 0$, define

$$\Gamma_\lambda^\alpha := \{u \in H_r^1(\mathbb{R}^3) : \Gamma_\lambda(u) \le \alpha\}$$

and

$$\mathcal{S}^d = \Big\{u \in H_r^1(\mathbb{R}^3) : \inf_{v \in \mathcal{S}_r} \|u - v\| \le d\Big\}.$$

Obviously, $\mathcal{S}_r \subset \mathcal{S}^d$, i.e. $\mathcal{S}^d \ne \phi$ for all $d > 0$. For some $0 < d < 1$, we will find a solution $u \in \mathcal{S}^d$ of problem (2.2) for sufficiently small $\lambda > 0$. The following proposition is crucial to obtain a suitable (PS)-sequence for Γ_λ and plays a key role in our proof. Choose

$$0 < d < \min\left\{\frac{1}{3}\left[\frac{3}{2}\mathcal{S}^3\kappa^{-1}\right]^{\frac{1}{4}}, \sqrt{3b}\right\}, \tag{4.1}$$

where κ is given in (1.4).

Proposition 4.1. *Let* $\{\lambda_i\}_{i=1}^{\infty}$ *be such that* $\lim_{i\to\infty}\lambda_i = 0$ *and for all* i, $\{u_{\lambda_i}\} \subset \mathcal{S}^d$ *with*

$$\lim_{i\to\infty} \Gamma_{\lambda_i}(u_{\lambda_i}) \le b \quad \text{and} \quad \lim_{i\to\infty} \Gamma'_{\lambda_i}(u_{\lambda_i}) = 0.$$

Then for d *small enough, there is* $u_0 \in \mathcal{S}_r$, *up to a subsequence, such that* $u_{\lambda_i} \to u_0$ *in* $H_r^1(\mathbb{R}^3)$.

Proof. For convenience, we write λ for λ_i. Since $u_\lambda \in \mathcal{S}^d$, there exist $U_\lambda \in \mathcal{S}_r$ and $v_\lambda \in H^1(\mathbb{R}^3)$ such that $u_\lambda = U_\lambda + v_\lambda$ with $\|v_\lambda\| \le d$. Since $\mathcal{S}_r$ is compact, up to a subsequence, there exist $U_0 \in \mathcal{S}_r$ and $v_0 \in H^1(\mathbb{R}^3)$, such that $U_\lambda \to U_0$ strongly in $H^1(\mathbb{R}^3)$, $v_\lambda \to v_0$ weakly in $H^1(\mathbb{R}^3)$, $\|v_0\| \le d$ and $v_\lambda \to v_0$ a.e. in $\mathbb{R}^3$. Let $u_0 = U_0 + v_0$, then $u_0 \in \mathcal{S}^d$ and $u_\lambda \to u_0$ weakly in $H^1(\mathbb{R}^3)$. It follows from $\lim_{i\to\infty}\Gamma'_\lambda(u_\lambda) = 0$ that $I'(u_0) = 0$. Now, we show $u_0 \not\equiv 0$. Otherwise, if $u_0 \equiv 0$, then $\|U_0\| = \|v_0\| \le d$. By (4.1), $\|\nabla U_0\| < \sqrt{3b}$. On the other hand, by $U_0 \in \mathcal{S}_r$ and the Pohozaev's identity, $\|\nabla U_0\| = \sqrt{3b}$, which is a contradiction. So $u_0 \not\equiv 0$ and $I(u_0) \ge b$. Meanwhile, thanks to Lemma 2.5, $\Gamma_\lambda(u_\lambda) = I(u_0) + I(u_\lambda - u_0) + o(1)$, then we have $I(u_\lambda - u_0) \le o(1)$. Thus, by (1.4) and the Sobolev' embedding theorem, $\|u_\lambda - u_0\|^2 \le \frac{2}{3}\kappa\mathcal{S}^{-3}\|u_\lambda - u_0\|^6 + o(1)$. If $\|u_\lambda - u_0\| \not\to 0$ as $\lambda \to 0$, up to a subsequence, we can get that $\|u_\lambda - u_0\| \ge [\frac{3}{2}\mathcal{S}^3\kappa^{-1}]^{\frac{1}{4}}$ for λ small. This is a contradiction. Thus, $u_\lambda \to u_0$ strongly in $H_r^1(\mathbb{R}^3)$. The proof is completed. □

By Proposition 4.1, there exist

$$0 < d < \min\left\{1, \frac{1}{3}\left(\frac{3}{2}\mathcal{S}^3\kappa^{-1}\right)^{\frac{1}{4}}, \sqrt{3b}\right\}$$

and $\omega > 0$, $\lambda_0 > 0$ such that $\|\Gamma'_\lambda(u)\| \ge \omega$ for $u \in \Gamma_\lambda^{D_\lambda} \cap (\mathcal{S}^d \backslash \mathcal{S}^{\frac{d}{2}})$ and $\lambda \in (0, \lambda_0)$. Then, we have the following proposition.

Proposition 4.2. *There exists* $\alpha > 0$ *such that for small* $\lambda > 0$,

$$\Gamma_\lambda(\gamma(s)) \ge C_\lambda - \alpha \quad \text{implies that } \gamma(s) \in \mathcal{S}^{\frac{d}{2}},$$

where $\gamma(s) = U(\frac{\cdot}{s}), s \in (0, t_0]$.

Proof. From a change of variables and the Pohozǎev identity,

$$\Gamma_\lambda(\gamma(s)) = \left(\frac{s}{2} - \frac{s^3}{6}\right)\int_{\mathbb{R}^3} |\nabla U|^2 \mathrm{d}x + \frac{\lambda}{4}\int_{\mathbb{R}^3} \phi_{\gamma(s)}|\gamma(s)|^2 \mathrm{d}x.$$

It follows from Lemma 2.1 that $\Gamma_\lambda(\gamma(s)) = (\frac{s}{2} - \frac{s^3}{6})\int_{\mathbb{R}^3} |\nabla U|^2\mathrm{d}x + O(\lambda^2)$. Note that

$$\max_{s\in[0,t_0]} \left(\frac{s}{2} - \frac{s^3}{6}\right)\int_{\mathbb{R}^3} |\nabla U|^2 \mathrm{d}x = b$$

and $C_\lambda \to b$ as $\lambda \to 0$, the conclusion follows. □

The next proposition assures the existence of a bounded Palais–Smale sequence for Γ_λ. Choose small $\alpha_0 > 0$ satisfying

$$\alpha_0 \le \min\left\{\frac{\alpha}{2}, \frac{1}{9}d\omega^2\right\}. \tag{4.2}$$

Noting that $\lim_{\lambda\to 0} C_\lambda = \lim_{\lambda\to 0} D_\lambda = b$, without loss of generality, we can assume that $D_\lambda < C_\lambda + \alpha_0 \le 2C_\lambda$ for $\lambda > 0$ small enough.

Proposition 4.3. *For any $\lambda > 0$ small enough, there exists $\{u_n\}_n \subset \Gamma_\lambda^{D_\lambda} \cap \mathcal{S}^d$ such that $\Gamma'_\lambda(u_n) \to 0$ as $n \to \infty$.*

Proof. Assume by contradiction, there exists $a(\lambda) > 0$ such that $|\Gamma'_\lambda(u)| \ge a(\lambda)$, $u \in \mathcal{S}^d \cap \Gamma_\lambda^{D_\lambda}$ for some small $\lambda > 0$. Then there exists a pseudo-gradient vector field T_λ in $H_r^1(\mathbb{R}^3)$ on a neighborhood Z_λ of $\mathcal{S}^d \cap \Gamma_\lambda^{D_\lambda}$ (cf. [29]), such that for any $u \in Z_\lambda$ satisfies

$$\|T_\lambda(u)\| \le 2\min\{1, |\Gamma'_\lambda(u)|\},$$

$$\langle \Gamma'_\lambda(u), T_\lambda(u)\rangle \ge \min\{1, |\Gamma'_\lambda(u)|\}|\Gamma'_\lambda(u)|.$$

Let η_λ be a Lipschitz continuous function on $H_r^1(\mathbb{R}^3)$ such that $0 \le \eta_\lambda \le 1$, $\eta_\lambda(u) \equiv 1$ if $u \in \mathcal{S}^d \cap \Gamma_\lambda^{D_\lambda}$ and $\eta_\lambda(u) = 0$ if $u \in H_r^1(\mathbb{R}^3) \setminus Z_\lambda$. Let ξ_λ be a Lipschiz continuous function on $\mathbb{R}$ such that $0 \le \xi_\lambda \le 1$, $\xi_\lambda(t) \equiv 1$ for $|t - C_\lambda| \le \frac{\alpha}{2}$ and $\xi_\lambda(t) = 0$ for $|t - C_\lambda| \ge \alpha$. Let

$$e_\lambda(u) = \begin{cases} -\eta_\lambda(u)\xi_\lambda(\Gamma_\lambda(u))T_\lambda(u), & \text{if } u \in Z_\lambda, \\ 0, & \text{if } u \in H_r^1(\mathbb{R}^3)\backslash Z_\lambda, \end{cases}$$

then for any $u \in H_r^1(\mathbb{R}^3)$, the following initial value problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\Phi_\lambda(u,t) = e_\lambda(\Phi_\lambda(u,t)), \\ \Phi_\lambda(u,0) = u \end{cases}$$

exists a unique global solution $\Phi_\lambda : H_r^1(\mathbb{R}^3) \times [0,\infty) \to H_r^1(\mathbb{R}^3)$ which satisfies

(1) $\Phi_\lambda(u,t) = u$, if $t = 0$ or $u \notin Z_\lambda$ or $|\Gamma_\lambda(u) - C_\lambda| \ge \alpha$,
(2) $\|\frac{\mathrm{d}}{\mathrm{d}t}\Phi_\lambda(u,t)\| \le 2$, for all u, t,
(3) $\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_\lambda(\Phi_\lambda(u,t)) \le 0$, for all u, t.

With arguments similar as those in [14], for any $s \in [0, t_0]$, there exists $t_s \ge 0$ such that

$$\Phi_\lambda(\gamma(s), t_s) \in \Gamma_\lambda^{C_\lambda - \alpha_0},$$

where γ is given in Proposition 4.2 and α_0 is given in (4.2). Let

$$T_1(s) := \inf\{t \ge 0 : \Phi_\lambda(\gamma(s), t) \in \Gamma_\lambda^{C_\lambda - \alpha_0}\}$$

and $\gamma_0(s) = \Phi_\lambda(\gamma(s), T_1(s))$, then γ_0 is well defined in $[0, t_0]$ and there holds $\Gamma_\lambda(\gamma_0(s)) \le C_\lambda - \alpha_0$ for $s \in [0, t_0]$. With the similar arguments in [12, 14], we can get that $\gamma_0(s)$ is continuous in $[0, t_0]$ and $\|\gamma_0(s)\| \leq \mathcal{C}_0 + d < \mathcal{C}_0 + 1$. Thus, $\gamma_0 \in \Upsilon_\lambda$ with $\max_{t\in[0,t_0]} \Gamma_\lambda(\gamma_0(t)) \leq C_\lambda - \alpha_0$, which is in contradiction with the definition of C_λ. Therefore, the proof is completed. □

Proof of Theorem 1.1 Concluded. It follows from Proposition 4.3 that there exists $\lambda_0 > 0$ such that for $\lambda \in (0, \lambda_0)$, there exists $\{u_n\} \in \Gamma_\lambda^{D_\lambda} \cap \mathcal{S}^d$ with $\Gamma'_\lambda(u_n) \to 0$ as $n \to \infty$. Assume that $u_n \to u_\lambda$ weakly in $H^1(\mathbb{R}^3)$, then $\Gamma'_\lambda(u_\lambda) = 0$. By the compactness of $\mathcal{S}_r$, we get that $u_\lambda \in \mathcal{S}^d$ and $\|u_n - u_\lambda\| \leq 3d$ for n large. In light of Lemmas 2.1 and 2.5,

$$\Gamma_\lambda(u_n) = \Gamma_\lambda(u_\lambda) + \Gamma_\lambda(u_n - u_\lambda) + o(1).$$

By the choice of d, it is easy to verify that $\Gamma_\lambda(u_n - u_\lambda) \geq 0$ for large n. So, $\Gamma_\lambda(u_\lambda) \leq D_\lambda$. Then $u_\lambda \in \Gamma_\lambda^{D_\lambda} \cap \mathcal{S}^d$ with $\Gamma'_\lambda(u_\lambda) = 0$. On the other hand, it is easy to know that $0 \notin \mathcal{S}^d$ for small $d > 0$. Thus choosing $d > 0$ small enough, u_λ is a nontrivial solution of (1.1). In the following, we consider the asymptotic behavior of u_λ as $\lambda \to 0$. Observe that

$$\Gamma_\lambda(u_\lambda) = I(u_\lambda) + \frac{\lambda}{4}\int_{\mathbb{R}^3} \phi_{u_\lambda} u_\lambda^2 \mathrm{d}x,$$

and that, for any $\varphi \in C_0^\infty(\mathbb{R}^3)$,

$$\Gamma'_\lambda(u_\lambda)\varphi = I'(u_\lambda)\varphi + \lambda\int_{\mathbb{R}^3} \phi_{u_\lambda} u_\lambda \varphi \mathrm{d}x.$$

Note that $u_\lambda \in \mathcal{S}^d$. Then, by Lemma 2.1 we get that

$$I(u_\lambda) \leq D_\lambda \quad \text{and} \quad I'(u_\lambda) \to 0 \ \text{ as } \lambda \to 0.$$

Assume that $u_\lambda \to u$ weakly in $H_r^1(\mathbb{R}^3)$, then $I'(u) = 0$. Recall that $D_\lambda \to b$ as $\lambda \to 0$ and $b \in (0, \frac{1}{3}\mathcal{S}^{\frac{3}{2}})$. It is easy to verify that $u_\lambda \to u$ strongly in $H_r^1(\mathbb{R}^3)$ as $\lambda \to 0$ and $I(u) \leq b$. On the other hand, it follows from $u_\lambda \in \mathcal{S}^d$ that $u \in \mathcal{S}^d$. Obviously, $0 \notin \mathcal{S}^d$ for d small enough. Hence, choosing $d > 0$ small enough, $u \not\equiv 0$ and $I(u) \geq b$. Therefore, $I(u) = b$, namely, u is a least energy solution of problem (2.3). □

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